DYNAMIC UTILITY MAXIMIZATION, WEALTH EFFECT AND DERIVATION OF THE CONSUMPTION FUNCTION: FURTHER CONSIDERED

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In a previous paper [1] I have analyzed a dynamic model of the household, in which the rational behavior of the consumer over time is formulated as a problem of the calculus of variations. That is, one should choose a saving plan so as to maximize a particular class of the utility integral

$$(1)\quad J[c,b]=\int_0^\infty [u(c)+v(b)]e^{-\beta t}dt$$ [1]

subject to the budget equation

$$(2)\quad Db=w+ib-c,\ \text{or}\ i\int_0^\infty ce^{-it}dt=ib_0+i\int_0^\infty we^{-it}dt.$$

The notation is restated below.

- c=real consumption
- b=real balance of the financial asset
- w=real wage rate
- i=real rate of interest
- β=rate of discount

The term $v(b)$ in (1) leads to the wealth effect[2] Under the assumption that the subjective rate of discount β is greater than the real rate of interest i we

have confirmed that there exists a unique optimal trajectory starting from a given initial value of the asset holdings b_0. Furthermore we have focused attention on the case where there exists a long-run stationary point having the saddle point property. The optimal saving plan depends upon w, i and β. So that, in general, the saving function is written as

(3) $Db=f(w,i,\beta)$.

In this note we shall contemplate specifying the form of $f(\cdot)$ by using the idea of "flexible accelerator" applied to the theory of investment by Lucas[3]. This approach will generate a saving function of a stock adjustment pattern. In what follows the stationary value of wealth, b^*, is regarded as the desired level of wealth. And c^* is the corresponding value of consumption to b^*. That is,

(4) $c^*=w+ib^*$.

The dynamic motion of the system is described by the differential equations

(2) $Db=w+ib-c$,

(5) $Dc=\dfrac{(\beta-i)u'(c)-v'(b)}{u''(c)}$.

(2) is the performance equation and (5) is the Euler equation. Expanding this system about the stationary solution (b^*,c^*), we obtain the approximate linear system

(6) $Db=i(b-b^*)-(c-c^*)$

(7) $Dc=-\dfrac{v''(b^*)}{u''(c^*)}(b-b^*)+(\beta-i)(c-c^*)$,

or

(8) $[Db\ Dc]=[b\ c]\begin{bmatrix} i & -v''/u'' \\ -1 & \beta-i \end{bmatrix}+[-ib^*+c^*\ \dfrac{v''}{u''}b^*-(\beta-i)c^*]$.

First, we solve the homogeneous part of the system

(9) $Dy=yA$,

where $y=[b\ c]$, $Dy=[Db\ Dc]$ and

$$A=\begin{bmatrix} i & -v''/u'' \\ -1 & \beta-i \end{bmatrix}.$$

Let the eigenvalues of A be α_1 and α_2. That is,

$\alpha_1=\frac{1}{2}[\beta-[\beta^2-4(i(\beta-i)-v''/u'')]^{1/2}]$,

$\alpha_2=\frac{1}{2}[\beta+[\beta^2-4(i(\beta-i)-v''/u'')]^{1/2}]$.

It is assumed that

(10) $i(\beta-i)-v''/u''<0$.

This assumption seems plausible, since $i(\beta-i)$ is negligible. Then $\alpha_1<0$ and $\alpha_2>0$. Moreover let

$$B=\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix},$$

where $[x_{11}\ x_{12}]$ is an eigenvector corresponding to α_1 and $[x_{21}\ x_{22}]$ is an eigenvector corresponding to α_2. Now define new variables

(11) $z=[z_1\ z_2]=yB^{-1}$.

Differentiating (11) yields

(12) $Dz=DyB^{-1}$.

Multiplying both sides of (9) on the right by B^{-1} yields

(13) $DyB^{-1}=yAB^{-1}$.

Substituting (12) into the left side and substituting $y=zB$ (obtained from (11)) into the right side yield

(14) $Dz=zBAB^{-1}$.

Here, BAB^{-1} can be written in the Jordan canonical form

$$(15)\quad BAB^{-1}=\begin{bmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{bmatrix}.$$

Thus, (14) reduces to

$$(16)\quad [Dz_1\ Dz_2]=[z_1\ z_2]\begin{bmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{bmatrix}.$$

Solving (16) gives

$$(17)\quad [z_1\ z_2]=[R_1e^{\alpha_1 t}\ R_2e^{\alpha_2 t}],$$

where R_1 and R_2 are constants. Hence,

$$(18)\quad [b\ c]=[R_1e^{\alpha_1 t}\ R_2e^{\alpha_2 t}]\begin{bmatrix}x_{11} & x_{12}\\ x_{21} & x_{22}\end{bmatrix}.$$

That is,

$$(19)\quad b=R_1x_{11}e^{\alpha_1 t}+R_2x_{21}e^{\alpha_2 t},$$

$$(20)\quad c=R_1x_{12}e^{\alpha_1 t}+R_2x_{22}e^{\alpha_2 t}.$$

The particular solutions, b^* and c^*, of the original

non-homogeneous equations added to (19) and (20) respectively make

$$(21)\quad b=R_1x_{11}e^{\alpha_1 t}+R_2x_{21}e^{\alpha_2 t}+b^*,$$

$$(22)\quad c=R_1x_{12}e^{\alpha_1 t}+R_2x_{22}e^{\alpha_2 t}+c^*.$$

From the initial conditions

$$(23)\quad R_1=\frac{x_{22}(b_o-b^*)-x_{21}(c_o-c^*)}{x_{11}x_{22}-x_{12}x_{21}},$$

$$(24)\quad R_2=\frac{-x_{12}(b_o-b^*)+x_{11}(c_o-c^*)}{x_{11}x_{22}-x_{12}x_{21}}.$$

In (21) we shall employ a stable solution

$$(25)\quad b=R_1x_{11}e^{\alpha_1 t}+b^*.$$

Differentiating (25) we have

$$(26)\quad Db=R_1x_{11}\alpha_1e^{\alpha_1 t}$$

$$=-\alpha_1(b^*-b).$$

(26) implies that the larger the difference between the actual wealth b and the desired one b*, the larger will be the current saving Db. The actual wealth b approaches the stationary level b* as time t goes to infinity. In the stationary state the propensity to save becomes zero. (26) is very akin to the "capital

stock adjustment principle" in the theory of investment. The larger the desired level of wealth, the larger will be the current level of saving. On the other hand, given desired wealth b* the larger the actual wealth, the smaller will be the current level of saving. That is,

$$(27)\quad \frac{\partial Db}{\partial b}=\alpha_1<0.$$

Of course this means the "wealth effect".

NOTES

1) $u(0)=0$, $u(\infty)=\infty$; $u'>0$, $u''<0$; $u'(0)=\infty$, $u'(\infty)=0$. $v(b)$ also satisfies these conditions.

2) Kurz[2] takes the wealth effects into consideration in the context of the neoclassical theory of optimal growth.

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