DYNAMIC UTILITY MAXIMIZATION AND THE DERIVATION OF CONSUMPTION FUNCTION*

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1. INTRODUCTION

EVER SINCE the appearance of Keynes' "General Theory", much of concern has been concentrated to the estimation of consumption function. It is well known that the validity of the liquid assets hypothesis, relative income hypothesis, permanent income hypothesis, life-cycle hypothesis and other hypotheses has been tested statistically in such an atmosphere. Needless to say the empirical study of this kind is extremely important for the construction of macroeconometric model of the national economy and its application. It, however, seems to me that little attention is paid to the theoretical or axiomatic study of the microeconomic foundation of aggregate consumption function, although such a study is of great importance for the construction of dynamic general equilibrium theory. In this paper we shall retrospect on the dynamic model of consumer behavior which has been built heretofore and extend it from a certain point of view.

2. SAVING THEORY IN PERSPECTIVE

Apart from the simple Fisheresque two-period analysis, the multi-period optimization behavior of an individual was first investigated by Strotz[6]. He analyzed a problem with a special type of budget constraint. So that his results for an "ocean voyager" are not necessarily directly useful for our object to determine the optimum propensity to consume(or save). In that sense we should say that the first admirable study in the saving theory was done by Yaari[10]. He dealt with Ramsey integral maximization problem with a finite horizon in terms of classical calculus of variations, emphasizing a significance of a bequest of accumulated assets to descendants. Douglas[2] supposed that the utility at each moment in time is generated not only by the consumption but also by the cash balances[1]. Arrow-Kurz[1] pointed out some shortcomings of the finite horizon type of formulation and discussed a problem of accumulating material assets, paying attention to some factors yielding consumption benefits such as public goods and external economies. Uzawa[7] analyzed an integral maximization problem with an endogenous rate of discount.

Furthermore Mills and Uzawa[8] established a remarkable new method different from the integral control approach. In that new theory the intertemporal preference ordering of the household is described in terms of Fisherian schedule of time preference. The rate of time preference[2] depends upon both of the absolute level of consumption and the relative rate of increase of it, provided the intertemporal preference ordering is separable and intertemporal marginal rate of substitution is continuous with respect to the variation in consumption paths. And if the intertemporal preference ordering is homothetic, then the absolute level of consumption vanishes out of arguments of the Fisherian function. The dynamic optimality condition is that the rate of time preference equals the instantaneous marginal rate of transformation(the real rate of return on assets possessed by the individual) under the budget constraint. Uzawa proved that there exists a unique optimal consumption path converging a long-run stationary point at which the propensity to consume equals unity when the intertemporal preference ordering is separable, and that an optimum average propensity to consume is kept to be unchanged over time when the preference relation is not only separable but also homothetic.

3. INTERTEMPORAL CHOICE AND TIME PREFERENCE

Before proceeding to a concrete consideration of the utility maximization problem we would review some important points on the concept of dynamic utility. The consumer behavior over time is concerned with an optimal allocation of utility involving a curtailment of the present consumption due to regard for the prospective felicity. Koopmans[5] first challenged the rigorous mathematical analysis of the structure of time preference in terms of an ordinal utility function in the tradition of Böhm-Bawerk. He divided a stream of utility over an infinite horizon between the immediate utility and the prospective utility, or more precisely under some postulates the utility function is written in the form

(1) $U({}_1x)=V[u_1(x_1),U_2({}_2x)]=V[u(x_1),U({}_2x)]$

where $V[u_1,U_2]$ is the aggregator.(The notation follows Koopmans' one.) He illustrated the aggregate utility $U({}_1x)$ by the indifference curves in the unit square relying upon the ordinal property of aggregator V and clarified some implications of the dynamic utility function. Moreover he referred to a remarkable fact that $U({}_1x)$ can be written in the form of the sum of discounted present values

$$(2)\quad U({}_1x)=\sum_{t=1}^{\infty}\alpha^{t-1}u(x_t),\quad 0<\alpha<1$$

under his postulates. The continuous version of (2) is well known as the Ramsey integral, which will play an important role in the later discussion. No doubt the exclusion of intertemporal complementarity of consumption is a drastic simplification as Hicks[4](Chap. 21) has already pointed out. Nevertheless it is too difficult to reject the operationality of additive utility function. In fact we do not know the general law of intertemporal complementary relation at all.

