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STABILIZATION THROUGH MONETARY POLICY IN A GROWING  
ECONOMY IN WHICH THE PERSISTENT INFLATION PREVAILS

MUTSUHIRO KATO\*

## 1. INTRODUCTION

IT IS OFTEN SAID that various economic theories can be divided between the positive theory and the normative one. The theory of optimal economic growth as a normative theory has been developed vigorously for the last two decades. On the contrary, little performances have been accumulated in the theory of stable economic growth. Recently J.L.Stein and K.Nagatani have contributed much to this field on the basis of the Keynes-Wicksell model of monetary growth (Stein-Nagatani[3]). Subsequently J.B.Taylor[4], R.J.Mackay[2] and M.G. Hadjimichalakis[1] have elaborated the Stein-Nagatani model. It, however, seems that the consideration of this problem is not necessarily done sufficiently.

In the present paper we shall study the monetary stabilization policy in the framework of the Keynes-

Wicksell model. First we construct a model of monetary growth. Next we ascertain that the monetary policy is effective to stabilize the system. And finally the speed of convergence of the dynamic system is discussed.

## 2. THE MODEL

Consider a competitive, fully employed economy where a single good  $Y$  is produced with the aid of fixed capital  $K$  and labor  $N$ . That is, the aggregative production function is written as

$$(1) Y=Y(K,N),$$

$$Y_1 > 0, Y_2 > 0; Y_{11} < 0, Y_{22} < 0$$

where  $Y_i$  means a derivative of  $Y$  with respect to the  $i$ -th argument.  $Y(\cdot)$  is homogeneous of degree one in  $K$  and  $N$ . The labor force grows at a constant rate  $n$ . That is,

$$(2) N=N_0 e^{nt}.$$

$N_0$  is an initial value of  $N$ . By virtue of the linear homogeneity of (1) the output per unit of capital  $y=Y/K$  is written as

$$(3) y=y(x),$$

$$y' > 0, y'' < 0$$

where  $x$  is labor per unit of capital  $N/K$ .  $y'(x)$  is the marginal product of labor. It is assumed that the marginal productivity of any factor is diminishing. The marginal product of capital  $r$  is written as

$$(4) r(x) = y(x) - xy'(x).$$

$$r' = -xy'' > 0$$

$r$  may be regarded as the rate of profit.

The investment behavior of the corporate sector is described by the following function.

$$(5) I/K = \Phi_1[r(x), i, q]$$

$$= \Phi_2[r(x), i, g(\pi)]$$

$$\Phi_1 > 0, \Phi_2 < 0, \Phi_3 > 0$$

In (5)  $I$  is the desired investment in the real term,  $K$  is the stock of capital in the real term (Assume that the life of  $K$  is perpetual.),  $i$  is the real rate of interest and  $q$  is the expected rate of change of the price level. The rate of investment is encouraged by the higher rate of return on capital assets and by the lower rate of interest. We introduce the following price expectations function.

$$(6) q = g(\pi)$$

$$g' > 0$$

Namely the expected rate of inflation  $q$  is positively related to the rate of price change currently experienced  $\pi$ .

The saving in the real term  $S$  is written as

$$(7) S = S[Y + DM/p, M/p + K, q].$$

In (7)  $M$  is the stock of money,  $p$  is the absolute price level of the product and  $D$  is a differential operator  $d/dt$ . The money is supplied at a proportionate growth rate  $\mu$  in the form of transfer payments. That is,

$$(8) DM/M = \mu.$$

M is the outside money, that is, the financial claims of the private sector upon the public sector. Thus in (7)  $Y+DM/p$  means the disposable income<sup>1/</sup>. The second argument  $M/p+K$  refers to the wealth effect. The effect of  $M/p$  on S is the so-called Pigou effect. The reason why K is taken into account is that the saving is done not only by the households but also by the business corporations. S is homogeneous of degree one in  $Y+DM/p$  and  $M/p+K$ . Then the saving per unit of capital  $S/K$  is written as

$$(9) S/K = S[y(x) + \mu m, m+1, q] \\ = S^*[y(x) + \mu m, m, g(\pi)], \\ S_1^* > 0, S_2^* < 0, S_3^* < 0$$

where m is the real money balance per unit of capital  $M/pK$ .

The nominal demand for money balance L is written as

$$(10) L = L[pY, i, r(x), M+pK, q].$$

L is homogeneous of degree one in pY and  $M+pK$ . So that, we have

$$(11) L/pK = L[y(x), i, r(x), m+1, q] \\ = L^*[y(x), i, r(x), m, g(\pi)]. \\ L_1^* > 0, L_2^* < 0, L_3^* < 0, L_4^* > 0, L_5^* < 0$$

An argument  $y(x)$  is related to the transaction motive. Arguments i and r (rates of return on alternative assets to money) are opportunity costs of holding wealth in the

form of cash balances. An argument  $m(+1)$  represents the complementary relation between the demand for money balances and the stock of wealth. The last argument  $g(\pi)$  implies that money is a risky asset during the period of inflation.

The change of price is solely generated by the market disequilibrium. The actual rate of inflation is positively related to the excess demand for products deflated by the stock of capital. That is,

$$(12) \pi = E[I/K - S/K].$$

$$E[0] = 0, E' > 0$$

It is assumed that the money market is always in equilibrium. Therefore,

$$(13) m = L*[\cdot].$$

We assume that the realized investment  $DK$  equals the desired investment  $I^2/$ . That is,

$$(14) DK = I \text{ or } DK/K = \bar{D}[\cdot].$$

### 3. THE MONEY SUPPLY RULE AS A STABILIZATION DEVICE

Although the proportional policy is much more superior, we adopt the derivative policy in this paper<sup>3/</sup>. That is,

$$(15) D\mu = -\alpha D\pi \text{ or } \mu = \mu_0 - \alpha\pi.$$

This rule implies that the monetary authority should carry out the vigorous tight money policy against the

vicious inflation.

