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Progress and Coordination: A Study of Mathematical Analyses of Dynamic Growths and Static Distributions

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Mathematical models of the growth and its counterpart degradation were given already in the last decade of the 18th century by Malthus.[1] His idea was that the population can grow exponentially with constant rate, while the production to feed it can only grow linearly with constant quantity, so the degradation of the standard of living or poverty will ensue. Quetelet[2] and Verhulst[3] supposed that the population increases acceleratedly at the first stage, and deceleratedly afterwards. Verhulst's formula of the logistic function, which shows that the speed of growth is proportional to the product of population itself and the resource remaining after the utilization in feeding the population, was applied by the biologist Pearl[4] and Reed[5] experimentally and by the statistician Yule[6] logically. Some of the recent statistical economists apply the above formula to the production developments.[7] Samuelson used this formula to depict the stabilities,[8] Tinbergen and Bos assumed the resource as the total demand including the foreign demand.[9] In a closed economy total demand would be the derivation from the balance of production and distributions. Rostow's process of developement of precondition, take off, and maturities are histori cal representation indexed with the number of automobiles.[10] Robinson's idea[11] of leaden, golden and platinum ages are related to the balances.

The present author proposes the connections between this formula and the static distribution of economic sizes.

The sigmoid curve and the appropriate logistic function, which has been taken as to be applicable in depicting statistical growths and developements, has been effectivelly used by the demographers, biologists and economists. It is the integrated form of the differential equation

$$\frac{dP}{dT}=aP(r-P)$$

in which P is the variable denoting the present status of human and animal population or economic production, variable T is the time, and the constant r is the limited resource or stock of materials, which may be transformed to the population or production. Thus $\frac{dP}{dT}$ is proportional to the product of the present population and the remaining resource. This differential equation can be solved in the following way:

$$\frac{dP}{P(r-P)}=a\ dT,\ \frac{dP}{P}-\frac{d(r-P)}{r-P}=ra\ \ dT,\ dln\frac{P}{r-P}=ra\ \ dT$$

and the following logistic formula is obtained as solution:

$$P=\frac{r}{1+e^{-\mathrm{raT}-c}},\ \text{ where c is a constant.}$$

This relations is known and applied by several authors. The present author's idea of transformation is as follows: when $e^{-\mathrm{raT}-c}<1$

$$P=r\left[1-e^{-(\mathrm{raT}+c)}+e^{-2(\mathrm{raT}+c)}-e^{-3(\mathrm{raT}+c)}+\cdots\right]$$

As raT+c and accordingly its multiples may be taken as sufficiently large the following formula is obtained approximately:

$$P=r-r^{-}e^{\ -\mathrm{raT}-c}$$

and derive $\frac{dP}{dT}=ar^2e^{-\mathrm{raT}-c}$

or $$\frac{dP}{dT}=ar^2e^{-c}e^{-\left(\pm\sqrt{arT}\right)^2}$$

Put $\sqrt{arT}=log\tau,\ e^{\sqrt{\mathrm{arT}}}=\tau,\ \tau=\geqq 1$

$-\sqrt{arT}=log\tau,\ e^{-\sqrt{\mathrm{arT}}}=\tau,\ \tau=1>x>0$

$\frac{\mathrm{dP}}{\mathrm{dT}}$ reveals a skew symmetric distribution curve which is very like to logarithmic normal distribution curve.

The further differentiation by the variable raT is:

$$2a^{\frac{3}{2}}r^{\frac{3}{4}}e^{-c}T^{\frac{1}{2}}e^{-\tau^2}$$

in which all except T can not be zero, $T=o$ or $logT=1$ should be the highest point of the distribution curve.

The logarithmic normal distributions of several economic status, as income, possessions and so forth, were known as Gibrats' law, which may be connected to Pareto's descriptions of the same status.

Gibrat's law[11] : $c.e^{-S}$ S: variable.

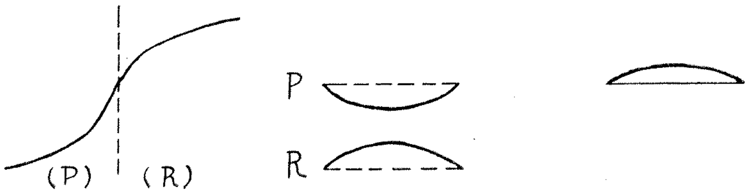
Pareto's description[12] : $b.U^{-v}$ U: variable.

