SEPARATED EFFECTS OF A COST FUNCTION AND A PRIOR DISTRIBUTION ON A SINGLE SAMPLING ATTRIBUTE PLAN

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1. INTRODUCTION

A discussion of the Bayesian sampling plan has resulted in claiming its advantages of including economic costs and prior information formally in its solution. Inclusion of them allows us to find an optimal sampling plan corresponding to the minimum expected total cost.

We are confronted, however, with a question in explicitly specifying $f_0(p)$ (a prior distribution for a lot fraction defective) and $l(n, x, p)$ (a cost function for decision on a lot). In the practical application of this plan, a decision maker may be interested in determining whether or not reductions in the sample size and/or the expected total cost can offset his effort to do research into and estimate parameters of $f_0(p)$ and $l(n, x, p)$.

This article aims to describe how to look into the separated effects of $f_0(p)$ and $l(n, x, p)$ which have different characteristics from each other for a decision maker. Section 2 introduces equations which determine a sampling plan (called model 2) without $l(n, x, p)$ using a beta distribution as $f_0(p)$ in its relation with a non-Bayesian sampling plan (called model 1). Then, section 3 deals with a sampling plan (called Model 3) for $l(n, x, p)$ and $f_0(p)$. Finally, section 4 sets forth two approaches to the above aim. The one is a standard sensitivity analysis using model 3. The second one proposed is that changes in the sampling size and the expected total cost between models 1 and 2 and between models 2 and 3 are looked on as the effects of $f_0(p)$ and $l(n, x, p)$ respectively.

2. A SINGLE SAMPLING ATTRIBUTE PLAN WITHOUT A COST FUNCTION FOR A BETA PRIOR DISTRIBUTION

First, consider JIS (Japanese Industrial Standard) Z9002 as an example of the non-Bayesian sampling plan. The sample size $n$ and the acceptance number $c$ are determined by solving the following equations:

\begin{equation}
\alpha = \sum_{x+c+1}^{n} \binom{n}{x} f_0^c (1-f_0)^{n-x},
\end{equation}

[1] proposes a new prior distribution for the sampling attribute plan.
where $\alpha$ is Producer's Risk, $\beta$ is Consumer's Risk, $p_0$ is Producer's Risk Point, $p_i$ is Consumer's Risk Point, $p_0 < p_i$, and $x$ is the number of defectives found in the sample. We now have model 1 that consists of $\alpha$, $\beta$, $p_0$ and $p_i$.

Next, suppose that a decision maker can use a beta distribution $f_\beta(p|k_1, k_2)$ as $f_\alpha(p)$ for the lot fraction defective $p$ in addition to information to determine $\alpha$, $\beta$. Then, $f_\alpha(p)$ is given by

\begin{equation}
\beta = \sum_{x=0}^{n} \binom{n}{x} p_0^x (1-p_0)^{n-x},
\end{equation}

where $0 < p < 1$, $0 < k_1 < k_2$ and the beta function $B(k_1, k_2)$ is given by

\begin{equation}
B(k_1, k_2) = \int_0^1 t^{k_1-1} (1-t)^{k_2-1} \, dt.
\end{equation}

The decision maker will decide whether or not to accept a lot based on a posterior distribution for $p$ after observing $x$. From Bayes' theorem, a posterior distribution $f_\beta(p)$ is given by

\begin{equation}
f_\beta(p) = f_\alpha(p) g(x|n, p) / \int_0^1 f_\alpha(p) g(x|n, p) \, dp = p^{k_1+x-1} (1-p)^{k_2+n-x-1} / B(k_1+x, k_2+n-x) = f_\beta(p|k_1+x, k_2+n-x),
\end{equation}

where $g(x|n, p)$ is the binomial distribution and given by

\begin{equation}
g(x|n, p) = \binom{n}{x} p^x (1-p)^{n-x}.
\end{equation}

If the decision maker desires the probability of rejecting good lots under $p_0$ to be at most $\alpha$ and that of accepting bad lots above $p_i$ to be at most $\beta$ on a posterior distribution, he will solve the following (7) and (8) with respect to $n$ and $x$ to obtain a satisfactory sampling plan for him:

\begin{equation}
\int_{p_0}^{p_*} f_\beta(p|k_1+x, k_2+n-x) = \alpha,
\end{equation}

\begin{equation}
\int_{p_i}^{1} f_\beta(p|k_1+x, k_2+n-x) = 1 - \int_{p_0}^{p_*} f_\beta(p|k_1+x, k_2+n-x) = \beta,
\end{equation}

where for a specific value, $\int_0^{p'} f_\beta(p|\alpha, \beta)$ is called the incomplete beta func-

