A SIMPLE MODEL OF THE BUSINESS CYCLES IN THE EXPANSION PHASE

by

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In this paper I try to see under what conditions a downturn may be brought about by the emergence of excess capacity as a result of overinvestment. For this purpose I have constructed a model on an assumption of the acceleration principle subject to a simple modification.

Let us assume that when current output exceeds a certain proportion r (less than unity) of capacity output, investment will be induced according to the familiar acceleration principle.

Our additional assumptions are as follows:

1) The economy produces one commodity so that it has *the* supply curve. This commodity is used to satisfy both consumption and investment needs (This assumption is made only to make our arguments simpler. See the supplement). The short-run supply curve is horizontal up to "practical capacity" (as Kalecki named it) and then rises upwards up to full capacity where it becomes perfectly inelastic. As productive capacity increases, the horizontal part is extended and the upward rising part is shifted to the right.

2) There is no lag in consumption so that investment gives immediate rise to a multiplier effect on income.

3) The investment decision lags one unit period of time.

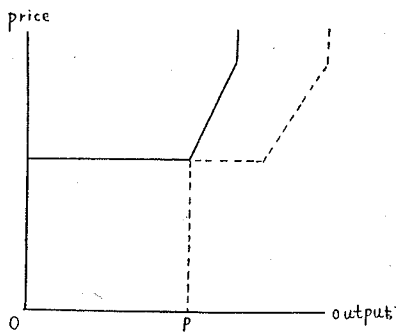

Figure 1.

The gestation period of investment goods (though we have made no distinction between consumption and investment goods) is also one unit period (*i. e.* investment goods produced in this period are put into use in the next period).

4) In every period demand will be matched exactly by supply (as Hicks assumes in his book).[1]

5) There is no technological progress during the whole expansionary phase.

6) The depreciation level is constant over time.

Our notations are as follows:

y_t National income (in real terms) in period t
c_t Consumption in period t
i_t Investment (net)
s the marginal propensity to save
v the accelerator
σ the coefficient of social productivity[2]
P_t practical capacity in period t ($\Delta P_{t+1}=\sigma i_t$)
d depreciation per period

Our basic assumption may be translated into mathematical terms as follows:

a) When actual gross output falls short of the certain proportion of capacity output, namely when

$$y_t+d \leqq rP_t\,,$$

then there will be no gross investment, *i. e.*

$$i_t=-d.$$

b) When actual gross output falls between rP and P, namely when

$$P_t \geqq y_t+d>rP_t\,,$$

there will be positive net investment. We can think of the investment

1) J. R. Hicks, *A Contribution to the Theory of the Trade Cycle,* London, 1951.
2) σ is not necessarily equal to the reciprocal of v. This concept was introduced by Domar in "Expansion and Employment," *American Economic Review,* March 1947.

function in two alternative forms:

$$\text{i.} \quad i_t = v(y_{t-1} - y_{t-2}).$$
$$\text{ii.} \quad i_t = v(y_{t-1} + d - rP_{t-1}).$$

This assumption may be justified by the fact that an economy is usually operated within the range of practical capacity all through the upswing as Kalecki repeatedly emphasized in his books though he used the argument against the validity of the acceleration principle.[3] The first alternative form corresponds to the investment function adopted by Samuelson and later by Hicks. The accelerator remains inflexible during the upswing in this model. Since we have introduced the effect of the degree of utilization of capital stock on the investment decision, this model of the inflexible accelerator seems to be less plausible than the second one. The latter is, in essence, a variation of Goodwin's flexible accelerator.[4] The accelerator is flexible in the sense that induced investment is also influenced by capital accumulation. We shall analyze the second model, which is more complex and interesting than the first, in this note. It must be noted that the model is non-linear at least in the left-tail of the investment function as shown in Figure 2.

