SEPARATED EFFECTS OF A COST FUNCTION AND A PRIOR DISTRIBUTION ON A SINGLE SAMPLING ATTRIBUTE PLAN

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1. INTRODUCTION

A discussion of the Bayesian sampling plan has resulted in claiming its advantages of including economic costs and prior information formally in its solution. Inclusion of them allows us to find an optimal sampling plan corresponding to the minimum expected total cost.

We are confronted, however, with a question in explicitly specifying of $f_0(p)$ (a prior distribution for a lot fraction defective) and $l(n, x, p)$ (a cost function for decision on a lot)[1]. In the practical application of this plan, a decision maker may be interested in determining whether or not reductions in the sample size and/or the expected total cost can offset his effort to do research into and estimate parameters of $f_0(p)$ and $l(n, x, p)$

This article aims to describe how to look into the separated effects of $f_0(p)$ and $l(n, x, p)$ which have different characteristics from each other for a decision maker. Section 2 introduces equations which determine a sampling plan (called model 2) without $l(n, x, p)$ using a beta distribution as $f_0(p)$ in its relation with a non-Bayesian sampling plan (called model 1). Then, section 3 deals with a sampling plan (called Model 3) for l(n, x, p) and $f_0(p)$. Finally, section 4 sets forth two approaches to the above aim. The one is a standard sensitivity analysis using model 3. The second one proposed is that changes in the sampling size and the expected total cost between models 1 and 2 and between models 2 and 3 are looked on as the effects of $f_0(p)$ and $l(n, x, p)$ respectively.

2. A SINGLE SAMPLING ATTRIBUTE PLAN WITHOUT A COST FUNCTION FOR A BETA PRIOR DISTRIBUTION

First, consider JIS (Japanese Industrial Standard) Z9002 as an example of the non-Bayesian sampling plan. The sample size n and the acceptance number c are determined by solving the following equations:

$$(1) \qquad \alpha = \sum_{x=c+1}^{n} \binom{n}{x} p_0^x (1-p_0)^{n-x},$$

[1] [1] proposes a new prior distribution for the sampling attribute plan.

$$(2) \qquad \beta=\sum_{x=0}^{c}\binom{n}{x}p_1^x(1-p_1)^{n-x},$$

where α is Producer's Risk, β is Consumer's Risk, p_0 is Producer's Risk Point, p_1 is Consumer's Risk Point, $p_0<p_1$, and x is the number of defectives found in the sample. We now have model 1 that consists of α, β, p_0 and p_1.

Next, suppose that a decision maker can use a beta distribution $f_p(p|k_1, k_2)$ as $f_0(p)$ for the lot fraction defective p in addition to information to determine α, β. Then, $f_0(p)$ is given by

$$(3) \qquad \begin{aligned} f_0(p) &= f_\beta(p|k_1, k_2) \\ &= p^{k_1-1}(1-p)^{k_2-1}/B(k_1, k_2), \end{aligned}$$

where $0<p<1$, $0<k_1<k_2$ and the beta function $B(k_1, k_2)$ is given by

$$(4) \qquad B(k_1, k_2)=\int_0^1 p^{k_1-1}(1-p)^{k_2-1}dp.$$

The decision maker will decide whether or not to accept a lot based on a posterior distribution for p after observing x. From Bayes' theorem, a posterior distribution $f_1(p)$ is given by

$$(5) \qquad \begin{aligned} f_1(p) &= f_0(p)\, g(x|n, p)\Big/\int_0^1 f_0(p)\, g(x|n, p)\, dp \\ &= p^{k_1+x-1}(1-p)^{k_2+n-x-1}/B(k_1+x, k_2+n-x) \\ &= f_\beta(p|k_1+x, k_2+n-x), \end{aligned}$$

where $g(x|p)$ is the binominal distribution and given by

$$(6) \qquad g(x|n, p)=\binom{n}{x}p^x(1-p)^{n-x}.$$

If the decision maker desires the probability of rejecting good lots under p_0 to be at most α and that of accepting bad lots above p_1 to be at most β on a posterior distribution, he will solve the following (7) and (8) with respect to n and x to obtain a satisfactory sampling plan for him :[2)]