It is assumed that the rate of discount (α in (2)) is constant in the usual analysis. This is, however, a considerably rigid formulation as was noted by Koopmans. A fairly general discussion about the rate of discount by Mills makes this point clear.[3] The dynamic utility as of time $t(>0)$ U_t can generally be represented as

$$(3)\quad U_t=\Phi(c_t,U_{t+\Delta t},\Delta t)\qquad \text{where } c \text{ is consumption}$$

provided Δt is sufficiently small. Then the rate of discount is defined by

$$(4)\quad -\lim_{\Delta t\to 0}\Phi_{23}=-\Phi_{23}(c_t,U_t,0)$$

It is obvious that the rate of discount depends upon consumption and prospective utility evaluated at time t. For example, an additive type of utility U_t with a constant discount rate β

(5) $U_t=\int_t^{\infty}u(c_\tau)e^{-\beta(\tau-t)}d\tau, \quad \tau\geqq t$

can be written as

(6) $U_t=\Phi(c_t,U_{t+\Delta t},\Delta t)$

$=u(c_t)\Delta t+e^{-\beta\Delta t}U_{t+\Delta t}$

for sufficiently small Δt

By straightforward calculation we have

(7) $-\Phi_{23}(c_t,U_t,0)=\beta$

Although it is a bold simplification to regard the rate of discount $-\Phi_{23}(c,U,0)$ as a constant, we will introduce this assumption for mathematical convenience.

Strotz[6] thought that the true dynamic theory of utility maximization had to take account of the continual revision of plan due to the change of the present date. We, however, ignore this somewhat complicated problem known as the "inconsistency" in the present paper.

4. RECONSIDERATION OF THE RAMSEY INTEGRAL MAXIMIZATION APPROACH

In this section we shall reconsider the ordinary Ramsey integral maximization model which is a starting point of our analysis. We would examine whether there is a unique optimal saving plan with a finite and an infinite horizon in what follows. The problem is choosing a consumption path so as to maximize

$$(8)\quad J=\int_0^{T \text{ or } \infty} u(c)e^{-\beta t}dt, \qquad 0<T<\infty$$

among feasible consumption plans which satisfy a differential equation

$$(9)\quad c+Db=w+ib$$

In (8) and (9)
c=real consumption
b=real asset holdings
w=real wage rate (const.)
i=real rate of interest (const.)
β=subjective rate of discount (const.)
u=felicity indicator (u(c) is a strict concave function.)

and D is a differential operator d/dt. The boundary condition in the finite plan case is that the saving balance with an initial value b(0) is accumulated up to a stipulated finite value b(T)(>b(0)). The Hamiltonian is

$$(10)\quad H=e^{-\beta t}[u(c)+\lambda(w+ib-c)]$$

where λ is an auxiliary variable. The necessary condition for maximum of H is

$$(11)\quad \lambda=u'(c)$$

The motion of $\lambda(\geqq 0)$ is governed by a differential equation

$$(12)\quad D\lambda=(\beta-i)\lambda$$

where $\lambda \neq 0$ on the optimum path. The property of a solution may be examined according as $\beta \gtreqless i$.

CASE I. $\beta > i$

A singular curve $D\lambda=0$ is characterized by $\lambda=0$. And $Db=0$ curve is negatively sloped in the (b,λ) plane. It intersects λ-axis at $\lambda=u'(w)$ and approaches b-axis as an asymptoto. The structure of a solution is typically illustrated in Figure 1 in which b is measured along the horizontal axis and λ is measured along the vertical one.

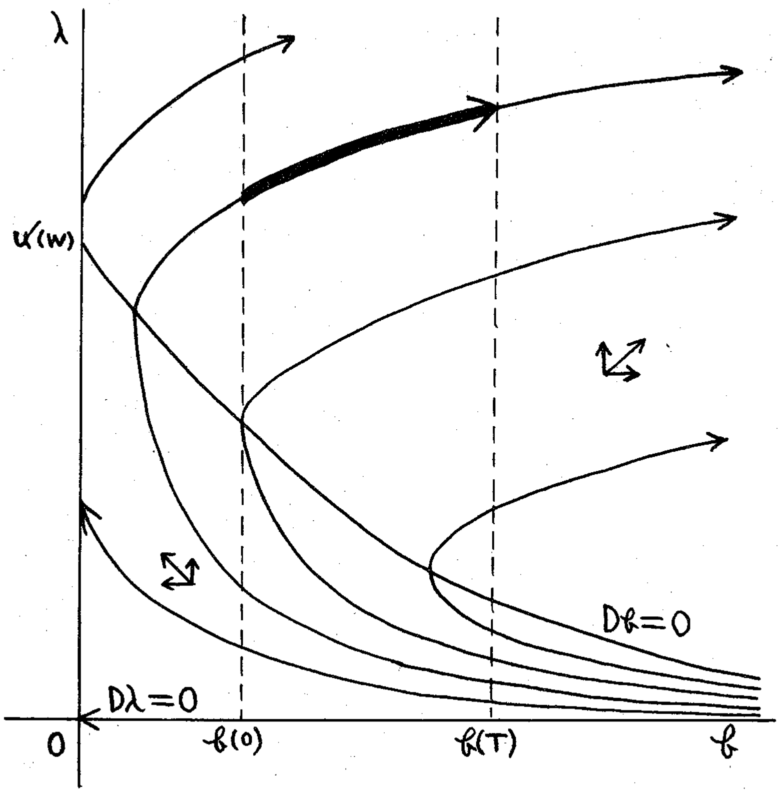

Figure 1.