#### 4. THE SHORT-RUN SOLUTION

By solving the market equations

$$(12) \pi = E[\Phi[r(x), i, g(\pi)] - S^*[y(x) + (\mu_0 - \alpha\pi)m, m, g(\pi)]]$$

and

$$(13) m = L^*[y(x), i, r(x), m, g(\pi)]$$

for  $\pi$  and  $i$  in terms of  $x$ ,  $m$ ,  $\mu_0$  and  $\alpha$  we have the short-run solutions

$$(16) \pi = \pi(x, m; \mu_0, \alpha)$$

and

$$(17) i = i(x, m; \mu_0, \alpha).$$

We require informations about the signs of the partial derivatives of  $\pi$  and  $i$  with respect to  $x$  and  $m$  in what follows. The following assumptions are useful in the succeeding analysis.

Assumption I. The wealth effect on the saving is negligible, that is,  $S_2^* \approx 0$ .

Assumption II. The investment is interest inelastic, that is,  $\Phi_2 \approx 0$ .

Differentiating (12) and (13) with respect to  $x$  yields

$$(18) \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} \pi_x \\ i_x \end{bmatrix} = \begin{bmatrix} E'(\Phi_1 r' - S_1^* y') \\ -(L_1^* y' + L_3^* r') \end{bmatrix},$$

where  $Z_{11} = 1 - E'(\Phi_3 g' + S_1^* \alpha m - S_3^* g')$ ,

$$Z_{12} = -E' \Phi_2,$$

$$Z_{21} = L_5^* g',$$

$$Z_{22} = L_2^*.$$

Let the coefficient determinant be  $\Delta$ . That is,

$$(19) \Delta = L_2^* [1 - E'(\Phi_3 g' + S_1^* \alpha m - S_3^* g')] + E' \Phi_2 L_5^* g'.$$

$\Delta$  is positive for sufficiently large value of  $\alpha$ . Solving (18) gives

$$(20) \pi_x = \frac{E'}{\Delta} [L_2^* (\Phi_1 r' - S_1^* y') - \Phi_2 (L_1^* y' + L_3^* r')]$$

and

$$(21) i_x = -\frac{1}{\Delta} [(L_1^* y' + L_3^* r') [1 - E'(\Phi_3 g' + S_1^* \alpha m - S_3^* g')] + E'(\Phi_1 r' - S_1^* y') L_5^* g'].$$

Obviously the signs of (20) and (21) are indeterminate. So that we introduce the following assumptions.

Assumption III. A rise in the labor-capital ratio raises the proportionate rate of change of the price level, that is,  $\pi_x > 0$ .

Assumption IV. A rise in the labor-capital ratio lowers the rate of interest, that is,  $i_x < 0$ .

A rise in  $x$  raises the rate of profit. The higher rate of profit encourages the investment and as a result the excess demand is generated in the product market. Thus  $\pi$  rises. Therefore the assumption III is justified. The assumption IV is justified in the following way. In (21) a term  $E'(\Phi_1 r' - S_1^* y') L_5^* g'$  is not important in the determination of the sign of  $i_x$ , since this term mainly reflects the situation in the product market. If the substitution effect dominates the income effect in the liquidity preference, then  $L_1^* y' + L_3^* r' < 0$ . So that  $i_x < 0$ , provided that the monetary policy is sufficiently vigorous.

Differentiating (12) and (13) with respect to  $m$  yields

$$(22) \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} \pi_m \\ i_m \end{bmatrix} = \begin{bmatrix} -(E' S_1^* \mu + S_2^*) \\ 1 - L_4^* \end{bmatrix}.$$

Solving (22) gives

$$(23) \pi_m = \frac{1}{\Delta} [-L_2^* (E' S_1^* \mu + S_2^*) + E' \Phi_2 (1 - L_4^*)]$$

and

$$(24) i_m = \frac{1}{\Delta} [(1 - L_4^*) [1 - E' (\Phi_3 g' + S_1^* \alpha m - S_3^* g')] + L_5^* g' (E' S_1^* \mu + S_2^*)].$$

If  $S_2^* \approx 0$  and  $\Phi_2 \approx 0$ , then  $\pi_m > 0$ . If  $S_2^* \approx 0$  and  $\alpha$  is sufficiently large, then  $i_m < 0$ . Thus, we establish the following lemma.

Lemma 1. Under the assumptions I and II a rise in the real balance per unit of capital raises the proportionate rate of change of the price level and lowers the rate of interest, so long as the monetary policy is carried out in a sufficiently vigorous manner.

## 5. THE EFFECTIVENESS OF STABILIZATION POLICY

The dynamic processes of the system are described by the following differential equations.

$$(25) \quad Dx/x = n - \Phi[r(x), i, g(\pi)]$$

$$(26) \quad Dm/m = \mu_0 - (1+\alpha)\pi - \Phi[r(x), i, g(\pi)]$$

The path such that both  $x$  and  $m$  are kept to be unchanged is called the balanced growth path. We denote the balanced growth path by  $(x^*, m^*)$ . Our major concern is whether the balanced growth path is dynamically stable and whether the system can be stabilized through the monetary policy when it is unstable. A rather restrictive answer is given by the following theorem.

THEOREM 1. Under the assumptions I, II, III and IV the system is locally stable, if the money supply policy of the derivative type is carried out in a sufficiently vigorous manner.

Proof. Taking the linear approximation system around the balanced growth path we have

$$(27) \begin{bmatrix} Dx \\ Dm \end{bmatrix} \approx \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} x-x^* \\ m-m^* \end{bmatrix},$$

where  $Q_{11} = -x^*(\Phi_1 r' + \Phi_2 i_x + \Phi_3 g' \pi_x)$ ,

$$Q_{12} = -x^*(\Phi_2 i_m + \Phi_3 g' \pi_m),$$

$$Q_{21} = -m^*[(1+\alpha+g' \Phi_3) \pi_x + \Phi_2 i_x + \Phi_1 r']$$

and  $Q_{22} = -m^*[(1+\alpha+g' \Phi_3) \pi_m + \Phi_2 i_m]$ .

