ON THE PROCEDURE FOR DYNAMIZATION OF GENERAL EQUILIBRIUM THEORY IN TERMS OF CALCULUS OF VARIATIONS*

MUTSUHIRO KATO

> "There is one concept, however, which plays a central role in the General Theory which is not static, and that is why the General Theory will not be fully satisfactory until it is brought into relation with Dynamics. While many of the restrictions which writers have tried recently to impose on static theory strike me as vexatious and wrongheaded, there is a more radical restriction which must be imposed but which is in fact less commonly imposed. Positive saving, which plays such a great role in the General Theory, is essentially a dynamic concept. This is fundamental. --------------------------- In static economics we must assume that saving is zero. This is not formally inconsistent, although it may well be inconsistent in any likely circumstances, with a positive rate of interest."
>
> R.F.Harrod,"Towards a Dynamic Economics" (1948),pp.10-11.

1. INTRODUCTION

ALMOST EVERY ECONOMIST must recognize that one of the most important moot problems in economics is to dynamize the general equilibrium theory. We nevertheless do not know attempts of dynamization with a good success except for Uzawa's remarkable work[1]. Some economists have the view that the neoclassical and von Neumann types of the multisector theory of economic growth are the dynamic version of the static general equilibrium theory. It is, however, too difficult to believe that this view is accepted by many economists. It seems that the crucial reason why the construction of dynamic general equilibrium theory has been delayed for a long while is due to the insufficient advancement of the dynamic theory of the individual unit (i.e. household, firm). Fortunately the theory of dynamic optimization behavior of the unit in terms of calculus of variations has been developed rapidly for the past ten years. So that we can, now, rely upon this performance for our purpose. Although the dynamic models of the household and firm of the integral control type are, needless to say, the simplest and boldest formulation, yet they are the best one available at present.

We shall proceed from the general(dynamics) to the special(statics) in the subsequent sections. After the discussion of the general equilibrium system a criterion for the classification of economic theories will be stated on the basis of our general equilibrium analysis. Especially the logical relationships among the general equilibrium theory, the Keynesian theory and the neoclassical theory of economic growth and the position of them in the history of economics will be clarified.

2. ASSUMPTIONS

Main assumptions of our system are as follows.

(I)Assumptions common to both the dynamic and static general equilibrium theories

1.There are not the international trade and capital movement.

2.There is not the government sector. So that there are not public goods and social overhead capital.

3.There is not money.

4.The perfect competition prevails in markets.

5.There are not externality in the productive and consumptive activities.

6.The labor service is regarded as a single grade.

(II)Assumptions in the dynamic general equilibrium theory

1.The product is only one.

2.The fixed factor(resource) of production is only one. We call this capital.

3.The firm does not issue debts. Namely the stock is an only financial asset.

4. There is not a stock market.
5. Stocks are issued in the form of par issue.
6. The product price and the wage rate are not stochastic variables.
7. Each variable is continuous with respect to time.
8. Each unit has an infinite planning horizon.
9. There is not the technical progress.

(III)Assumption in the static general equilibrium theory

1. There are many products, many fixed factors of production and many financial assets(including bonds and deposit).

3. NOTATION

The following symbols will be employed in the present paper.

(I)Symbols used in the dynamic general equilibrium theory

1.Household

c_i=real consumption outlay of the i-th household

a_i=number of stock certificates owned by the i-th household

b_i=real stock holdings of the i-th household

β=rate of discount of the future utility(const.)

2.Firm

Q_j=real output of gross product of the j-th firm

K_j=amount of the fixed resource of production (capital) held by the j-th firm

N_j^D=demand for the variable resource of production (labor service) of the j-th firm

I_j=real gross investment of the j-th firm

k_j=capital-employment ratio of the j-th firm

r_j=rate of return of the j-th firm

δ=rate of depreciation(const.)

3.Prices

p=price of product($\neq$ const.)

π=rate of increase of the product price($\neq$ const.)

w=wage rate($\neq$ const.)

ω=rate of increase of the wage rate($\neq$ const.)

p^S=nominal par of a stock certificate(const.)

α=yield of the stock(const.)

ρ=real yield of the stock($\neq$ const.)

4.Aggregative variables

C=aggregate consumption

I=aggregate gross investment

Q=aggregate gross output

N^D=aggregate demand for labor service

N^S=aggregate labor supply(number of households)(const.)

a=total number of stock certificates possessed by all households

K=aggregate stock of capital

n=number of firms(const.)

5.Other symbol

D=differential operator d/dt

(II)Symbols used in the static general equilibrium theory

1.Household

x_i=consumption vector of the i-th household (column vector)

X_i=consumption set of the i-th household

r_i=salary and asset income except dividend of the i-th household

2.Firm

y_j=production vector of the j-th firm (column vector)

Y_j=production set of the j-th firm

π_j=profit of the j-th firm

m_j=fixed costs of the j-th firm

3.Prices

p=price vector (row vector)

4.Aggregative variables

x=total consumption vector

y=total production vector

z(=x-y)=excess demand vector

r=salary and asset income except dividend as a whole
m=fixed costs as a whole
π=total profit
s=number of households
n=number of firms
5.Other symbols
ℓ=number of commodities(number of products is ℓ less one, the ℓ-th commodity is labor service)
R^ℓ=commodity space(ℓ-dimensional Euclidean space)

4. DYNAMIC THEORY OF THE HOUSEHOLD

The representative household (the i-th household) in the dynamic world always supplies a unit quantity of labor service earning money wage w and moreover receives dividend $\alpha p^s a_i$ as a reward of holding equities. He allocates his income between the current consumption expenditure pc_i and the purchase of new stocks $p^s Da_i$(saving) in view of a certain dynamic optimality criterion.[2] It is assumed that the instantaneous utility is generated not only by the consumption but also by the real balance of stocks $b_i = p^s a_i/p$. Ignoring the intertemporal complementarity of consumption[3], dependence of the rate of discount on the consumption and utility[4] and the possibility of continual planning revision due to the change of the present date[5], we introduce a criterion functional

$$(1)\quad \int_0^\infty [u_i(c_i)+v_i(b_i)]e^{-\beta t}dt$$

where $u_i(\cdot)$and $v_i(\cdot)$ are strict concave utility functions satisfying the Inada conditions on derivatives. Our utility integral is a straightforward extension of the familiar Ramsey integral.[6] It is inconvenient to adopt the simple Ramsey integral model because of its unfavorable property of a solution.[7] The budget constraint equation is

(2) $p^s Da_i = w + \alpha p^s a_i - pc_i$

in the nominal expression or

(3) $Db_i = w/p + (\alpha - \pi) b_i - c_i = w/p + \rho b_i - c_i$

in the real expression. Thus our problem is to choose a consumption plan so as to maximize (1) among the feasible plans satisfying (3). This problem is a typical one of the calculus of variations. Let the Hamiltonian form be

(4) $H_i = e^{-\beta t}[u_i(c_i) + v_i(b_i) + \lambda_i Db_i]$

where λ_i is an auxiliary variable. The relationship between λ_i and c_i is given by

(5) $\lambda_i = u_i'(c_i)$

since H_i must be maximal with respect to consumption. By maximum principle λ_i must satisfy the following auxiliary equation.

(6) $D\lambda_i = (\beta - \rho)\lambda_i - v_i'(b_i)$

We assume that the subjective rate of discount β is greater than the real yield of the stock ρ.[8] A system of equations (3),(5) and (6) determines an optimal trajectory starting with a given initial value $b_i(0)$. Before explaining the structure of a solution geometrically in terms of the phase diagram we shall solve two differential equations algebraically. The solution to (3) is

(7) $b_i(t) = [b_i(0) + \int_0^t (w/p - c_i) e^{-\int_0^s \rho d\tau} ds] e^{\int_0^t \rho ds}$

and the solution to (6) is

$$(8)\ \lambda_i(t)=[\lambda_i(0)-\int_0^t v_i'(b_i)e^{-\int_0^s(\beta-\rho)d\tau}ds]e^{\int_0^t(\beta-\rho)ds}$$

The motion of a pair of b(t) and $\lambda(t)$ is explicitly described by (7) and (8). Since the real wage rate w/p and real yield $\rho=\alpha-\pi$ are not unchanged over time, we can not illustrate a precise phase diagram. We can, however, understand the structure of a solution to a certain extent by introducing the static expectation provisionally. Under this assumption three cases are distinguishable.

