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Switched Knockout Options : Numerical Valuation

Kunio Hanada
Toshikazu Kimura*

In modern financial markets, various option contracts have been introduced and traded. Knockout options are kinds of exotic contingent claims whose right to exercise is nullified when the underlying asset price hits a knockout boundary. Beginning with a mathematical model of Merton in 1973, some extended models have been developed for the knockout options, under a common assumption that the knockout boundary exists in the whole trading interval. In this paper, however, we consider a new European knockout option whose knockout boundary exists only in a certain part of the trading interval, so that we call it a *switched knockout option*. Extensive numerical experiments show that the switched knockout options have quite different properties from the ordinary knockout as well as vanilla options, especially on the sensitivity with volatility.

Keywords : finance; investment analysis; stochastic; switched knockout options; European call/put; numerical valuation

1. Introduction

In modern financial markets, various option contracts have been introduced and traded. For a vanilla European option, the payoff at exercise can be determined by the spot price of the underlying asset, independently on its past history in the trading interval. The so-called *exotic* or *path-dependent* options have values that depend on the history of the asset price in some non-trivial way. Among various exotic options, we focus on a knockout option with an incomplete boundary in this paper.

Knockout options are contingent claims whose right to exercise is nullified when the underlying asset value crosses a certain value. The set of those values over the trading interval is called a knockout boundary. Knockout options are classified as either *up-and-out* or *down-and-out* options by the relative position between initial values of the asset price and the knockout boundary. Of

**Correspondence : T. Kimura, Division of Management Informatics, Graduate School of Economics & Business Administration, Hokkaido University, Nishi 7, Kita 9, Kita-ku, Sapporo 060-0809, Japan.*
E-mail : kimura@econ. hokudai. ac. jp

course, they are classified into two basic types, *i.e., call* or *put*. Hence, there are totally four different types in knockout options: When the initial price is below the knockout boundary, there are up-and-out calls and puts. On the other hand, when the initial price is above the knockout boundary, there are down-and-out calls and puts.

In Merton [10], he has first studied a basic mathematical model of down-and-out European knockout options to obtain closed pricing formulas under an assumption that the knockout boundary is an exponential function of remaining time to maturity. Rubinstein and Reiner [14] and Rich [12] developed pricing formulas for all types of the basic knockout options. Rich also examined comparative statistics for these formulas. In addition, more general knockout options have been proposed by many researchers: Cox and Rubinstein [3] dealt with a down-and-out European knockout option with a rebate, whose holder can receive a specified amount of money if the boundary is crossed. Kunitomo and Ikeda [8] and Geman and Yor [4] obtained pricing formulas for knockout options with two separate boundaries that are located above and below the asset price at the initial time. Roberts and Shortland [13] analyzed the option price under an assumption that both of the drift and volatility parameters are functions of time. Linetsky [9] proposed a new-type knockout option called a step option, which is not instantaneously nullified when the asset price hits the knockout boundary. These basic and generalized knockout options above have exponential knockout boundaries. Recently, Hanada and Kimura [5] developed an approximate pricing formula for a knockout option with a general class of non-exponential knockout boundaries.

All of the previous results are based on a common assumption that the knockout boundary exists in the whole trading interval from initial time to maturity. In this paper, however, we consider an incomplete knockout boundary that exists only in certain parts of the trading interval. In other words, there is a toggled *switch* in the knockout boundary; this option is equivalent to a vanilla or an ordinary knockout option according as the switch is *off* or *on*. Hence, we call it a *switched knockout option* in this paper. Obviously, the vanilla and ordinary knockout options are special cases of our switched knockout option. From extensive numerical experiments, we will see that features of the switched knockout option are certainly in the middle of these options. In principle, a switched boundary can be applied to any option. From the practical point of view, this means that the switched knockout option greatly extends the applicability of option contracts in the management of financial risks. From the theoretical point of view, the analysis of switched knockout options is also important: We will see in numerical experiments that the switching effects are dominant only in the up-and-out calls and down-and-out puts among

the four different switched knockout options, and that a certain symmetric relation exists between these two options. These results indicate the necessity and possibility of developing tractable approximate pricing formulas for the up-and-out calls and down-and-out puts.

This paper is organized as follows: First, we mathematically specify the switched knockout option to show that its price at arbitrary time satisfies a partial differential equation together with some boundary conditions. Secondly, we numerically solved this equation by the Crank-Nicolson method to examine general properties of switched knockout options. To avoid redundancy, we are mainly concerned with the analysis of the up-and-out call option, but we also refer to some general properties of other three types shortly. Finally, we give a brief summary of the paper and a few remarks on future developments.