Obviously there is not an optimal consumption plan with an infinite time horizon. On the contrary there exist many feasible plans with a finite horizon satisfying the boundary conditions as is clearly shown in the phase diagram. There must be an only plan such that the time required is just precisely T among those plans.[4] That is an optimal plan (starting with given initial value b(0)) with a finite horizon T, which is shown by a heavy arrowed path in Figure 1. This solution, however, is inadequate in view of our object to construct the dynamic general equilibrium theory, since the consumption gradually decreases along the optimal trajectory namely such a motion is inconsistent with a general feature of the process of economic growth.

CASE II. $\beta = i$

In this case $D\lambda$ is always zero for any value of λ. Therefore the phase diagram is depicted as Figure 2.

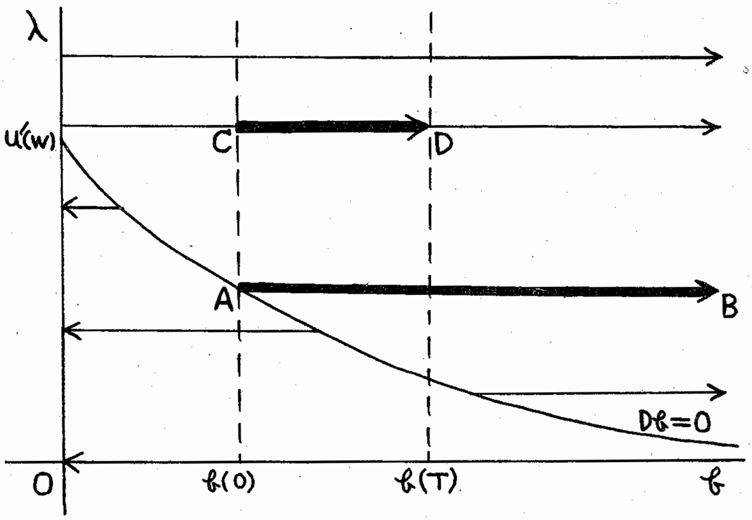

Figure 2.

It follows from Figure 2 that there exists a unique optimal plan with an infinite horizon which is indicated by a heavy arrowed path AB. (It is obvious that a plan with smaller $\lambda = u'(c)$ is more favorable.) A trajectory AB satisfies the transversality condition

$$(13)\quad \lim_{t\to\infty} e^{-\beta t}\lambda = 0$$

An optimal plan with a finite horizon T is indicated by a heavy arrowed path CD.

CASE III. $\beta < i$

In this case the phase diagram is illustrated in Figure 3.

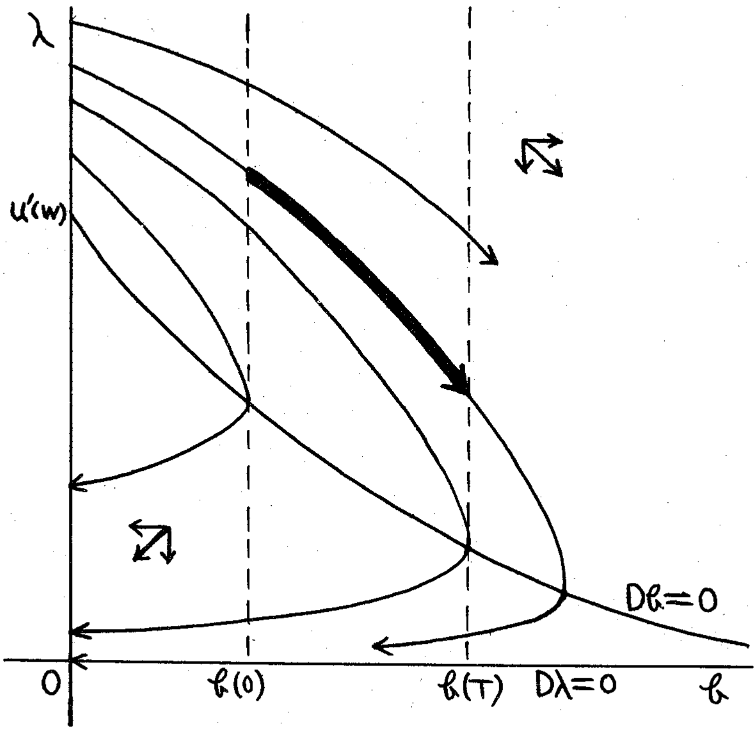

Figure 3

There is not an optimal plan with an infinite horizon. An optimal plan with a finite horizon T is indicated by a heavy arrowed curve. Obviously the real consumption increases along this path.

