A NOTE ON THE EXISTENCE THEOREM OF ACTIVITY EQUILIBRIUM

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1. Introduction

Economic equilibrium is defined as a Pareto optimum in many cases. In this note, the definition of Pareto optimum will be extended to broader cases, and the existence of which will also be proved. With the aid of this theorem an equilibrium of generalized Edgeworth's box-diagram will be proved to exist[1] under less restrictive assumptions[2].

2. Existence of Activity Equilibrium

We consider the following abstract economic model. It consists of:

$\mathscr{X}_i$: the ith activity set where complete preordering $\lesssim_i$ is defined ($i \in I \equiv \{1, \cdots, n\}$)[3,4],

T_0: possible activity set of the economy, which is a subset of $\prod_{i \in I} \mathscr{X}_i$[5].

Definition 1. $x \in T_0$ is an activity equilibrium if there is no $y \in T_0$ such that $y > x$[6].

We can easily prove the following

Lemma *$x \in T_0$ is an activity equilibrium if and only if $y \gtrsim x$ implies $y \sim x$ for all $y \in T_0$*[7].

Assumption 1. $T_0 \neq \phi$ and T_0 is bounded in $\prod_{i \in I} \mathscr{X}_i$[8].

* I am grateful to Takashi Ito, Faculty of Science, Mathematics, Hokkaido University, for his comments.

1) As for the generalization of Edgeworth's box-diagram, see [3].
2) see [3].
3) The same terminology and notations as [1] will be used with respect to preordering.
4) The notation "$\equiv$" should be read as "is defined".
5) $\prod_{\lambda \in \Lambda} S_\lambda$ is the Cartesian product of S_λ where λ runs among Λ.
6) $x = (x_1, \cdots, x_n) \gtrsim y = (y_1, \cdots, y_n)$ if and only if $x_i \gtrsim_i y_i$ for all $i \in I$. $x \gtrsim y \equiv y \lesssim x$. $x > y \equiv x \gtrsim y$ and not $y \gtrsim x$.
7) $x \sim y \equiv x \gtrsim y$ and $x \lesssim y$.
8) Given a preordering $\leqslant$ defined on E, a set $X \subset E$ is bounded (with respect to $\leqslant$) if there exists x_0 such that $x_0 \geqslant x$ for all $x \in X$.

Assumption 2. For any non-empty bounded $X \subset \mathscr{X}_i$,

$$\underset{\lesssim_i}{\mathrm{Max}}\, X \neq \phi^{9)}.$$

Assumption 3. Every bounded[10] monotone[11] sequence $\{x_i^q\}$ in $\mathscr{X}_i$ has a maximum[12] with respect to $\lesssim_i$.

Assumptions 2 and 3 are equivalent to each other. In fact,

Theorem 1. *Assumption 2 holds if and only if Assumption 3 holds.*

Proof. It is easily seen that the condition is necessary since the bounded monotone sequence $\{x_i^q\}$ itself is a bounded set in $\mathscr{X}_i$. Suppose, next, the condition is not sufficient. Let the bounded set in $\mathscr{X}_i$ be X. Then for each $x_i^q \in X$, there exists $x_i^{q+1} \in X$ such that $x_i^{(q+1)} >_i x_i^q$. The sequence $\{x_i^q\}$ thus formed is bounded and monotone. Hence there exists positive integer q_0 such that $x_i^{q_0} \gtrsim_i x^q$ for all $q = 1, 2, \cdots$, which is a contradiction. Q.E.D.

Theorem 2. *Given $w \in T_0$, there exists an activity equilibrium x such that $x \gtrsim w$ under the assumptions 1 and 2 (or equivalently 3) for all $i \in I$.*

Proof. We define T_ν ($\nu \in I$) by induction as follows:

$$T_1 \equiv \{y \in T_0 \cap \Pi(w) | P_{r_1}(y) \in \underset{\lesssim_1}{\mathrm{Max}}\, P_{r_1}(T_0 \cap \Pi(w))\}^{13),14)}$$

$$T_\nu \equiv \{y \in T_{\nu-1} | P_{r\nu}(y) \in \underset{\lesssim_\nu}{\mathrm{Max}}\, P_{r_\nu}(T_{\nu-1}\} \qquad (2 \leqq \nu \leqq n)$$

Hence $T_{\nu-1} \supset T \neq \phi$ for $\nu \in I$ by the assumptions.

An element $x \in T_n$ will be shown to be an activity equilibrium.

Suppose $y \in T_0$ and $y \gtrsim x$. Then $P_{r_\nu}(y) \in \underset{\lesssim_\nu}{\mathrm{Max}}\, P_{r\nu}(T_{\nu-1})$ provided $y \in T_{\nu-1}$. By induction, $y \in T_n$. Therefore, $y_i \sim_i x_i$ for $i \in I$, i.e. $y \sim x$. x is an activity equilibrium by the lemma. Q.E.D.

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9) Given a complete preordering $\leqslant$ defined on E and $X \subset E$, $\underset{\leqslant}{\mathrm{Max}}\, X \equiv \{x \in X | x \geqslant x'$ for all $x' \in X\}$.

