Spectrum of Time Operators

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Abstract

Let H be a self-adjoint operator on a complex Hilbert space $\mathcal{H}$. A symmetric operator T on $\mathcal{H}$ is called a time operator of H if, for all $t \in \mathbb{R}$, $e^{-itH}D(T) \subset D(T)$ ($D(T)$ denotes the domain of T) and $Te^{-itH}\psi = e^{-itH}(T+t)\psi$, $\forall t \in \mathbb{R}, \forall \psi \in D(T)$. In this paper, spectral properties of T are investigated. The following results are obtained: (i) If H is bounded below, then $\sigma(T)$, the spectrum of T, is either $\mathbb{C}$ (the set of complex numbers) or $\{z \in \mathbb{C} | \mathrm{Im}\, z \geq 0\}$. (ii) If H is bounded above, then $\sigma(T)$ is either $\mathbb{C}$ or $\{z \in \mathbb{C} | \mathrm{Im}\, z \leq 0\}$. (iii) If H is bounded, then $\sigma(T) = \mathbb{C}$. The spectrum of time operators of free Hamiltonians for both nonrelativistic and relativistic particles is exactly identified. Moreover spectral analysis is made on a generalized time operator.

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1 Introduction

In the paper [6], Schmüdgen studied a pair (T, H) of a symmetric operator T and a self-adjoint operator H on a complex Hilbert space $\mathcal{H}$ (in the notation there, $T = P, H = -Q$) such that, for all $t \in \mathbb{R}$, $e^{-itH}D(T) \subset D(T)$ ($D(T)$ denotes the domain of T) and

$$Te^{-itH}\psi = e^{-itH}(T + t)\psi, \ \forall t \in \mathbb{R}, \forall \psi \in D(T). \tag{1.1}$$

This is a stronger version of the representation of the canonical commutation relation (CCR) with one degree of freedom, since (1.1) implies that

$$\langle T\phi, H\psi \rangle - \langle H\phi, T\psi \rangle = \langle \phi, i\psi \rangle, \quad \psi, \phi \in D(T) \cap D(H), \tag{1.2}$$

i.e., T and H satisfy the CCR in the sense of sesquilinear form on $D(H) \cap D(T)$ and hence, in particular, $TH - HT = i$ on $D(HT) \cap D(TH)$, the CCR in the original sense. We call (1.1) the *weak Weyl relation* (WWR).

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About twenty years later, Miyamoto [3] used the WWR to introduce a proper concept of time operator in quantum mechanics. Namely a symmetric operator T on $\mathcal{H}$ is called a *time operator* of H if (T, H) obeys the WWR (1.1) (in [3], (1.1) is called the *T-weak Weyl relation*). We remark that, in this terminology, one has in mind the case where, in application to quantum mechanics, H is the Hamiltonian of a quantum system. Some fundamental properties of the pair (T, H) were investigated in [3].

The work of Miyamoto [3] was extended by the present author in a previous paper [2], where a generalized version of the WWR (1.1), called a *generalized weak Weyl relation*, is given and, in terms of it, a concept of *generalized time operator* is introduced. We remark that a time operator as well as a generalized one of a given self-adjoint operator H is not unique [2, Proposition 2.6, §11]. Physically the set of generalized time operators associated with a self-adjoint operator H (a Hamiltonian) can be regarded as a class of symmetric operators which play a role in controlling decays (in time) of survival probabilities as well as time-energy uncertainty relations [2, 3].