2. Mathematical formulation

We use the same assumptions as those in the Black-Scholes model [1] except for knockout boundaries: Assume that the capital market is well-defined and follows the efficient market hypothesis. Let $S(t)$ denote the underlying asset price at time t and let $T(\geq 0)$ be the maturity. Then, under a risk-neutral probability measure **P**, the process $\{S(t)\ ;\ 0 \leq t \leq T\}$ satisfies the stochastic differential equation

$$\frac{dS(t)}{S(t)} = rdt + \sigma dW(t), \quad 0 \leq t \leq T, \tag{1}$$

where r is the risk-free interest rate and σ is the volatility of the process $S(\cdot)$, all of which are assumed to be positive constants. In (1), $\{W(t);\ 0 \leq t \leq T\}$ is the standard Brownian motion process under the measure **P**, so that the process $S(\cdot)$ becomes a geometric Brownian motion. Also, assume that the option price written on $S(t)$, say V, is a function of $S(t)$ and t, *i.e.*, $V \equiv V(S(t), t)$ for $S(t) > 0$ and $0 \leq t \leq T$. From these assumptions and Itô's lemma, we have the partial differential equation

$$\frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 V(S(t),t)}{\partial S(t)^2} + rS(t)\frac{\partial V(S(t),t)}{\partial S(t)} - rV(S(t),t) + \frac{\partial V(S(t),t)}{\partial t} = 0, \tag{2}$$

$$S(t) > 0, \quad 0 \leq t \leq T;$$

see Harrison and Pliska [6] or Øksendal [11].

For a vanilla call option with the exercise price $K(>0)$, the option price V satisfies the terminate condition

$$V(S(T),T)=\max(S(T)-K,0), \tag{3}$$

together with the boundary conditions

$$\lim_{\xi\to\infty}\frac{V(\xi,t)}{\xi-Ke^{-r(T-t)}}=1, \quad 0\le t\le T \tag{4}$$

and

$$\lim_{\xi\to 0}V(\xi,t)=0, \quad 0\le t\le T. \tag{5}$$

For a switched knockout option, however, these boundary conditions should be modified as follows: Let I_{on} be the set of time intervals where the nullified switch is on, and let $I_{\text{off}}\equiv[0,\ T]\setminus I_{\text{on}}$. Let $B(t)$ be the value of knockout boundary at time t and assume that $B(t)>0$ for $t\in[0,\ T]$. Then, for the up-and-out call type, the boundary conditions should be

$$\begin{aligned}
&\lim_{\xi\to\infty}\frac{V(\xi,t)}{\xi-Ke^{-r(T-t)}}=1, && t\in I_{\text{off}},\\
&V(\xi,t)=0, && (\xi,t)\in[B(t),\infty)\times I_{\text{on}},\\
&\lim_{\xi\to 0}V(\xi,t)=0, && 0\le t\le T,
\end{aligned} \tag{6}$$

whereas, for the down-and-out call type, the boundary conditions are given by

$$\begin{aligned}
&\lim_{\xi\to\infty}\frac{V(\xi,t)}{\xi-Ke^{-r(T-t)}}=1, && 0\le t\le T,\\
&\lim_{\xi\to 0}V(\xi,t)=0, && t\in I_{\text{off}},\\
&V(\xi,t)=0, && (\xi,t)\in[0,B(t)]\times I_{\text{on}}.
\end{aligned} \tag{7}$$

Similarly, we can formulate the price of the switched knockout puts with the exercise price K: The terminate condition at time $t=T$ is given by

$$V(S(T),T)=\max(K-S(T),0). \tag{8}$$

The boundary conditions are, for the up-and-out put type,

$$\begin{aligned}
&\lim_{\xi\to\infty}V(\xi,t)=0, && t\in I_{\text{off}},\\
&V(\xi,t)=0, && (\xi,t)\in[B(t),\infty)\times I_{\text{on}},
\end{aligned} \tag{9}$$

$$\lim_{\xi \to 0} V(\xi, t) = Ke^{-r(T-t)}, \qquad 0 \le t \le T,$$

and for the down-and-out put type,

$$\begin{aligned} &\lim_{\xi \to \infty} V(\xi, t) = 0, && 0 \le t \le T, \\ &\lim_{\xi \to 0} V(\xi, t) = Ke^{-r(T-t)}, && t \in I_{\text{off}}, \\ &V(\xi, t) = 0, && (\xi, t) \in [0, B(t)] \times I_{\text{on}}. \end{aligned} \tag{10}$$