The solutions to (27) are formally written as

$$(28) \begin{bmatrix} x \\ m \end{bmatrix} = \begin{bmatrix} x^* \\ m^* \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} e^{\beta_1 t} \\ e^{\beta_2 t} \end{bmatrix},$$

where  $\beta_1$  and  $\beta_2$  are roots to the following characteristic equation.

$$(29) \beta^2 - (Q_{11} + Q_{22})\beta + Q_{11}Q_{22} - Q_{12}Q_{21} = 0$$

If the following Routh-Hurwitz conditions are satisfied, then the system is stable, that is,  $\lim_{t \rightarrow \infty} x = x^*$  and  $\lim_{t \rightarrow \infty} m = m^*$ .

$$(R-H I) \quad Q_{11} + Q_{22} < 0$$

$$(R-H II) \quad \begin{vmatrix} -(Q_{11} + Q_{22}) & 0 \\ 1 & Q_{11}Q_{22} - Q_{12}Q_{21} \end{vmatrix} > 0$$

The signs of  $Q_{11}$ ,  $Q_{12}$ ,  $Q_{21}$ ,  $Q_{22}$  are negative. Thus, the condition (R-H I) is satisfied. We examine the sign of an element  $Q_{11}Q_{22}-Q_{12}Q_{21}$ .

$$(30) \quad Q_{11}Q_{22}-Q_{12}Q_{21} = x^*m^*(1+\alpha)[\pi_m r' \Phi_1 + (\pi_m i_x - \pi_x i_m) \Phi_2] \\ \approx x^*m^*(1+\alpha)\pi_m r' \Phi_1 > 0$$

Thus, the condition (R-H II) is satisfied. (Proof is complete.)

Remark 1. There is not the cyclical phenomena in the linear approximation system described above. The reason is as follows. The discriminant to (29) is

$$(31) \quad (Q_{11}+Q_{22})^2 - 4(Q_{11}Q_{22}-Q_{12}Q_{21}) = (Q_{11}-Q_{22})^2 + 4Q_{12}Q_{21}.$$

This is obviously positive. Therefore  $\beta_1$  and  $\beta_2$  are different real numbers. Thus the solutions (28) are not spirals, that is, the business cycles do not occur. ( $\beta_1, \beta_2 < 0$ , since the Routh-Hurwicz conditions are met.)

In THEOREM 1 we have sought to solve the differential equations explicitly. Since we can not solve (25) and (26), the local stability of the system has been examined. However, we can consider the global stability of the system by applying the second method of Lyapunov unless the explicit solutions matter. The following general stability theorem shows that the monetary policy can converge the dynamic path starting with any initial point to the balanced growth path.

THEOREM 2. Under the assumptions I, II, III and IV the system is globally stable, if the monetary policy

is sufficiently vigorous.

Proof. Define a function

$$(32) V(x, m, t) = (x - x^*)^2 + (m - m^*)^2.$$

Obviously  $V(x^*, m^*, t) = 0$ , and  $V(x, m, t) > 0$  unless  $x = x^*$  and  $m = m^*$ . We consider the sign of the derivative DV.

$$(33) DV = 2(x - x^*)Dx + 2(m - m^*)Dm \\ = 2(x - x^*)x[n - \Phi(r(x), i, g(\pi))] \\ + 2(m - m^*)m[\mu_0 - (1 + \alpha)\pi - \Phi(r(x), i, g(\pi))]$$

On the balanced growth path

$$(34) x = x^* \text{ and } n - \Phi(\cdot) = 0$$

and

$$(35) m = m^* \text{ and } \mu_0 - (1 + \alpha)\pi - \Phi(\cdot) = 0.$$

Hence,

$$(36) DV(x^*, m^*, t) = 0.$$

Now,

$$(37) \text{ If } x > x^*, \text{ then } n - \Phi(\cdot) < 0 \\ \text{and if } x < x^*, \text{ then } n - \Phi(\cdot) > 0, \\ \text{that is, } (x - x^*)[n - \Phi(\cdot)] < 0 \text{ unless } x = x^*,$$

since

$$(38) \frac{\partial [n - \Phi(\cdot)]}{\partial x} = -(\Phi_1 r' + \Phi_2 i_x + \Phi_3 g' \pi_x) < 0.$$

Moreover,

- (39) If  $m > m^*$ , then  $\mu_0 - (1+\alpha)\pi - \bar{\Phi}(\cdot) < 0$   
 and if  $m < m^*$ , then  $\mu_0 - (1+\alpha)\pi - \bar{\Phi}(\cdot) > 0$ ,  
 that is,  $(m-m^*)[\mu_0 - (1+\alpha)\pi - \bar{\Phi}(\cdot)] < 0$  unless  $m=m^*$ ,

since

$$(40) \frac{\partial [\mu_0 - (1+\alpha)\pi - \bar{\Phi}(\cdot)]}{\partial m} = -[(1+\alpha + \bar{\Phi}_3 g')\pi_m + \bar{\Phi}_2 i_m] < 0.$$

Therefore,

- (41)  $DV(x, m, t) < 0$  unless  $x=x^*$  and  $m=m^*$ .

Namely,  $V = (x-x^*)^2 + (m-m^*)^2$  is a Lyapunov function. Hence by the Lyapunov stability theorem the system is stable in the large. (Proof is complete.)

## 6. THE STABILITY OF THE PRICE MECHANISM

So far we have analyzed the relationship between the monetary policy and the stability of the system. In this section the influence of the monetary policy upon the rate of inflation is argued. From the previous discussion we can easily insist upon the following proposition.