$$(7) \qquad \int_0^{p_0} f_1(p)=\int_0^{p_0} f_\beta(p|k_1+x, k_2+n-x)=\alpha,$$

$$(8) \qquad \begin{aligned} \int_{p_1}^1 f_1(p) &= \int_{p_1}^1 f_\beta(p|k_1+x, k_2+n-x) \\ &= 1-\int_0^{p_1} f_\beta(p|k_1+x, k_2+n-x)=\beta, \end{aligned}$$

where for a specific value, $p'\int_0^{p'} f_\beta(p|\circ, \circ)$ is called the incomplete beta func-

2) We go along with the same idea as [5] which designs the sampling variables plan for a normal prior distribution.

tion ratio, whose values are given by [9], so that we can find n and c, given α, β, p_0, p_1, k_1, and k_2. But in order to compare (7) and (8) with (1) and (2), we use the relation between the partial sum of a binominal expansion and the incomplete beta function ratio[3] which is given by

$$(9)\qquad \sum_{x=0}^{m} g(x|n, p') = 1 - \int_0^{p'} f_\beta(p|m+1, n-m) \qquad \text{for any } m(\leqq n)\,.$$

Then, we rewrite (7) and (8) as below:

$$(10)\qquad \int_0^{p_0} f_1(p) = \sum_{x=0}^{c+k_1-1} \binom{n+k_1+k_2-1}{x} p_0^x (1-p_0)^{n+k_1+k_2-1-x} = \alpha\,,$$

$$(11)\qquad \int_{p_1}^{1} f_1(p) = 1 - \sum_{x=0}^{c+k_1-1} \binom{n+k_1+k_2-1}{x} p_1^x (1-p_1)^{n+k_1+k_2-1-x} = \beta\,,$$

where such x above the summation notation that satisfies (7) and (8) is replaced by c. We propose here model 2 that consists of α, β, p_0, p_1 and $f_0(p)$.

Comparison of (10) and (11) with (1) and (2) shows us that if $\alpha=\beta$, a prior distribution reduces the sample size by k_1+k_2-1 and the acceptance number by k_1-1. In general ($\alpha\neq\beta$), n and c are reduced by k_1+k_2-1 and k_1-1 respectively from the sampling plan having exchanged values of α and β in the original equations (1) and (2). As mentioned above, we can find n and c in them in the general case of α and β by means of [4]. The equations having exchanged values of α and β in (1) and (2) give almost the same value of n as the value of n in the original equations and a little smaller value of c than the value of c in the original equations. So we may approximately consider the effect of $l(n, x, p)$ in the case of $\alpha=\beta$. In any case, the larger are the parameters of $f_0(p)$, k_1 and k_2, the larger is a reduction in the sample size.

3. A SINGLE ATTRIBUTE SAMPLING PLAN BASED ON A COST FUNCTION AND A BETA PRIOR DISTRIBUTION

In this section we take $l(n, x, p)$ into the model in addition to $f_0(p)$.

$$(12)\qquad l(n, x, p) = ap(N-n) + rx + ns \qquad \text{for } x \leqq c\,,$$

$$(13)\qquad = rpN + Ns \qquad \text{for } x > c\,,$$

where N is the lot size. We use three parameters having following interpretation[4]:

- a: cost of a defective item accepted,
- r: cost repairing a defective item found in sampling and testing,
- s: cost of sampling and testing.

3) See [8].

4) We use the same cost function as [7].

If the consumer represents the next production process, the cost parameter a may consist of costs of rework or costs of assembling, dis-assembling and so on. If items are finished goods, the cost parameter a may include even losses of good will in the markets adding to repairing cost or replaced costs. What happens to rejected lots and to defective items in either accepted or rejected lots are represented below:

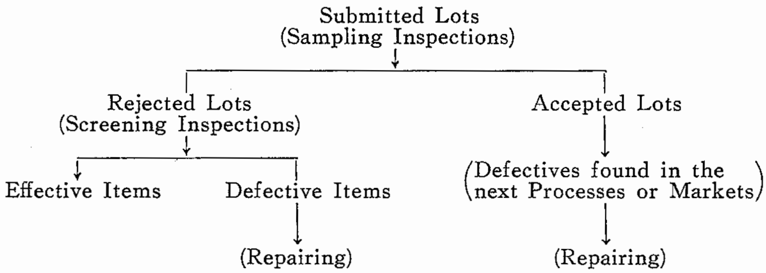


Fig. 1.