In this paper, we investigate spectral properties of (generalized) time operators. We first recall the definition of the spectrum of a linear operator A on $\mathcal{H}$. The resolvent set of A, denoted $\rho(A)$, is defined to be the set of complex numbers z satisfying the following three conditions: (i) $A - z$ is injective ; (ii) $\mathrm{Ran}(A - z)$, the range of $A - z$, is dense in $\mathcal{H}$; (iii) $(A - z)^{-1}$ with $D((A - z)^{-1}) = \mathrm{Ran}(A - z)$ is bounded. Then the spectrum of A, denoted $\sigma(A)$, is defined by $\sigma(A) := \mathbb{C} \setminus \rho(A)$, where $\mathbb{C}$ is the set of complex numbers. It follows that, if A is closable, then $\sigma(\bar{A}) = \sigma(A)$, where $\bar{A}$ is the closure of A, and $\mathrm{Ran}(\bar{A} - z) = \mathcal{H}$ for all $z \in \rho(\bar{A}) = \rho(A)$. In particular, for all symmetric operators S on $\mathcal{H}$, $\sigma(S) = \sigma(\bar{S})$ and $\mathrm{Ran}(\bar{S} - z) = \mathcal{H}$ for all $z \in \rho(\bar{S}) = \rho(S)$.

One of the motivations for this work comes from the following fact:

Theorem 1.1 ([3], [2, Theorem 2.8]) *If H is a self-adjoint operator on $\mathcal{H}$ and semi-bounded (i.e., bounded below or bounded above), then no time operator T of H can be essentially self-adjoint .*

This theorem combined with a general theorem [5, Theorem X.1] implies that, in the case where H is semi-bounded, the spectrum $\sigma(T)$ of T $(= \sigma(\overline{T}))$ is one of the following three sets:

(i) $\mathbb{C}$.

(ii) $\overline{\Pi}_+$, the closure of the upper half-plane $\Pi_+ := \{z \in \mathbb{C} | \mathrm{Im}\, z > 0\}$.

(iii) $\overline{\Pi}_-$, the closure of the lower half-plane $\Pi_+ := \{z \in \mathbb{C} | \mathrm{Im}\, z < 0\}$.

Then it is interesting to examine which one is realized, depending on properties of H.

The outline of the present paper is as follows. In Section 2, we prove a theorem on the spectrum of time operators (Theorem 2.1). In Section 3 we consider time operators on direct sums of Hilbert spaces. In Section 4, we identify the spectrum of concrete time operators, including the Aharonov-Bohm time operator [1] and time operators of a relativistic Schrödinger operator. In Section 5, we prove a theorem similar to Theorem 2.1 in the case where T is a generalized time operator.

2 Main Result

In this section we prove the following theorem:

Theorem 2.1 *Let H be a self-adjoint operator on $\mathcal{H}$ and T be a time operator of H. Then the following* (i)—(iii) *hold:*

(i) *If H is bounded below, then $\sigma(T)$ is either $\mathbb{C}$ or $\overline{\Pi}_+$.*

(ii) *If H is bounded above, then $\sigma(T)$ is either $\mathbb{C}$ or $\overline{\Pi}_-$.*

(iii) *If H is bounded, then $\sigma(T) = \mathbb{C}$.*

Remark 2.1 The time operator T has no eigenvalues , i.e., the point spectrum $\sigma_{\mathrm{p}}(T)$ of T is an empty set [3, Corollary 4.2].

Remark 2.2 In the case where $\sigma(T) = \overline{\Pi}_+$ or $\overline{\Pi}_-$, $\overline{T}$ is maximally symmetric [5, p.141].

Throughout the rest of this section, T represents a time operator of H. The following lemma is a key fact to prove Theorem 2.1.

Lemma 2.2 *Suppose that H is bounded below. Then, for all $\beta > 0$, $e^{-\beta H}D(\overline{T}) \subset D(\overline{T})$ and, for all $\psi \in D(\overline{T})$*

$$\overline{T}e^{-\beta H}\psi = e^{-\beta H}(\overline{T} - i\beta)\psi. \tag{2.1}$$

Proof. Apply [2, Theorem 6.2]. ∎

We denote by T^* the adjoint of T.