**THEOREM 3.** Under the assumptions I, II, III and IV the rate of change of the price level converges to a finite value so long as the monetary policy is applied in a sufficiently vigorous manner. The more vigorous the monetary policy, the smaller will

be the ultimate value of the rate of change of the price level.

Proof. From THEOREM 2 any path tends to move toward the balanced growth path. On the balanced growth path  $Dx=0$  and  $Dm=0$  hold. From (25) and (26) the proportionate rate of change of the price level on the balanced growth path  $\pi^*$  becomes

$$(42) \pi^* = \frac{\mu_0 - n}{1 + \alpha} = \text{const.}$$

Hence,  $\lim_{t \rightarrow \infty} \pi = \pi^* = \text{const.}$  It is obvious that the larger  $\alpha$ , the smaller will be  $\pi^*$ . (Proof is complete.)

Remark 1. Stein-Nagatani consider the stability of the price mechanism by examining the magnitude of a derivative  $\partial \pi / \partial \pi$  in the short-run. Given  $x$  and  $m$ , by differentiation in (12) and (13) we obtain

$$(43) \frac{\partial \pi}{\partial \pi} = 1 - \Delta / L_2^* > 1$$

if  $\alpha$  is sufficiently large. This result seems paradoxical. However, we must note that the values of  $x$  and  $m$  are kept to be constant in the calculation of (43). So that an implication of (43) should not be regarded as a property of the dynamic processes. In fact (43) is insignificant on the balanced growth path, since we can not calculate the derivative  $\partial \pi / \partial \pi$  because of constancy of  $\pi (= \pi^*)$ . Therefore (43) never implies the price instability of the system.

Remark 2. Hadjimichalakis insists that a derivative  $\partial \pi / \partial p$  should be used in the stability analysis of the price mechanism in place of  $\partial \pi / \partial \pi$ .<sup>4/</sup> Given  $x$  and  $m$ , we

have

$$(44) \frac{\partial \pi}{\partial p} = 0.$$

This implies that the rate of inflation does not depend on the absolute price level in the short-run. Hadjimichalakis says that if  $\partial \pi / \partial p$  is negative, then the price mechanism is stable. This means the stability not of  $\pi$  but of  $p$ . The stability of  $p$ , however, does not seem important in the analysis of economic growth. In the usual Keynes-Wicksell model  $p$  such that  $\pi = 0$  does not necessarily exist. Therefore,  $\partial \pi / \partial p < 0$  does not necessarily guarantee the price stability. Thus the consideration of (44) is not relevant for the stability analysis of the price mechanism.

## 7. THE EFFECT OF MONETARY POLICY ON THE REAL ASPECTS

The influences of the monetary policy on such real variables as the employment-capital ratio, rate of return on capital assets and real wage rate on the balanced growth path are clarified by the following theorem.

**THEOREM 4.** On the balanced growth path the more vigorous the monetary policy, the larger will be the labor-capital ratio and the rate of profit and the smaller will be the real wage rate as long as the persistent inflation prevails. As a special case the monetary policy is neutral, provided the price stability is realized.

Proof. Set (25) and (26) equal to zero. And differ-

entiate those equations with respect to  $\alpha$  to get

$$(45) \begin{bmatrix} -Q_{11}/x^* & -Q_{12}/x^* \\ -Q_{21}/m^* & -Q_{22}/m^* \end{bmatrix} \begin{bmatrix} x_{\alpha}^* \\ m_{\alpha}^* \end{bmatrix} = \begin{bmatrix} 0 \\ -\pi^* \end{bmatrix}. \quad (\text{See (27)})$$

Solving these equations gives

$$(46) x_{\alpha}^* = \frac{\pi^* (\bar{\Phi}_2^i m + \bar{\Phi}_3 g' \pi_m)}{(1+\alpha) [\pi_m \bar{\Phi}_1 r' + (\pi_m^i x - \pi_x^i m) \bar{\Phi}_2]} \\ \approx \frac{\pi^* (\bar{\Phi}_2^i m + \bar{\Phi}_3 g' \pi_m)}{(1+\alpha) \pi_m \bar{\Phi}_1 r'}$$

Hence,

$$(47) \text{sign}(x_{\alpha}^*) = \text{sign}(\pi^*), \text{ where } \pi^* = \frac{\mu_0 - n}{1+\alpha}.$$

If  $\pi^* > 0$ , then  $x_{\alpha}^* > 0$ . The larger  $\alpha$ , the larger will be the rate of profit  $r(x)$  and the smaller will be the real wage rate  $y'(x)$ , since  $r'(x) > 0$  and  $y''(x) < 0$ . If  $\pi^* = 0$ , then  $x_{\alpha}^* = 0$ . So that the rate of profit and real wage rate are unchanged. Namely, the magnitude of  $\alpha$  can not affect the real variables. (Proof is complete.)

## 8. THE SPEED OF CONVERGENCE AND THE MONETARY POLICY

In this section somewhat different types of the policy are discussed. The rate of monetary expansion  $\mu$  is adopted as a control variable, while  $\alpha$  has been the control variable in the foregoing sections. In fact the

monetary authority can not expand the money balance at extremely high rates or at extremely low rates by various social restrictions. Let the highest admissible rate of monetary expansion be  $\mu^{**}$  and the lowest admissible one be  $\mu^*$ . That is, the admissible control of  $\mu$  is described by an inequality

$$(48) \mu^* \leq \mu \leq \mu^{**}$$

or by a set notation

$$(49) \mu \in U = \{\mu \mid \mu^* \leq \mu \leq \mu^{**}\}.$$

Of course,  $\mu = \mu(t)$  is piecewise continuous in a some appropriate interval.