The general form of such a cost function is given by [2].

First, we try to find c, given n based upon [3]. As we consider $x=np$ on the average, by equating (12) with (13) and solving them for p, we determine the break even value of p which is denoted by p_b.

$$(14)\qquad p_b = s/(a-r)\,.$$

We still assume that a decision maker can use a prior distribution $f_0(p)$. From (5) the average of the posterior distribution $E_1(p)$ is given by

$$(15)\qquad E_1(p) = (k_1+x)/(k_1+k_2+n)\,.$$

If (14)$\geqq$(15), the decision maker accepts a lot at a cost of (12), otherwise he rejects a lot at that of (13). Therefore, c and p_b must satisfy the following inequation:

$$(16)\qquad (k_1+c)/(k_1+k_2+n) \leqq p_b < (k_1+c+1)/(k_1+k_2+n)\,.$$

By solving (16) for c, we get

$$(17)\qquad (k_1+k_2+n)\,p_b-(k_1+1) < c \leqq (k_1+k_2+n)\,p_b-k_1\,.$$

Next, we must find the optimal sample size n^*. While the asymtotic formular by [2][5] allows us to obtain the approximate value of n^*, we use a computer[6] to evaluate ETC (the expected total cost) and find n^* corresponding to the minimum ETC. For given n and c, we define ETC, given by

5) [10] uses this asymtotic formula to get the approximate value of n^* for a beta prior distribution and a cost function.

6) ETC is computed in Hokudai University Computing Center.

$$\begin{aligned}(18)\qquad &\sum_{x=0}^{c} g(x)\int_0^1 \left\{ap(N-n)+rx+ns\right\} f_1(p)\,dp + \sum_{x=c+1}^{n} g(x)\int_0^1 (rpN+Ns) f_1(p)\,dp \\ &= \sum_{x=0}^{c} g(x)\left\{aE_1(p)(N-n)+rx\right\} + \sum_{x=c+1}^{n} g(x)\left\{rE_1(p)N+(N-n)s\right\} + ns,\end{aligned}$$

where $g(x)$ is the marginal distribution of x, given by

$$(19)\qquad g(x)=\int_0^1 f_0(p)\, g(x|n,p)\,dp\,.$$

We have now model 3 that consists of $f_0(p)$ and $l(n, x, p)$.

4. SEPARATING THE CHANGES OF THE SAMPLE SIZE AND THE EXPECTED TOTAL COST INTO THE EFFECTS OF A COST FUNCTION AND A PRIOR DISTRIBUTION RESPECTIVELY

In this section we consider the separated effects of $f_0(p)$ and $l(n, x, p)$ using two different approaches. They are shown by some numerical examples at $N=100$.

THE FIRST APPROACH USING THE STANDARD SENSITIBITY ANALYSIS

First, we fix the parameters of $l(n, x, p)$ at $a=36$, $r=10$ and $S=5$ (The unit may be yen). And the parameters of $f_0(p)$ are changed from ($k_1=1$, $k_2=7$) to ($k_1=1$, $k_2=9$). On the other side, for the fixed parameters of $f_0(p)$, $k_1=1$ and $k_2=8$, the parameters of $l(n, x, p)$ are changed from ($a=34$, $r=10$, $S=5$) to ($a=38$, $r=10$, $S=5$). Each case is given by the case number specified in Tables 1 and 2.