U may be expressed as e^{W} or $b.U^{-v}=b.e^{-Wv}$, W taken as variable and assuming $Wv=S$, the above two description become same.

Dr. Benoit Mandelbrot of the Harvard University[13] generalized Pareto's law to apply in all fields of science as "universal" law somewhat extravagantly, but yet didn't connected it with Gibrats' law. Mandelbrot's notion of "probability that the fortunes return to their initial state after a time greater than $t=K.t^{-\frac{1}{2}}$ (which is the law of Pareto with exponent of 1/2") might be explained as above. It reveals the highest point of growth rate and most probable points of existence.

By such mathematical relations it might be supposed that the economic (production) growths are related to the distributions of firms and individual persons classified in their sizes of equipments, possessions, earnings and costs. How should be explained by these connections? Big enterprises, possessors, and earners might make researches, experiments and investments more than the small ones. But their numbers might be small (though the number might be counted by the laborers or capital amounts). Number of smaller or moderate firms and earners might be bigger. Which are the forerunners? Those are the problems of which some authors are doing now.[14] [15]

A supposition might be given as the explanation. The sigmoid growth



curve can be divided into one phase of exponentially ascending positivly and ocher phase of that of negative. Compare those two phases to the linear trend line. The former curve is concave and the latter convex. These two suggest the forms of distribution, one divergent and the other convergent. The distributinn of firm size might be divergent in the prior stage of development (small number of optimal sized firms), and convergent in the posterior stage (majority of optimal sized firms). This assumption may seemingly contradict to the fact that in the last stage of developement some or amalgamations of firms combined into big monopolistic firms, yet the big firms always have many connected plants or "undertakers" or jobbers under them.

The present author started as an agronomist and then became an economist, so it is convenient to quote examples from the history of agriculture. Ancient primitive economies were those of self sufficing villages. The majority of the people worked in the fields which were divided into small holdings. It has been the historian's problem that whether those primitive communities were democratic or hierarchic, i.e. were there no controlling powers, or were there chiefs and lords who controlled the community and collected poll taxes and rents.

There were several epoques in world history in which large scale farm managements dominated. The first historical epoque was that of Roman "latifundia" which used slaves, and the second began in the "Estates" of English industrial revolution with rural laborers, and the third the new continent "plantations" which introduced negro slaves. Recent global movements of enlargements of farm sizes by using big machines should be added. American ways are yet the family farm system and Russian ways are coercion of cellectivizations of family farms.

Those big farms increase profits in early stages but the smaller (family) farms can follow to the innovations, by introducing smaller machines and more intensive cultivation, and compete with the big ones.

Enlargement of factories and big businesses have been thought to be the eternal tendency, but the trends might be only one phase of business cycles, as Schumpeter revealed. There need to be big combining equipment, but there may be small undertakers manufacturing the various parts. The Japanese industrial prosperity has been presumably derived from this system.

Separations of employing capitalists and employed labore classes are only the first phase of industrial developments. The author studied the distributions of farm numbers referring to their average costs ofrice pro duction. They are the good example of the logarithmic normal distribution as Maruyama proved. (16) (17) Who are the cheap producers? Presumably they are specialized (commercial) rice paddy farmers, while those rice producers mainly for self consuming purposes are rather costly producers. Farmers who raise rice under inferior (climate and soil) conditions are supposedely of the later initiation, and sometimes meet with the bad (climate and pest damages) conditions. Presumably the half of the recorded maximum and zero of the yields per unit area is the average of the total yields. Dr. Morita (18) used the logistic formula to know the maximum rice production by the increase data, which seems now somewhat low, for the maximum yield per unit area became very high.

Postscript.

The present author conceived these ideas rather long, but has been thinking that they should be revealed in mathematically functional form skillfully. He reported them in some conference against the critic of unlimited growths.

This papers were examined and corrected by Professor K. Yamamoto of the Hokkaido University, who told me the transformation of Verhulst's formula changed to $P=r-re^{-arT-c}$ reveals the curve of only ascending, perhaps as Gomperz Curve, though its derivative shows both ascending and descending growths. The author thank Professor Yamamoto and expect his further study. The relation of Pareto's and Gibrat's forms might be my own device. (May 7, 1964• Sapporo, Japan.)

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