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A NOTE ON THE HAWKINS-SIMON CONDITIONS*

M. KATO, G. MATSUMOTO, and T. SAKAI

The object of the present note is to provide an alternative proof of the theorem that the Hawkins-Simon conditions are necessary and sufficient for the static Leontief system to have positive solutions. Originally, the theorem is proved by Hawkins and Simon [2], and subsequently by Morishima [5], and Nikaido [6], [7] in a more general setting. Most of the books referring to the theorem, however, confine their remarks to the two-sector case for the convenience of a diagrammatic exposition. The following proof will be performed in the wake of the original version by Hawkins and Simon.

Let us consider the system of nonhomogeneous linear equations:

\[
\sum_{j=1}^{m} a_{ij} x_j = k_i \quad (i = 1, \ldots, m)
\]

with \(a_{ij} \leq 0\) for all \(i \neq j\). Then, the theorem to be proved is as follows:

**Theorem:** A necessary and sufficient condition that the \(x_i\) satisfying (1) be all positive for any positive \(k_i > 0\) is that all principal minors of the square matrix \(A = (a_{ij})\) are positive, i.e.,

\[
\begin{vmatrix}
  a_{11} & \cdots & a_{1s} \\
  \vdots & \ddots & \vdots \\
  a_{s1} & \cdots & a_{ss}
\end{vmatrix} > 0 \quad (s = 1, \ldots, m)
\]

**Proof.** We first prove that the Hawkins-Simon conditions (2) is necessary for the system (1) to have solutions \(x_i > 0\) for any \(k_i > 0\), and then that (2) is sufficient.

(Necessity) If the system (1) has positive solutions, the first equation of the system (1) can be rewritten as

\[
a_{11} = \frac{1}{x_1} (k_1 - \sum_{j=2}^{m} a_{1j} x_j) > 0.
\]

This implies that (2) holds when \(s = 1\).

By using the first equation of the system (1), \(x_1\) can be eliminated from the remaining equations. Thus, we obtain the subsystem of equations:

\[
\sum_{j=2}^{m} \left( a_{ij} - \frac{a_{11}}{a_{11}} a_{1j} \right) x_j = k_i - \frac{a_{11}}{a_{11}} k_1 \quad (i = 2, \ldots, m)
\]

*) The authors are indebted to Professor T. Shirai for suggestions which led them to write this note.

1) See, for example, Dorfman, Samuelson, and Sollow ([1], Chap. 9), Kuenne ([3], Chap. 6), and Morishima ([4], Chap. 2).
For simplicity, the system (4) is denoted by

\[ \sum_{j=2}^{m} a_{ij}^{(2)} x_j = k_i^{(1)} \quad (i = 2, \ldots, m) \]

In view of the conditions that \( a_{ij} \leq 0 \) for all \( i \neq j \), and \( a_{11} > 0 \), we conclude that

\[ a_{ij}^{(2)} \leq 0 \quad \text{for all} \quad i = j, \quad \text{and} \quad k_i^{(1)} > 0. \]

If the system (1) has positive solutions, the first equation of the subsystem (5) can be rewritten as

\[ a_{ij}^{(2)} = \frac{1}{k_i} \left( k_i^{(1)} - \sum_{j=2}^{m} a_{ij}^{(2)} x_j \right) > 0. \]

It is easily seen that this implies that the condition (2) holds for \( s=2 \), because

\[ a_{i1}^{(2)} = \frac{1}{a_{11}} \left| \begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right|. \]

Similarly, by using the first equation of the subsystem (5), \( x_2 \) (the first variable of the subsystem) can be eliminated from the remaining \((m-2)\) equations. Thus, we obtain the second subsystem of \((m-2)\) equations whose off-diagonal coefficients and nonhomogeneous terms have the same sign as those of the system (1) or (5). In general, by using the first equation of the \( s \)-th subsystem of \((m-s)\) equations, \( x_{s+1} \) (the first variable of the \( s \)-th subsystem) can be eliminated out of the remaining \((m-s-1)\) equations. Thus, we shall obtain the \((s+1)\)-th subsystem of \((m-s-1)\) equations whose off-diagonal coefficients and nonhomogeneous terms have the same sign as those of earlier subsystems.

If the system (1) has positive solutions, the first equation of the \((s-1)\)-th subsystem can be rewritten as

\[ a_{s1}^{(s-1)} = \frac{1}{x_s} \left( k_s^{(s-1)} - \sum_{j=s+1}^{m} a_{sj}^{(s-1)} x_j \right) > 0, \quad (s=2, \ldots, m) \]

The above procedure of elimination is equivalent to such operations that a multiple of the elements of one row of the coefficient matrix \( (a_{ij}) \) is added to the corresponding elements of other rows. According to the well-known property of determinants, this does not alter the values of the principal minors of \( (a_{ij}) \). Thus, we have

\[ \left| \begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1s} \\ a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ a_{s1} & a_{s2} & \cdots & a_{ss} \end{array} \right| = \left| \begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1s} \\ 0 & a_{21}^{(1)} & \cdots & a_{2s}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{ss}^{(s-1)} \end{array} \right| \]

\[ = a_{11} a_{22}^{(1)} a_{33}^{(2)} \cdots a_{s-1,s-1}^{(s-3)} a_{ss}^{(s-1)} \quad (s=2, \ldots, m) \]
This completes the proof of necessity.

\textit{(Sufficiency)} In view of (10), we have
\begin{equation}
\begin{align*}
(a_{11} > 0, \quad a_{22}^{(1)} > 0, \quad \ldots, a_{m-1,m-1}^{(m-2)} > 0, \quad \text{and} \quad a_{mm}^{(m-1)} > 0),
\end{align*}
\end{equation}
if all principal minors of the matrix \((a_{ij})\) are positive.

To begin with, let us consider the last \((i.e., \text{the (m-1)-th})\) subsystem:
\begin{equation}
\begin{align*}
a_{mm}^{(m-1)} x_m = k_m^{(m-1)},
\end{align*}
\end{equation}
where \(k_m^{(m-1)} > 0\).

By virtue of (11), we can divide both sides of the equation (12) by \(a_{mm}^{(m-1)}\) and obtain
\begin{equation}
\begin{align*}
x_m > 0.
\end{align*}
\end{equation}

In order to work out the proof by mathematical induction, suppose that we have already obtained solutions:
\begin{equation}
\begin{align*}
x_m > 0, \quad x_{m-1} > 0, \quad \ldots, x_{s+1} > 0.
\end{align*}
\end{equation}

Now, let us consider the first equation of the \((s-1)-th\) subsystem:
\begin{equation}
\begin{align*}
a_{ss}^{(s-1)} x_s = k_s^{(s-1)} - \sum_{j=s+1}^{m} a_{sj}^{(s-1)} x_j,
\end{align*}
\end{equation}
where the nonhomogeneous term \(k_s^{(s-1)}\) is positive and the off-diagonal coefficients \(a_{sj}^{(s-1)}\) are nonpositive. By (14), the right-hand-side of the equation is positive. By virtue of (11), we can divide both sides of the equation by \(a_{ss}^{(s-1)}\) and obtain
\begin{equation}
\begin{align*}
x_s > 0.
\end{align*}
\end{equation}

Thus the proof is complete, Q. E. D.

References
\begin{enumerate}
\item Morishima, M., \textit{Sangyo Renkan Ron Nyumon}, (Introduction to Input-Output Analysis) (Sobunsha, 1956).
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