Lemma 2.3 *Suppose that H is bounded below. Then, for all $\beta > 0$, $e^{-\beta H}D(T^*) \subset D(T^*)$ and, for all $\psi \in D(T^*)$*

$$T^*e^{-\beta H}\psi = e^{-\beta H}(T^* - i\beta)\psi. \tag{2.2}$$

Proof. Lemma 2.2 implies that $e^{-\beta H}(\overline{T} - i\beta) \subset \overline{T}e^{-\beta H}$. We have $(\overline{T})^* = T^*$. For each bounded linear operator A on $\mathcal{H}$ with $D(A) = \mathcal{H}$ and all densely defined linear operators B on $\mathcal{H}$, $(AB)^* = B^*A^*$. Using these facts, one can show that $e^{-\beta H}T^* \subset (T^* + i\beta)e^{-\beta H}$. Thus the desired result follows. ∎

Proof of Theorem 2.1

(i) By the fact on the spectrum of T mentioned after Theorem 1.1, we need only to show that the case $\sigma(T) = \overline{\Pi}_-$ is excluded. For this purpose, suppose that $\sigma(T) = \overline{\Pi}_-$. Then $\Pi_+ = \rho(T) = \rho(\overline{T})$.

In general, we have for all $z \in \mathbb{C} \setminus \mathbb{R}$ the orthogonal decomposition

$$\mathcal{H} = \ker(T^* - z^*) \oplus \mathrm{Ran}(\overline{T} - z) \tag{2.3}$$

Applying this structure with $z = i \in \Pi_+$, we obtain $\ker(T^* + i) = \{0\}$. Since T is not essentially self-adjoint by Theorem 1.1, it follows that $\ker(T^* - i) \neq \{0\}$. Hence there exists a non-zero vector $\psi \in D(T^*)$ such that $T^*\psi = i\psi$. Then, by Lemma 2.3, $i(1-\beta) \in \sigma_{\mathrm{p}}(T^*)$.

Since $\beta > 0$ is arbitrary, we can take it to be $1 < \beta$. Then $\gamma := i(1-\beta) \in \Pi_-$. Taking $z = \gamma^*$ in (2.3), we have the orthogonal decomposition

$$\mathcal{H} = \ker(T^* - \gamma) \oplus \mathrm{Ran}(\overline{T} - \gamma^*).$$

Hence $\mathrm{Ran}(\overline{T} - \gamma^*)$ is not dense in $\mathcal{H}$. Therefore $\gamma^* \in \sigma(\overline{T}) = \sigma(T)$, i.e., $i(\beta - 1) \in \sigma(T)$. But $i(\beta - 1) \in \Pi_+$. This is a contradiction. Thus $\sigma(T) \neq \overline{\Pi}_-$.

(ii) If H is bounded above, then $\widehat{H} := -H$ is bounded below. It is easy to see that $\widehat{T} := -T$ is a time operator of $\widehat{H}$. Hence, by part (i), $\sigma(\widehat{T}) = \mathbb{C}$ or $\overline{\Pi}_+$. On the other hand, $\sigma(T) = \{-\lambda | \lambda \in \sigma(\widehat{T})\}$, which implies that $\sigma(T) = \mathbb{C}$ or $\overline{\Pi}_-$.

(iii) This follows from (i) and (ii). ∎

In the next section we analyze the spectrum of nontrivial examples of time operators. Here we present only simple examples.

Example 2.1 We denote by $\hat{r}$ the multiplication operator on $L^2([0,\infty))$ by the variable $r \in [0,\infty)$: $(\hat{r}g)(r) := rg(r)$, a.e.$r \in [0,\infty), g \in D(\hat{r})$. The operator $\hat{r}$ is self-adjoint and nonnegative.

Let p_0 be an operator on $L^2([0,\infty))$ defined as follows:

$$D(p_0) := C_0^\infty((0,\infty)), \tag{2.4}$$

$$(p_0 g)(r) := -ig'(r), \quad g \in D(p_0), \tag{2.5}$$

where, for an open set $\Omega \subset \mathbb{R}^n$ $(n \in \mathbb{N})$, $C_0^\infty(\Omega)$ denotes the set of infinitely differentiable functions on Ω with compact support in Ω. Then it is easy to see that $-p_0$ is a time operator of $\hat{r}$ and that

$$\sigma(-p_0) = \overline{\Pi}_+.$$

Hence this is an example which illustrates one of the case of Theorem 2.1-(i).

