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On Convexity of A System of Linear Interval Equations

Yoshihiro TANAKA

In this paper, a necessary and sufficient condition for the solution set of a system of linear interval equations to be convex is given. It is also discussed the relation between the convexity and feasibility of an optimization problem. Finally, extended Leontief systems are proposed as an application of our results.

1. Introduction

We are concerned with the system

$$\tilde{A}\tilde{x} = \tilde{b} \qquad (\tilde{A} \in A,\ \tilde{b} \in b), \tag{1}$$

where A is a regular interval matrix (i.e., $\forall \tilde{A} \in A$ are nonsingular) and b is an interval vector, which originated from a practical situation where the coefficients and the right-hand vector cannot be sharply defined. However, a large part of researchers in this field aimed at evaluating the hull of the solution set of (1) efficiently [3], so there are few literatures deal with the solution set. Among them, Oettli [5] showed the intersection of the solution set with each orthant is a convex polytope. Recently Rohn [8] proposed a necessary and sufficient condition for the solution set to be nonconvex by utilizing solutions of the auxiliary equations.

In this paper, we propose a necessary and sufficient condition for the solution set to be convex under the condition that the coefficient interval matrix is inverse positive, which is usually assumed in considering solution methods to (1) [3]. The proposed condition can be easily examined under the inverse positiveness as contrasted with the Rohn's result. We also propose extended Leontief systems as an application of our results.

The terminology follows Neumaier [3]; we will summarize it briefly for convenience sake. Let $\mathbf{IR}$, $\mathbf{IR}^n$, and $\mathbf{IR}^{n\times n}$ be the set of real closed intervals, n-dimensional interval vectors and $n \times n$ interval matrices, respectively. For each $A = [\underline{A}, \overline{A}] \in \mathbf{IR}^{n\times n}$, we write $A_c := (\overline{A} + \underline{A})/2$ for the midpoint matrix, $\Delta := (\overline{A} - \underline{A})/2$ for the radius matrix, and $|A| := \sup\{|\tilde{A}| \mid \tilde{A} \in A\}$. Note that sum, difference and inverse for $A, B \in \mathbf{IR}^{n\times n}$ are defined by

$$A \pm B := \mathscr{H}\{\tilde{A} \pm \tilde{B} \,|\, A \in \tilde{A},\ \tilde{B} \in B\},$$

$$A^{-1} := \mathscr{H}\{\tilde{A}^{-1} \,|\, \tilde{A} \in A\},$$

where $\mathscr{H}$ denotes the interval hull (i.e., the smallest hyperrectangle which contains the given set). We call $A \in \mathbf{IR}^{n\times n}$ regular when each $\tilde{A} \in A$ is regular.

2. Results for Inverse Positive Matrices

Oettli and Prager [6] gave the following explicit expression for the solution set X of (1):

$$X = \{x \,|\, |A_c x - b_c| \leq \Delta |x| + \delta\}, \tag{2}$$

where $|x|_i = |x_i|$, $i = 1,\ldots,n$, from which it is shown that the intersection of X with each orthant is a convex polytope [5].

Let Y be the set of 2^n vectors such that $Y = \{y \in \mathbf{R}^n \,|\, |y_j| = 1, j = 1,\ldots,n\}$ and T_y be diag(y). Rohn [9] considers the following equation associated with (1):

$$A_c x - b_c = T_y (\Delta |x| + \delta), \tag{3}$$

which has a unique solution $x_y \in X$ for each $y \in Y$ constructing conv X in total. The main theorem in [8] is as follows.

Theorem 1 [8].