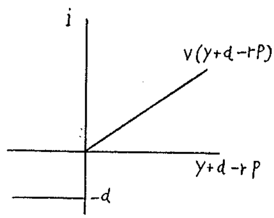

Figure 2.

c) When actual gross output goes beyond capacity output, consequent price rises will affect the whole accelerator mechanism. Besides as I have shown in another paper[5] (if my reasoning is correct), the analysis in real terms alone will break down. We must consider what will happen once the economy hits this moving ceiling of full capacity. I cannot enter into this important problem in this paper.

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3) M. Kalecki, *Essays in the Theory of Economic Fluctions*, London, 1939; idem., *Theory of Economic Dynamics*, London, 1954.

4) Cf. R. M. Goodwin, "Secular and Cyclical Aspects of the Multiplier and the Accelerator," in *Income, Employment and Public Policy*, New York, 1948, pp. 108–132; idem., "The Nonlinear Accelerator and the Persistence of Business Cycles," *Econometrica*, January 1949, pp. 1–17; also see N. Kaldor, "The Relation of Economic Growth and Cyclical Fluctmations," *Economic Journal*, March 1954, for a simple numerical example.

5) Kazuo Sato, "A Reformulation of the Theory of the Multiplier," Hokkaido University Economic Review, April 1955.

The flexible accelerator model

The basic set of equations is as follows:

$y_t = c_t + i_t$ (income equation)

$c_t = (1-s)y_t + k$ (consumption function; k constant)

$i_t = v(y_{t-1} + d - rP_{t-1})$ (investment function)

$\Delta P_{t+1} = \sigma i_t$ (capacity effect of investment)

At the recovery actual output becomes larger relative to capacity and falls into the region where the accelerator is again put into action. We start our analysis from this point of recovery.

Eliminating c and i, we get the following two equations:

$$\begin{cases} vrP_{t-1} = -sy_t + vy_{t-1} + k + vd \\ P_{t+1} - P_t + r\sigma vP_{t-1} = \sigma vy_{t-1} + \sigma vd \end{cases}$$

Finally we get one equation in y:

$$sy_{t+2} - (v+s)y_{t+1} + v(1+r\sigma s)y_t = r\sigma vk$$

The equilibrium solution is obtained by putting $y_t = \bar{y}$:

$$\bar{y} = \frac{k}{s}.$$

Substituting $y'_t = y_t - \bar{y}$ into our equation, we get a homogeneous equation:

$$sy'_{t+2} - (v+s)y'_{t+1} + v(1+r\sigma s)y'_t = 0$$

Its characteristic equation is gven as follows:

$$s\lambda^2 - (v+s)\lambda + v(1+r\sigma s) = 0.$$

$$\lambda = \frac{1}{2s}\left[(s+v) \pm \sqrt{(s+v)^2 - 4sv(1+rs\sigma)}\right]$$

$$= \frac{1}{2s}\left[(s+v) \pm \sqrt{(s-v)^2 - 4s^2 r\sigma v}\right]$$

First let us examine how s and v affect the value of λ with r and σ taken as given.

i) λ real roots. $(s-v)^2 \geqq 4s^2 r\sigma v$.

$$s - v > 2s\sqrt{r\sigma v} \quad \rightarrow \quad s > \frac{v}{1-2\sqrt{r\sigma v}} \quad \text{where } 1 > 2\sqrt{r\sigma v}$$

$$\text{or } s - v < -2s\sqrt{r\sigma v} \quad \rightarrow \quad s < \frac{v}{1+2\sqrt{r\sigma v}}$$

In Figure 3, the first inequality is represented by the region B and the second inequality by the region A.