3. Features of the switched knockout options

3.1 Preliminaries

In general, it is quite diffcult to obtain an analytical solution of the partial differential equation (2) together with such complex conditions as described above. The purpose of this paper is, however, not to obtain closed-form pricing formulas, but to examine general properties of the switched knockout options, in particular, the differences from the associated options without the nullified switch. Hence, we use a numerical method for the examination. In our numerical experiments, we used the Crank-Nicolson method for solving (2) with the terminate and boundary conditions. The Crank-Nicolson method has been known as a most accurate implicit finite-difference method; see Courtadon [2] for details. Also, see Hull [7] and Wilmott *et al*. [15] for the general theory of finite-difference methods for option pricing.

To keep the original form of the knockout boundary as it is and to avoid the complication, we directly apply the Crank-Nicolson method to (2) without using any transformation of variables in the calculation.

For convenience, we set the initial time to be $t = 0$ and the maturity to be $T = 1$. As a computational requirement, we restrict the state space of $S(t)$ for all t in an interval $[0, S_{\max}]$ with $S_{\max} \equiv 1,000$ and divide this interval into 10,000 fragments with equal widths. Also, the time interval $[0, 1]$ is divided into 500 fragments. For the option parameters, we use $K = 100$ and $r = 0.05$ in all cases, and $\sigma = 0.3$ if not clearly mentioned. For the knockout boundary function, we use a constant-valued boundary

$$B(t) = \begin{cases} B, & t \in I_{\text{on}} \\ S_{\max}, & t \in I_{\text{off}}, \end{cases} \tag{11}$$

where both I_{on} and I_{off} are compact sets in $[0, 1]$.

3.2 Up-and-out call

Figures 1 and 2 illustrate the curves of the up-and-out call price $V(S(0), 0)$ as a function of $S(0)$ for several knockout boundaries with $B = 140$ or $B = 180$, where the intervals $I_{on} = \emptyset$ (empty set) and $I_{on} = [0, 1]$ are added for compari-

