CHARACTER-THEORETIC TRANSFERS

by

Tomoyuki Yoshida*

Department of Mathematics, Hokkaido University, Sapporo, Japan

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1. Introduction

In the previous paper [15], we considered the determinants of characters. The purpose of this paper is to study further about the determinants of induced characters and to apply them to some transfer theorems. For groups $H \leq G$, we define the character theoretic transfer $T_H^G : \hat{H} \longrightarrow \hat{G}$ by $T_H^G(\lambda) = \det(\lambda^G + 1_H^G)$ for all $\lambda \in \hat{H}$. Elementary properties about induced representations yield some properties of character theoretic transfers. For example, T_H^G is a homomorphism of $\hat{H}$ to $\hat{G}$ and to tell the truth, it is the dual of the ordinary transfer of G/G' to H/H'. The Mackey decomposition theorem about the restrictions of induced representations is translated as follows:

<u>Proposition 2.2</u>. Let $H, K \leq G$ and $\lambda \in \hat{H}$. Then

$$T^G(\lambda)_{|K} = \prod_g T^K(\lambda^{g^{-1}}{}_{|K \cap H^g}),$$

where g runs over a complete set of representatives of (H, K)-double cosets of G.

This result shall be used most frequently in this paper. Over and above these we state the concept of Λ-stabilities, Tate theorem, etc., in Section 2.

In Section 4, applying the character theoretic transfers, we shall prove some transfer theorems which are our purpose. Section 3 is the preparations for Section 4 and we introduce the concept of singularities. Since a character theoretic transfer is the dual of an ordinary transfer, we can rewrite the proofs of known transfer theorems by the use of character theoretic transfers. We give a typical example. For what type of p-groups P does the condition

$$(1) \quad P \cap G' = P \cap N_G(P)'$$

for every finite group G with Sylow p-subgroup P holds ? By Tate theorem (Proposition 2.5), the condition (1) is equivalent to

$$(2) \quad P \cap G^pG' = P \cap N^pN', \text{ where } N = N_G(P).$$

Suppose (1) is false and take element x of $P \cap G^pG' - N^pN'$ of minimal order. Then there is a linear character λ of N of order p such that $\lambda(x) \neq 1$. Clearly, $T^G(\lambda)(x) = 1$. We shall apply Mackey decomposition to $N\backslash G/P$. As $\lambda(x) \neq 1$, we have that

$$(3) \quad \text{there is } g \in G - N \text{ such that } T_S^P(\mu)(x) \neq 1,$$
$$\text{where } S = P \cap P^g \text{ and } \mu = \lambda^{g^{-1}}|_S .$$

(By Lemma 2.6, this is equivalent to the fact that (1) does not hold). Now, assume $y = x^{ipu} \in S$ for an integer i and an element u of P. Then it follows from the minimality of the order of x that $gyg^{-1} \in P \cap N^pN' \leq \operatorname{Ker} \lambda$, and so

we have $\mu(y) = 1$. Hence we proved that if (1) does not hold, then there is a proper subgroup S of P such that

(4)
(a) μ is a linear character of S of order p,
(b) $T_S^P(\mu)(x) \neq 1$, and
(c) whenever $x^{ipu} \in S$ for some integer i and some element u of P, $x^{ipu} \in \operatorname{Ker} \mu$.

Assume first that the p-group P is regular. Then considering the dual of the proof by Wielandt, we have that the regular p-group P has no proper subgroup S of P satisfying the condition (4). For the reason, we say that the subgroup S satisfying (4) is a <u>singular subgroup</u> in P. Now the contradiction yields Wielandt theorem : (1) holds for regular Sylow p-subgroup P. Does the converse hold ? We wish to know about p-groups which have no proper singularities. Very happily there exists a far larger class of p-groups which have no proper singular singular subgroups.

<u>Proposition 3.7(1)</u> A p-group P has no proper singular subgroup if and only if P has no quotient group isomorphic to $Z_p \wr Z_p$.

Hence we have

<u>Theorem 4.2</u>. If a Sylow p-subgroup P of the finite group G

has no quotient group isomorphic to $Z_p \wr Z_p$, then $P \cap G' = P \cap N_G(P)'$.

