# OPTIMAL CAPITAL ADJUSTMENT: AN ANALYSIS WHERE DELIVERY LAG ENTERS EXPLICITLY

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## INTRODUCTION

Much of the current interest in the adjustment cost models stems from their ability to provide a rigorous theoretical justification for the use of "flexible accelerator" and/or distributed lags in econometric studies of investment behavior.[1] It is well-known that this justification depends on the assumption that the cost of adjustment function is convex. Most authors insist that such cost is internal to the firm and arises in the form of foregone output for at least two reasons. First of all, the firm loses output by diverting resources from production activity to investment planning activity. Second, again in the form of lost output, there is a kind of installation cost which arises from the alteration of production lines or layout. The adjustment cost, therefore, occurs in the range of the internal technical relationship between the output and the internal adjustment of capital goods.

To my inference, however, most adjustment cost literature uses continuous time analysis on the assumption that it is possible for the firm to shorten the delivery lags into zero, if it would like to pay more delivery price, and therefore that the internal technical relationship and the adjustment cost function are always defined at zero delivery lag.

Although such an assumption is certainly important and convenient, it is necessary to examine a more general delivery price. Thalberg [12] and Maccini [6] present the models of optimal capital adjustment with delivery lag on the assumption that the firm faces a certain capital good market in which a kind of adjustment delay or backlog prevails, and that the delivery price of capital goods depends on that delay. In this paper, we shall make a similar approach in order to give light upon the important nature of capital as a fixed input. Further more we suppose that the

1) Some authors have contributed to the theoretical derivation of "flexible accelerator" within the framework of the adjustment costs. For example, Eisner and Strotz [1], Lucas [4] [5], Treadway [14] [15] [16] [17], and Gould [3].

internal technical relationship between the output and the internal adjustment of capital depends on the delivery lag as well as the level of investment planning activity. This is a natural extension of the internal adjustment cost function.[1)]

## I.

In the spirit of adjustment cost models, we postulate that the rate of output, $Q$, is governed by the following technical relationship:

$$Q = f(K, L) - C(\theta, I) \tag{1}$$

where $K$, $L$, $C$, $\theta$ and $I$ denote the total number of units of capital stock, the total number of units of labor, the internal adjustment cost measured by foregone output, the delivery lag, and the rate of investment order, respectively. In this paper, the delivery lag is defined as the period of time between the placement of an order for capital goods and the final delivery or installation of them. For convenience, we shall assume that the lag appearing in the technical relationship is the same as the adjustment delay prevailing in the capital good market under consideration, and that the lag is beyond its control and is expected to remain constant forever.

Our internal adjustment cost is separable from the production function in a large sense (or 'general production function' in Treadway's sense [15]). The production function in a narrow sense, $f(\cdot)$, posesses positive marginal products and is strictly concave, i. e.,

$$f_K, f_L > 0$$
$$f_{KK}, f_{LL} < 0 \quad f_{KK}f_{LL} - f_{LK}^2 > 0,$$

here we are assuming that the two inputs are complimentary:

$$f_{KL} = f_{LK} > 0.$$

We suppose the internal adjustment cost function takes the form:

$$C(\theta, I)/\theta = a(I/\theta),$$

that is to say, the cost function is homogeneous of degree one. Further more we suppose $a(\cdot)$ has the properties:

$$a(0) = 0, \quad a' > 0, \quad a'' > 0 \quad \text{for all} \quad I \geqq 0,$$

that is to say, the internal adjustment cost per delivery lag is strictly convex in $I/\theta$, the investment per delivery lag.

1) On another occasion, this author surveys the models of adjustment costs, in some critical manner. See Sakai [8]. One of interesting isseus pointed out by him will be discussed in this paper.

On the other hand, we suppose that the delivery price depends on the conditions, especially on the period of construction of the capital good, and that it is a decreasing function of that period, the delivery lag.[1] If the delivery price is independent of the rate of investment order, the firm can order as many capital goods as it wants at the given delivery price, and can install them after $\theta$ period.

Denoting the delivery price by $q(\theta)$, we define the total adjustment cost function[2]

$$(2) \qquad G(\theta, I) = a(I/\theta)\theta + q(\theta) I = g(I/\theta)\theta$$

where $g(\cdot)$ has the strict convexity

$$g(0) = 0, \quad g' > 0, \quad g'' > 0 \quad \text{for all} \quad I \geqq 0.$$

Through this paper, we shall assume that the firm can employ as many labor inputs as it wants at the given wage rate, and that it can borrow or lend in a perfect financial market at the given interest rate, $r$. All the variables are measured in terms of products of the firm, or in other words, the product price of it is normalized into one. Further more we shall exclude uncertainty and any dynamic expectations about future.