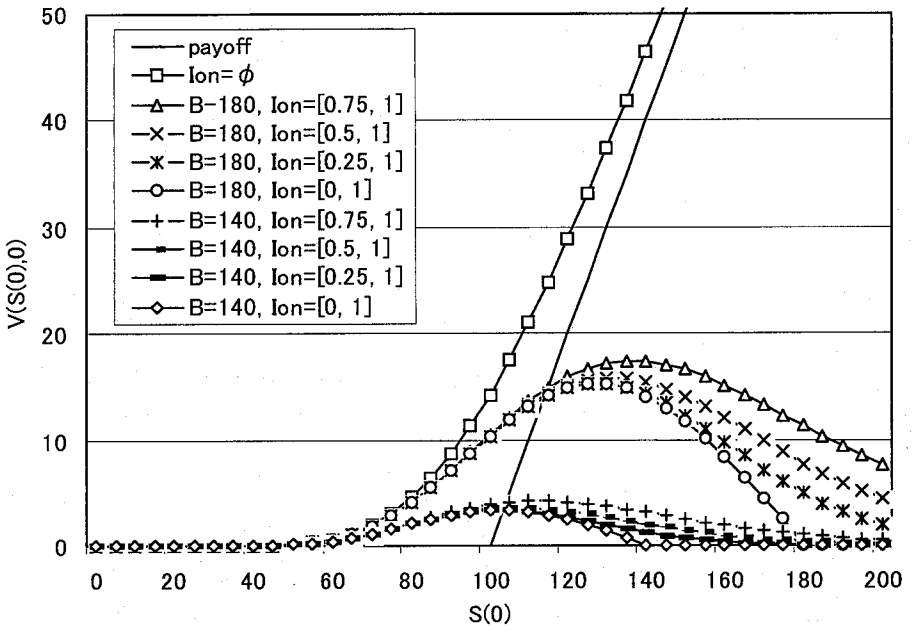

Figure 1 . Prices of the up-and-out calls : Latter-part cases

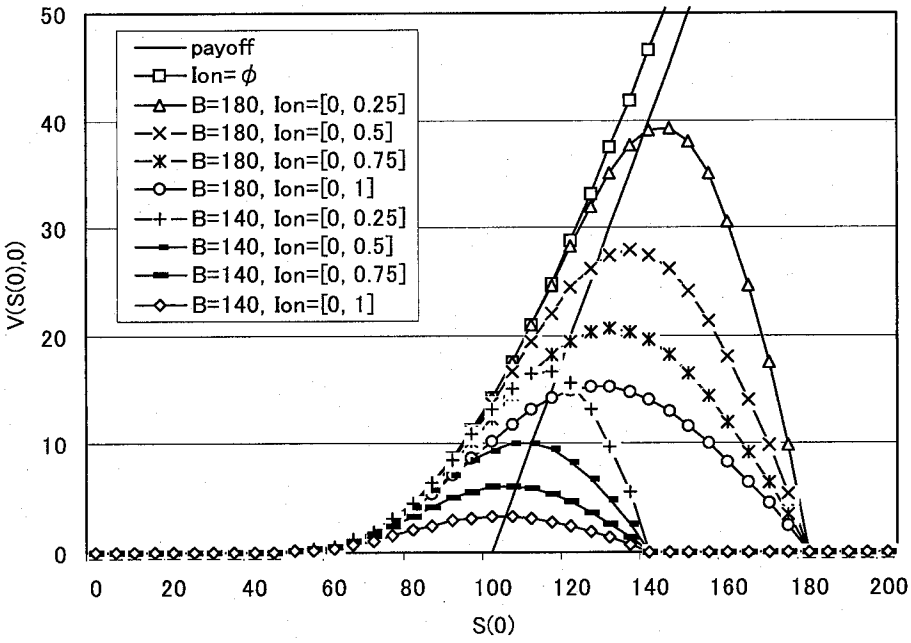

Figure 2 . Prices of the up-and-out calls: Former-part cases

sons, which represent the vanilla and ordinary knockout options, respectively. Clearly, the prices of these extreme cases give upper and lower bounds for V of the switched knockout options. In Figure 1, the knockout boundaries exist in latter parts of the trading interval, whereas in Figure 2 they exist in former parts. From these figures, we see that there are significant differences between these two cases: The option prices for the former-part cases are higher and more sensitive to the length of I_{on} than those for the latter-part cases. No doubt, this result is due to the assumption that the process $S(\cdot)$ follows a geometric Brownian motion with continuous sample paths. In actual markets, it is reasonable to place a knockout boundary at a latter part of the trading interval for hedging risk in future. In this sense, switched knockout options with latter-part boundaries can be attractive alternatives to the vanilla option. Another marked difference is the value of each option price when $S(0) \geq B$. That is, the option prices for the latter-part cases have positive values, while those for the former-part cases are always 0.

To see the effects of volatility to option prices, we compute the prices of switched knockout options with $\sigma = 0.2,\ 0.3,\ 0.4$. Figures 3 and 4 illustrate the curves of $V(S(0),0)$ as a function $S(0)$ when $I_{on} = [0.5,\ 1]$ and $I_{on} = [0,\ 0.5]$, respectively. As in Figures 1 and 2, we have used $B = 140$ or $B = 180$. For the vanilla option, it is well known that the price is monotonously increasing with σ, *i.e.*, $\partial V/\partial\sigma > 0$ for all $\sigma > 0$. However, we see from Figures 3 and 4 that this property does not hold for switched knockout options:

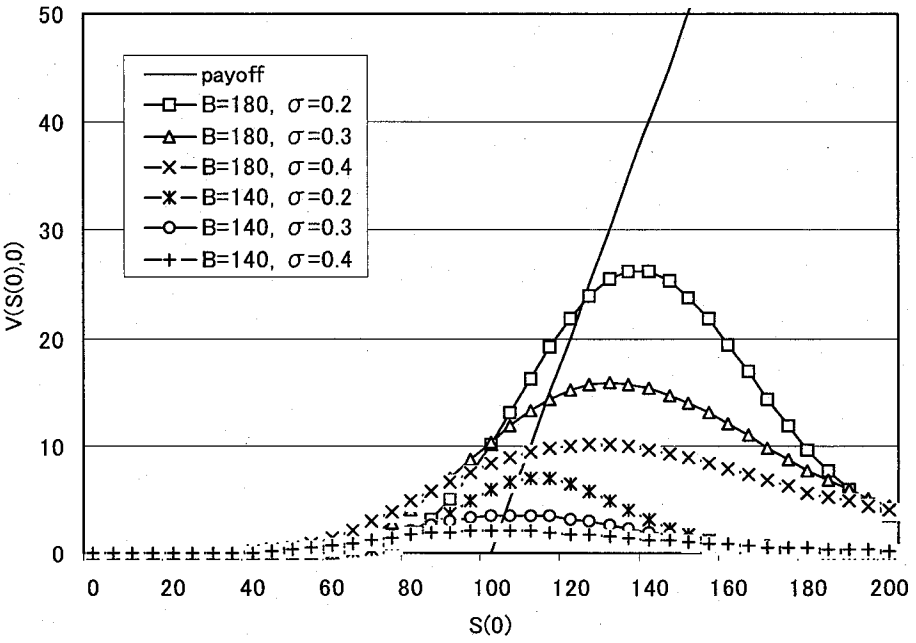

Figure 3 . Prices of the up-and-out calls (I_{on}= [0.5, 1])

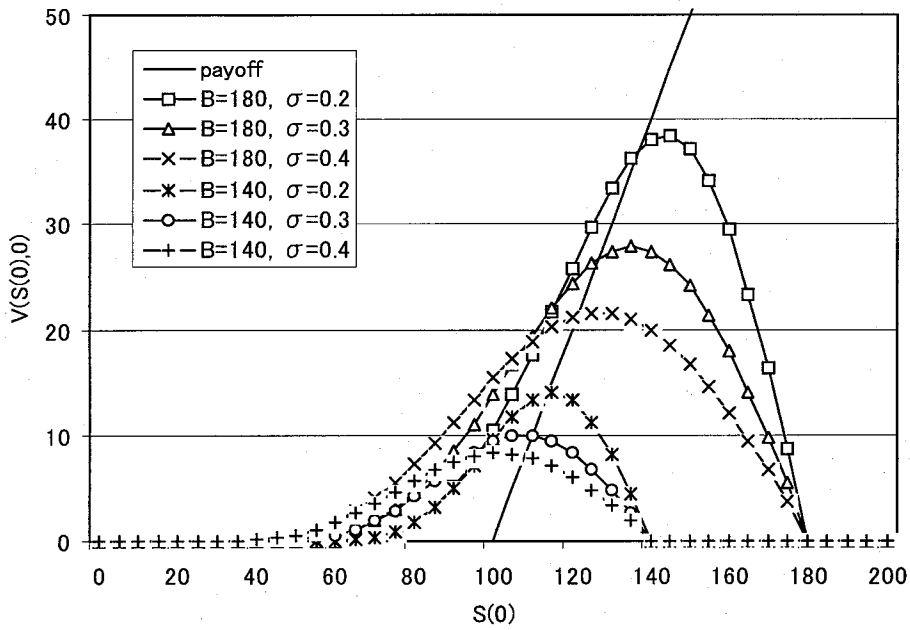

Figure 4 . Prices of the up-and-out calls (I_{on}= [0, 0.5])

Roughly speaking, for all $\sigma > 0$, $\partial V/\partial\sigma > 0$ when $S(0) << K$ and $\partial V/\partial\sigma < 0$ when $S(0) >> K$. This result indicates that a new scheme for risk hedging should be invented for switched knockout options.

3.3 Down-and-out call

Figures 5 and 6 illustrate the curves of the down-and-out call price $V(S(0), 0)$ as a function of $S(0)$ for latter-part and former-part knockout boundaries with $B = 80$, respectively. From these figures, we see that the down-and-out call has properties similar to those for the associated vanilla call option. This result is, in some sense, reasonable because of the similarity on the boundary position. That is, both down-and-out and vanilla calls have the knockout boundaries in the direction that the option value is decreasing. This observation can be certified by another experiment where we changed the values of σ, which shows that the price for the down-and-out call is, unlike the up-and-out call, an increasing function of volatility just as in the vanilla call. We also see that the option prices for the latter-part cases are a little bit more sensitive to the length of I_{on} than those for the former-part cases, although there are almost no differences.