This theorem is more useful than Wielandt theorem. For example, it is conjectured that almost all unipotent groups of simple groups of Lie type satisfy the hypothesis of the theorem.

We can similarly generalize some other wellknown transfer theorems, too. For example, observing the proof of the above theorem, we know that even the assumption that P is a Sylow groups is not necessary. (See Theorem 4.3 and 4.6). When we try such generalizations, we need to introduce the functor Φ_r^* on p-groups and the concept of subgroups of <u>Sylow type</u> (Definition 3.1). Replacing the word "weakly closed" to " Sylow type", Hall-Wielandt theorem holds again (Theorem 4.3, 4.4, 4.5 and 4.6).

Finally, we consider the relation of transfers and fusion. By the focal subgroup theorem,the complete description of fusion determines the structure of the focal subgroup. Clearly, the converse does not hold, but the knowledge about transfers shall give some important informations about fusion. In his paper [1], Alperin proved that local subgroups control the global fusion. As a corollary, he had that if P is a Sylow p-subgroup of a group G , then

$$(5) \quad P \cap G' = \langle [F, N_G(F)] \mid F \leq P \rangle .$$

We shall try to add some conditions to the subgroups F, that is, we wish to find a family which controls transfers. If $A \leq P$, then there is $g \in G$ and $B = A^g \leq P$ such that $N_P(B)$ is a Sylow p-subgroup of $N_G(B)$. Hence we have that the collection of the pairs $(F, N_G(F))$ such that the subgroup F of P contains a G-conjugate of $N_P(A)$ for some $A \leq P$ is a conjugation family. About families which control transfers, the comparable condition with this is the fact that F contains a G-conjugate of a singular subgroup in P (Theorem 4.8). Similarly as Goldschmidt [8], we can add further conditions to F (Theorem 4.9). As the search for singular subgroups is easier than for conjugation families, these theorem is useful to determine the structure of some local subgroups in simple groups with given Sylow 2-subgroups. Such examples are given in Section 5.

Our notation and terminology are almost standard and follow that of [9]. The notation G and p denotes always a finite group and a prime, respectively. For a character χ of a finite group G, the determinant of χ is defined by $(\det \chi)(y) = \det \rho(y)$, where ρ is a representation with character χ. Then $\det \chi$ is a linear character of G. We extend the definition of det to generalized characters, so

det is a homomorphism of the additive group of generalized characters to the multiplicative group of linear characters. Some of the terms, which are not so standard and are used in this paper are :

$T^G(\theta) = T^G_H(\theta)$	$= \det(\theta^G + \theta(1)\, 1^G_H)$;
X^Y	$= \{x^y \mid x \in X,\ y \in Y\}$;
X_Y	$= \bigcap\{X^y \mid y \in Y\}$;
Λ-stable	see Definition 2.2 ;
$G\hat{} = \hat{G}$	the set of linear characters of G ;
$R^X_Y = R_Y$	the restriction $\hat{X} \longrightarrow \hat{Y}$ for $Y \leq X$;
$\hat{G}_p$	a Sylow p-subgroup of $\hat{G}$;
$G'(p)$	$= G'O^p(G)$;
$G'_r(p)$	$= G'G^{p^r}$
Sylow type	see Definition 3.1 ;
$\Phi^*(P) = \Phi^*_1(P)$	$= \bigcap\{\Phi(M) \mid M \leq P,\ \lvert P:M\rvert \leq p\}$;
$\Phi^*_r(P)$	see Definition 3.2 ;
weakly regular p-group	see Definition 3.3 ;
(weak) p-singularity	see Definition 3.4 ;
R_g	$= R \cap H^g$, see Proposition 4.1 ;
λ'_g	$= \lambda^{-1}\lvert_{R_g} \cdot \lambda^{g^{-1}}\lvert_{R_g}$, see Proposition 4.1.
$[A, B; n]$	$= [[A, B; n-1], B]$, $[A, B; 0] = A$;
Bender groups	$L_2(q)$, $Sz(q)$, $U_3(q)$, $q = 2^n \geq 4$;

Steinberg modules absolutely irreducible GF(2)B-modules of a Bender group B belonging to the block of defect 0;

St(B) the semidirect product of the Steinberg module by a Bender group.

2. Character-theoretic transfers

In this section, we shall study some basic properties of character-theoretic transfers.