Example 2.2 Let $L > 0$ and $V_L := (-L/2, L/2) \subset \mathbb{R}$. We denote by $\hat{x}_L$ the multiplication operator on $L^2(V_L)$ by the variable $x \in V_L$. Then $\hat{x}_L$ is a bounded self-adjoint operator. We define an operator p_L as follows:

$$D(p_L) := C_0^\infty(V_L),$$
$$p_L f := -if', \quad f \in D(p_L).$$

Then it is easy to see that $-p_L$ is a time operator of $\hat{x}_L$ and

$$\sigma(-p_L) = \mathbb{C}.$$

Hence this is an example which illustrates Theorem 2.1-(iii). It should be remarked that p_L has uncountably many self-adjoint extensions [4, pp.257–259].

3 Time Operators on Direct Sum Hilbert Spaces

In applications, time operators on direct sum Hilbert spaces may be useful. We briefly discuss this aspect here. Let H_j $(j = 1, 2)$ be a self-adjoint operator on a complex Hilbert space $\mathcal{H}_j$ which has a time operator T_j. Let

$$\mathcal{H} := \mathcal{H}_1 \oplus \mathcal{H}_2. \tag{3.1}$$

Then

$$T := T_1 \oplus T_2 \tag{3.2}$$

is a time operator of $H_1 \oplus H_2$ [2, Proposition 2.14].

Theorem 3.1 *Let H_j, T_j and T be as above. Then:*

(i) *If H_1 is bounded below and H_2 is bounded above, then $\sigma(T) = \mathbb{C}$.*

(ii) *If one of H_1 and H_2 is bounded, then $\sigma(T) = \mathbb{C}$.*

Proof. (i) By Theorem 2.1, $\sigma(T_1) = \mathbb{C}$ or $\overline{\Pi}_+$, and $\sigma(T_2) = \mathbb{C}$ or $\overline{\Pi}_-$. By a general theorem, we have $\sigma(T) = \sigma(T_1) \cup \sigma(T_2)$. Hence, in each case, we have $\sigma(T) = \mathbb{C}$.

(ii) In this case, we can apply Theorem 2.1-(iii) to conclude that one of $\sigma(T_1)$ and $\sigma(T_2)$ is equal to $\mathbb{C}$. Thus the desired result follows. ∎

Remark 3.1 In each case of Theorem 3.1-(i) and (ii), $H_1 \oplus H_2$ can be unbounded both above and below.

Example 3.1 Let

$$\mathcal{H}_L := L^2([0, \infty)) \oplus L^2(V_L),$$

$\hat{r}, p_0$ be as in Example 2.1 and $\hat{x}_L, p_L$ be as in Example 2.2. Then $H_L := \hat{r} \oplus \hat{x}_L$ on $\mathcal{H}_L$ is self-adjoint and bounded below (but unbounded above). Moreover $T_L := (-p_0) \oplus (-p_L)$ is a time operator of H_L and $\sigma(T_L) = \mathbb{C}$. Thus this example shows that the spectrum of a time operator of a self-adjoint operator which is bounded below, but unbounded above, can be equal to $\mathbb{C}$.

4 Examples

4.1 Time operators of the free Hamiltonian of a nonrelativistic particle

Let Δ be the n-dimensional generalized Laplacian acting in $L^2(\mathbb{R}^n_x)$ $(n \in \mathbb{N})$, where $\mathbb{R}^n_x := \{x = (x_1, \cdots, x_n) | x_j \in \mathbb{R}, j = 1, \cdots, n\}$, and

$$H_0 := -\frac{\Delta}{2m} \tag{4.1}$$

with a constant $m > 0$. In the context of quantum mechanics, H_0 represents the free Hamiltonian of a nonrelativistic particle with mass m in the n-dimensional space $\mathbb{R}^n_x$.