It is apparent that the labor input must be chosen to satisfy:

$$(3) \qquad f_L(K(t), L(t)) = w \qquad t \geqq 0$$

where $w$ denotes the real wage rate. Of course, the second order condition is

$$(4) \qquad f_{LL}(K(t), L(t)) < 0 \qquad t \geqq 0.$$

The implicit function theorem thus implies the existence of a short-run labor demand function:

$$(4) \qquad \hat{L}(K(t), w).$$

Now we define a new composite function:

$$(5) \qquad F(K(t), w) = f(K(t), \hat{L}(K(t), w) - w\hat{L}(K(t), w).$$

It is apparent that

$$(6\text{–a}) \qquad F_K = f_K > 0$$

1) Thalberg [12] rationalized this assumption. He says, "…the shorter period of construction, the more expensive is the production of a certain amount of capital, in the sense that more man-hours are required". (pp. 100) See also Thalberg [13] and Maccini [6].

2) $G(\cdot)$ has the properties
$G_I = g' = a' + q > 0, \quad G_{II} = g''/\theta = a''/\theta > 0,$
$G_\theta = g - g'I/\theta = a - a'I/\theta + q' < 0, \quad G_{\theta\theta} = g''I^2/\theta^3 = a''I^2/\theta^3 + q'' > 0.$

(6–b) $$F_w = -\hat{L} < 0$$

(6–c) $$F_{KK} = [f_{KK}f_{LL} - f_{KL}^2]/f_{LL} < 0$$

(6–d) $$F_{Kw} = f_{KL}/f_{LL} = F_{wK} < 0$$

(6–e) $$F_{ww} = -1/f_{LL} > 0\,.$$

## II.

We suppose that the firm's objective is the maximization of the present value of net cash flow. This problem can now be formulated, to choose $I(t)$ for $t \geqq 0$, in order to maximize:

(7) $$V = \int_0^\infty e^{-rt}\Big[F(K(t), w) - g\,(I(t)/\theta)\,\theta\Big]dt,$$

subject to

(8) $$K(t) = K_0(t) \qquad 0 \leqq t < \theta,$$

and

(9) $$\dot{K}(t) = I(t-\theta) \qquad t \geqq \theta\,.$$

That is to say, the firm formulates its investment plan at time zero, but it will not receive deliveries of capital goods until time $\theta$. So the initial conditions include the firm's capital stock from zero to time $\theta$. On the other hand, the performance equation of capital goods is defined after time $\theta$. Hence we must obtain the solutions for optimal control systems having time delays in the control variable.[1)]

To solve this problem, we form the Hamiltonian[2)]

(10) $$\begin{aligned} H = {} & e^{-rt}\Big[F(K(t), w) - g\,(I(t)/\theta)\,\theta + z(t)\,I(t-\theta)\Big] \\ & + e^{-r(t+\theta)}\Big[F(K(t+\theta), w) - g\,(I(t+\theta)/\theta)\,\theta + z(t+\theta)\,I(t)\Big]\,, \end{aligned}$$

wher $z(t)$ is the imputed value of investment.

The optimality conditions are:

(11) $$z(t+\theta) = g'\,(I(t)/\theta)\,e^{r\theta}$$

1) Soliman [11] presents a necessary condition for optimality of control system having time delays in the control vector. For simplicity, we consider a control problem with a constant time delay corresponding to the single capital good case. However, the result can be easily extended to systems with variable time delays and general N "quasi-fixed" inputs. The latter case of N "quasi-fixed" input is examined by Treadway [17] and Mortensen [7], without allowing the delivery lags.

2) See Soliman, op. cit. El-Hodiri and others [2] discuss the allocation of output among consumption and two types of capital investment with different *gestation periods*. Their mathematical technique is identical with ours. While Maccini presents the Hamiltonian expression unlike (10), it is easier to use one like (10) to calculate more complex problems. See Maccini, op. cit. (pp. 273)

(12) $$\dot{z}(t+\theta)=rz(t+\theta)-F_K(K(t+\theta),w)$$

(13) $$\dot{K}(t+\theta)=I(t)$$

(14) $$\lim_{t\to\infty} z(t)\,e^{-rt}=0$$

(15) $$K(t)=K_0(t) \qquad 0\leqq t<\theta.$$

Condition (11) states that the imputed value of investment must be equated to its marginal cost. The marginal cost of investment, $g'$, is the sum of the marginal internal adjustment cost, $a'$, and the delivery price, $q(\theta)$, where the compound interest factor, $r\theta$, appears to take into account the fact that the marginal cost of investment is incurred at the time of ordering whereas the imputed value of investment is being evaluated at the time of delivery. Integrating (12) directly and taking account of (14), we obtain