In the foregoing analysis, saving plans satisfying the budget restraint (9) and maximum principle (12) as necessary conditions are visualized in terms of phase diagram. It remains to show that the necessary conditions are also sufficient, that such plans (with a finite horizon T) are indeed optimal. Let $[\hat{c},\hat{b}]$ be a plan satisfying the conditions of (9),(12) and the boundary conditions. Let $[c,b]$ be any feasible plan, that is, any plan satisfying (9) and the boundary conditions. The difference between the value of utility integral on the optimal trajectory and that on any feasible trajectory is written in the form

$$(14)\quad J(\hat{c})-J(c)=\int_0^T u(\hat{c})e^{-\beta t}dt-\int_0^T u(c)e^{-\beta t}dt$$
$$=\int_0^T [u(\hat{c})-u(c)-u'(\hat{c})(\hat{c}-c)]e^{-\beta t}dt$$
$$+\int_0^T u'(\hat{c})(\hat{c}-c)e^{-\beta t}dt$$

By using the performance equation (9) the second term can be rewritten as follows.

$$(15)\quad \int_0^T u'(\hat{c})(\hat{c}-c)e^{-\beta t}dt=\int_0^T u'(\hat{c})i(\hat{b}-b)e^{-\beta t}dt$$
$$-\int_0^T u'(\hat{c})(D\hat{b}-Db)e^{-\beta t}dt$$

Integrating the second term by parts

$$=\int_0^T u'(\hat{c})i(\hat{b}-b)e^{-\beta t}dt$$

$$+\int_0^T (\hat{b}-b)[u''(\hat{c})D\hat{c}-\beta u'(\hat{c})]e^{-\beta t}dt$$
$$-[u'(\hat{c})(\hat{b}-b)e^{-\beta t}]_0^T$$

We can eliminate $D\hat{c}$ by using the Euler equation

$$(16)\quad D\hat{c}=(\beta-i)\frac{u'(\hat{c})}{u''(\hat{c})}$$

which is obtained by substituting (11) into (12). Thus expression (15) reduces to

$$(17)\quad \int_0^T u'(\hat{c})(\hat{c}-c)e^{-\beta t}dt=-[u'(\hat{c})(\hat{b}-b)e^{-\beta t}]_0^T$$

Hence (14) becomes

$$(18)\quad J(\hat{c})-J(c)=\int_0^T [u(\hat{c})-u(c)-u'(\hat{c})(\hat{c}-c)]e^{-\beta t}dt$$
$$-[u'(\hat{c})(\hat{b}-b)e^{-\beta t}]_0^T$$
$$=\int_0^T [u(\hat{c})-u(c)-u'(\hat{c})(\hat{c}-c)]e^{-\beta t}dt$$
$$-u'(\hat{c}(T))(\hat{b}(T)-b(T))e^{-\beta T}$$
$$+u'(\hat{c}(0))(\hat{b}(0)-b(0))$$

The first term is positive by virtue of the strict concavity of the utility function and the second and third terms vanish by the boundary conditions. Therefore

$$(19)\quad J(\hat{c})>J(c),\qquad \text{or } J(\hat{c})=\max_c J(c)$$

completing the proof of sufficiency. It is obvious that the optimal plan $[\hat{c},\hat{b}]$ is unique (by virtue of the strict concavity of the utility function).

So far we have considered the maximization problem of the so-called Ramsey integral (as the simplest utility integral) subject to the budget constraint of the household. The main results obtained are as follows.

1. There always exists a unique optimal saving plan with a finite horizon whether the rate of discount β is greater than the rate of interest i or not.
2. On the contrary there exists a unique optimal saving plan with an infinite horizon only if $\beta = i$. This path with a constant consumption, however, has not a long-run stationary equilibrium point. It follows therefore that the asset is accumulated unlimitedly.

It is inconvenient for a certain object that the saving model with an infinite horizon has not a long-run stationary equilibrium point toward which a dynamic path converges. So that we shall introduce a new utility integral and examine its dynamic property in the subsequent sections.