Let us call the larger root λ_1. Then,

$$\lambda_1 = \frac{1}{2s}\left[s+v+\sqrt{(s-v)^2-4s^2 r\sigma v}\right].$$

In order that λ_1 be greater than unity,

$$\sqrt{(s-v)^2-4s^2 r\sigma v} > s-v\,.$$

In the region A, where $s-v<0$, the inequality does hold. Therefore $\lambda_1>1$ in the region A. In the region B, where $s-v>0$, the inequality does not hold, *i. e.* $\lambda_1<1$ in the region B.

ii) complex roots. $(s-v)^2<4s^2 r\sigma v.$

$$\frac{v}{1+2\sqrt{r\sigma v}} < s < \frac{v}{1-2\sqrt{r\sigma v}}\,.$$

$$|\lambda|^2 = \lambda_1\lambda_2 = \frac{v}{s}(1+rs\sigma)\,.$$

$$|\lambda|>1 \ \ if\ \frac{v}{s}(1+rs\sigma)>1 \ \text{ or } \ s<\frac{v}{1-r\sigma v}\,,$$

$$|\lambda|<1 \ \ if\ \frac{v}{s}(1+rs\sigma)<1 \ \text{ or } \ s>\frac{v}{1-r\sigma v}\,.$$

In Figure 3 the absolute value of the complex roots is greater than one in the region C and less than one in the region D.

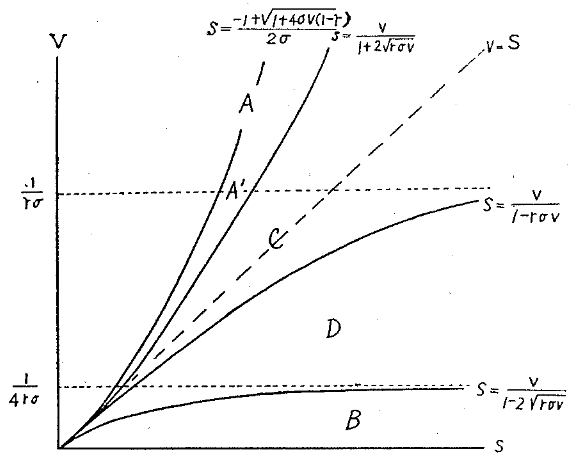

Figure 3.

So far we have made our analysis as if the economy always remains in the region where the accelerator acts in an ideal way, that is, neither overcapacity nor undercapacity. However, this cannot always be the case. As a matter of fact, we can distinguish three alternative paths of development.

i) Natural death of the boom. The initial stage of capaital shortage is followed by a strong capacity effect of investment and capital stock runs into excess, thus stifling investment outlets. The upswing will thus be brought to an end and the boom dies a natural death.

ii) Steady or balanced growth. The degree of utilization of capital equipment remains always moderately high (namely between 1 and r), thus keeping investment opportunities open. The economy continues to grow steadily until scarcity of reoources (labor and raw materials) sets in.

iii) Hitting the ceiling. When the multiplier effect is stronger relative to the capacity effect, the economy has its output go beyond practical capacity. Production is carried out in what Hicks calls the full employment zone (but which should be called the full capacity zone). This is also a situation of post-war inflation. The usual business cycle analysis in real terms is not strong in explaining how this situation leads to a downturn. Effective demand will be absorbed partly in higher prices and partly in smaller output than "real" analysis demands. This will not at all weaken investment inducements, but rather strengthen them. The whole mechanism needs more detailed analysis and special attention should be paid to empirical evidence. I cannot enter into this important problem in this note.

Now we must analyze what combinations of coefficient values lead to these three alternative cases. By a few simple numerical examples we can find that they depend in a great degree on the initial conditions, so that even if a steady growth is potentially possible, it may happen that capacity catches up with actual capital requirements, thus killing the boom prematurely. We have to satisfy ourselves with more general analysis.

Suppose that we have the case of steady growth. Then, the relation

$$P_t \geqq y_t + d > rP_t$$

must always hold. Or

$$1 \geqq \frac{y_t + d}{P_t} > r\,.$$

Tending t to infinity and applying our assumed functional relationship between y and P, we get

$$\lim_{t\to\infty}\frac{y_t+d}{P_t} = \lim_{t\to\infty}\frac{y_t}{P_t} = \frac{rv}{v-s\lambda}\,.$$