TABLE 1

$a=36$, $r=10$, $S=5$

$k_1=1$, $k_2=7$	Case 1
$k_1=1$, $k_2=8$	Case 2
$k_1=1$, $k_2=9$	Case 3

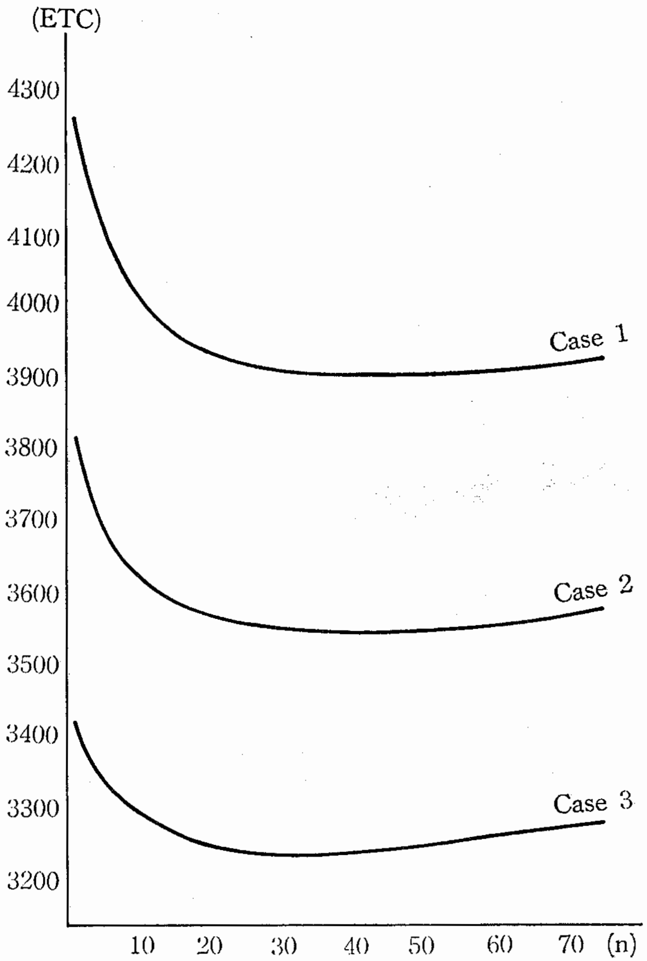
TABLE 2

$k_1=1$, $k_2=8$

$a=34$, $r=10$, $s=5$	Case 4
$a=35$, $r=10$, $s=5$	Case 5
$a=36$, $r=10$, $s=5$	Case 6 (Case 2)
$a=37$, $r=10$, $s=5$	Case 7
$a=38$, $r=10$, $s=5$	Case 8

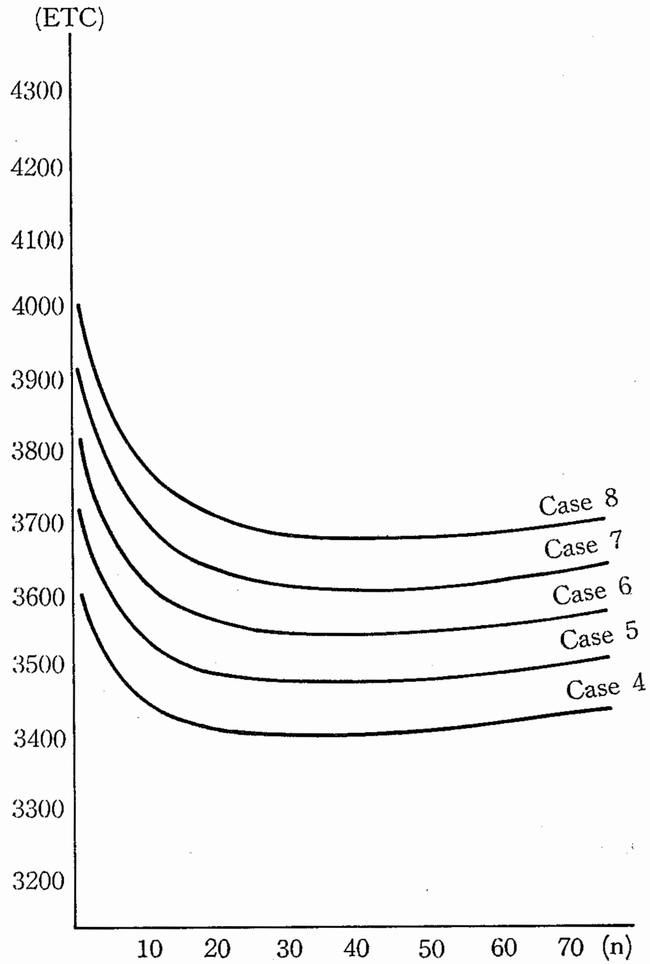
Figures 2 and 3 show the changes of ETC due to the effects of the parameters of $f_0(p)$ and $l(n, x, p)$ respectively. For each case in Tables 3 and 4 show the optimal sampling plan (n^*, c^*) and the corresponding minimum ETC.