CASE I

Two singular curves $Db_i=0$ and $D\lambda_i=0$ do not intersect. There exists a unique optimal path (a heavy arrowed curve) on which the holding asset is accumulated unlimitedly as is typically illustrated in Figure 1.

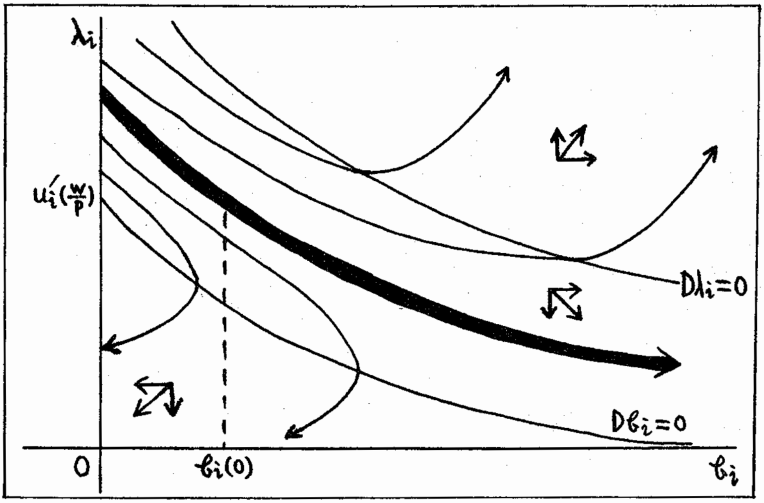


Figure 1.

CASE II

Two singular curves intersect once. The structure of a solution in this case is illustrated in Figure 2.

A unique optimal path (a heavy arrowed curve) approaches a long-run stationary equilibrium which is a saddle point.

CASE III

Two singular curves intersect twice (or more times). There also exists a unique optimal path (a heavy arrowed curve) converging a long-run equilibrium point as is depicted in Figure 3.

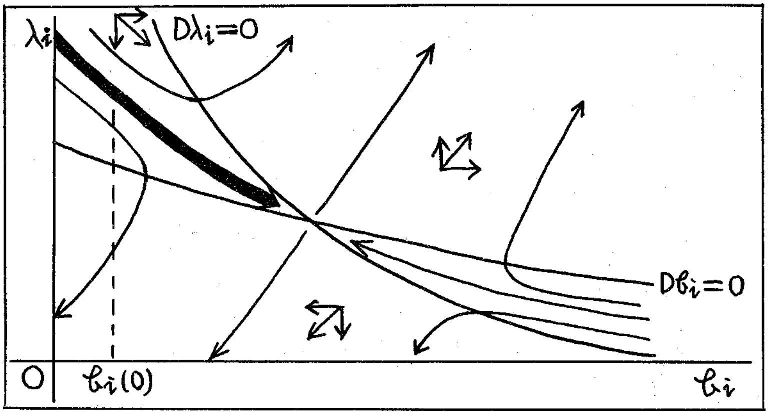


Figure 2.

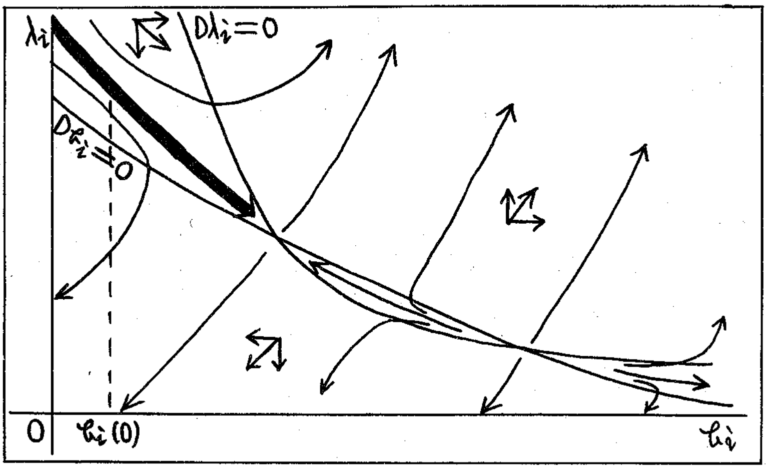


Figure 3.

In above any cases the consumption and the balance of stocks on the optimal path increase respectively and the transversality condition

$$(9)\ \lim_{t\to\infty} e^{-\beta t}\lambda_i = 0$$

is satisfied. In the present paper we assume CASE II or 3, since CASE I is not consistent with our dynamic model of the firm. As a result of above variational analysis we have the consumption function of the individual household

$$(10)\ c_i = c_i(\pi, \omega; \alpha, \beta)$$

which means that the optimal consumption plan depends on π, ω, α and β.[9]

5. STATIC THEORY OF THE HOUSEHOLD[10]

We obtain the static theory of the household by assuming that the planning horizon of the household is zero. Now it is easy to extend the model to the case of many consumption goods by utilizing concepts of vector and set in linear space, since the time variable disappears from the model. In the static world, of course, the household does not save and hence income is consumed entirely, that is, the propensity to consume equals unity. The balance of financial assets is regarded as a given constant. Therefore the utility of the saving balance also becomes a constant losing the role of variable.

The representative household earns the wage by supplying his labor service and moreover receives the salary, interest and dividend. The salary means the reward paid to the entrepreneur, manager and technician. The budget constraint equation of the i-th household is written in the form

(11) $px_i = r_i + \sum_{j=1}^{n} \theta_{ij} \pi_j$

where θ_{ij} is the fraction of the issued stock of the j-th firm that the i-th household owns. In (11) the second term of the right-hand side means the dividend income of the i-th household. (All of the profit of the corporate sector is paid to the household sector as the dividend in the static world, since the corporate sector does not invest.) The price system p is written as $[p_1 --- p_{\ell-1} p_\ell](\geq 0)$ where $p_1, ---, p_{\ell-1}$ represent prices of consumption goods and p_ℓ represents wage rate. Obviously $p \in \Omega = \{v \mid v \geqq 0, v \in R^\ell\}$ (Ω is the nonnegative quadrant in the commodity space.) The consumption vector x_i is written as $[x_1^i --- x_{\ell-1}^i x_\ell^i]'$ where $x_1^i > 0, ---, x_{\ell-1}^i > 0$ represent the demand for consumption goods and $x_\ell^i < 0$ represents the supply of labor service. The consumption vector is feasible only in a certain domain in the commodity space from the physiological point of view. This subset in the commodity space is the consumption set X_i. It is assumed that X_i is closed, convex, connected and lower bounded. Any binary relation between consumption vectors which belong to the consumption set is the complete preordering in the sense of Debreu[8]. (The complete preordering is a preference relation which satisfies the reflexivity, transitivity and completeness.) If the consumption set is a connected subset involving the complete preordering in the commodity space and satisfies the continuity assumption on preferences, then there is a continuous ordinal utility (or order preserving) function on the consumption set, and vice versa. (The utility function is a correspondence between the indifference class and the real number. Of course the utility function is an increasing function.) We assume that no saturation consumption exists for every consumer. So that the income is exhausted entirely.[11] Further we assume that the preference relation satisfies the strict convexity condition, in other words, the utility function is strictly quasiconcave. (This condition assures us the uniquness of the optimal consumption vector.[12])

As pointed out already, since the saving balance loses its role as a variable in the static theory, the utility function as the objective function is simply written in the form

(12) $u_i = u_i(x_i)$

In (12) the labor supply has disutility while consumption activity yields positive utility. Thus the rational behavior of the household is choosing an optimal consumption vector $x_i \in \mathbf{X}_i$ so as to maximize his utility indicator (12) subject to the budget restraint equation (11). Geometrically an optimal consumption vector is a point of contact of an indifference class and the budget constraint hyperplane which is orthogonal to the given price system p. Finally in our model the market value of the initial endowment of commodities is not regarded as an income and, in addition, the reservation demand for labor service is not contained in the demand for labor, since the reservation demand does not appear directly in the market. Of course the absurd assumption that the household owns the capital stock and supplies its service to the firm is not adopted in our model.