Definition 2.1. For $H \leq G$ and $\theta \in \mathrm{ch}\, H$, we set

$$T^G(\theta) = T^G_H(\theta) = \det(\theta^G - \theta(1)1^G_H).$$

The mapping $T^G_H : \mathrm{ch}\, H \longrightarrow \hat{G}$ is called a character-theoretic transfer.

If there is no danger of confusion we shall denote the restrictions of T^G_H to subgroups of $\hat{H}$ by the same notation T^G_H.

Lemma 2.1. Let $H \leq G$ and $\theta, \theta_1, \theta_2 \in \mathrm{ch}\, H$. Then the following hold :

(1) $T^G(\theta) = T^G(\det \theta)$.

(2) $T^G(\theta_1 \theta_2) = T^G(\theta_1)^{d_2}\, T^G(\theta_2)^{d_1}$, where $d_i = \theta_i(1)$.

In particular, $T^G|_{\hat{H}}$ is a homomorphism of $\hat{H}$ to $\hat{G}$.

(3) If $H \leq K \leq G$, then $T^G_K(T^K_H(\theta)) = T^G_H(\theta)$.

(4) If $\chi \in \mathrm{ch}\, G$, then $T^G(\chi_{|H}) = (\det \chi)^{|G:H|}$.

Proof. We prove (1). We may assume that θ is a character of H. Let ρ be a matrix representation of H with character θ and degree d. We extend the definition of ρ to all of G

by setting $\rho(y)$ equal to the $d \times d$ 0 matrix for all y in $G - H$. Let x_i, $1 \leq i \leq n$, be a complete set of right coset representatives of H in G. Then the induced representation $\rho^G : G \longrightarrow GL(nd, C)$ is defined by the rule

$$\rho^G(y) = (\rho(x_i\, y\, x_j^{-1})), \quad y \in G .$$

Thus $\rho^G(y)$ is an $n \times n$ matrix of blocks whose (i, j)-th entry is the $d \times d$ matrix $\rho(x_i\, y\, x_j^{-1})$. See [9, 4.4].

Let $y \in G$. For each i, there is a unique coset $H\, x_{\bar{i}}$ containing $x_i\, y$. If $j \neq \bar{i}$, then $\rho(x_i\, y\, x_j^{-1}) = 0$. Thus it follows that

$$(\det \rho^G)(y) = (\mathrm{sgn}(y))^d \prod_i \det(\rho(x_i\, y\, x_{\bar{i}}^{-1})),$$

where $\mathrm{sgn} = \det 1_H^G$. Replacing ρ by $\det \rho$, we have

$$\begin{aligned}(\det(\det \rho)^G)(y) &= \mathrm{sgn}(y) \prod_i \det((\det \rho)(x_i\, y\, x_{\bar{i}}^{-1})) \\ &= \mathrm{sgn}(y) \prod_i (\det \rho)(x_i\, y\, x_{\bar{i}}^{-1}) \\ &= (\mathrm{sgn}(y))^{1-d} (\det \rho^G)(y).\end{aligned}$$

Thus by the definition of T^G, we have $T^G(\theta) = T^G(\det \theta)$.

The remainder of this lemma follow from (1) and the fact that $\det(\theta_1\, \theta_2) = (\det \theta_1)^{d_2} (\det \theta_2)^{d_1}$, $(\theta^K)^G = \theta^G$, $(\chi_{|H})^G = \chi \cdot 1_H^G$.

In the remainder of this paper, we consider only the restriction $T_H^G : \hat{H} \longrightarrow \hat{G}$. This is a homomorphism.

<u>Proposition 2.2</u> (<u>Mackey decomposition</u>).

(1) Let $H, K \leq G$ and $\lambda \in \hat{H}$. Let $G = \sum_i Hg_iK$ be a decomposition of G to (H, K)-double cosets. For each i, set $K_i = K \cap H^{g_i}$ and $\lambda_i = \lambda^{g_i^{-1}}{}_{|K_i}$. Then

$$T^G(\lambda)_{|K} = \prod_i T^K(\lambda_i).$$

(2) Let $H \leq G$, $x \in G$ and $\lambda \in \hat{H}$. Let $G = \sum_i Hg_i\langle x\rangle$ be a decomposition of G to $(H, \langle x\rangle)$-double cosets. Set $r_i = |\langle x\rangle : \langle x\rangle \cap H^{g_i}|$. Then

$$T^G(\lambda)(x) = \prod_i \lambda(g_i\, x^{r_i}\, g_i^{-1}).$$

<u>Proof</u>. Mackey decomposition theorem ([4, (44.2)]) yields

$$\lambda^G{}_{|K} = \sum_i \lambda_i{}^K, \quad 1^G_{H|K} = \sum_i 1^K_{K_i}.$$

Thus (1) follows from the definition.