(16) $$z(t+\theta)=\int_{t+\theta}^{\infty}F_K(K(s),w)\,e^{-r(s-(t+\theta))}\,ds,$$

i.e., the imputed value of investment, and consequently the marginal cost of investment, must be positive and set equal to the present value of future marginal products of capital. Condition (14) is a transversality condition and it states that the imputed value of investment must approach zero as time recedes indefinitely. Finally, condition (13) and (15) are repetition of (9) and (8).

Eliminating $z$ and $\dot{z}$, the model reduces to two differential equations in $I$ and $K$:

(17) $$\dot{I}(t)=\Big[rg'(I(t)/\theta)-e^{-r\theta}F_K(K(t+\theta),w)\Big]\theta/g''(I(t)/\theta)$$

(13) $$\dot{K}(t+\theta)=I(t)$$

together with (14) and (15). But a simple substitution enables us to reduce the above two differential equations to a standard second order differential equation which contains no time notation:

(18) $$\ddot{K}=\Big[rg'(\dot{K}/\theta)-e^{-r\theta}F_K(K,w)\Big]\theta/g''(\dot{K}/\theta).$$

The stationary point $(K^*,0)$ in the $(K,\dot{K})$-plane is defined by

(19) $$F_K(K^*,w)=re^{r\theta}g'(0/\theta)$$

with

(20–a) $$\partial K^*/\partial\theta=re^{r\theta}\Big[q'(\theta)+rg'(0/\theta)\Big]\Big/F_{KK}(K^*,w)\gtreqless 0$$
$$\text{as}\quad q'(\theta)+rg'(0/\theta)\lesseqgtr 0$$

(20–b) $$\partial K^*/\partial r=(1+r\theta)e^{r\theta}g'(0/\theta)/F_{KK}(K^*,w)<0$$

(20–c) $$\partial K^*/\partial w=-F_{Kw}(K^*,w)/F_{KK}(K^*,w)<0.$$

A linear approximation of (18) in a neighborhood of $(K^*,0)$ yields:

(21) $$\ddot{K} - r\dot{K} + M^*[K - K^*] = 0$$

where

$$M^* = \theta F_{KK}(K^*, w)/e^{r\theta} g''(0/\theta) < 0.$$

So the differential equation (21) has the roots

(22) $$\alpha = r/2 \pm \left[(r/2)^2 - M^*\right]^{1/2}.$$

These roots are both real and of opposite signs, indicating a saddle point configuration in the neighborhood of $(K^*, 0)$. Thus there exists one path that approaches the equilibrium point asymptotically as shown by the heavy arrows in Figure 1. The motion of the linearized system is expressed by the well-known "flexible accelerator":

(23) $$\dot{K} = \alpha[K - K^*] \qquad \alpha < 0.$$

This is the optimal capital adjustment mechanism we are anxious to derive. This adjustment mechanism can be characterized by a *distributed lag* with the relative weight on the equilibrium stock of capital, $K^*$, increasing with time:

(24) $$K(t) = (1 - e^{\alpha t})K^* + e^{\alpha t} K_0(\theta).$$

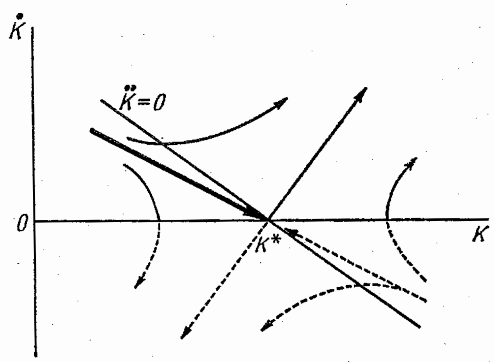


**Figure 1.**

## III.

In order to illuminate the effects of a change in the delivery lag on the rate of capital adjustment, we shall see, first of all, the effect on the $(\ddot{K} = 0)$ singular curve. The singular curve being defined by

(25) $$F_K(K, w) = re^{r\theta} g'(\dot{K}/\theta),$$

we can see that the slope of the singular curve is negative at least in a neighborhood of $(K^*, 0)$:

(26) $$\emptyset(K;\, \theta, r, w) = \partial\dot{K}/\partial K\Big|_{\ddot{K}=0} = \theta F_{KK}(K, w)/re^{r\theta} g'(\dot{K}/\theta) < 0.$$

If the slope $\emptyset(K; \theta, r, w)$ is approximately equal to $\emptyset(K^*; \theta, r, w)$, we might insist that

$$(27) \qquad \partial\emptyset(K; \theta, r, w)/\partial\theta = \emptyset_{K^*}(K^*; \theta, r, w)\,\partial K^*/\partial\theta + \emptyset_\theta(K^*; \theta, r, w).$$

However, we can not obtain the information of the effect on the singular curve of a change in the delivery lag, because we have no information about the third derivative of $F(\cdot)$. If $\emptyset_{K^*}=0$, the above comparative dynamic analysis yields a stronger result *via* simple calculation. In this case, we have

$$(28) \qquad \begin{aligned} \partial\emptyset(K; \theta, r, w)/\partial\theta &= \emptyset_\theta(K^*; \theta, r, w) \\ &= F_{KK}(K^*, w)\Big[1-r\theta-\theta q'(\theta)/g'(0/\theta)\Big]\Big/re^{r\theta}g'(0/\theta). \end{aligned}$$

But the sign of the terms in parenthesis is positive or negative, depending on the elasticity of the marginal adjustment cost with respect to $\theta$:

$$\begin{aligned} &1-r\theta-\theta q'(\theta)/g'(0/\theta) \\ =\,&1-\Big[q'(\theta)+rg'(0/\theta)\Big]\theta/g'(0/\theta) \\ =\,&1-r\theta+E^* \quad \text{where} \quad E^* = -(\partial g'/\partial\theta)(\theta/g'). \end{aligned}$$

Hence the result is

$$(29) \qquad \begin{aligned} &\emptyset_\theta = F_{KK}(K^*, w)\Big[1-r\theta+E^*\Big]\Big/re^{r\theta}g'(0/\theta) \lesseqgtr 0 \\ &\text{as} \quad 1-r\theta+E^* \gtreqless 0. \end{aligned}$$

There remains one more channel through which a change in the delivery lag has an effect on the rate of adjustment. From the result of (20–a), it is apparent that[1)]

$$(30) \qquad \partial K^*/\partial\theta \gtreqless 0 \quad \text{as} \quad E^*-r\theta \gtreqless 0$$

Now we conclude the analysis by noticing that an increase in the delivery lag will make the investment spread over a comparatively longer stretch of time if it reduces the marginal adjustment cost in full degree *via* reduction in the delivery price of capital goods. Or in other words, the firm's rate of investment (or of planned deliveries of capital goods) changes in the opposite direction as the change in the equilibrium stock of capital for small $t$. This is summarized in (31):

$$(31) \qquad I(t-\theta) = \alpha\Big[K(t)-K^*\Big] \qquad \alpha<0$$

with

$$\left.\begin{aligned} &\partial K^*/\partial\theta > 0 \\ &\partial\alpha/\partial\theta < 0 \end{aligned}\right. \qquad \text{for large } E^*$$

1) $q'+rg' = -(E^*-r\theta)\,g'/\theta$

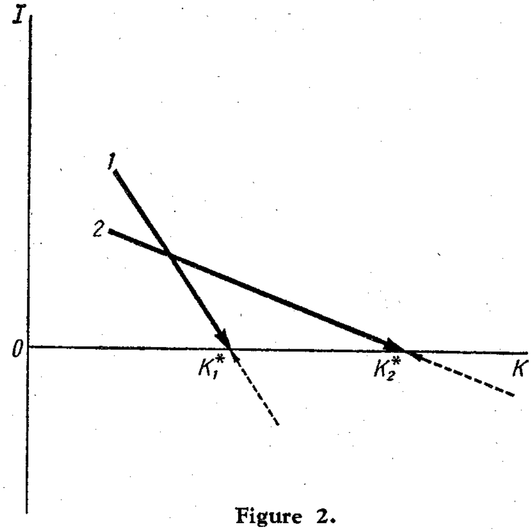


**Figure 2.**

and is depicted in Figure 2. The path 1 corresponds to the old equilibrium point $(K_1^*, 0)$, and has the steeper slope. The path 2, on the other hand, comes to appear as a result of a change in the delivery lag. The capital stock on this path will approach $(K_2^*, 0)$. While we are not interested in some results of changes in other parameters, it is easy to deduce some corollaries common to the models of adjustment cost by employing similar analytical techniques.

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