6. DYNAMIC THEORY OF THE FIRM

The firm is a collection of various scarce resources (e.g. material resources such as factory, machinery, office building etc. and immaterial ones such as sales network, managerial and R&D abilities, know-how, good-will etc.) which are called the fixed resource (or factor) of production or more simply capital. The entrepreneur seeks to manage his firm so as to maximize profit by an optimal employment policy in the short-run and so as to maximize the value of firm by an optimal investment policy in the long-run. In the dynamic world the stockholder has an interest not in the dividend but in the net cash flow which is defined by

$$(13)\quad pQ_j - wN_j^D - pI_j$$

Before deriving the optimality condition we must describe the short-run fixity of capital. The short-run fixity of capital means the fact that time and costs are required to accumulate the fixed capital.[13] The delivery lag or the gestation period of capital, the first aspect of the fixity of capital, is ignored in our analysis.[14] The costs due to the fixity of capital are usually called the costs of adjustment. The costs of adjustment consist of the planning costs and training costs and so on. The planning costs mean the costs which are incurred in making the investment project. The training costs mean the costs which are incurred in training workers to operate the new equipment. It is assumed that these costs of adjustment are represented in the form of foregone output. More specifically we formulate the supply of product Q_j as

$$(14)\quad Q_j = F^j(K_j, N_j^D) - C_j(I_j)$$

That is the supply of product equals the output $F^j(\cdot)$ less the adjustment cost $C_j(I_j)$. In (14) the production function $F^j(\cdot)$ is, for the time being, linear homogeneous and the adjustment cost function $C_j(\cdot)$ is strictly convex.[15] The reader must pay attention to the feature of our model that the technical constraint (14) is additively separable and the adjustment cost is internal cost.

The optimization behavior of the firm can be divided into two steps. The first step is the maximization of the net cash flow at every moment with respect to the amount of employment. That is the entrepreneur hires workers so as to maximize (13) for arbitrary stock of capital K_j and investment plan I_j. The necessary condition for maximum of (13) is

$$(15)\ \partial F^j(\cdot)/\partial N_j^D = w/p$$

From this we have a tentative labor demand function

$$(16)\ N_j^D = \Phi_j(K_j, w/p)$$

By the Euler theorem on the homogeneous function the capital-employment ratio k_j becomes a function of real wage rate. That is

$$(17)\ k_j = k_j(w/p)$$

The second step of optimization is the maximization of a sum of discounted present values of prospective net cash flows

$$(18)\ \int_0^\infty [pQ_j - wN_j^D - pI_j]e^{-\alpha t}dt$$

with respect to investment. We regard the rate of discount α as the yield of stock. In our model the gross investment is financed by the issue of new stocks, reserved profit and depreciation allowance, since the issue of new debts is not taken into account. The relationship between a control variable I_j and a state variable K_j is given by a performance equation with a term of proportional capital decay

$$(19)\ DK_j = I_j - \delta K_j$$

In order to derive the investment function and the path of accumulation of fixed capital dynamically optimal we form the Hamiltonian function

$$(20)\quad H_j = e^{-\alpha t}[pQ_j - wN_j^D - pI_j + p\lambda_j DK_j]$$

$$= e^{-\alpha t}[pF^j(K_j, \Phi_j(\cdot)) - pC_j(I_j) - w\Phi_j(\cdot) - pI_j + p\lambda_j(I_j - \delta K_j)]$$

where λ_j is an auxiliary variable. The necessary condition for maximum of H_j is given by

$$(21)\quad \lambda_j = 1 + C_j'(I_j)$$

(21) is described geometrically in Figure 4.

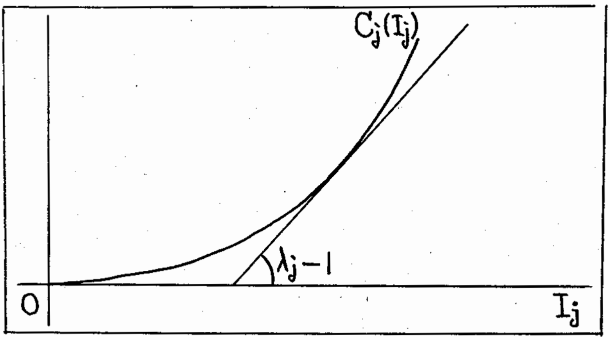


Figure 4.

By maximum principle we have

$$(22)\quad D\lambda_j = (\rho + \delta)\lambda_j - \partial F^j/\partial K_j$$

$$= (\rho + \delta)\lambda_j - f_j'[k_j(w/p)]$$

where $f_j(k_j) = F^j(\cdot)/N_j^D$ is a well known per capita production function.

We must solve (19) and (22) to obtain the optimal investment plan. The solution to (19) is

$$(23)\quad K_j(t) = (K_j(0) + \int_0^t I_j(s)e^{\delta s}ds)e^{-\delta t}$$

and the solution to (22) is

$$(24)\ \lambda_j(t)=\left(\lambda_j(0)-\int_0^t f_j'[k_j(w/p)]e^{-\int_0^s(\rho+\delta)d\tau}ds\right)e^{\int_0^t(\rho+\delta)ds}$$

The motion of K_j and λ_j under the static expectation is visualized in Figure 5.

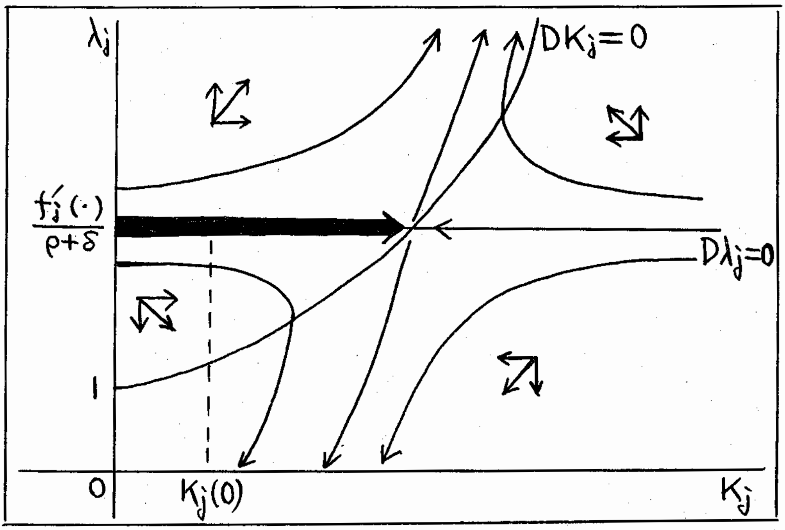


Figure 5.

A unique optimal trajectory satisfying the transversality condition $\lim_{t\to\infty} e^{-\alpha t}p\lambda_j=0$ is indicated by a heavy arrowed line. Obviously the optimal investment plan depends on π, ω, α and δ. That is we obtain the gross investment function of the individual firm

$$(25)\ I_j=I_j(\pi,\omega;\alpha,\delta)$$

Along with (25) the path of accumulation of the fixed capital is also determined simultaneously. So that we write K_j as

(26) $K_j = K_j(\pi, \omega; \alpha, \delta)$

By substituting (26) into (16) we get the labor demand function

(27) $N_j^D = N_j^D(\pi, \omega; \alpha, \delta)$

Thus we get the gross product supply function

(28) $Q_j = F^j(K_j, N_j^D) - C_j(I_j)$

$= Q_j(\pi, \omega; \alpha, \delta)$

As a digression we can easily confirm that the rate of profit

(29) $r_j = \dfrac{pQ_j - wN_j^D}{pK_j}$

depends on π, ω, α and δ. (In the literature the return or profit $pQ_j - wN_j^D$ is sometimes called the quasi-rent.)