If the time required for equilibrating adjustment is too long, a new policy which shortens the departure from the balanced growth path as faster as possible will be required even if the system is stable. We are concerned with a policy which minimizes the distance between an actual path and the balanced growth path in what follows. That is, the problem is choosing an optimal rate of monetary expansion so as to minimize

$$(50) J = \int_0^T [(x-x^*)^2 + (m-m^*)^2] dt,$$

where  $T$  is a planning horizon of the authority<sup>5/</sup>. The motions of state variables are expressed by the performance equations

$$(25) Dx = x[n - \Phi(r(x), i, g(\pi))]$$

and

$$(51) Dm = m[\mu - \pi - \Phi(r(x), i, g(\pi))].$$

Let the Hamiltonian function be

$$(52) H(x, m; \varphi_1, \varphi_2; \mu) = -[(x - x^*)^2 + (m - m^*)^2] + \varphi_1 Dx + \varphi_2 Dm,$$

where  $\varphi_1$  and  $\varphi_2$  are auxiliary variables. The answer to this question is given by the following maximum principle.

**THEOREM 5.** The necessary condition for existence of an optimal control  $\mu$  and corresponding optimal trajectories  $x$  and  $m$  of the control problem of minimizing (50) subject to (49), (25) and (51) is that there are continuous functions  $\varphi_1(t) \neq 0$  and  $\varphi_2(t) \neq 0$  such that

$$(I) \quad \begin{aligned} Dx/x &= n - \Phi(\cdot) \\ Dm/m &= \mu - \pi - \Phi(\cdot), \end{aligned} \quad (\text{performance equations})$$

$$(II) \quad \begin{aligned} D\varphi_1 &= 2(x - x^*) - \varphi_1 [n - \Phi(\cdot) - x(\Phi_1 r' + \Phi_2 i_x + \Phi_3 g' \pi_x)] \\ &\quad + \varphi_2 m [(1 + \Phi_3 g') \pi_x + \Phi_1 r' + \Phi_2 i_x] \\ D\varphi_2 &= 2(m - m^*) + \varphi_1 x (\Phi_2 i_m + \Phi_3 g' \pi_m) \\ &\quad - \varphi_2 [\mu - \pi - \Phi(\cdot) - m [(1 + \Phi_3 g') \pi_m \\ &\quad \quad \quad + \Phi_2 i_m]], \end{aligned}$$

(auxiliary equations)

$$(III) \quad \text{Max}_{\mu \in U} H(x, m; \varphi_1, \varphi_2; \mu),$$

that is,

If  $\varphi_2 > 0$ , then  $\mu = \mu^{**}$

If  $\varphi_2 < 0$ , then  $\mu = \mu^*$

If  $\varphi_2 = 0$ , then  $\mu$  is arbitrary in the set

$$U = \{\mu \mid \mu^* \leq \mu \leq \mu^{**}\}.$$

Proof. See a textbook on the Pontryagin control theory.

We would explain the optimal plan in terms of a phase diagram. The solutions to the performance equations are illustrated in Fig. 1, where the rate of monetary expansion  $\mu$  is fixed at an arbitrary level.

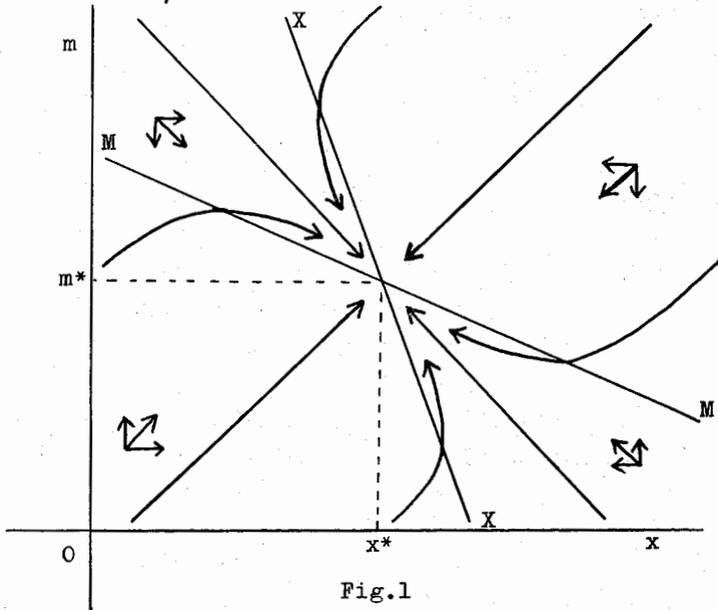


Fig.1

In Fig.1 the XX curve and the MM curve are the curves such that  $Dx=0$  and  $Dm=0$  respectively and a point of intersection of them represents the balanced growth path. The slopes of the singular curves are

$$(53) \left. \frac{dm}{dx} \right|_{XX} = - \frac{\Phi_1 r' + \Phi_2 i_x + \Phi_3 g' \pi_x}{\Phi_2 i_m + \Phi_3 g' \pi_m} < 0$$

and

$$(54) \left. \frac{dm}{dx} \right|_{MM} = - \frac{(1 + \Phi_3 g') \pi_x + \Phi_1 r' + \Phi_2 i_x}{(1 + \Phi_3 g') \pi_m + \Phi_2 i_m} < 0.$$

(It is assumed that  $\pi_m > 0$  and  $i_m < 0$  in this section.)  
The directions of the motions of  $x$  and  $m$  are shown by the following inequalities.

$$(55) \frac{\partial(Dx/x)}{\partial x} = -(\Phi_1 r' + \Phi_2 i_x + \Phi_3 g' \pi_x) < 0$$

$$(56) \frac{\partial(Dm/m)}{\partial m} = -[(1 + \Phi_3 g') \pi_m + \Phi_2 i_m] < 0$$

Obviously the  $XX$  curve is not affected by the variation in  $\mu$ . The  $MM$  curve shifts upward when  $\mu$  increases. This inference is due to the following lemma.

Lemma 2. A rise in the rate of monetary expansion lowers the labor-capital ratio on the balanced growth path.