5. INTRODUCTION OF THE UTILITY OF ASSET BALANCE

Now we shall introduce a functional

$$(20)\quad J(c,b)=\int_0^\infty [u(c)+v(b)]e^{-\beta t}dt$$

instead of (8).[5] It is fairly realistic to assume that the utility is generated not only by the current consumption but also by the real balance of financial assets (securities) in the dynamic world. For the time being the instantaneous utility at every moment is represented in the separable and additive form as is described in (20).[6] It is, moreover, assumed that the subjective rate of discount is greater than the rate of return on assets in the later discussion. Although rather restrictive, yet this assumption seems to be more valid than the previous stiff assumption $\beta=i$ (in the infinite horizon case) in view of the myopic imprudence of a human being. The Hamiltonian form is

$$(21)\quad H=e^{-\beta t}[u(c)+v(b)+\lambda Db]$$

The relationship between λ and c is

$$(22)\quad \lambda=u'(c)$$

since H must be maximal with respect to c. By maximum principle there must be $\lambda\neq 0$ satisfying the differential equation

$$(23)\quad D\lambda=(\beta-i)\lambda-v'(b)$$

(23) is, of course, equivalent to the Euler equation

$$(24)\quad Dc=\frac{(\beta-i)u'(c)-v'(b)}{u''(c)}$$

In addition the transversality condition

$$(25)\quad \lim_{t\to\infty} e^{-\beta t}\lambda=0$$

must be satisfied. We would examine whether a unique optimal trajectory starting with a given initial value b(0) exists. A singular curve $Db=0$ becomes

$$(26)\quad c=w+ib$$

And another singular curve $D\lambda=0$ becomes

(27) $\lambda=\frac{v'(b)}{\beta-i}$

Both $Db=0$ and $D\lambda=0$ curves are negatively sloped by virtue of the concavity of $u(c)$ and $v(b)$. It is, however, indeterminate whether $Db=0$ and $D\lambda=0$ curves intersect. Possible three representative cases are illustrated in Figure 4,5 and 6.

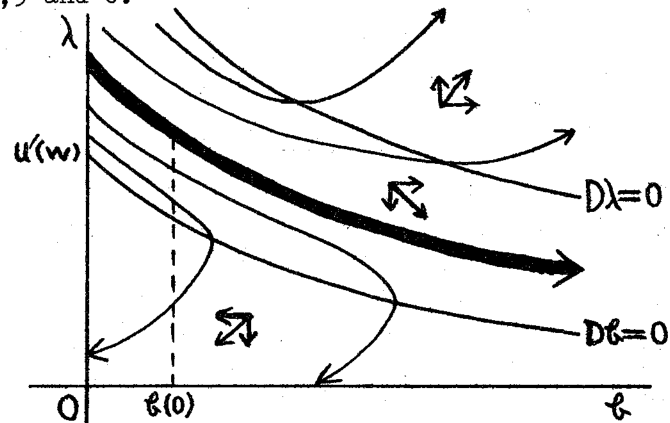

Figure 4.

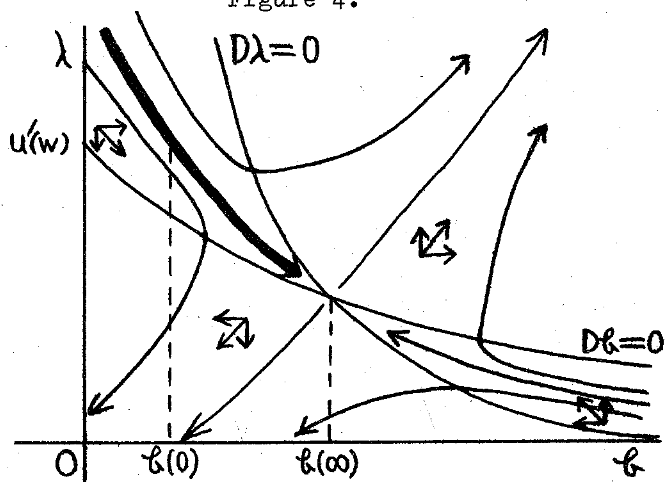

Figure 5.

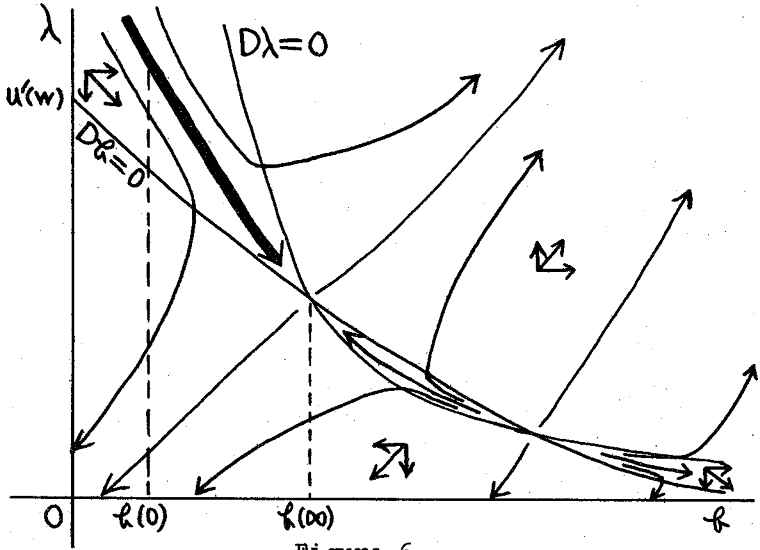

Figure 6.