So far we have considered the case of constant returns to scale. The case where the law of constant returns to scale does not hold is investigated by Treadway [29]. By applying his method to our model we can confirm that there exists a unique optimal plan not only in the case of decreasing returns to scale but also in the case of weak increasing returns to scale. However, there is not any optimal solution if the law of strong increasing returns to scale prevails in the economy.

7. STATIC THEORY OF THE FIRM

The static theory of the firm is obtained by assuming that there is not the future in the planning horizon. So that the investment, capital accumulation and depreciation are not treated and, moreover, the total profit is paid to the stockholder as dividend in the static theory. In other words, the investment goods is not included in the goods the firm produces and the concept of net cash flow in the dynamic theory results in the dividend or profit.

It is assumed that the firm produces many consumption goods jointly in our model. The production vector of the j-th firm is written as $y_j=[y_1^j \text{---} y_{\ell-1}^j y_\ell^j]'$. In this vector $y_1^j>0, \text{---}, y_{\ell-1}^j>0$ represent the supply of consumption goods (net output that is output less intermediate input) and $y_\ell^j<0$ represents the demand for labor service. The subset Y_j in the commodity space R^ℓ such that the production vector is technologically feasible in that set is called the production set. It is assumed that the production set is closed, strictly convex and upper bounded. So that the optimal production vector is uniquely determined. Since the amount of fixed factors of production is given in the static world, phenomena of constant and increasing returns to scale can not take place. Thus our convexity and upper boundedness assumptions are fully justified. Y_j has properties such that $0\in Y_j$, $Y_j \cap \Omega=0$ where $\Omega=\{v \mid v\geqq 0, v\in R^\ell\}$ and $Y_j \cap (-Y_j)=0$.

Given the price vector p, the entrepreneur seeks to maximize the payment of dividend

$$(30) \quad \pi_j=py_j-m_j$$

under the technical constraint. That is he chooses $y_j\in Y_j$ so as to maximize π_j. In (30) m_j means the fixed cost and it consists of interest cost, payment of salary and so on.[16] Geometrically the optimal production vector is given by the point of contact between the production set and the hyperplane with normal p. Although the maximized profit must be nonnegative, every p does not necessarily assure us the nonnegative profit. The set of price vectors such that the maximized profit is nonnegative is written as T_j'. That is

$$(31)\quad T_j'=\{p \mid p\geq 0, p\in R^{\ell}, \max \pi_j \geqq 0\}\subset R^{\ell}$$

T_j' is a closed cone with vertex O(the origin).

8. DYNAMIC GENERAL EQUILIBRIUM

So far we have analyzed the optimization behavior of the individual unit in both the dynamic and static levels. And we have derived behavioral functions of each unit. The aggregate demand and supply are equated through price mechanism in markets. In this section the dynamic competitive equilibrium is defined and the determination of the relative price (real wage rate) is discussed. **In such** an equilibrium the dynamic allocation of resource (time shape of capital accumulation) and the income distribution are determined.

Let us now aggregate the behavioral functions of each unit. We have the following aggregate functions.

$$(32)\quad C=\Sigma_{i=1}^{N^S} c_i(\pi,\omega;\alpha,\beta)=C(\pi,\omega;\alpha,\beta)$$

$$(33)\quad I=\Sigma_{j=1}^{n} I_j(\pi,\omega;\alpha,\delta)=I(\pi,\omega;\alpha,\delta)$$

$$(34)\quad Q=\Sigma_{j=1}^{n} Q_j(\pi,\omega;\alpha,\delta)=Q(\pi,\omega;\alpha,\delta)$$

$$(35)\quad N^D=\Sigma_{j=1}^{n} N_j^D(\pi,\omega;\alpha,\delta)=N^D(\pi,\omega;\alpha,\delta)$$

Thus the dynamic equilibrium of the product market is given by

(36) $Q(\pi,\omega;\alpha,\delta)=C(\pi,\omega;\alpha,\beta)+I(\pi,\omega;\alpha,\delta)$

The dynamic equilibrium of the labor market is given by

(37) $N^S=N^D(\pi,\omega;\alpha,\delta)$

Unknowns in a system of equations (36) and (37) are π and ω.[17] Absolute prices of p and w, however, are not determined since Walras' law

(38) $p[C+I-Q]+w[N^D-N^S]=0$

$$\text{where } p(t)=p_0\exp[\int_0^t\pi(s)ds]$$
$$w(t)=w_0\exp[\int_0^t\omega(s)ds]$$

holds. Walras' law (38) is derived as follows. An identity

(39) $w+\alpha p^S a_i=pc_i+p^S Da_i$

holds for the i-th household. Summing (39) over all households yields

(40) $wN^S+\alpha p^S a=pC+p^S Da$

On the other hand, an identity

(41) $pQ_j=wN_j^D+(\text{Internal Reserve})_j+(\text{Dividend})_j+p\delta K_j$

holds for the j-th firm. Summing (41) over all firms yields

(42) $pQ=wN^D+(\text{Internal Reserve})+(\text{Dividend})+p\delta K$

Combining (40) and (42) we have (38), since the payments of dividend always equal the receipts of dividend $\alpha p^S a$ from our assumption. Thus we obtain the equilibrium price ratio w/p. The distributive aspect of the equilibrium is shown in Table 1.

TABLE 1.

<table>
<tr><td colspan="5">Gross National Product pQ</td></tr>
<tr><td rowspan="4">Wage Cost
wN^D</td><td colspan="4">Gross Profit $pQ-wN^D$</td></tr>
<tr><td colspan="3">Net Profit $pQ-wN^D-p\delta K$</td><td>Depreciation
$p\delta K$</td></tr>
<tr><td colspan="2">Dividend</td><td>Internal Reserve</td><td>Depreciation Allowance</td></tr>
<tr><td>Net Cash Flow
$pQ-wN^D-pI$</td><td>Issue of New Stocks</td><td colspan="2">Gross Savings of Firms</td></tr>
<tr><td colspan="2">Consumption pC</td><td colspan="3">Gross Investment pI</td></tr>
<tr><td colspan="2"></td><td colspan="2">Net Investment pDK</td><td>Replacement Investment</td></tr>
<tr><td colspan="4">Net National Product</td><td rowspan="3"></td></tr>
<tr><td>Wage Income
wN^S</td><td colspan="2">Dividend Income $\alpha p^S a$</td><td rowspan="2"></td></tr>
<tr><td colspan="3">Income of Households</td></tr>
<tr><td colspan="2">Consumption pC</td><td>Savings of Households
$p^S Da$</td><td>Net Savings of Firms</td><td>Depreciation Allowance</td></tr>
<tr><td colspan="2">Consumption</td><td colspan="3">Gross Savings</td></tr>
</table>

The precise definition of the dynamic competitive equilibrium in the decentralized market economy is as follows. (A superscript "*" indicates an equilibrium value.)