Proof. By differentiating a system of equations  $Dx=0$  and  $Dm=0$  with respect to  $\mu$  we have

$$(57) \begin{bmatrix} G_1 & G_2 \\ G_1 + \pi_x & G_2 + \pi_m \end{bmatrix} \begin{bmatrix} x^* \\ m^* \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

where  $G_1 = \Phi_1 r' + \Phi_2 i_x + \Phi_3 g' \pi_x > 0$

and  $G_2 = \Phi_2 i_m + \Phi_3 g' \pi_m > 0$ .

Let the coefficient determinant be  $G$ . That is,

$$(58) G = G_1 \pi_m - G_2 \pi_x.$$

The stability of the system implies that

$$(59) \left. \frac{dm}{dx} \right|_{MM} > \left. \frac{dm}{dx} \right|_{XX},$$

where

$$(60) \left. \frac{dm}{dx} \right|_{MM} = -\frac{G_1 + \pi_x}{G_2 + \pi_m} \text{ and } \left. \frac{dm}{dx} \right|_{XX} = -\frac{G_1}{G_2}.$$

Hence,

$$(61) G > 0.$$

Thus,

$$(62) x_\mu^* = -\frac{G_2}{G} < 0.$$

(Proof is complete.)

The optimal trajectories of the state variables are depicted in Fig. 2, where a singular curve  $M^*M^*$  corresponding to an optimal control  $\mu^*$  and another one  $M^{**}M^{**}$  corresponding to an optimal control  $\mu^{**}$  are indicated. In Figure 2 a point  $A(x_0, m_0)$  is an initial point.

Finally consider the case where the integrand always equals unity. The functional to minimize takes the form

$$(63) J = \int_0^T dt = T.$$

This problem is a time optimal problem. The maximum principle gives the following theorem.

THEOREM 6. The necessary condition of the control prob-

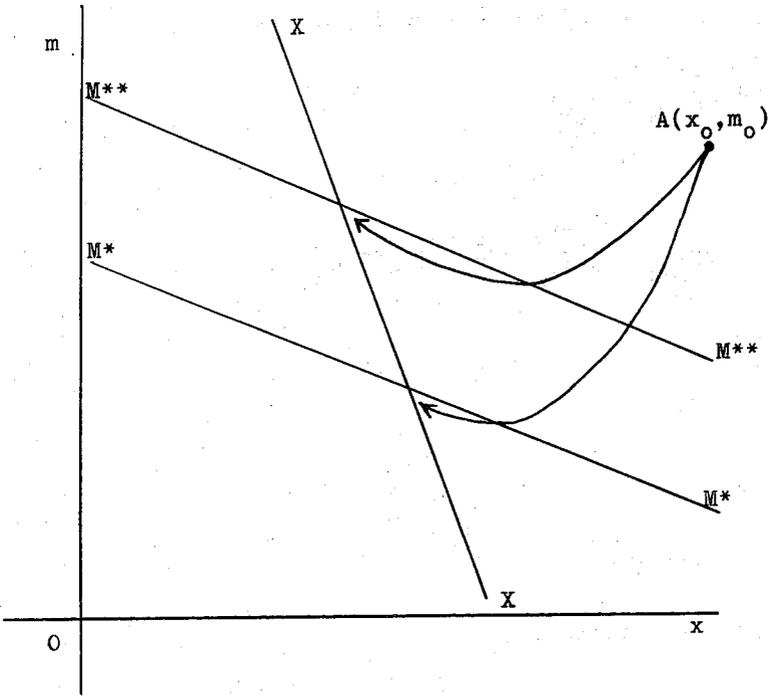


Fig. 2 (The case where there is no switching of  $\mu$  during the planning period.)

lem of minimizing the time required subject to (49), (25) and (51) is that there are continuous functions  $\varphi_1 \not\equiv 0$  and  $\varphi_2 \not\equiv 0$  such that

$$(I) \quad Dx/x = n - \Phi(\cdot)$$

$$Dm/m = \mu - \pi - \Phi(\cdot),$$

$$(II) \quad D\varphi_1 = -\varphi_1 [n - \Phi(\cdot) - x(\Phi_1 r' + \Phi_2 i_x + \Phi_3 g' \pi_x)] \\ + \varphi_2 m [(1 + \Phi_3 g') \pi_x + \Phi_1 r' + \Phi_2 i_x]$$

$$D\varphi_2 = \varphi_1 x (\Phi_2 i_m + \Phi_3 g' \pi_m) \\ - \varphi_2 [\mu - \pi - \Phi(\cdot) - m[(1 + \Phi_3 g') \pi_m \\ + \Phi_2 i_m]],$$

$$(III) \max_{\mu \in U} H(x, m; \varphi_1, \varphi_2; \mu) = \varphi_1 Dx + \varphi_2 Dm,$$

that is,

If  $\varphi_2 > 0$ , then  $\mu = \mu^{**}$

If  $\varphi_2 < 0$ , then  $\mu = \mu^*$

If  $\varphi_2 = 0$ , then  $\mu = \text{arbitrary}$ .

Proof. See a textbook on the Pontryagin control theory.

An only difference between the optimal solutions to the two control problems is the auxiliary equations. So that, Fig. 2 also applies to THEOREM 6.

#### NOTES

- \*) The author expresses thanks to Prof. Hayakawa, Prof. Shirai, Prof. Kobayashi, Mr. Sakai and Mr. Matsu moto for their comments and suggestions in the advanced seminar.
- 1) An attempt which adopts the disposable income instead of the real output as an argument in the consumption function is performed by R.J.Mackay[2]. Such an attempt is exceptional in the Keynes-Wicksell model, although it is prevalent in the neoclassical theory of monetary growth.
- 2) Other writers adopt alternative assumptions. Stein-Nagatani assume that

$$DK/K = a(I/K) + (1-a)(S/K),$$

where  $1 > a > 0$ , when  $\pi > 0$   
 $a = 0$ , when  $\pi \leq 0$ .