In Figure 4 two singular curves do not intersect. Therefore there is not a stationary equilibrium point. There exists a unique optimal path (heavy arrowed curve) on which asset balance b is accumulated unlimitedly. In Figure 5 and 6 two singular curves intersect once and twice respectively. In both of Figure 5 and 6 there exists a unique optimal path converging a long-run stationary equilibrium point which is a saddle point. We assume the case of Figure 5 or 6 in what follows. Obviously the optimal consumption plan depends upon w, i and β, that is

$$(28)\quad c=c(w,i,\beta)$$

(28) is the consumption function of the individual household. Obviously optimum propensity to consume approaches unity unlimitedly.

6. NUMERICAL EXAMPLE

Let us examine the structure of solution by a numerical example. Assume that

(29) $u(c)=10\log c$
(30) $v(b)=0.75\sqrt{b}$
(31) $w=5$
(32) $i=0.05$
(33) $\beta=0.1$

Thus we have

(34) $c=5+0.05b$

as an equation of $Db=0$ curve and

(35) $c=\frac{4}{3}\sqrt{b}$

as an equation of $Dc=0$ curve. A phase diagram of this system is depicted in Figure 7. An optimal path is indicated by a heavy arrowed curve,

7. DERIVATION OF THE EULER EQUATION

We would derive the Euler equation (24) for convenience of later discussion. Let the functional (20) be

(36) $J[b,Db]=\int_0^{\infty}[u(c)+v(b)]e^{-\beta t}dt$

Define

(37) $b(\varepsilon)=b^*+\varepsilon\varphi(t)$

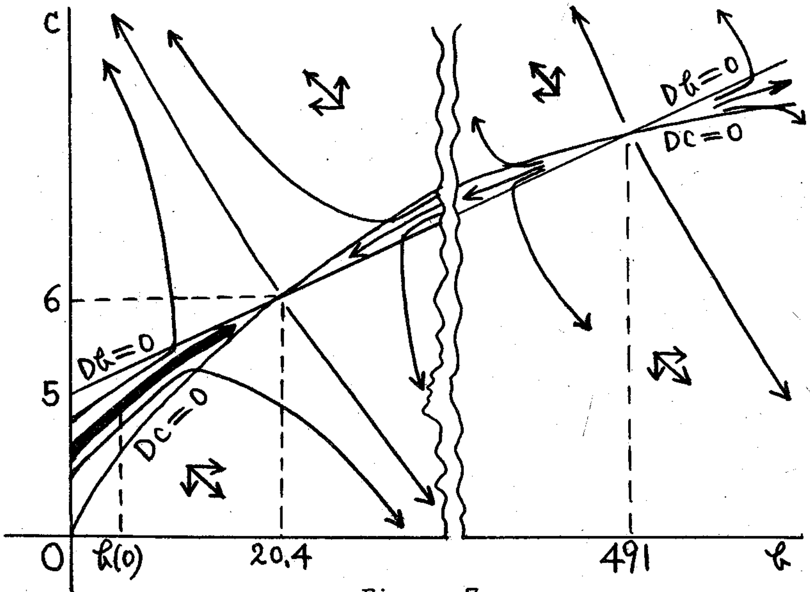

Figure 7.

where b* is an extremal and $\varphi(t)$ is an arbitrary differentiable function which satisfies

$$(38)\quad \varphi(0)=0, \qquad \lim_{t\to\infty}\varphi(t)=0$$

We assume that $|\varepsilon|$ is a small real number. Of course

$$(39)\quad J[b^*,Db^*] \geqq J[b(\varepsilon),Db(\varepsilon)]$$

holds. The necessary condition for maximum of the functional (36) is

$$(40)\quad \delta J = \left.\frac{\partial J[b(\varepsilon),Db(\varepsilon)]}{\partial \varepsilon}\right|_{\varepsilon=0}$$

where δJ means the first variations. Let us calculate

(40).