\footnote{We go along with the same idea as [5] which designs the sampling variables plan for a normal prior distribution.}
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tion ratio, whose values are given by [9], so that we can find \( n \) and \( c \), given \( \alpha, \beta, p_0, p_1, k_1, \) and \( k_2 \). But in order to compare (7) and (8) with (1) and (2), we use the relation between the partial sum of a binominal expansion and the incomplete beta function ratio\(^6\) which is given by

\[
\sum_{x=0}^{m} g(x|n, p') = 1 - \int_0^{p'} f_{\alpha}(p|m+1, n-m) \quad \text{for any } m(\leq n).
\]

Then, we rewrite (7) and (8) as below:

\[
\int_0^{p_1} f_1(p) = \sum_{x=0}^{\epsilon+k_1-1} \left( n+k_1+k_2-1 \right) p_0(1-p_0)^{n+k_1+k_2-1-x} = \alpha,
\]

\[
\int_{p_1}^{1} f_1(p) = 1 - \sum_{x=0}^{\epsilon+k_1-1} \left( n+k_1+k_2-1 \right) p_0(1-p_0)^{n+k_1+k_2-1-x} = \beta,
\]

where such \( x \) above the summation notation that satisfies (7) and (8) is replaced by \( c \). We propose here model 2 that consists of \( \alpha, \beta, p_0, p_1 \) and \( f_0(p) \).

Comparison of (10) and (11) with (1) and (2) shows us that if \( \alpha=\beta \), a prior distribution reduces the sample size by \( k_1+k_2-1 \) and the acceptance number by \( k_1-1 \). In general \( (\alpha=\beta) \), \( n \) and \( c \) are reduced by \( k_1+k_2-1 \) and \( k_1-1 \) respectively from the sampling plan having exchanged values of \( \alpha \) and \( \beta \) in the original equations (1) and (2). As mentioned above, we can find \( n \) and \( c \) in them in the general case of \( \alpha \) and \( \beta \) by means of [4]. The equations having exchanged values of \( \alpha \) and \( \beta \) in (1) and (2) give almost the same value of \( n \) as the value of \( n \) in the original equations and a little smaller value of \( c \) than the value of \( c \) in the original equations. So we may approximately consider the effect of \( l(n, x, p) \) in the case of \( \alpha=\beta \). In any case, the larger are the parameters of \( f_0(p) \), \( k_1 \) and \( k_2 \), the larger is a reduction in the sample size.

3. A SINGLE ATTRIBUTE SAMPLING PLAN BASED ON A COST FUNCTION AND A BETA PRIOR DISTRIBUTION

In this section we take \( l(n, x, p) \) into the model in addition to \( f_0(p) \).

\[
l(n, x, p) = ap(N-n)+rx+ns \quad \text{for } x \leq c,
\]

\[
l(n, x, p) = rpN+Ns \quad \text{for } x > c,
\]

where \( N \) is the lot size. We use three parameters having following interpretation\(^3\):

- \( a: \) cost of a defective item accepted,
- \( r: \) cost repairing a defective item found in sampling and testing,
- \( s: \) cost of sampling and testing.

\(^3\) See [8].

\(^4\) We use the same cost function as [7].
If the consumer represents the next production process, the cost parameter \( a \) may consist of costs of rework or costs of assembling, dis-assembling and so on. If items are finished goods, the cost parameter \( a \) may include even losses of good will in the markets adding to repairing cost or replaced costs. What happens to rejected lots and to defective items in either accepted or rejected lots are represented below:

![Diagram]

The general form of such a cost function is given by [2].