1.HOUSEHOLD

$$\int_0^{\infty}[u_i(c_i^*)+v_i(b_i)]e^{-\beta t}dt=\max_{c_i}\int_0^{\infty}[u_i(c_i)+v_i(b_i)]e^{-\beta t}dt$$

$p^*c_i+p^SDa_i=w^*+\alpha p^Sa_i$, $b_i=p^Sa_i/p^*$, for each i

2.FIRM

$$\int_0^{\infty}[p^*\{F^j(K_j,N_j^D{}^*)-C_j(I_j^*)\}-w^*N_j^D{}^*-p^*I_j^*]e^{-\alpha t}dt$$

$$=\max_{N_j^D,I_j}\int_0^{\infty}[p^*\{F^j(K_j,N_j^D)-C_j(I_j)\}-w^*N_j^D-p^*I_j]e^{-\alpha t}dt$$

$Q_j^*=F^j(K_j,N_j^D{}^*)-C_j(I_j^*)$, $DK_j=I_j-\delta K_j$, for each j

3.MARKETS

$C^*+I^*=Q^*$, $C^*=\sum_i c_i^*$, $I^*=\sum_j I_j^*$, $Q^*=\sum_j Q_j^*$

$N^D{}^*=N^S$, $N^D{}^*=\sum_j N_j^D{}^*$

4.PRICES

$$p^*=p_0\exp[\int_0^t\pi^*(s)ds]>0$$

$$w^*=w_0\exp[\int_0^t\omega^*(s)ds]>0$$

9. STATIC GENERAL EQUILIBRIUM

The static version of the competitive system dynamically considered in the last section is described in this section. The concept of equilibrium in the static world has been studied by many economists including L.Walras, V.Pareto, J.R.Hicks, P.A.Samuelson, G.Debreu, K.J.Arrow, W.Hildenbrand et. al. along with the concept of optimum for the past one hundred years. Our system is, needless to say, essentially the same formulation as Arrow-Debreu model.[18] The static competitive equilibrium, p^* and $z^*=x^*-y^*$, is defined as follows.

1.HOUSEHOLD

$$u_i(x_i^*)=\max_{x_i} u_i(x_i),\quad x_i \in X_i,\quad p^* x_i = r_i + \sum_{j=1}^{n} \theta_{ij} \pi_j^*,\quad \text{for each } i$$

2.FIRM

$$p^* y_j^* - m_j = \max_{y_j} [p^* y_j - m_j],\quad y_j \in Y_j,\quad \text{for each } j$$

3.MARKETS

$$p^* z^* = p^*(x^* - y^*) = 0,\quad z^* \leqq 0,\quad x^* = \sum_{i=1}^{s} x_i^*,\quad y^* = \sum_{j=1}^{n} y_j^*$$

4.PRICES

$$p^* \in T = \{p \mid p \in R^{\ell}, p \geq 0, \sum_{k=1}^{\ell} p_k = 1, \pi_j^* \geqq 0, \text{ for each } j\}$$

$pz=0$ represents Walras' law. This is obtained as follows. Summing the budget restraint equations over all households yields

(43) $px=r+\pi$

Summing an identity

(44) $\pi_j = py_j - m_j$

over all firms yields

(45) $\pi = py - m$

Substituting (45) for (43) yields Walras' law, since r=m. A set T is obtained by adding a simplex condition $\Sigma_k p_k=1$ to

$$(46)\quad T'=T_1'\cap T_2'\cap\cdots\cap T_n'$$

A normalization of the price system makes the proof of the existence of an equilibrium utilizing the Kakutani fixed point theorem possible. (The proof is not shown here. It can be performed along Arrow-Debreu line.) Of course, we can not determine absolute prices in the system. In other words, we can determine a relative price system alone. The price ratio among various goods was called "value" in the classical tradition.

The distributive aspect of the static competitive equilibrium is shown in Table 2.

Table 2.

<table>
<tr><td colspan="3">National Income</td></tr>
<tr><td>Wage Income
$-p_\ell x_\ell$</td><td>Salary and Asset Income except Dividend $r=\Sigma_i r_i$</td><td>Dividend Income
$\pi=\Sigma_i\Sigma_j\theta_{ij}\pi_j$</td></tr>
<tr><td colspan="3">Expenditure(Demand for Consumption Goods) $\sum_{k=1}^{\ell-1} p_k x_k$</td></tr>
<tr><td colspan="3">Output(Supply of Consumption Goods) $\sum_{k=1}^{\ell-1} p_k y_k$</td></tr>
<tr><td>Wage Cost
$-p_\ell y_\ell$</td><td>Fixed Costs
$m=\Sigma_j m_j$</td><td>Profit
py-m</td></tr>
</table>

10. DYNAMIC OPTIMUM

The welfare implications of the dynamic growth processes have been considered, in the main, in the theory of optimal growth for these fifteen years. As for the one-sector theory contributions by P.A.Samuelson, D.Cass, T.C. Koopmans, C.C.von Weizsäcker, et. al., as for the two-sector theory contributions by H.Uzawa et. al. and as for the multisector theory contributions by DOSSO, R.Radner, M.Morishima, L.Mckenzie et. al. are especially remarkable. Since these so-called turnpike theorems, however, are not based on an explicit analysis of the behavior of individual units, it is yet ambiguous whether the dynamic equilibrium is Pareto optimal and an arbitrarily given dynamic Pareto optimum can be realized by means of the market mechanism. Although Malinvaud[21],[22] form a notable exception in the point that his model preserves the thought of general equilibrium theory to some extent, his analysis is fairly formal and abstract. After all we must dynamize Arrow's and Debreu's basic theorems of welfare economics[19] in the framework of general equilibrium system. However, this work is extremely difficult. I do not know whether the proof of the basic theorems of welfare economics dynamically formulated is possible. In this section we give a definition alone of the dynamic Pareto optimum.

A state $[c_i^o, I_j^o, Q_j^o, N_j^{Do}]$ is a dynamic optimum, if the following three conditions are met.

1.Market equilibrium

$$C^o+I^o=Q^o,\quad C^o=\sum_i c_i^o,\quad I^o=\sum_j I_j^o,\quad Q^o=\sum_j Q_j^o$$

$$N^{Do}=N^S,\quad N^{Do}=\sum_j N_j^{Do}$$

2.It is impossible to increase the utility integral of one or more households without decreasing the utility integral of other households.

3.$c_i^o>0$, for all i

$Q_j^o = F^j(K_j, N_j^{Do}) - C_j(I_j^o)$, for all j

(That is c_i^o is feasible physiologically and I_j^o, Q_j^o and N_j^{Do} are feasible technologically.)

Needless to say, this definition is an immediate extension of Debreu's one.

11. STATIC OPTIMUM

The concept of static optimum is well known already in the field of welfare economics. The definition of the static Pareto optimum is as follows. A state $[x_i^o, y_j^o]$ is a static optimum if the following three conditions are met.

1.Market equilibrium

$$z^o = x^o - y^o \leqq 0,\quad x^o = \sum_i x_i^o,\quad y^o = \sum_j y_j^o$$

2.It is impossible to make one household better off without making another one worse off.

3.$x_i^o \in X_i$, for all i

$y_j^o \in Y_j$, for all j

The relationship between the competitive equilibrium and the optimum in the framework of statics is given by the basic theorems of welfare economics ("an equilibrium is an optimum" and "given an optimum, there is an equilibrium").

12. CLASSIFICATION OF ECONOMIC THEORIES

Now we can classify various economic theories on the basis of above discussion. Economic theories can be divided into the three categories. The first category is a system which takes no account of the firm as a unit different from the household. The second is a system which takes account of the firm as a unit different from the household. And the last is a system which does not formulate the rational behavior of the unit. These three categories can be divided between the static theory and the dynamic theory respectively. We have Table 3. by applying this criterion to existing theories.

Table 3.

<table>
<tr><th>Criterion</th><th>Statics</th><th>Dynamics</th></tr>
<tr><td>A system which takes no account of the firm as a unit different from the household</td><td></td><td>Neoclassical Theory of Economic Growth</td></tr>
<tr><td rowspan="2">A system which takes account of the firm as a unit different from the household</td><td rowspan="2">Static General Equilibrium Theory</td><td>Dynamic General Equilibrium Theory</td></tr>
<tr><td>Keynesian Theory of Economic Growth</td></tr>
<tr><td rowspan="2">A system which does not formulate the rational behavior of the unit</td><td rowspan="2">Static Leontief Model</td><td>Dynamic Leontief Model</td></tr>
<tr><td>von Neumann Growth Model</td></tr>
</table>

We would explain this classification table in detail.