$$(41)\quad \delta J=\varepsilon\int_0^{\infty}[u'(c)\Big|_{\varepsilon=0}(i\varphi-D\varphi)+v'(b(\varepsilon))\Big|_{\varepsilon=0}\varphi]e^{-\beta t}dt$$

$$=\varepsilon\int_0^{\infty}[u'(c)\Big|_{b=b^*}(i\varphi-D\varphi)+v'(b^*)\varphi]e^{-\beta t}dt$$

$$\left(\text{since } b(\varepsilon)\Big|_{\varepsilon=0}=b^*\right)$$

integrating by parts

$$=\varepsilon\int_0^{\infty}[iu'(c)\Big|_{b=b^*}+u''(c)Dc\Big|_{b=b^*}-\beta u'(c)\Big|_{b=b^*}+v'(b^*)]\varphi(t)e^{-\beta t}dt$$

$$=0$$

By a lemma of the calculus of variations this reduces to

$$(42)\quad iu'(c)+u''(c)Dc-\beta u'(c)+v'(b)=u''(c)Dc-(\beta-i)u'(c)+v'(b)=0$$

(dropping " $\Big|_{b=b^*}$ " for typographical convenience)

Thus we have obtained the Euler equation.

8. COMPARATIVE DYNAMICS

To do the comparative dynamic analysis of the optimum path we have to solve (9) and (42).

Since the asset accumulation equation (9) is a linear differential equation, we can easily solve this. The solution is

(43) $b(t)=\frac{w}{i}(e^{it}-1)-e^{it}[\int_0^t c(s)e^{-is}ds-b(0)]$

The integral of the Euler equation may be obtained by the theory of the calculus of variations. First we assume the existence of an optimum control variable. Let it be c* and a corresponding extremal be b*. Obviously

(44) $Db^*=w+ib^*-c^*$

We perturb c* as

(45) $c(\varepsilon)=c^*+\varepsilon\eta(t)$

where η is a perturbation term of c* and is a piecewise continuous function. $c(\varepsilon)$ must be admissible for a small real number ε. A perturbation of b* due to that of c* is written as

(46) $b(\varepsilon)=b^*+\varepsilon\varphi(t)$

where $\varphi(0)=\lim_{t\to\infty}\varphi(t)=0$. And needless to say

(47) $J[b^*,c^*]\geqq J[b(\varepsilon),c(\varepsilon)]$

holds. $b(\varepsilon)$ and $c(\varepsilon)$ satisfy

(48) $Db(\varepsilon)=w+ib(\varepsilon)-c(\varepsilon)$
$=w+i(b^*+\varepsilon\varphi)-(c^*+\varepsilon\eta)$ (by (46) and (45))
$=Db^*+i\varepsilon\varphi-\varepsilon\eta$ (by (44))
$=Db^*+\varepsilon D\varphi$ (by differentiating (46))

Thus we have

(49) $D\varphi - i\varphi + \eta = 0$

The solution to this equation is

(50) $\varphi(t) = -e^{it}\int_0^t \eta(s)e^{-is}ds$

Hence the perturbation term η must satisfy

(51) $\int_0^\infty \eta(t)e^{-it}dt = 0$

The functional J[b,c] takes its maximum value if

(52) $\left.\frac{\partial J[b(\varepsilon),c(\varepsilon)]}{\partial \varepsilon}\right|_{\varepsilon=0} = 0$

Let us calculate (52).

$$(53)\ \left.\frac{\partial J}{\partial \varepsilon}\right|_{\varepsilon=0} = \int_0^\infty [u'(c(\varepsilon))\Big|_{\varepsilon=0}\eta + \frac{1}{i}v'(b(\varepsilon))\Big|_{\varepsilon=0}(\eta + D\varphi)]e^{-\beta t}dt$$

$$= \int_0^\infty [u'(c^*) + \frac{1}{i}v'(b^*)]\eta e^{-\beta t}dt + \frac{1}{i}\int_0^\infty v'(b^*)D\varphi e^{-\beta t}dt$$

integrating by parts

$$= \int_0^\infty [u'(c^*) + \frac{1}{i}v'(b^*)]e^{-\beta t}e^{it}\eta e^{-it}dt$$

$$-\frac{1}{i}\int_0^\infty [v''(b^*)Db^* - \beta v'(b^*)]\varphi e^{-\beta t}dt$$

$$= 0$$

Thus we have

(54) $[u'(c) + \frac{1}{i}v'(b)]e^{-(\beta - i)t} = \text{const.}$

(55) $v''(b)Db - \beta v'(b) = 0$

by lemmas of the calculus of variations. By differentiating (54) with respect to time and taking (55) into account we can obtain the Euler equation.

Now we can do the comparative dynamic analysis of our system on the basis of above results.

1. The effect of a change in w

By differentiating (43) and (54) with respect to w, we have

$$(56)\quad \frac{\partial b}{\partial w}=\frac{1}{i}(e^{it}-1)-e^{it}\int_0^t \frac{\partial c}{\partial w}e^{-is}ds$$

$$(57)\quad [u''(c)\frac{\partial c}{\partial w}+\frac{1}{i}v''(b)\frac{\partial b}{\partial w}]e^{-(\beta-i)t}=0$$

Formally we can examine the effect of a change in w from the system of equations (56) and (57).