First, we try to find \( c \), given \( n \) based upon [3]. As we consider \( x=np \) on the average, by equating (12) with (13) and solving them for \( p \), we determine the break even value of \( p \) which is denoted by \( p_0 \).

\[
(14) \quad p_0 = \frac{s}{(a-r)}.
\]

We still assume that a decision maker can use a prior distribution \( f_0(p) \). From (5) the average of the posterior distribution \( E_1(p) \) is given by

\[
(15) \quad E_1(p) = \frac{(k_1+x)}{(k_1+k_2+n)}.
\]

If (14) \( \geq \) (15), the decision maker accepts a lot at a cost of (12), otherwise he rejects a lot at that of (13). Therefore, \( c \) and \( p_0 \) must satisfy the following inequation:

\[
(16) \quad \frac{(k_1+c)}{(k_1+k_2+n)} \leq p_0 < \frac{(k_1+c+1)}{(k_1+k_2+n)}.
\]

By solving (16) for \( c \), we get

\[
(17) \quad (k_1+k_2+n) p_0 -(k_1+1) < c \leq (k_1+k_2+n) p_0 - k_1.
\]

Next, we must find the optimal sample size \( n^* \). While the asymptotic formular by [2] allows us to obtain the approximate value of \( n^* \), we use a computer to evaluate ETC (the expected total cost) and find \( n^* \) corresponding to the minimum ETC. For given \( n \) and \( c \), we define ETC, given by

\[5\] [10] uses this asymptotic formula to get the approximate value of \( n^* \) for a beta prior distribution and a cost function.

\[6\] ETC is computed in Hokudai University Computing Center.
\[
\sum_{x=0}^{s} g(x) \left\{ ap(n-n) + rx + ns \right\} f_i(p) dp + \sum_{x=s+1}^{n} g(x) \left\{ rpN + Ns \right\} f_i(p) dp
\]

\[
= \sum_{x=0}^{s} g(x) \left\{ aE_i(p)(N-n) + rx \right\} + \sum_{x=s+1}^{n} g(x) \left\{ rE_i(p)N + (N-n)s \right\} + ns,
\]

where \( g(x) \) is the marginal distribution of \( x \), given by

\[
g(x) = \int_{p} f_i(p) g(x|n, p) dp.
\]

We have now model 3 that consists of \( f_i(p) \) and \( l(n, x, p) \).


In this section we consider the separated effects of \( f_i(p) \) and \( l(n, x, p) \) using two different approaches. They are shown by some numerical examples at \( N=100 \).

THE FIRST APPROACH USING THE STANDARD SENSITIVITY ANALYSIS

First, we fix the parameters of \( l(n, x, p) \) at \( a=36, \ R=10 \) and \( S=5 \) (The unit may be yen). And the parameters of \( f_i(p) \) are changed from \( (k_1=1, k_2=7) \) to \( (k_1=1, k_2=9) \). On the other side, for the fixed parameters of \( f_i(p) \), \( k_1=1 \) and \( k_2=8 \), the parameters of \( l(n, x, p) \) are changed from \( (a=34, r=10, S=5) \) to \( (a=38, r=10, S=5) \). Each case is given by the case number specified in Tables 1 and 2.

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Table 2</th>
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<tr>
<td>( a=36, r=10, S=5 )</td>
<td>( k_1=1, k_2=8 )</td>
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<td>( k_1=1, k_2=7 )</td>
<td>Case 1</td>
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<td>( k_1=1, k_2=8 )</td>
<td>Case 2</td>
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<td>( k_1=1, k_2=9 )</td>
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Figures 2 and 3 show the changes of ETC due to the effects of the parameters of \( f_i(p) \) and \( l(n, x, p) \) respectively. For each case in Tables 3 and 4 show the optimal sampling plan \( (n^*, c^*) \) and the corresponding minimum ETC.

The change in \( E_i(p) \) from case 1 to case 3 is 0.0250 \( (1/8 - 1/10) \), while the change in \( p_i \) from case 4 to case 8 is 0.0297 \( (5/(34-10) - 5/(38-10)) \). Though the former has a little smaller change than the latter, we read.
Fig. 2.

Case 1

Case 2

Case 3

Fig. 3.

Case 5

Case 6

Case 7

Case 8
that the effect of the former on ETC and the optimal sampling plan is fairly large. So, under the situation considered here, the effect of $f_0(p)$ on the optimal sampling plan is relatively large as compared with that of $l(n, x, p)$.