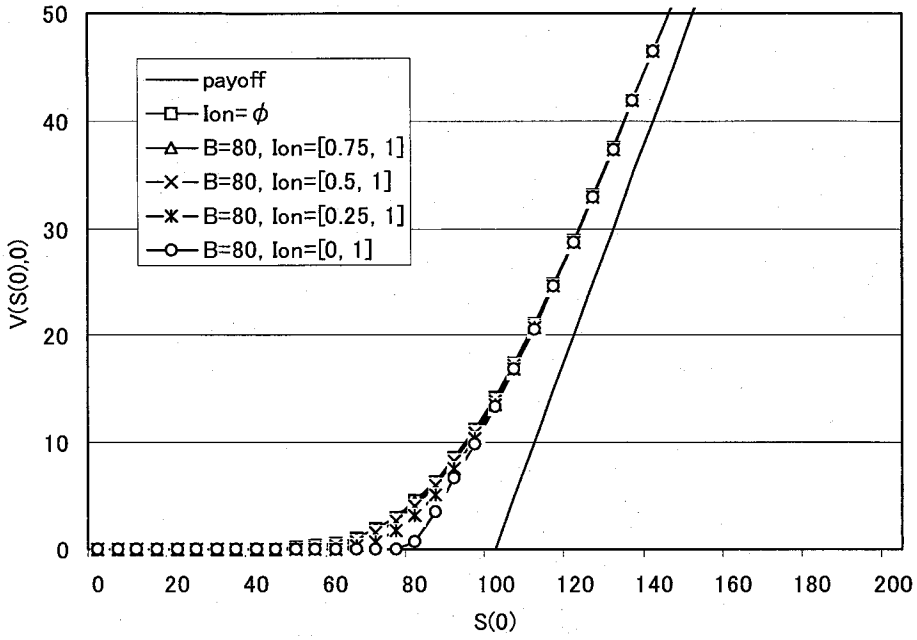

Figure 5 . Prices of the down-and-out calls: Latter-part cases

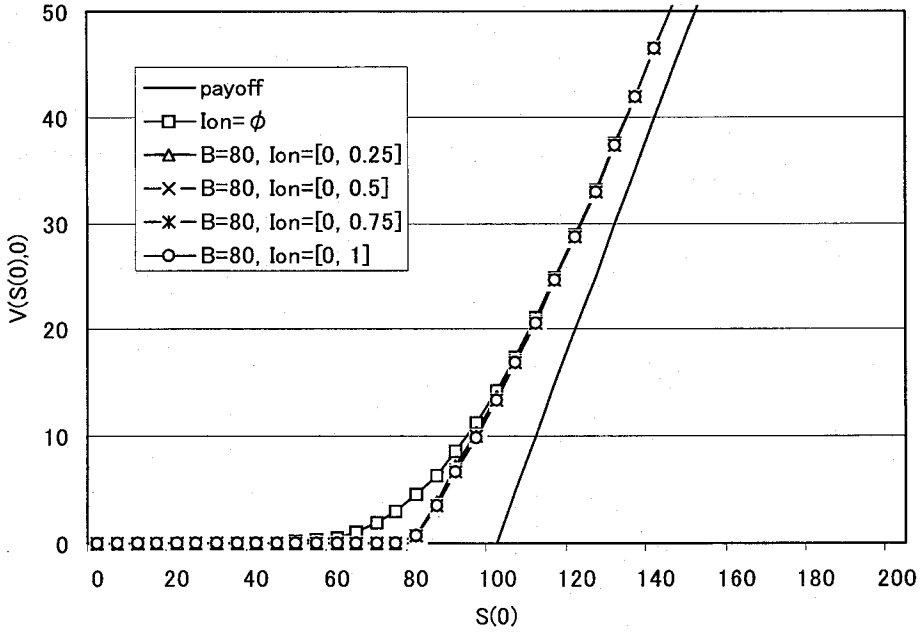

Figure 6 . Prices of the down-and-out calls: Former-part cases

3.4 Down-and-out put

Figures 7 and 8 show the price curves of the down-and-out put for latter-part and former-part knockout boundaries with $B=40$ or $B=60$, respectively. Comparing these figures with Figures 1 and 2, we observe that except for

some trivial differences, general properties of the down-and-out put and the up-and-out call are almost symmetric about the line $S(0)=K$. This result clearly reflects the symmetry of the payoff lines for call and put options; see Equations (3) and (8).