Let $A \in \mathbf{IR}^{n\times n}$ be regular. Then the solution set X of (1) is nonconvex if and only if there exist $y, z \in Y$, $i, j \in \{1,\ldots,n\}$ such that $y_i = z_i$, $(x_y)_j (x_z)_j < 0$, $\Delta_{ij} > 0$. □

By substituting $x = x^+ - x^-$, $|x| = x^+ + x^-$ ($x_i^+ = x_i$, $x_i^- = 0$ if $x_i \geq 0$, and $x_i^+ = 0$, $x_i^- = -x_i$ if $x_i < 0$) into (3) and setting $A_c - T_y\Delta = A_{ye}$, $A_c + T_y\Delta = A_{yf}$, $b_y = b_c + T_y\delta$, we get

$$x^+ = A_{ye}^{-1} A_{yf} x^- + A_{ye}^{-1} b_y. \tag{4}$$

Since $A_{ye}^{-1} A_{yf}$ is a *P-matrix*, i.e., all its principal minors are positive, each x_y uniquely exists and can be obtained by solving the above linear complementary problem (see [9]). Rohn's results are, however, rather conceptual because it is troublesome to solve (3) for all $y \in Y$.

Let us consider the case where the coefficient matrix A is inverse positive. This assumption is representatively satisfied when the coefficient matrix $A \in \mathbf{IR}^{n\times n}$ is an *M-matrix* in the sense of [3], or when the spectral condition $\rho(A_c^{-1}\Delta) < 1$ is satisfied and

$$A_c^{-1}\Delta\,(I - A_c^{-1}\Delta)^{-1}|A_c^{-1}| \leq |A_c^{-1}|$$

holds.

We begin with the following facts, which are special cases of convexness and nonconvexness of X, respectively.

Fact 1.

Let $A \in \boldsymbol{IR}^{n\times n}$ be such that $A^{-1} \geq 0$. Then the solution set X of (1) is convex if $b \in \boldsymbol{IR}^n$ satisfies $b \geq 0$ or $b \leq 0$.

Proof.

From the properties of inverse positive matrices, the solution set $X = \{x \,|\, x = \tilde{A}^{-1}\tilde{b},\ \tilde{A} \in A,\ \tilde{b} \in b\} \geq 0$ or $X \leq 0$. Since X is part of a single orthant, it is convex from [5]. □

Remark 1.

Consider (1) with

$$A = \begin{pmatrix} 1 & [-\frac{1}{2}, 0] \\ [-\frac{1}{2}, 0] & 1 \end{pmatrix} \quad \text{and } b = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Then A^{-1}($B \in A^{-1}$ may not be expressed by any $\tilde{A}^{-1}$, $\tilde{A} \in A$) is calculated as

$$A^{-1} = \begin{pmatrix} [1, \frac{4}{3}] & [0, \frac{2}{3}] \\ [0, \frac{2}{3}] & [1, \frac{4}{3}] \end{pmatrix},$$

and X is shown to be convex. In fact, X is a triangle whose vertexes are $(0, -2)$, $(1, -2)$, and $(1, -\frac{3}{2})$. However, for the same A and $b = (1, -3)^T$, X consists of two triangles whose vertexes are $(-\frac{1}{2}, -3)^T$, $(-\frac{1}{2}, -\frac{13}{4})^T$, $(0, -3)^T$, $(1, -\frac{5}{2})^T$, and $(1, -3)^T$. In this case, X is nonconvex.

On the other hand, the next fact holds with a slightly restrictive condition imposed on A.

Fact 2.

Let $A \in \boldsymbol{IR}^{n\times n}$ $(n \geq 2)$ be such that A is regular, $A^{-1} \geq 0$, and $\Delta > 0$. Then the solution set X is nonconvex if $0 \in int\ b$.

Proof.

It follows from [9, Theorem 4.7] that $\overline{x} = x_y$, $\underline{x} = x_{-y}$ for $y = e = (1,\dots,1)^T$. Since $\overline{x} \geq A^{-1}e > 0$, $\underline{x} \leq -\tilde{A}^{-1}e < 0$ for all $\tilde{A} \in A$, we have only to show $x_z \neq 0$ for

$z \neq e, -e$ due to Theorem 1. If $x_z = 0$, then $A_{ye}^{-1} b_y = 0$ by (4), which contradicts the regularness of A_{ye}. □

For the sake of simplicity, we suppose the following assumption.

Assumption (N):

For each $y \in Y$, the solution of (4) satisfies $x_i^+ > 0$ or $x_i^- > 0$.