**THE SECOND APPROACH PROPOSED**

Next, we take the second approach to see the effects of $f_0(p)$ and $l(n, x, p)$ by means of models 1, 2 and 3 not like the first approach using only model 3. Model 3 includes both $f_0(p)$ and $l(n, x, p)$; Model 2 includes only $f_0(p)$; Model 1 includes neither $f_0(p)$ nor $l(n, x, p)$. The one that operates on the optimal sampling plan through $p_b$ is $l(n, x, p)$. So we interpret that $p_b$ and $p_1$, which are included in models 1 and 2, are used by the decision maker as the substitute for $p_b$, since he cannot explicitly specify $l(n, x, p)$ for any reason. But he may specify $l(n, x, p)$, consequently determine $p_b$ at his expense to do research into it. Therefore, we regard models 1 and 2 as those which give a sub-optimal solution. If the decision maker determines $p_0$ and $p_1$ between which $p_b$ is put, his determination on $p_b$ and $p_1$ is considered appropriate.

Now, we consider the situation described by case 2 ($a=36, r=10, S=5, k_1=1$ and $k_2=8$) to show the numerical examples of the second approach. Model 3 gives the optimal solution ($n^*=35$, $c^*=7$, ETC=3731) as previously shown. Under this situation the decision maker, using models 1 or 2, may determine $p_0=0.10$ and $p_1=0.25$. They are appropriate because $p_0 < p_b < p_1$. Or he may determine $p_0=0.02$ and $p_1=0.12$ which are not appropriate because $p_0 < p_b < p_1$.

Suppose $a=0.05$ and $\beta=0.10$ that are the general case of JISZ9002. For the appropriate case model 1 gives $n=49$ and $c=8$ (exactly $\alpha=0.05187, \beta=0.10457$). Since model 1 gives $n=48$ and $c=7$ in (1) and (2) in which $\alpha=0.10$ and $\beta=0.05$ (exactly $\alpha=0.10207, \beta=0.06114$), model 2 gives $n=40$ ($=48-8-1+1$) and $c=7$ ($=7-1+1$). For the inappropriate case model 1 gives $n=41$ and $c=2$ (exactly $\alpha=0.04857, \beta=0.11562$). Model 2 gives $n=33$ and $c=1$ because model 1 having $\alpha=0.10$ and $\beta=0.05$ gives $n=41, c=1$.

The solution that model 3 gives is denoted by $M_0$. The solutions by
models 1 and 2 for the appropriate \( p_0 \) and \( p_1 \) are denoted by \( M_1 \) and \( M_2 \) and those for the inappropriate \( p_0 \) and \( p_1 \) by \( M'_1 \) and \( M'_2 \) respectively. Those solutions are characterized by the sub-optimal or optimal sample size and the corresponding ETC, whereby we represent them in Figs. 4 and 5.

A vertical difference between \( M'_1 \) and \( M_1 \) is regarded as a loss due to the inappropriate \( p_0 \) and \( p_1 \) in model 1. Reductions in ETC (20) and \( n \) (9) between \( M_1 \) and \( M_2 \) are due to \( f_0(p) \). Those (13 and 5 respectively) between \( M_2 \) and \( M_3 \) are due to \( l(n, x, p) \). We call the former "information effect" and the latter "cost effect".

5. SUMMARY AND CONCLUSION

Using the sampling attribute plan without a cost function for a prior distribution, which is proposed in section 2, we see the following;

(1) Prior information can reduce the sample size, while, if Producer's Risk Point and Consumer's Risk Point are appropriate, it also reduces ETC.

(2) If we take a beta distribution \( f_0(p|k_1, k_2) \) as a prior distribution it reduces the sampling size by about \( k_1 + k_2 - 1 \) (exactly \( k_1 + k_2 - 1 \) in the case
of $a = \beta$.

By putting this sampling plan between the non-Bayesian sampling plan
and the sampling plan based on both $f_0(p)$ and $l(n, x, p)$, we see the separated
effects of $f_0(p)$ and $l(n, x, p)$. They have different characteristics for a de-
cision maker. For instance, though he has much information about $p$, it
may be difficult for him to specify $l(n, x, p)$. In fact, if the items are final
goods, the parameter $a$ includes a loss of goodwill which may be fairly
large and not easy to estimate.

It is important to decompose a reduction in the sampling size and ETC
into the effects of $f_0(p)$ and $l(n, x, p)$ for the decision maker who is interested
in a degree of the specification in the sampling model. It is because, while
specifying parameters in the model lessons ETC, it requires costs and efforts
to do research into and make an estimate of them. To see the separated
effects of $f_0(p)$ and $l(n, x, c)$ is useful in view of the above reason. The
second approach to see them seems to be more useful because the practical
situation implies the decision maker’s choice among some models which
reflect a degree of the specification.

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