It is well known that H_0 is a nonnegative self-adjoint operator. We denote by $\hat{x}_j$ the multiplication operator on $L^2(\mathbb{R}^n_x)$ by the j-th variable $x_j \in \mathbb{R}^n_x$ and set

$$\hat{p}_j := -iD_j \tag{4.2}$$

with D_j being the generalized partial differential operator in the variable x_j on $L^2(\mathbb{R}^n_x)$. It is easy to see that $\hat{x}_j$ and $\hat{p}_j$ are injective.

We introduce

$$\Omega_j := \{k = (k_1, \cdots, k_n) \in \mathbb{R}^n_k | k_j \neq 0\}, \quad j = 1, \cdots, n. \tag{4.3}$$

For a real-valued, Borel measurable function G on $\mathbb{R}^n_k$ which is continuous on Ω_j, we define a linear operator on $L^2(\mathbb{R}^n_x)$ by

$$G(\hat{p}) := \mathcal{F}^{-1} G \mathcal{F}, \tag{4.4}$$

where $\mathcal{F} : L^2(\mathbb{R}^n_x) \to L^2(\mathbb{R}^n_k)$ is the Fourier transform:

$$(\mathcal{F}f)(k) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n_x} f(x) e^{-ikx} dx, \quad f \in L^2(\mathbb{R}^n_x),\, k = (k_1, \cdots, k_n) \in \mathbb{R}^n_k, \tag{4.5}$$

in the L^2-sense, $\hat{p} := (\hat{p}_1, \cdots, \hat{p}_n)$ and G on the right hand side of (4.4) represents the multiplication operator on $L^2(\mathbb{R}^n_k)$ by the function G. Since the Lebesgue measure of the set $\mathbb{R}^n_k \setminus \Omega_j$ is zero, it follows that $G(\hat{p})$ is self-adjoint.

For each $j = 1, \cdots, n$, one can define a linear operator on $L^2(\mathbb{R}^n_x)$ by

$$T_j(G) := \frac{m}{2} \left(\hat{x}_j \hat{p}_j^{-1} + \hat{p}_j^{-1} \hat{x}_j \right) + G(\hat{p}) \tag{4.6}$$

with domain

$$D(T_j(G)) := \mathcal{F}^{-1} C_0^\infty(\Omega_j), \tag{4.7}$$

where $C_0^\infty(\Omega_j)$ denotes the set of infinitely many differentiable functions on Ω_j with compact support in Ω_j. It is easy to see that $T_j(G)$ is a symmetric operator on $L^2(\mathbb{R}^n_x)$.

Lemma 4.1 *The operator $T_j(G)$ is a time operator of H_0.*

Proof. We write

$$T_j(G) = T_j + G(\hat{p})$$

with

$$T_j := \frac{m}{2} \left(\hat{x}_j \hat{p}_j^{-1} + \hat{p}_j^{-1} \hat{x}_j \right). \tag{4.8}$$

The operator T_j is a time operator of H_0 ([3], [2, §10]). By using the Fourier transform, one can show that $e^{-itH_0} G(\hat{p}) \subset G(\hat{p}) e^{-itH_0}$ for all $t \in \mathbb{R}$. Hence, by applying [2, Proposition 2.6], $T_j(G)$ is a time operator of H_0. ∎

Remark 4.1 The operator T_j is called the *Aharonov-Bohm time operator* [1]. Hence $T_j(G)$ is a perturbed Aharonov-Bohm time operator.

As for the spectrum of $T_j(G)$, we have the following theorem:

Theorem 4.2 $\sigma(T_j(G)) = \overline{\Pi}_+,\ j = 1, \cdots, n.$

Proof. By Theorem 2.1-(i) and (2.3), we need only to show that $\ker(T_j(G)^* - i) = \{0\}$. Let $f \in \ker(T_j(G)^* - i)$. Then $T_j(G)^* f = if$. This implies the equation

$$D_{k_j} \hat{f}(k) = \left(\frac{1}{2k_j} + \frac{k_j}{m} + \frac{i}{m} k_j G(k) \right) \hat{f}(k)$$

in the sense of distributions on Ω_j, where $\hat{f} := \mathcal{F} f$ and D_{k_j} is the generalized partial differential operator in the variable k_j. Hence