Thus,

$$1 \geqq \frac{rv}{v-s\lambda} > r\,.$$

This condition can be reduced to

$$v - s\lambda \geqq rv$$

or

$$\frac{v}{s}(1-r) \geqq \lambda\,.$$

In a similar way, the case (iii) produces a condition

$$y_t + d > P_t\,,$$

which is reduced to

$$\frac{rv}{v-s\lambda} > 1$$

or

$$\frac{v}{s} > \lambda > \frac{v}{s}(1-r)\,.$$

The case (i) gives rise to a condition

$$rP_t \geqq y_t + d$$

which is reduced to

$$r > \frac{rv}{v-s\lambda}$$

or

$$\lambda > \frac{v}{s}\,.$$

To summarize,

$$\begin{cases} \lambda > \dfrac{v}{s} & \text{case (i)} \\ \lambda < \dfrac{v}{s}(1-r) & \text{case (ii)} \\ \dfrac{v}{s} > \lambda > \dfrac{v}{s}(1-r) & \text{case (iii)} \end{cases}$$

This condition can be reduced to

$$\begin{cases} s > v & \text{case (i)} \\ s > \dfrac{-1+\sqrt{1+4\sigma v(1-r)}}{2\sigma} & \text{case (ii)} \\ s < \dfrac{-1+\sqrt{1+4\sigma v(1-r)}}{2\sigma} & \text{case (iii)} \end{cases}$$

Here we are of course assuming λ to be real. For if λ's are complex, output cannot continue expanding; it stops growing sooner or later and the downturn is inevitable, though the economy may run into the full capacity zone before all this takes place because of strong expansionary forces in spite of complex λ's. Thus, we may consider the entire regions of B, C, and D as belonging to the case (i). The case (ii), namely of steady or balanced growth is possible in an outer strip of the region A which is defined by an inequality

$$\frac{v}{1+2\sqrt{r\sigma v}} > s > \frac{-1+\sqrt{1+4\sigma v(1-r)}}{2\sigma}.$$

This strip is called A' in Figure 3.

Now it is evident from inspecting Figure 3 that even if there is no physical limitation of resources, "steady growth" in our sense does not seem probable because plausible values of s call for a very limited range of values of v. This conclusion is of much interest because many business cycle analysts seem to have made an implicit assumption that production does not go beyond practical capacity until and unless the physical ceiling of full employment is hit.

We can make another observation as to the effects of changes in r and σ. When σ increases, regions A, B, C, D, and A' change in the direction of arrows in Figure 4. Apparently an increased capacity effect has a dampening effect. However, the region A' is little affected except that it is rotated upwards.

An increase in r has a similar effect. These results correborates our intuitive anticipations.

Our final conclusion is this: once we take into account the effect of the degree of capital utilization over the conventional acceleration principle, it becomes clear that the upswing is most likely to die a natural death or to hit a ceiling of full capacity rather than full employment.

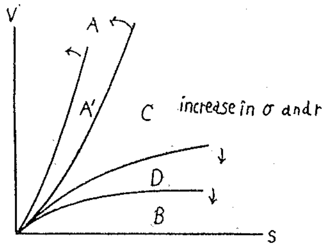

Figure 4.

Supplement

We can easily construct a two-industry model in which the economy is subdivided into consumption goods and investment goods industries. We retain our assumptions in the text and represent the consumption goods industry by the superscript or subscript 1 and investment goods industry by the superscript or subscript 2. Then, we get the following set of equations:

$$\begin{cases} y_t = c_t + i_t \\ c_t = (1-s)y_t + k \\ i_t = i_t^1 + i_t^2 \\ i_t^1 = v_1(c_{t-1} - r_1 P^1_{t-1}) \\ i_t^2 = v_2(i_{t-1} + d - r_2 P^2_{t-1}) \\ \Delta P^1_{t-1} = \sigma_1 i_t^1 \\ \Delta P^2_{t-1} = \sigma_2 i_t^2 \end{cases}$$

By solving this system, we can see that developments in the two industries may be different from each other and that the boom may be brought to an end due to excess capacity in either of the industries.