Let $K = \langle x\rangle$. We define K_i and λ_i as in (1). Then $r_i = |K : K_i|$. Since K is cyclic, for each i, K has a character μ_i such that $\mu_{i|K_i} = \lambda_i$. Thus $T^K(\lambda_i) = \mu_i{}^{|K:K_i|} = \mu_i{}^{r_i}$ by Lemma 2.1(4). Since $x^{r_i} \in K_i$, (2) follows from (1).

<u>Corollary 2.2.1</u>. Let $H \leq G$, $x \in G$ and $\lambda \in \hat{H}$. Assume $T^G(\lambda)(x) \neq 1$, then $\langle x\rangle^G \cap H \not\subseteq \operatorname{Ker} \lambda$, where $\langle x\rangle^G = \{y^g \mid y \in \langle x\rangle, g \in G\}$.

<u>Corollary 2.2.2</u>. Let $H \trianglelefteq G$ and λ be a self-conjugate linear character of H. Then for $x \in G$,

$$T^G(\lambda)(x) = \lambda(x^{|G:H|}).$$

<u>Corollary 2.2.3</u>. A character-theoretic transfer is the dual of an ordinary transfer.

The proofs of these corollaries are easy. Proposition 2.2 shall be frequently used after this. In view of Corollary 2.2.3, this is comparable to [2, XII,,Prop. 9.1].

In order to translate results given by the use of character-theoretic transfers to results about groups, we use some properties of abelian groups and the dual groups (particularly, the duality theorem). In the remainder of this paper, we use the following notation .

Let X be any finite group and $Y \leq X$. Then R_Y^X (or simply R_Y) denotes the restriction homomorphism $\hat{X} \longrightarrow \hat{Y}$. For $\Lambda \leq \hat{X}$, we set $\Lambda^{\perp} = \{x \in X \mid \lambda(x) = 1$ for all $\lambda \in \Lambda\}$. $\hat{X}_p$ denotes a unique Sylow p-subgroup of $\hat{X}$. We set $X'(p) = X'O^p(X)$ and $X'_r(p) = X'X^{p^r}$ for $r \geq 1$. If there is no danger of confusion, $R^X_{Y|\hat{X}_p}$ and $T^X_{Y|\hat{Y}_p}$ are again denoted by R_Y^X and T_Y^X .

<u>Definition 2.2</u>. Let $K \leq H \leq G$ and $\Lambda \leq \hat{H}$. Then K is called <u>Λ-stable</u> provided $R_K^H(\Lambda) \leq R_K^G \circ T_H^G(\Lambda)$.

<u>Lemma 2.3</u>. Let $K \leq H \leq G$ and $\Lambda \leq \hat{H}$. Then K is Λ-stable if and only if $K \cap T^G(\Lambda)^{\perp} \leq \Lambda^{\perp}$. In particular, if K is Λ-stable and the exponent of Λ is p^r, then $K \cap G'_r(p) \leq \Lambda^{\perp}$.

Proof. The first statement follows from the duality theorem. The second follows from the fact that T^G is a homomorphism.

Lemma 2.4. Let H be a subgroup of G of index prime to p. Then the following hold :

(1) The composition $T_H^G \circ R_H^G : \hat{G}_p \longrightarrow \hat{H}_p \longrightarrow \hat{G}_p$ is an isomorphism and $\hat{H}_p = (\mathrm{Im}\, R_H) \times (\mathrm{Ker}\, T^G)$. Furthermore, $(H \cap G'(p))/H'(p)$ is a direct factor of $H/H'(p)$.