10) A sequence $\{x^q\}$ in preordered space E is bounded if the set $\{x^q |\ q = 1, 2, \cdots\}$ is bounded.

11) A sequence $\{x^q\}$ in preordered space E is monotone if $x^q \leqslant x^{q+1}$ for $q = 1, 2, 3, \cdots$.

12) A sequence $\{x^q\}$ in preordered space E has a maximum if $\underset{\leqslant}{\mathrm{Max}}\, \{x^q |\ q = 1, 2, \cdots\} \neq \phi$.

13) $\Pi(w) \equiv \prod_{i \in I} \mathscr{X}_i^{w_i}$, where $\mathscr{X}_i^{w_i} = \{x_i \in \mathscr{X}_i | x_i \gtrsim_i w_i\}$.

14) $P_{r_i}(x)$ is the projection function from $\prod_{i \in I} \mathscr{X}_i$ into $\mathscr{X}_i$.

3. Existence of Exchange Equilibrium

In what follows we apply the above techniques to prove the existence of exchange equilibrium which is a generalization of the notion of the contract curve.

We consider the following economic model. It consists of n economic units. Each has its preference preordering $\lesssim_i$ and the same commodity space Ω[15] which corresponds to $\mathscr{X}_i$ in the previous section.

Let T be the transformation set which is a subset of R^l.
$\Gamma_0 \equiv \{x \in \Omega^n | \sum_{i \in I} x_i \in T\}$[16], which corresponds to T_0 in 2.

We introduce the following.

Definition 2. $x \in \Gamma_0$ is an exchange equilibrium if there exists no $y \in \Gamma_0$ such that $y > x$.

Therefore the exchange equilibrium is the activity equilibrium when the activity sets are confined to finite Euclidean spacs. But the converse may not be true.

Assumption 4. T is closed and bounded from above with respect to $\leqq$.

This assumption is less restrictive than to assume that T is compact.

Assumption 5. For any $x_i \in \Omega$, both Ω^{x_i} and Ω_{x_i} are closed[17] for all $i \in I$.

This assumption enables us to consider a continuous utility function u_i on Ω[18].

Theorem 3. *Given $w \in \Gamma_0$, there exists an exchange equilibrium x such that $x \gtrsim w$ under the assumptions* 4 and 5.

Proof. We define Γ_ν similarly as T_ν that is defined in 2, replacing $T$with Γ.

Let f be $f(x) = \sum_{i \in I} x_i$ for all $x \in \Omega^n$. f is a continuous function. $\Gamma_0 = \Omega^n \cap f^{-1}(T)$ implies that Γ_0 is closed by assumption 4. Γ is bounded from below and above with respect to $\leqq$ by assumption 4 and bence compact. $\Gamma_0 \cap \Pi(w)$ is also compact because $\Pi(w)$ is closed by assump-

15) $\Omega \equiv \{x \in R^l | x \geqq 0\}$, where R^l is l-dimensional Euclidean space and $\geqq$ denotes the usual vector inequality.

16) $\Omega^n \equiv \prod_{i \in I} \Omega$

17) Ω^{x_i} is defined in footnote 7). $\Omega_{x_i} \equiv \{x \in \Omega | x \lesssim_i x_i\}$

18) See [1] p. 56.

tion 5. Projection function[14] is continuous and hence $P_{r_1}(\Gamma_0 \cap \Pi(w))$ is compact[19].

$\underset{\leqq_1}{\mathrm{Max}}\, P_{r_1}(\Gamma_0 \cap \Pi(w)) \neq \phi$ by assumption 5. Therefor Γ_1 is not empty.

Moreover Γ_1 is compact and hence $\Gamma_\nu(\nu \in I)$ is compact and non-empty. Similar arguments as 2 prove that any element of $\Gamma_{\dot{n}}$ has the required property. Q.E.D.

4. Economic Interpretation

The above stated theorem 3, has usual concrete meaning of box-diagram when $n=2$, $l=2$, and T consists of one element. In general there exist n economic units and l commodities. Exchange equilibrium is the state where no exchange between economic units can increase utility of some economic unit without sacrificing some other's welfare. However the objects of economic activity in case of theorem 3 are confined to finite kinds of commodities. The theorem 2 removes this restriction. We can take any economic activity into account. Moreover, they are allowed to contain psychological or political activities, which do not belong to the economic activities in the usual sense. Similar interpretation can be done with respect to the activity equilibrium, i.e. there is no economic activity in the economy that increases every individval's welfare and at least for some one the state is strictly preferable. These notions are characteric of Pareto Optimum.

References

[1] G. Debreu "*Theory of value*" John wiley & Sons, Inc. 1959.
[2] N. Bourbaki "*Topologie générale*" Livre III chap. I et II deuxième édition.
[3] Kaneyuki Yamamoto and Ryosuke Hotaka "A Generalization of Box Diagram" Journal of Pre-Medical Course, Sapporo Medical College 3, 1962.

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19) See [2] p. 95, Theorem 2.