The change in $E_0(p)$ from case 1 to case 3 is 0.0250 (1/8−1/10), while the change in p_b from case 4 to case 8 is 0.0297 (5/(34−10)−5/(38−10)). Though the former has a little smaller change than the latter, we read



(ETC)
4300
4200
4100
4000
3900
3800
3700
3600
3500
3400
3300
3200
Case 1
Case 2
Case 3
10
20
30
40
50
60
70
(n)

Fig. 2.



(ETC)
4300
4200
4100
4000
3900
3800
3700
3600
3500
3400
3300
3200
Case 8
Case 7
Case 6
Case 5
Case 4
10
20
30
40
50
60
70
(n)

Fig. 3.

TABLE 3

	min ETC (n^*, c^*)
Case 1	4066 (41, 8)
Case 2	3731 (35, 7)
Case 3	3443 (29, 6)

TABLE 4

	min ETC (n^*, c^*)
Case 4	3596 (31, 7)
Case 5	3665 (33, 7)
Case 6	3731 (35, 7)
Case 7	3795 (37, 7)
Case 8	3855 (38, 7)

that the effect of the former on ETC and the optimal sampling plan is fairly large. So, under the situation considered here, the effect of $f_0(p)$ on the optimal sampling plan is relatively large as compared with that of $l(n, x, p)$.

THE SECOND APPROACH PROPOSED

Next, we take the second approach to see the effects of $f_0(p)$ and $l(n, x, p)$ by means of models 1, 2 and 3 not like the first approach using only model 3. Model 3 includes both $f_0(p)$ and $l(n, x, p)$; Model 2 includes only $f_0(p)$; Model 1 includes neither $f_0(p)$ nor $l(n, x, p)$. The one that operates on the optimal sampling plan through p_b is $l(n, x, p)$. So we interpret that p_0 and p_1, which are included in models 1 and 2, are used by the decision maker as the substitute for p_b, since he cannot explicitly specify $l(n, x, p)$ for any reason. But he may specify $l(n, x, p)$, consequently determine p_b at his expense to do research into it. Therefore, we regard models 1 and 2 as those which give a sub-optimal solution. If the decision maker determines p_0 and p_1 between which p_b is put, his determination on p_0 and p_1 is considered appropriate.

Now, we consider the situation described by case 2 ($a=36$, $r=10$, $S=5$, $k_1=1$ and $k_2=8$) to show the numerical examples of the second approach. Model 3 gives the optimal solution ($n^*=35$, $c^*=7$, ETC$=3731$) as previously shown. Under this situation the decision maker, using models 1 or 2, may determine $p_0=0.10$ and $p_1=0.25$. They are appropriate because $p_0<p_b<p_1$. Or he may determine $p_0=0.02$ and $p_1=0.12$ which are not appropriate because $p_0<p_1<p_b$.

Suppose $\alpha=0.05$ and $\beta=0.10$ that are the general case of JISZ9002. For the appropriate case model 1 gives $n=49$ and $c=8$ (exactly $\alpha=0.05187$, $\beta=0.10457$). Since model 1 gives $n=48$ and $c=7$ in (1) and (2) in which $\alpha=0.10$ and $\beta=0.05$ (exactly $\alpha=0.10207$, $\beta=0.06114$), model 2 gives $n=40$ $(=48-\ 8-1+1)$ and $c=7$ $(=7-1+1)$. For the inappropriate case model 1 gives $n=41$ and $c=2$ (exactly $\alpha=0.04857$, $\beta=0.11562$). Model 2 gives $n=33$ and $c=1$ because model 1 having $\alpha=0.10$ and $\beta=0.05$ gives $n=41$, $c=1$.