$$\hat{f}(k) = c(k_1, \cdots, k_{j-1}, k_{j+1}, \cdots, k_n) \sqrt{|k_j|} e^{k_j^2/(2m)} e^{iG_j(k)/m}, \quad k \in \Omega_j$$

where $c(k_1, \cdots, k_{j-1}, k_{j+1}, \cdots, k_n) \in \mathbb{C}$ is independent of k_j and G_j is a differentiable function on Ω_j such that $\partial G_j(k)/\partial k_j = k_j G(k),\ k \in \Omega_j$. Since $\hat{f}$ is in $L^2(\mathbb{R}^n_k)$, it follows that $c(k_1, \cdots, k_{j-1}, k_{j+1}, \cdots, k_n) = 0$ (a.e.). Hence $f = 0$. Thus $\ker(T_j(G)^* - i) = \{0\}$. ∎

4.2 Time operators of the free Hamiltonian of a relativistic particle

A Hamiltonian of a free relativistic particle with mass $m \geq 0$ moving in $\mathbb{R}^n_x$ is given by

$$H_{\text{rel}} := \sqrt{-\Delta + m^2} \tag{4.9}$$

acting in $L^2(\mathbb{R}^n_x)$. It is shown that the operator

$$T_j^{\text{rel}}(G) := \sqrt{-\Delta + m^2}\, \hat{p}_j^{-1} \hat{x}_j + \hat{x}_j \sqrt{-\Delta + m^2}\, \hat{p}_j^{-1} + G(\hat{p}) \tag{4.10}$$

with $D(T_j^{\text{rel}}(G)) := \mathcal{F}^{-1} C_0^\infty(\Omega_j)$ is a time operator of H_{rel} [2, Example 11.4].

Theorem 4.3 $\sigma(T_j^{\text{rel}}(G)) = \overline{\Pi}_+,\ j = 1, \cdots, n.$

Proof. As in Theorem 4.2, we need only to show that $\ker(T_j^{\text{rel}}(G)^* - i) = \{0\}$. Let $f \in \ker(T_j^{\text{rel}}(G)^* - i)$ and $\omega(k) := \sqrt{k^2 + m^2}$, $k \in \mathbb{R}^n_k$. Then

$$D_{k_j} \hat{f}(k) = \frac{1}{2} \left(\frac{k_j}{\omega(k)} - \frac{k_j}{\omega(k)} \left(\frac{\partial}{\partial k_j} \frac{\omega(k)}{k_j} \right) - \frac{k_j G(k)}{\omega(k) i} \right) \hat{f}(k)$$

in the sense of distributions on Ω_j. Hence

$$\hat{f}(k) = c(k_1, \cdots, k_{j-1}, k_{j+1}, \cdots, k_n) \sqrt{\frac{|k_j|}{\omega(k)}} e^{\omega(k)/2} e^{iF_j(k)/2}, \quad k \in \Omega_j$$

where $c(k_1, \cdots, k_{j-1}, k_{j+1}, \cdots, k_n) \in \mathbb{C}$ is independent of k_j and F_j is a differentiable function on Ω_j such that $\partial F_j(k)/\partial k_j = k_j G(k)/\omega(k),\ k \in \Omega_j$. Since $\hat{f}$ is in $L^2(\mathbb{R}^n_k)$, it follows that $c(k_1, \cdots, k_{j-1}, k_{j+1}, \cdots, k_n) = 0$ (a.e.). Hence $f = 0$. Thus $\ker(T_j^{\text{rel}}(G)^* - i) = \{0\}$. ∎

5 A Class of Generalized Time Operators

In this section we consider spectral properties of a class of generalized time operators. Let H be a self-adjoint operator on a complex Hilbert space $\mathcal{H}$ and T be a symmetric operator on $\mathcal{H}$.