2. The effect of a change in i

By differentiating (43) and (54) with respect to i, we have

$$(58)\quad \frac{\partial b}{\partial i}=w[\frac{e^{it}t}{i}-\frac{(e^{it}-1)}{i^2}]-e^{it}t[\int_0^t ce^{-is}ds-b(0)]$$
$$-e^{it}\int_0^t[\frac{\partial c}{\partial i}-cs]e^{-is}ds$$

$$(59)\quad [u''(c)\frac{\partial c}{\partial i}+\frac{v''(b)}{i}\frac{\partial b}{\partial i}-\frac{v'(b)}{i^2}]e^{-(\beta-i)t}$$
$$+[u'(c)+\frac{1}{i}v'(b)]te^{-(\beta-i)t}=0$$

We can also examine the effect of a change in i from (58) and (59).

9. LONG-RUN STATIONARY EQUILIBRIUM

Finally we would consider the effect of a change in exogenous variables in the long-run stationary equilibrium point. The long-run equilibrium is described by the following equations.

(60) $c(\infty)=w+ib(\infty)$

(61) $(\beta-i)u'(c(\infty))=v'(b(\infty))$

In this analysis a determinant

$$(62)\quad \begin{vmatrix} 1 & -i \\ (\beta-i)u''(c(\infty)) & -v''(b(\infty)) \end{vmatrix} = -v''+i(\beta-i)u''$$

plays a crucial role. The sign of this determinant, however, is indeterminate. Therefore we can not predict the effect of a change in w, i and β in the stationary state.

NOTES

*)I am indebted to Prof.Hayakawa, Prof.Kobayashi, Prof. Shirai, Mr.Sakai, and Mr.Matsumoto for their valuable comments and suggestions. All remaining errors, however, are the sole responsibility of me.

1)Douglas assumes that the cash facilitates the coordination of receipts and expenditures.

2)The rate of time preference corresponds to the concept of the marginal efficiency of investment in the dynamic theory of firm.

3)See Uzawa[9]. Unfortunately Mills'papers are not available.

4)The uniqueness of the optimal solution will be proved in the latter part of this section.

5)We assume that u(c) and v(b) satisfy the Inada's derivative conditions.(Of course u(c) and v(b) are strictly concave.)

In this place we would briefly refer to the existence of an extremal of the variational problem of the form (20) whose feature is an infinite upper limit on the integral. Consider any function $b(t,\varepsilon) = b^*(t)+\varepsilon\varphi(t)$ where ε is sufficiently close to zero. If $b^*(t)$ is an admissible function which maximizes J[b], then $b^*(t)$ must satisfy the Euler equation and, in addition,

$$(*)\quad \lim_{t\to\infty}\left\{\frac{\partial}{\partial \dot{b}}[u(c)+v(b)]e^{-\beta t}\right\}\varphi(t)=\lim_{t\to\infty}\left\{-u'(c)e^{-\beta t}\right\}\varphi(t)=0$$

We must distinguish two cases.

CASE 1.

If $\lim_{t\to\infty} b^*(t)$ is a finite constant, that is, $b^*(t)$ approaches a stationary equilibrium value, then c also approaches a stationary value. It follows, therefore, that $\lim_{t\to\infty}\left\{-u'(c)e^{-\beta t}\right\}=0$. On the other hand, it is necessary that $\lim_{t\to\infty}\varphi(t)=0$ if $b(t,\varepsilon)$ is to be admissible. Thus (*) is satisfied. We shall adopt this case in the present paper.

CASE 2.

If $\lim_{t\to\infty} b^*(t)=\infty$, then $\varphi(t)$ is arbitrary and, in particular, it need not be true that $\lim_{t\to\infty}\varphi(t)=0$. So that it must be true that $\lim_{t\to\infty}\left\{-u'(c)e^{-\beta t}\right\}=0$. That is c must not approach zero. However we, of course, require that $|\varphi(t)|$ is bounded. We shall rule out this case. $\lim_{t\to\infty} u'(c)e^{-\beta t}=0$ is well known as the transversality condition.

See Hadley-Kemp[3], pp.37-40.

6)In Appendix I of my paper "A Reconsideration of the

Neoclassical Theory of Economic Growth"(the last issue of this journal) we have formulated the utility integral in the general form

$$(**) \quad \int_0^{\infty} u(c,b)e^{-\beta t}dt$$

under an assumption $\beta \cong i$. But this utility function is somewhat perverse. So that we would like to reject the functional (**).

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