1.The neoclassical theory

The word "Neoclassical Theory" is used in various senses in the literature. We, however, confine the use of this word to the case of so-called neoclassical theory of economic growth. In the neoclassical world an individual is not only a household but also a firm, in other words, not only a worker but also an entrepreneur. So that the individual is autarkic as if he were like Robinson Crusoe. Namely the individual produces output by utilizing his capital stock and his own labor, and consumes a part of output produced and invests the rest. The division of products between consumption and investment depends on his intertemporal preference ordering. The saving is done in the form of real assets. Of course the form of investment function and saving function is completely identical. Since the economy is not divided between the household sector and the corporate sector, there are not any market (the product market, labor market and financial market) and therefore any price. Thus the neoclassical world is not the modern capitalist economy. Although it is sometimes pointed out that the substitutability between factors of production and flexibility of prices are essential to the neoclassical growth theory, such a view is beside the mark. (We will refer to the substitutability between productive factors again in the discussion of the Keynesian theory of economic growth.)

In the usual neoclassical analysis of economic growth the consideration of the microeconomic foundation is ignored except Cass-Yaari[6] and Uzawa[30],[34]. We shall construct the neoclassical growth model with special reference to its microeconomic foundation.[20] Let the i-th individual's production function be

(47) $q_i = f_i(k_i)$

where q_i is output and k_i is capital stock. $f_i(\bullet)$ satisfies the Inada conditions. It is assumed that the individual has a unit quantity of labor service. The output produced is divided between consumption c_i and investment (saving) Dk_i.

(48) $Dk_i = q_i - c_i$

The performance equation (48) is the budget constraint equation. The individual should maximize a utility integral

(49) $\int_0^\infty u_i(c_i)\exp(-\beta_i t)dt$

subject to (48). The utility function $u_i(\cdot)$ is strictly concave and satisfies the Inada conditions. His dynamic optimization behavior is examined as follows. Form the Hamiltonian expression

(50) $H_i=\exp(-\beta_i t)[u_i(c_i)+\lambda_i(f_i(k_i)-c_i)]$

where λ_i is the auxiliary variable. The necessary condition for maximum of H_i is

(51) $\lambda_i=u_i'(c_i)$

The motion of λ_i is given by the following auxiliary equation.

(52) $D\lambda_i=-[f_i'(k_i)-\beta_i]\lambda_i$

On the optimal path $\lambda_i \not\equiv 0$. Further the present value of λ_i must converge to zero ultimately.

(53) $\lim_{t\to\infty}\exp(-\beta_i t)\lambda_i=0$

Thus we have a phase diagram (Figure 6.).

There exists a unique optimal path indicated by a heavy arrowed curve. The optimal plan of consumption and that of capital accumulation depend on the subjective rate of discount β_i. Therefore we obtain the consumption function

(54) $c_i=c_i(\beta_i)$

Since the accumulation path of capital is also a function of β_i

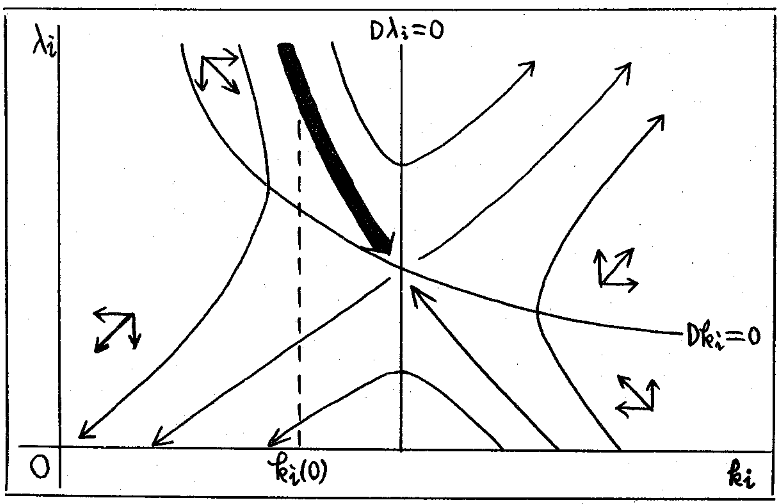


Figure 6.

(55) $k_i = k_i(\beta_i)$

Hence we have the product supply function

(56) $q_i = f_i[k_i(\beta_i)] = q_i(\beta_i)$

The investment function or saving function is written as

(57) $Dk_i = q_i(\beta_i) - c_i(\beta_i) = \varphi_i(\beta_i)$

The above is the microeconomic aspect of the neoclassical growth model of the Solow type. The aggregate behavioral functions are easily obtained. That is

(58) $C = \sum_{i=1}^{N} c_i(\beta_i) = C(\beta_1, \beta_2, ---, \beta_N)$

(59) $Q = \sum_i q_i(\beta_i) = Q(\beta_1, \beta_2, ---, \beta_N)$

(60) $DK = \sum_i Dk_i = \sum_i \varphi_i(\beta_i) = \Phi(\beta_1, \beta_2, ---, \beta_N)$, where $K = \sum_i k_i$

N is the population (number of individuals) and it is kept to be unchanged over time. Of course

(61) $C+DK=Q,\ (Q=\sum_i f_i(k_i))$

holds.[21]

2. The dynamic general equilibrium theory and the Keynesian theory of economic growth

In these theories it is recognized that the economy is divided between the corporate sector and the household sector. So that there are markets to bridge both sectors. Namely the economy is an interdependent organic entity in which many units are closely connected with one another through transactions in markets. There always exists a possibility of market disequilibrium in such an economy. Prices always change in such a way that markets are cleared, that is, the demand and supply are equated apart from their effectiveness. The difference between the dynamic general equilibrium theory and the Keynesian growth theory is that the former analyzes the dynamic optimization behavior of individual units, while the latter does not do it. Recent Keynesian models of economic growth have the feature that the substitutable aggregate production function with aggregate capital stock is assumed and the market disequilibrium generates the price change. An important conclusion of such a study is that the substitutability between factors of production and the flexibility of prices in markets do not necessarily assure us the stability of the balanced growth path (long-run equilibrium).

13. A NOTE ON THE HISTORY OF ECONOMICS

As a consequence of above discussion we reach a point of view on the methodology of the history of economics. Our method is essentially based on Mr. Kuhn's one.[22] We know four (or five) paradigms of economics at present.

The two of them were buried already and the rest is yet surviving. The former is the physiocracy and the classical economics and the latter is the static general equilibrium theory and the Keynesian macrodynamic theory (and, in addition, the dynamic general equilibrium theory). We would briefly explain these paradigms in what follows.

The first paradigm in economics is physiocracy. Quesnay's "Tableau économique" was the first systematic model of the national economy in which the structure of circulation was explicitly described. The Quesnay model, however, did not deal with the working of price mechanism in detail, although he was a supporter of the free enterprise system and free competition.

The classical school concentrated their effort on the study of distribution in the capital accumulation process in the period of the industrial revolution. The classical economics did not formulate the maximization behavior of the individual unit, although their concern was completely in the market system. As is well known the classical economists (A.Smith, T.Malthus, D.Ricardo, J.S.Mill, K.Marx et. al.) adopted the hypothesis of the "labor theory of value". But since this labor theory of value had not the sufficient validity as was noticed already by Ricardo and Mill, the scientific revolution in the 1870's necessarily arose.