We call the operator T a *generalized time operator* of H if $e^{-itH}D(T) \subset D(T)$ for all $t \in \mathbb{R}$ and there exists a bounded self-adjoint operator $C \neq 0$ on $\mathcal{H}$ with $D(C) = \mathcal{H}$ such that

$$Te^{-itH}\psi = e^{-itH}(T + tC)\psi, \quad \psi \in D(T). \tag{5.1}$$

We call C the *noncommutative factor* for (H, T).

The following facts are known:

Theorem 5.1 *Let T be a generalized time operator of H with noncommutative factor C.*

(i)([2, Theorem 2.8]) *Let H be semi-bounded and*

$$CT \subset TC. \tag{5.2}$$

Then T is not essentially self-adjoint .

(ii)([2, Corollary 5.3-(ii)]) *If $C \geq 0$ or $C \leq 0$, then $\sigma_{\mathrm{p}}(\overline{T}|[D(\overline{T}) \cap (\ker C)^{\perp}]) = \emptyset$.*

(iii)([2, Theorem 6.2-(ii)]) *Let H be bounded below. Then, for all $\beta > 0$, $e^{-\beta H}D(\overline{T}) \subset D(\overline{T})$ and*

$$\overline{T}e^{-\beta H}\psi - e^{-\beta H}\overline{T}\psi = -i\beta e^{-\beta H}C\psi, \quad \psi \in D(\overline{T}). \tag{5.3}$$

(iv)([2, Proposition 6.4, Corollary 6.6]) *The operators H and C strongly commute (i.e., their spectral measures commute) and H is reduced by $\overline{\mathrm{Ran}(C)}$.*

In what follows, T is a generalized time operator of H with noncommutative factor C satisfying (5.2).

5.1 The case where C has a non-zero eigenvalue

We first consider the case where C has a non-zero eigenavlue $a \in \mathbb{R} \setminus \{0\}$, i.e.,

$$\mathcal{K}_a := \ker(C - a) \neq \{0\}. \tag{5.4}$$

We have the orthogonal decomposition

$$\mathcal{H} = \mathcal{K}_a \oplus \mathcal{K}_a^{\perp}. \tag{5.5}$$

Relation (5.2) implies that

$$C\overline{T} \subset \overline{T}C. \tag{5.6}$$

Then it follows that $\overline{T}$ is reduced by $\mathcal{K}_a$ and hence by $\mathcal{K}_a^{\perp}$. We denote the reduced part of $\overline{T}$ to $\mathcal{K}_a$ and $\mathcal{K}_a^{\perp}$ by $\overline{T}_a$ and $\overline{T}_a^{\perp}$ respectively. Hence we have

$$\overline{T} = \overline{T}_a \oplus \overline{T}_a^{\perp} \tag{5.7}$$

relative to the orthogonal decomposition (5.5). Therefore

$$\sigma(T) = \sigma(\overline{T}) = \sigma(\overline{T}_a) \cup \sigma(\overline{T}_a^{\perp}). \tag{5.8}$$

By the strong commutativity of H and C (Theorem 5.1-(iv)), H also is reduced by $\mathcal{K}_a$. We denote the reduced part of H to $\mathcal{K}_a$ by H_a.

Lemma 5.2 *The operator $\overline{T}_a$ is a time operator of H_a/a.*

Proof. Let $\psi \in D(\overline{T}_a) = D(\overline{T}) \cap \mathcal{K}_a$. Then, for all $t \in \mathbb{R}$, $e^{-itH_a}\psi = e^{-itH}\psi \in D(\overline{T}) \cap \mathcal{K}_a = D(\overline{T}_a)$ and, by (5.1),

$$\overline{T}_a e^{-itH_a}\psi = e^{-itH_a}(\overline{T}_a + ta)\psi.$$

Thus the desired result follows. ∎

Theorem 5.3 *Let T be a generalized time operator of H with noncommutative factor C satisfying (5.2). Then the following* (i)—(v) *hold:*

(i) *If H_a is bounded below and $a > 0$, then $\sigma(\overline{T}_a)$ is either $\mathbb{C}$ or $\overline{\Pi}_+$.*

(ii) *If H_a is bounded above and $a < 0$, then $\sigma(\overline{T}_a)$ is either $\mathbb{C}$ or $\overline{\Pi}_+$.*