The assumption asserts that the solution of (4) does not degenerate and is usually assumed.

Lemma 2.

Let $A \in \boldsymbol{IR}^{n \times n}$ $(n \geq 2)$ be such that A is regular, $A^{-1} \geq 0$, and $\Delta > 0$. Suppose also that Assumption (N) holds. Then the solution set X is convex if and only if $x_i^1 x_i^2 \geq 0$, $i = 1,\ldots,n$, for any x_i^1, $x_i^2 \in X$.

Proof.

Since the intersection of X with each orthant is a convex polytope (see [6]), sufficiency of the condition is trivial.

To prove necessity, suppose that there exist x^1, $x^2 \in X$ such that $x_j^1 x_j^2 < 0$. Then $\overline{x}_j \underline{x}_j < 0$ holds since $\overline{x}_j \geq \max\{x_j^1, x_j^2\}$ and $\underline{x}_j \leq \min\{x_j^1, x_j^2\}$. On the other hand, it follows $\overline{x} = x_y$, $\underline{x} = x_{-y}$ for $y = e$. Hence we obtain from Assumption (N) either $(x_z)_j (x_y)_j < 0$ or $(x_z)_j (x_{-y})_j < 0$, which implies nonconvexness of X by Theorem 1. □

Note that under the hypothesis of the lemma, $A^H b$, the exact hull of X can be determined by x_y and x_{-y}.

Now we can establish the following theorems.

Theorem 3.

Let $A \in \boldsymbol{IR}^{n \times n}$ $(n \geq 2)$ be such that A is regular, $A^{-1} \geq 0$, and $\Delta > 0$. Suppose also that Assumption (N) holds. Then the solution set X is convex if and only if $(x_y)_i (x_{-y})_i \geq 0$, $i = 1,\ldots,n$, for $y = e$.

Proof.

Directly from the proof of Lemma 2. □

Theorem 4.

In the case X is convex in Theorem 3 and the spectral condition $\rho(A_c^{-1}\Delta) < 1$

is satified, it holds

$$\overline{x} = (A_c^{-1} + A_c^{-1}\Delta\,(I - A_c^{-1}\Delta)^{-1}A_c^{-1})^{-1}\overline{b},$$

and

$$\underline{x} = (A_c^{-1} - A_c^{-1}\Delta\,(I - A_c^{-1}\Delta)^{-1}A_c^{-1})^{-1}\underline{b},$$

Proof.

Immediate from the formulas following (4.5) and Theorem 4.5 [9]. □

3. Discussion

The inverse positiveness of matrices is typically important for slightly extended Leontief systems

$$Ax + c = x, \tag{5}$$

where $A \in \mathbf{IR}^{n\times n}$ is an input interval matrix, $x \in \mathbf{IR}^n$ and $c \in \mathbf{IR}^n$ are, respectively, output intervals and demand intervals. We should remember that the ordinary Leontief system is one of the most important models of industrial interdependence known as input-output, and is the simplest form of Walrasian general equilibrium. (cf.[2]). Recently Leontief systems have played a more important role in connection to polyhedral combinatorics, logic and expert systems.

Theorem 5.

Let $A \in \boldsymbol{IR}^{n\times n}$, $x \in \boldsymbol{IR}^n$ and $c \in \boldsymbol{IR}^n$. The extended Leontief systems (5) has the solution set $X \geq 0$ for any $c \geq 0$ if and only if generalized Hawkins-Simon conditions $(I - A)^{-1} \geq 0$ hold.

Proof.

Immediate from the Hawkins-Simon conditions for the ordinary case $(\Delta\,(A) = O,\ \delta\,(c) = 0)$. □

Theorem 6 (Corollary of Theorem 1[7]).