(J.L.Stein, "Money and Capacity Growth", JPE, 1966 assumes that  $I = DK$ .) Hadjimichalakis and Rose (H.Rose, "Unemployment in a Theory of Growth", IER, 1966) as-

sume that

$$DK=S.$$

- 3) The proportional policy is expressed in the form

$$D\mu = -\gamma\pi.$$

- 4) Hadjimichalakis calculates  $\partial\pi/\partial p$  in his short-run system. His definition of the short-run is characterized by  $x=N/K=\text{const.}$  and  $K=\text{const.}$  (p. 563). However, obviously  $x=\text{const.}$  and  $K=\text{const.}$  contradict each other. His calculation of  $\partial\pi/\partial p$  crucially depends upon the assumption that  $m=M/pK=\text{const.}$  in the short-run. If  $x=\text{const.}$  and  $m=\text{const.}$  in the Hadjimichalakis model, then  $\partial\pi/\partial p=0$  holds. Thus, Mr. Hadjimichalakis' analysis has little significance.

- 5) The control problems handled in this section are first considered by T. Tokita[5] in the framework of the neoclassical theory of monetary growth involving the variable rate of employment. His criterion functional to minimize is

$$\int_{t_0}^{t_1} [(k-k^*)^2 + (w-w^*)^2] dt,$$

where  $k$  is the capital-labor ratio and  $w$  is the real wage rate.

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## REFERENCES

- [1] Hadjimichalakis, M.G., "On the Effectiveness of Monetary Policy as a Stabilization Device", Review of Economic Studies, XL, 1973, pp. 561-70.
- [2] Mackay, R.J., "Stein and Nagatani on Stabilization Policies in a Growing Economy", Review of Economic Studies, XL, 1973, pp.571-78.
- [3] Stein, J.L. and Nagatani, K., "Stabilization Policies in a Growing Economy", Review of Economic Studies, XXXVI, April, 1969, pp.165-83.
- [4] Taylor, J.B., "Stabilization Policies in a Growing Economy: A Comment", Review of Economic Studies, XXXIX, October, 1972, pp.515-19.
- [5] Tokita, T., "A New Monetary Stabilizing Policy in a Growing Economy", Economic Studies Quarterly, XXI, August, 1970, pp.36-43. (in Japanese).

## APPENDIX

In what follows the stability of the neoclassical growth system is analyzed for the convenience of comparison with the Keynesian growth system described in the text.

## A. THE MODEL

Consider the following non-monetary neoclassical system.

$$(a) Y=Y(K,N)$$

$$(b) DK=sY, \quad (0 \leq s \leq \bar{s} < 1)$$

$$(c) DN/N=n$$

In (b)  $s$  is a constant saving ratio and  $\bar{s}$  is its upper limit. ( $s(t)$  is a piecewise continuous function.) Other symbols are the same as the Keynesian model. By virtue of the linear homogeneity of production function (a) we have

$$(d) y=y(x),$$

where  $y=Y/K$  and  $x=N/K$ . The performance equation of the system is

$$(e) Dx=x(n-sy(x)).$$

This is a non-linear differential equation. So that we can not obtain an explicit solution by solving (e) directly. As Lancaster points out there are two main approaches to the analysis of stability, explicit solu-

tion and the Liapunov method<sup>1/</sup>. In the stability analysis of the neoclassical growth model the explicit solution method gives the local stability theorem and the second method of Liapunov gives the global stability theorem. They are in turn shown in the successive sections.

#### B. THE STABILITY IN THE SMALL

THEOREM A. The system (e) is stable in the small.

Proof. Expanding the system near the balanced growth path  $x^*$ , we obtain the approximate linear system

$$(f) \quad Dx \approx [-sx^*y'(x^*)](x-x^*).$$

Solving this equation gives

$$(g) \quad x \approx (x_0 - x^*) \exp[(-sx^*y'(x^*))t] + x^*,$$

where  $x_0$  is an initial point of  $x$ . Obviously

$$(h) \quad -sx^*y'(x^*) < 0.$$

Hence,  $\lim_{t \rightarrow \infty} x = x^*$ , that is, the system is locally stable.

(Proof is complete.)

#### C. THE STABILITY IN THE LARGE

THEOREM B. The system (e) is stable in the large.

Proof. Define a function

$$(i) \quad V(u) = u^2,$$

where  $u = x^* - x$ . Then  $V \geq 0$  and  $V > 0$  if and only if  $x \neq x^*$ . We shall show that  $V(u)$  is a Liapunov function.

$$\begin{aligned}
 (j) \quad DV(u) &= 2uDu \\
 &= 2u(-Dx) \\
 &= -2ux(n - sy(x)) \\
 &= -2ux(n - sy(x^* - u)) \\
 &< -2ux[n - s(y(x^*) - uy'(x^*))] \\
 &\quad \text{(by strict concavity of } y(x)) \\
 &= -2sxu^2 y'(x^*) \\
 &< 0 \quad \text{unless } x = x^*
 \end{aligned}$$

Therefore, the system is globally stable by the Liapunov stability theorem. (Proof is complete.)<sup>2/</sup>

#### D. THE SPEED OF CONVERGENCE

Consider a control problem of minimizing an integral

$$(k) \quad J = \int_0^T (x - x^*)^2 dt.$$

The Hamiltonian expression is

$$(l) \quad H = -(x - x^*)^2 + \psi x[n - sy(x)],$$

where  $\psi$  is an auxiliary variable. The Pontryagin maximum principle gives the following theorem.