The solution that model 3 gives is denoted by M_3. The solutions by

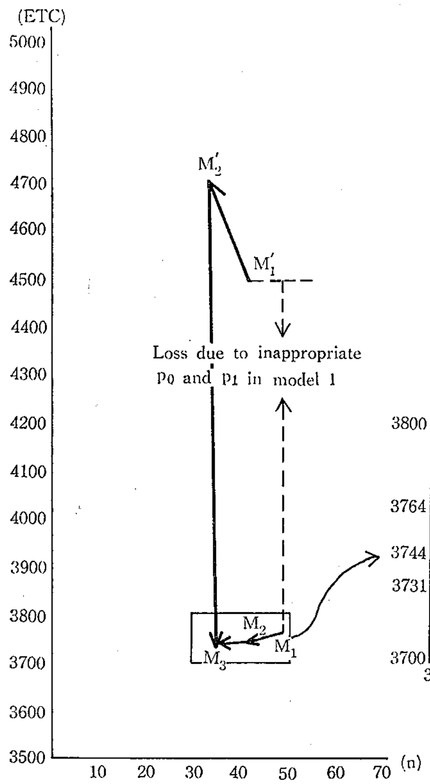


Fig. 4.

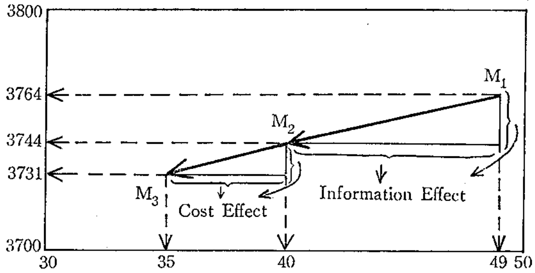


Fig. 5.

models 1 and 2 for the appropriate p_0 and p_1 are denoted by M_1 and M_2 and those for the inappropriate p_0 and p_1 by M_1' and M_2' respectively. Those solutions are characterized by the sub-optimal or optimal sample size and the corresponding ETC, whereby we represent them in Figs. 4 and 5.

A vertical difference between M_1' and M_1 is regarded as a loss due to the inappropriate p_0 and p_1 in model 1. Reductions in ETC (20) and n (9) between M_1 and M_2 are due to $f_0(p)$. Those (13 and 5 respectively) between M_2 and M_3 are due to $l(n, x, p)$. We call the former "information effect" and the latter "cost effect".

5. SUMMARY AND CONCLUSION

Using the sampling attribute plan without a cost function for a prior distribution, which is proposed in section 2, we see the following;

(1) Prior information can reduce the sample size, while, if Producer's Risk Point and Consumer's Risk Point are appropriate, it also reduces ETC.

(2) If we take a beta distribution $f_\beta(p|k_1, k_2)$ as a prior distribution it reduces the sampling size by about k_1+k_2-1 (exactly k_1+k_2-1 in the case

of $\alpha=\beta$).

By putting this sampling plan between the non-Bayesian sampling plan and the sampling plan based on both $f_0(p)$ and $l(n, x, p)$, we see the separated effects of $f_0(p)$ and $l(n, x, p)$. They have different characteristics for a decision maker. For instance, though he has much information about p, it may be difficult for him to specify $l(n, x, p)$. In fact, if the items are final goods, the parameter a includes a loss of good will which may be fairly large and not easy to estimate.

It is important to decompose a reduction in the sampling size and ETC into the effects of $f_0(p)$ and $l(n, x, p)$ for the decision maker who is interested in a degree of the specification in the sampling model. It is because, while specifying parameters in the model lessons ETC, it requires costs and efforts to do research into and make an estimate of them. To see the separated effects of $f_0(p)$ and $l(n, x, c)$ is useful in view of the above reason. The second approach to see them seems to be more useful because the practical situation implies the decision maker's choice among some models which reflect a degree of the specification.

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