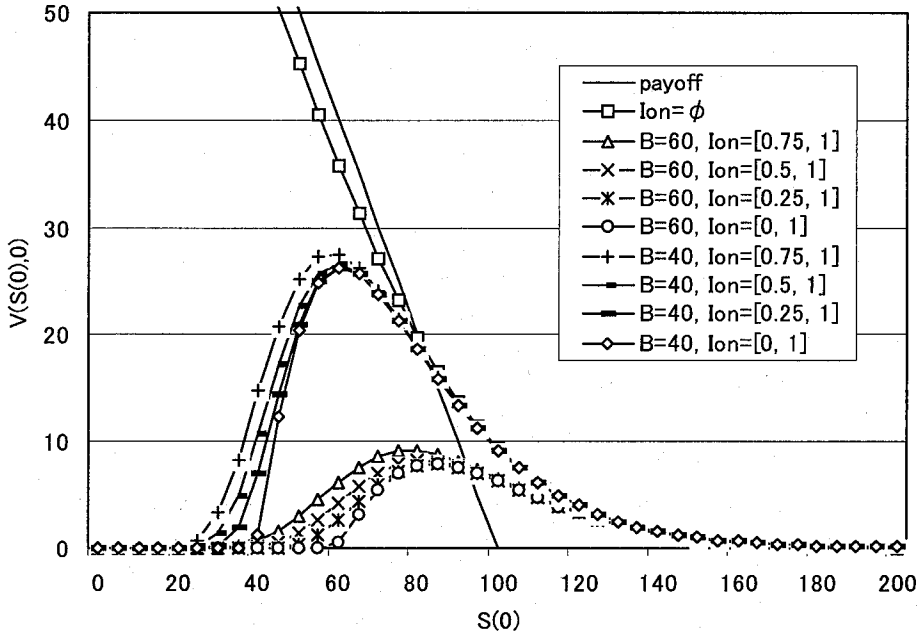

Figure 7 . Prices of the down-and-out puts: Latter-part cases

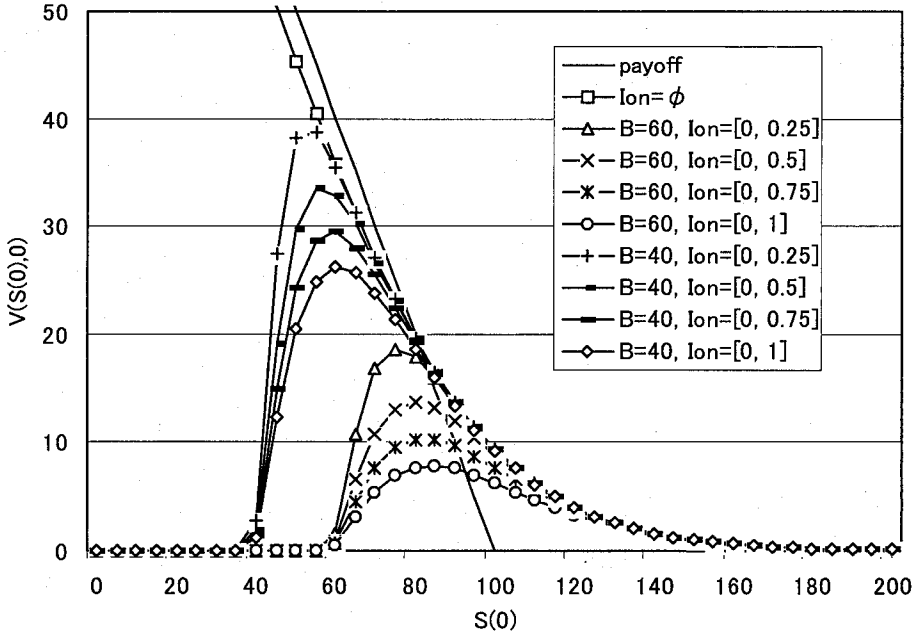

Figure 8 . Prices of the down-and-out puts: Former-part cases

3.5 Up-and-out put

A similar symmetric relation can be also found between the prices of up-and-out put and down-and-out call switched knockout options. Figures 9 and 10 show the price curves of the up-and-out put for latter-part and former-part