Walras surmounted defects of the analysis of the market economy in the classical economics by theorizing the rational behavior of the firm and the household as a problem of constrained maximization. Namely Walras constructed the foundation of the static general equilibrium theory of the multimarket. In this respect contributions by Menger and Jevons were insufficient, since they did not deal with the theory of firm. In many textbooks of the history of economics this scientific revolution is called the "marginal revolution." This term, however, is somewhat inadequate. We shall use the word the "WALRASIAN REVOLU-

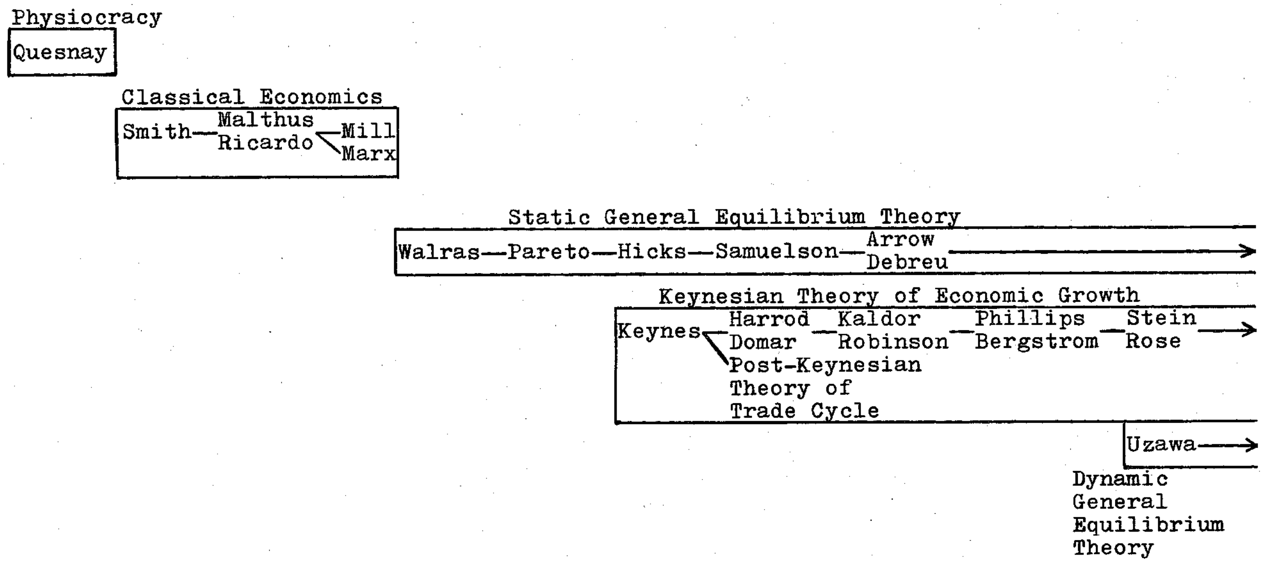
TION" instead of the marginal revolution. It should be noted that the analysis of the maximization behavior of the individual unit had already been performed to some extent by H.Gossen (the case of household), A.Cournot (the case of firm) and D.Lardner (the case of firm) before the Walrasian revolution. But they could not reach an idea of the determination of prices in the market mechanism. Although Walras resolved the paradox of value, he and his followers failed in theorizing the dynamic aspect of the market economy, that is, in the analysis of capital accumulation and economic growth which was an important part of the classical economics. This fact gave rise to the crisis of economics in 1930's.

Since the static general equilibrium theory does not involve the analysis of investment and saving, we can not analyze the unemployment. Keynes focused attention on the dynamic behavior of the firm (investment behavior) and that of the household (saving behavior) and achieved the Keynesian revolution. Since the analysis of Keynes himself was insufficient, much of effort for true dynamization has been made by R.F.Harrod, E.D.Domar, N.Kaldor, J. Robinson, A.W.Phillips, A.R.Bergstrom, A.C.Enthoven, J.L. Stein, H.Rose, H.Uzawa et. al. for these forty years.[23] This stream of the development of the Keynesian economics is called the Keynesian theory of economic growth. A feature of this Keynesian growth theory is that the microeconomic analysis of the behavior of the individual unit is not performed sufficiently. Formulating the dynamic maximization behavior of the unit rigorously in the Keynesian framework is building the dynamic general equilibrium theory. The dynamization of the general equilibrium theory has been delayed, although many economists have hoped it for a long time. At last Uzawa's epochmaking papers, however, have appeared in 1969. He has achieved the dynamization of the theory of unit in terms of calculus of variations. Since the monetary aspect of the theory, however, is extremely weak in Uzawa's (and also in my) setting, the further development is hoped for. (This fact applys similarly to the case of static general equilibrium system.) Moreover it remains to prove the existence of a dynamic equilibrium and to investigate welfare implications of it.

Finally we are now in a new sort of crisis. The scope of analysis of our economics is confined, in the main, to the market system. So that we can not deal with sufficiently some serious problems such as pollution, environmental disruption, externality and the necessity of supply of public goods and accumulation of social overhead capital which emerge outside the market mechanism. In other words we recognize severely that the price system can not resolve all economic problems. That is we must achieve a new scientific revolution. This is, needless to say, an extremely difficult work. But we must not advance avoiding it.

The locus of the evolution of economics is summarized in Table 4.

Table 4. Paradigms in the History of Economics



NOTES

*) I have greatly benefited from Prof. Uzawa's lecture at the graduate school of the University of Hokkaido and from frequent discussions with Prof. Hayakawa, Prof. Shirai, Prof. Kobayashi, Mr. Sakai and Mr. Matsumoto. All remaining errors are the sole responsibility of me.

1) Uzawa[31],[33],[35].

2) There are two criteria. One is the utility integral maximization approach used in the present paper and another is the rate of time preference approach originated by Mills and Uzawa. (See for example Uzawa[34].)

3) This assumption may be justified to some extent, since we do not know the general law of intertemporal complementary relation at all.

4) See Koopmans[16] and Mills' discussion cited in Uzawa [37]. (Unfortunately Mills' papers are not available.) Uzawa[32] deals with a saving model with an endogenous rate of discount. In general the rate of discount depending on the utility and consumption makes the computation complicated.

5) See Strotz[28].

6) Somewhat detailed explanation of our saving model is given in Kato[15].

7) A simple Ramsey integral

$$\int_0^\infty u_i(c_i)e^{-\beta t}dt$$

maximization model does not yield an optimal solution except for the case where $\beta=\rho$. However, a finite horizon type of formulation

$$\int_0^T u_i(c_i)e^{-\beta t}dt, \quad 0<T<\infty$$

always has an optimal solution. The finite horizon model with a bequest motive is examined by Yaari[42]. See Kato[15].

8) We have no grounds for believing the validity of this assumption. This, however, may be justified in view of myopic imprudence of the consumer.

9) Since

$$p(t)=p_0 \exp[\int_0^t \pi(s)ds]$$

$$w(t)=w_0 \exp[\int_0^t \omega(s)ds]$$

π and ω can play the role of arguments.

10) Debreu[8] is utilized in the description of the static general equilibrium system.

11) Thus the budget constraint equation can not take the form of inequality.

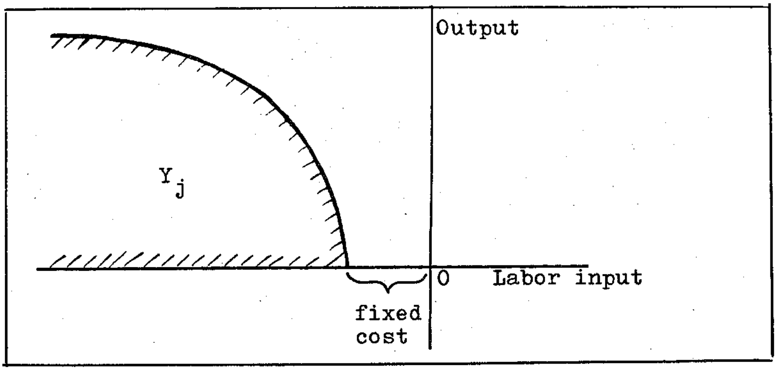
12) This fact is proved in Arrow[1].