**THEOREM C.** The necessary condition for existence of an optimal control  $s$  and a corresponding optimal trajectory of a state variable  $x$  of the control problem of minimizing the functional (k) subject to the performance equation (e) is that there is a continuous function  $\psi(t) \neq 0$  such that

- (I)  $Dx/x = n - sy(x)$ ,
- (II)  $D\varphi = 2(x - x^*) - \varphi[n - sy(x) - sxy'(x)]$ ,
- (III)  $\text{Max}_s H(x, \varphi, s)$ ,

that is,

If  $\varphi > 0$ , then  $s = 0$

If  $\varphi < 0$ , then  $s = \bar{s}$

If  $\varphi = 0$ , then  $s = \text{arbitrary} (0 \leq s \leq \bar{s} < 1)$ .

Proof. See a textbook on the Pontryagin control theory.

The structure of solutions to equations (I) and (II) is illustrated in Fig. A. (Note that  $x^*$  depends upon the value of  $s$ .)

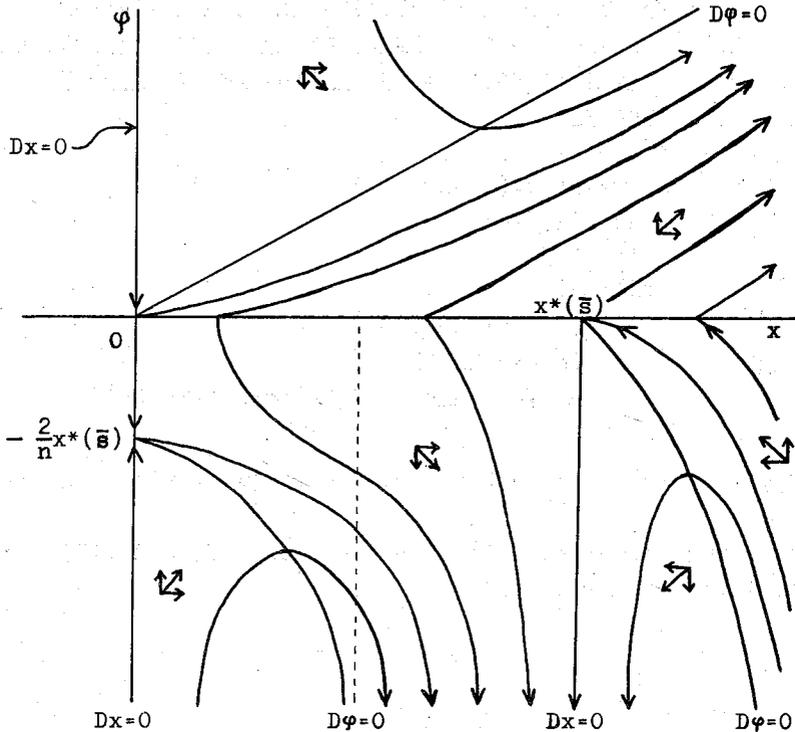


Fig. A

Finally the time optimal problem is investigated.  
The criterion functional is

$$(m) J = \int_0^T dt.$$

The Hamiltonian is

$$(n) H = \varphi x [n - sy(x)].$$

Then we obtain the following theorem.

THEOREM D. The necessary condition for existence of an optimal control  $s$  and a corresponding optimal trajectory of a state variable  $x$  of the control problem of minimizing the time required (m) subject to (e) is that there is a continuous function  $\varphi(t) \neq 0$  such that

- (I)  $Dx/x = n - sy(x)$ ,
- (II)  $D\varphi/\varphi = -[n - sy(x) - sxy'(x)]$ ,
- (III)  $\text{Max}_s H(x, \varphi, s)$ ,

that is,

If  $\varphi > 0$ , then  $s = 0$

If  $\varphi < 0$ , then  $s = \bar{s}$

If  $\varphi = 0$ , then  $s = \text{arbitrary}$ . ( $0 \leq s \leq \bar{s} < 1$ )

Proof. See a textbook on the Pontryagin control theory.

The structure of solutions to equations (I) and (II) is illustrated in Fig. B.

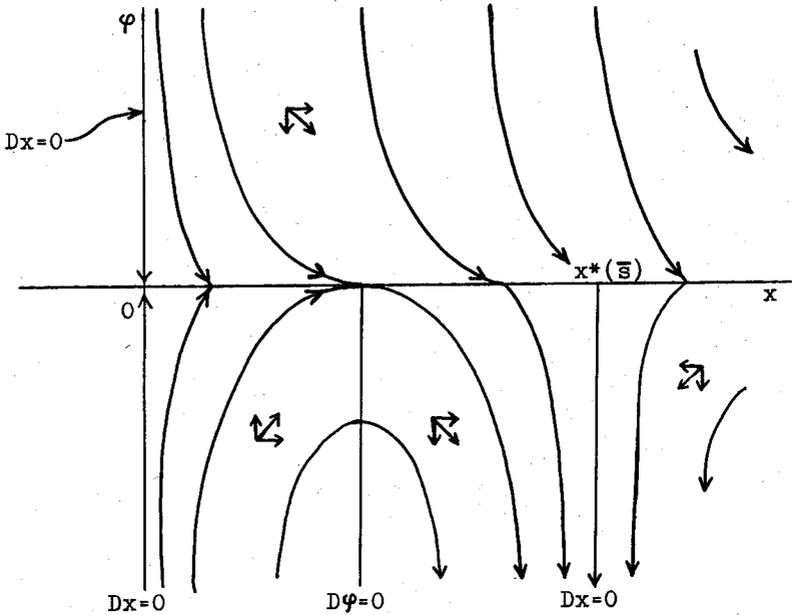


Fig. B

NOTES

- 1) K.Lancaster[2], p.197.
- 2) This proof follows that of E.Burmeister and A.R. Dobell[1], p.28.

REFERENCES

- [1] Burmeister, E. and Dobell, A.R., "Mathematical Theories of Economic Growth", Macmillan, 1970.
- [2] Lancaster, K., "Mathematical Economics", Macmillan, 1968.