Let $A \in \boldsymbol{IR}^{n\times n}$, $x \in \boldsymbol{IR}^n$ and $c \in \boldsymbol{IR}^n$. The extended Leontief systems (5) has the solution set $X \geq 0$ for $c = [\underline{c}, \overline{c}]$ if and only if all its extremal subsystems, i.e., its i-th equation is $((I - \overline{A})\,x)_i = \overline{c}_i$ or $((I - \underline{A})\,x)_i = \underline{c}_i$, have nonnegative solutions. □

Rohn [7] considered the problem

$$\max\,\{\tilde{c}^T x \,|\, \tilde{A}x = \tilde{b},\ x \geq 0\}, \tag{6}$$

where $\tilde{A} \in A$, $\tilde{b} \in b$, $\tilde{c} \in c$, and gives necessary and sufficient conditions for strong feasibility of it. Recall that we call an interval linear system $Ax = b$ *strongly feasible* when each subsystem is feasible. The following theorem can be easily applied to the extended Leontief systems (5).

Theorem 7.

Let $A \in \boldsymbol{IR}^{n \times n}$ $(n \geq 2)$ *be such that* A *is regular,* $A^{-1} \geq 0$, *and* $\Delta > 0$. *Suppose also that Assumption (N) holds and there exists an* $x > 0$ *in* X. *Then the interval linear system* $Ax = b$ *is strongly feasible if and only if the solution set* X *is convex.*

Proof.

Necessity of the condition is obvious by Theorem 6.

Sufficiency is proved by Lemma 2 since X is confined in the orthant $\{x \mid x > 0\}$ under the hypothesis. □

More recently Rohn [10] has considered the class of linear programming problem (6) with $A \in \mathbf{IR}^{m \times n}$ $(n \geq m)$ including *Leontief substitution models*. Let B be an m-tuple of integers from $\{1,\ldots,n\}$. He calls the problem (6) *B-stable* if each problem (6) has a nondegenerate basic optimal solution with basic variables x_j, $j \in B$ and calls it *strongly B-stable* if additionally, each such an optimal solution is unique. Let $Z_B = \{z \in \mathbf{B}^n \mid |z_j| = 1 \text{ for } j \in B,\ z_j = 1 \text{ for } j \notin B,\ j = 1,\ldots,n\}$, and define $A_{yz} = A_c - T_y \Delta T_z$ and $c_z = c_c + T_z \gamma$ $(c = [c_c - \gamma,\ c_c + \gamma])$.

Theorem 8[10].

The problem (6) is [strongly] B-stable if and only if for each $y \in Y$, $z \in Z_B$ *the LP problem*

$$\max\{c_z^T x \mid A_{yz} x = b_y,\ x \geq 0\} \tag{7}$$

has a [unique] nondegenerate basic optimal solution with basic variables x_j, $j \in B$. □

B-stability, which has a close connection with Assumption (N), can be verified in finite calculations from Theorem 8.

Finally, it is noted that convexity of X is not a necessary nor sufficient condition for $A^H b = A^G b$, where $A^H b = \mathscr{H}\{ A^{-1} b \mid \tilde{A} \in A\}$ and $A^G b$ denotes the hull by Gauss elimination [1], [3], i.e.,

$$A^G := U_A^F L_A^F,\ A^G b = U_A^F L_A^F b,$$

where forward substitution is defined by

$$x_i := \frac{y_i - L_{i1}x_1 - \cdot\cdot\cdot\cdot - L_{i,i-1}x_{i-1}}{L_{ii}},$$

and backward substitution is defined by

$$y_i := \frac{b_i - U_{in}y_n - \cdot\cdot\cdot\cdot - U_{i,i+1}y_{i+1}}{U_{ii}}.$$

Indeed, it is easily calculated that $A^H b = ([0, 1], [-2, -\frac{3}{2}])^T$, $A^G b = ([-\frac{1}{3}, 1], [-\frac{8}{3}, -\frac{3}{2}])^T$ for the example indicaded by Remark 1 with $b = (1, -2)^T$.

4. Conclusion

We have proposed an efficient condition for the solution set of linear interval equations to be convex under mild assumptions. In our framework, convexity of the solution set has been shown to be necessary and sufficient for feasibility of the optimization problem. We also have proposed extended Leontief systems as an application.

Associate Professor of Economics, Hokkaido University

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