13) The investment theory ignoring the short-run fixity of capital is sometimes called the neoclassical theory of investment. Such a theory has the feature that investment at an infinite rate at the initial point is required to raise a given initial stock of capital to a desired level. See Arrow[2]. Jorgenson[13] is also included in the neoclassical category in a wider sense.

14) Recent Maccini's paper deals with the effect of the delivery lag on the optimal amount of investment. See Maccini[20].

15) Gould[11], Lucas[18],[19], Maccini[20], Sakai[25], Treadway[29], Uzawa[31],[33],[35],[39],[40] are other types of formulation of the adjustment cost function. Lucas[19] attempts an extension to the case of many capital goods.

16) Some economists represent the amount of fixed costs by the distance between the production set and the origin. (The case of one output is illustrated.) This view, however, is obviously inadequate.



17) We can rewrite (36) and (37) as

$$(36)^* \quad Q(p,w;\alpha,\delta)=C(p,w;\alpha,\beta)+I(p,w;\alpha,\delta)$$

$$(37)^* \quad N^S=N^D(p,w;\alpha,\delta)$$

except for the initial point. Market equilibrium equations at the initial point determine π_0 and ω_0.

18) Arrow-Debreu[3].

19) Arrow[1], Debreu[7],[8].

20) The description of this section is based on Kato[14].

21) The proof of sufficiency is as follows. We indicate variables on the optimal path by "*". Variables without the asterisk are feasible ones. Our object is to show that

$$(a) \quad \int_0^\infty u_i(c_i^*)\exp(-\beta_i t)dt > \int_0^\infty u_i(c_i)\exp(-\beta_i t)dt$$

Compute the difference between two utility integrals in (a).

$$(b) \quad \int_0^\infty u_i(c_i^*)\exp(-\beta_i t)dt-\int_0^\infty u_i(c_i)\exp(-\beta_i t)dt$$
$$=\int_0^\infty [u_i(c_i^*)-u_i(c_i)-u_i'(c_i^*)(c_i^*-c_i)]\exp(-\beta_i t)dt$$
$$+\int_0^\infty u_i'(c_i^*)(c_i^*-c_i)\exp(-\beta_i t)dt$$

By using (48) the second term can be rewritten as follows.

$$\text{(c)}\quad \int_0^\infty u_i'(c_i^*)(c_i^*-c_i)\exp(-\beta_i t)dt=\int_0^\infty u_i'(c_i^*)[f_i(k_i^*)-f_i(k_i)]$$

$$\times\exp(-\beta_i t)dt-\int_0^\infty u_i'(c_i^*)(Dk_i^*-Dk_i)\exp(-\beta_i t)dt$$

Integrating the second term by parts yields

$$\text{(d)}\quad \int_0^\infty u_i'(c_i^*)(c_i^*-c_i)\exp(-\beta_i t)dt=\int_0^\infty u_i'(c_i^*)[f_i(k_i^*)-f_i(k_i)]$$

$$\times\exp(-\beta_i t)dt$$

$$+\int_0^\infty(k_i^*-k_i)[u_i''(c_i^*)Dc_i^*-\beta_i u_i'(c_i^*)]\exp(-\beta_i t)dt$$

$$-[u_i'(c_i^*)(k_i^*-k_i)\exp(-\beta_i t)]_0^\infty$$

Substituting the Euler equation into the second term yields

$$\text{(e)}\quad \int_0^\infty u_i'(c_i^*)(c_i^*-c_i)\exp(-\beta_i t)dt=\int_0^\infty u_i'(c_i^*)[f_i(k_i^*)-f_i(k_i)$$

$$-f_i'(k_i^*)(k_i^*-k_i)]\exp(-\beta_i t)dt$$

$$-[u_i'(c_i^*)(k_i^*-k_i)\exp(-\beta_i t)]_0^\infty$$

Hence (b) is written as

$$\text{(f)}\quad \int_0^\infty u_i(c_i^*)\exp(-\beta_i t)dt-\int_0^\infty u_i(c_i)\exp(-\beta_i t)dt$$

$$=\int_0^\infty[u_i(c_i^*)-u_i(c_i)-u_i'(c_i^*)(c_i^*-c_i)]\exp(-\beta_i t)dt$$

$$+\int_0^\infty u_i'(c_i^*)[f_i(k_i^*)-f_i(k_i)-f_i'(k_i^*)(k_i^*-k_i)]\exp(-\beta_i t)dt$$

$$-[u_i'(c_i^*)(k_i^*-k_i)\exp(-\beta_i t)]_0^\infty$$

The first and the second terms are positive by virtue of the strict concavity of $u_i(c_i)$ and $f_i(k_i)$. The third term

$$\text{(g)}\ [u_i'(c_i^*)(k_i^*-k_i)\exp(-\beta_i t)]_0^\infty=\lim_{t\to\infty}(k_i^*-k_i)\exp(-\beta_i t)u_i'(c_i^*)$$
$$-u_i'(c_i^*(0))[k_i^*(0)-k_i(0)]$$

vanishes by virtue of the transversality condition. Thus (a) holds.

Next we would restate our model by Uzawa's approach which does not rely on the utility integral. The rate of time preference δ_i is written as

$$\text{(h)}\ \delta_i=\delta_i(Dc_i/c_i)$$

if the intertemporal preference ordering is not only separable but also homothetic. (The Ramsey integral (49) is homothetic if and only if $u_i(c_i)$ takes the form $u_i(c_i)=-Ac^{1-\eta}(\eta>1)$. Then δ_i is written as $\delta_i=\beta_i+\eta Dc_i/c_i$. See Uzawa[40], p.23, footnote 2.) The dynamic optimality condition is

$$\text{(i)}\ \delta_i=f_i'(k_i)$$

This corresponds to the Euler equation. Let us derive differential equations of the output q_i and the average propensity to consume $x_i=c_i/q_i$ to analyze the structure of a solution by the phase diagram. We can easily get

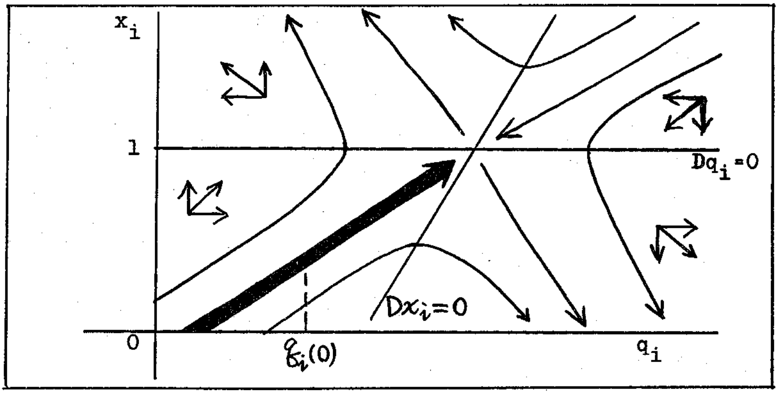
$$\text{(j)}\ Dq_i/q_i=f_i'(k_i)(1-x_i)$$

This corresponds to the budget constraint equation. Another equation is

$$\text{(k)}\ Dx_i/x_i=Dc_i/c_i-f_i'(k_i)(1-x_i)$$

$$=Dc_i/c_i-\delta_i(Dc_i/c_i)(1-x_i)$$

A singular curve $Dq_i=0$ is represented by $x_i=1$. And the configulation of the Fisherian function $\delta_i(\cdot)$ makes the slope of another singular curve $Dx_i=0$ positive. The phase diagram of this system is pictured in the following figure.



An optimal path starting with a given initial value $q_i(0)=f_i(k_i(0))$ is indicated by a heavy arrowed curve. (See Uzawa[34].)

22) Kuhn[17].

23) Phillips[23], Bergstrom[4],[5], Enthoven[9], Stein [26],[27], Rose[24],Uzawa[36],[38],[39]. Besides for example Inada[12],Williamson[41],Fujino[10] and so on.

(University of Hokkaido)

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