(iii) *If H_a is bounded above and $a > 0$, then $\sigma(\overline{T}_a)$ is either $\mathbb{C}$ or $\overline{\Pi}_-$.*

(iv) *If H_a is bounded below and $a < 0$, then $\sigma(\overline{T}_a) = \mathbb{C}$ or $\overline{\Pi}_-$.*

(v) *If H_a is bounded, then $\sigma(\overline{T}_a) = \mathbb{C}$.*

Proof. In (i) and (ii), H_a/a is bounded below. Hence, Lemma 5.2 and Theorem 2.1-(i) yield the results stated in (i) and (ii). Similarly other cases follow. ∎

5.2 The case where Ran(C) is closed

We next consider the case where Ran(C) is closed. Then $\mathcal{H}$ is decomposed as

$$\mathcal{H} = \ker C \oplus \mathrm{Ran}(C). \tag{5.9}$$

By the closed graph theorem, there exists a constant $M > 0$ such that

$$\|C\psi\| \geq M\|\psi\|, \quad \psi \in (\ker C)^{\perp} = \mathrm{Ran}(C). \tag{5.10}$$

The operators T and C are reduced by Ran(C). We denote by $\widetilde{T}$ and $\widetilde{C}$ the reduced part of T and C to Ran(C) respectively. The operator $\widetilde{T}$ is symmetric and $\widetilde{C}$ is a bounded self-adjoint operator on Ran(C) which is bijective with $\widetilde{C}^{-1}$ bounded. It follows from (5.2) that

$$\widetilde{C}\widetilde{T} \subset \widetilde{T}\widetilde{C}. \tag{5.11}$$

Lemma 5.4 *The operator*

$$T_C := \widetilde{C}^{-1}\widetilde{T} = \widetilde{T}\widetilde{C}^{-1}|D(\widetilde{T}) \tag{5.12}$$

on Ran(C) *is a symmetric operator.*

Proof. We have $D(T_C) = D(T) \cap \text{Ran}(C)$. Hence $D(T_C)$ is dense in Ran(C). Relation (5.11) implies that

$$\widetilde{C}^{-1}\widetilde{T} \subset \widetilde{T}\widetilde{C}^{-1}. \tag{5.13}$$

Hence $T_C^* = \widetilde{T}^*\widetilde{C}^{-1} \supset \widetilde{T}\widetilde{C}^{-1} \supset T_C$. Thus the desired result follows. ∎

By Theorem 5.1-(iv), H is reduced by Ran(C). We denote the reduced part of H to Ran(C) by $\widetilde{H}$.

Theorem 5.5 *The operator* T_C *is a time operator of* $\widetilde{H}$ *and the following* (i)—(iii) *hold:*

(i) *If* $\widetilde{H}$ *is bounded below, then* $\sigma(T_C)$ *is either* $\mathbb{C}$ *or* $\overline{\Pi}_+$.

(ii) *If* $\widetilde{H}$ *is bounded above, then* $\sigma(T_C)$ *is either* $\mathbb{C}$ *or* $\overline{\Pi}_-$.

(iii) *If* $\widetilde{H}$ *is bounded, then* $\sigma(T_C) = \mathbb{C}$.

Proof Since $D(T_C) = D(T) \cap \text{Ran}(C)$, it follows that $e^{-it\widetilde{H}}D(T_C) \subset D(T_C)$ for all $t \in \mathbb{R}$. Using (5.1), one can see that T_C satisfies

$$T_C e^{-it\widetilde{H}}\psi = e^{-it\widetilde{H}}(T_C + t)\psi, \quad \psi \in D(T_C).$$

These facts and Lemma 5.4 imply that T_C is a time operator of $\widetilde{H}$. Then (i)—(iii) follow from application of Theorem 2.1. ∎

Example 5.1 A simple example is given by the case where $C \neq 0$ is an orthogonal projection. Then $T_C = \widetilde{T}$. To construct such examples, see [2, §11].

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