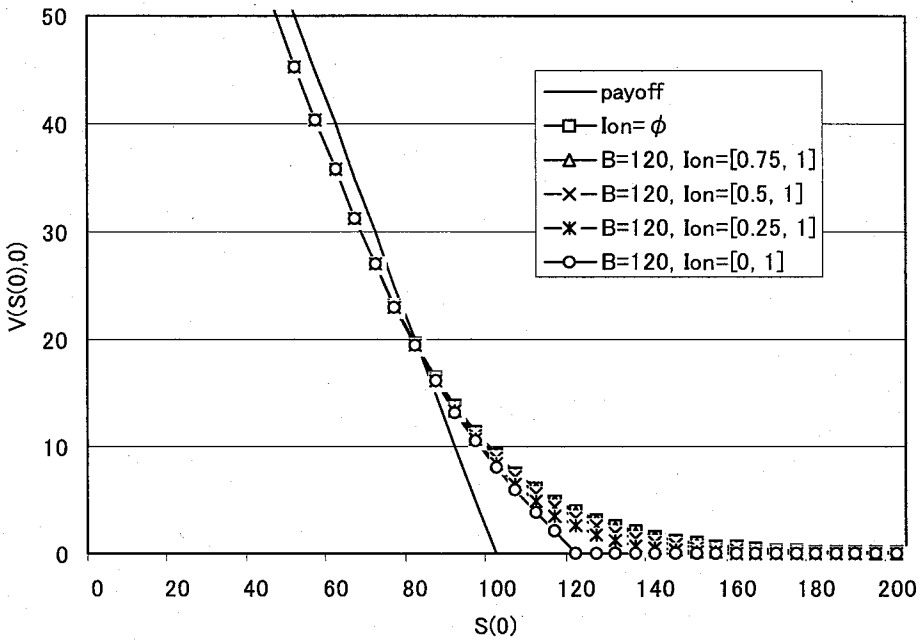

Figure 9 . Prices of the up-and-out puts: Latter-part cases

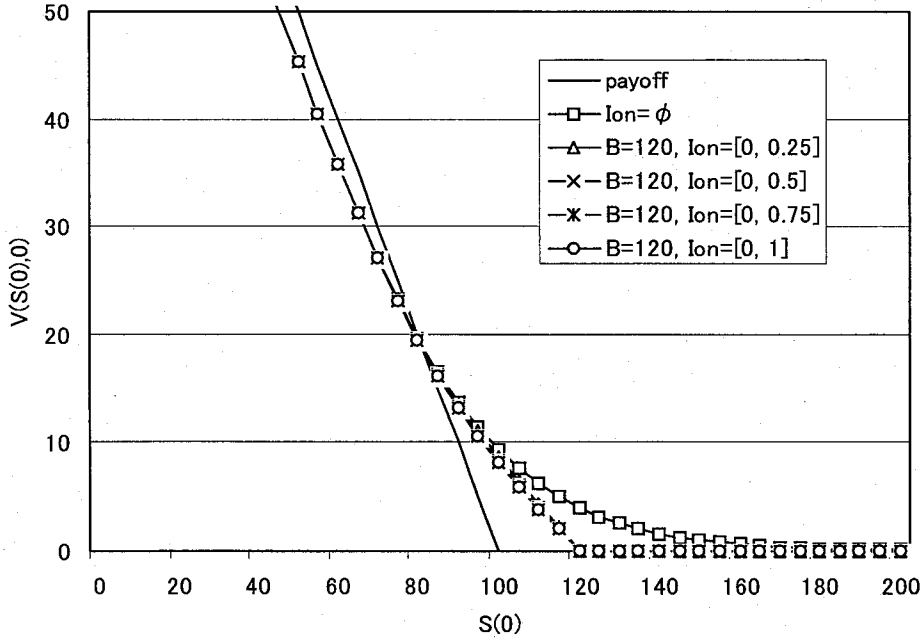

Figure10. Prices of the up-and-out puts: Former-part cases

knockout boundaries with $B=120$, respectively. Comparing these figures with Figures 5 and 6, we can certainly check the symmetry. As shown in the down-and-out call, the up-and-out put also has a similarity to the associated vanilla put.

4. Conclusion

In this paper, we have introduced the switched knockout option whose boundary feature is in the middle of the vanilla and ordinary knockout options. From extensive numerical experiments, we saw that the position of I_{on} in the trading interval significantly affects the option price, and that the sign of the hedge parameter $\partial V/\partial \sigma$ varies depending on $S(0)$. Also, we saw that there are some similar and symmetric relations among the four types of switched knock-out options. That is, the up-and-out call and the down-and-out put are almost symmetric about the line $S(0)=K$. An almost same symmetry can be found between the down-and-out call and the up-and-out put. In addition, we saw that the down-and-out call (up-and-out put) shows a striking similarity to the associated vanilla call (put).

A future direction of this research is to examine the cases that

- I_{on} contains many disjoint intervals,
- two knockout boundaries are located above and below $S(0)$,
- the knockout boundary is either a certain function of time t and $S(t)$ or a random variable.

Another future direction is to develop an approximate pricing formula for the switched knockout option ; see Hanada and Kimura [5] for a related research.

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Doctoral Student, Hokkaido University
Professor, Hokkaido University

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