(2) Let $K \leq H$, $r \geq 1$ and $\Lambda \leq \Omega_r(\hat{H}_p)$. Assume that $\Lambda^\perp \leq H \cap G_r'(p)$ and that whenever $\lambda \in \Lambda$ and $T^G(\lambda) = 1$, then $\lambda_{|K} = 1$. Then $T^G(\lambda)_{|K} = \lambda_{|K}^{|G:H|}$ for each $\lambda \in \Lambda$. In particular, K is Λ-stable and $K \cap G_r'(p) = K \cap \Lambda^\perp$.

Proof. (1) Set $R = R_H^G : \hat{G}_p \longrightarrow \hat{H}_p$ and $T = T_H^G : \hat{H}_p \longrightarrow \hat{G}_p$. By Lemma 2.1(4), $T(\mu_{|H}) = \mu^{|G:H|}$ for $\mu \in \hat{G}_p$. Since $|G : H|$ is prime to p, $T \circ R$ is an isomorphism. Thus R is a monomorphism, T is an epimorphism and $(\mathrm{Im}\, R) \cap (\mathrm{Ker}\, T) = 1$. Since $\mathrm{Im}\, R \simeq \hat{G}_p \simeq \hat{H}_p/\mathrm{Ker}\, T$, we have that $\hat{H}_p = (\mathrm{Im}\, R) \times (\mathrm{Ker}\, T)$. By the duality theorem of abelian groups, $(\mathrm{Im}\, R)^\perp/H'(p) = (H \cap G'(p))/H'(p)$ is a direct factor of $H/H'(p)$, as required.

(2) Let $\lambda \in \Lambda$. By (1), $\lambda = \mu_{|H} \cdot \nu$ for some $\mu \in \hat{G}_p$ and $\nu \in \mathrm{Ker}\, T$. Since $\Lambda \geq R(\Omega_r(\hat{G}_p))$, we have $\mu_{|H} \in \Lambda$ and so $\nu \in \Lambda \cap \mathrm{Ker}\, T$. By the assumption, $\nu_{|K} = 1$. Thus $T(\lambda)_{|K} = \lambda_{|K}^{|G:H|}$ by Lemma 2.1 (4). Since $|G : H|$ is prime to p, we have $R_K \circ T^G(\Lambda) = R_K(\Lambda)$, so K is Λ-stable. By Lemma 2.3, we conclude that $K \cap G_r'(p) = K \cap \Lambda^\perp$, as required.

<u>Proposition 2.5</u> (Tate-Thompson [13]). Let H be a subgroup of G of index prime to p and $K \leq H \cap G'(p)$. Assume that $H \cap G_1'(p) = H_1'(p)K$. Then the following hold :

(1) $H \cap G'(p) = H'(p)K$.

(2) If $K \trianglelefteq H$ and $K \leq O^p(G)$, then $H \cap O^p(G) = O^p(H)K$.

<u>Proof</u>. (1) Set $\Lambda = (H/H'(p)K)^\wedge$. Since $K \leq H \cap G'(p)$, we have that $\Lambda \geq R_H^G(\hat{G}_p)$. By the assumption and Lemma 2.3, H is $\Omega_1(\Lambda)$-stable. Thus $R_H\, T^G(\Omega_1(\Lambda)) = \Omega_1(\Lambda)$. By Lemma 2.4(1), $\Omega_1(\Lambda) \cap \operatorname{Ker} T^G = 1$, so $\Lambda \cap \operatorname{Ker} T^G = 1$. By Lemma 2.4(1), $\Lambda = \operatorname{Im} R_H^G$. Thus (1) follows from the duality theorem.

(2) This is proved by the similar way as in [13]. Use [16, Theorem 5] for [13, Lemma 2].

<u>Corollary 2.5.1</u>. Let $K \leq P \leq H \leq G$ and P be a Sylow p-subgroup of G. Assume that $P \cap G_1'(p) = (P \cap H_1'(p))K$. Then the following hold :

(1) If $K \leq G'$, then $P \cap G' = (P \cap H')K$.

(2) If $K \leq O^p(G)$ and $K \trianglelefteq P$, then $P \cap O^p(G) = (P \cap O^p(H))K$.

<u>Lemma 2.6</u>. Suppose H is a subgroup of G of index prime to p and $R_H^G(\Omega_r(\hat{G}_p)) \leq \Lambda \leq \Omega_r(\hat{H}_p)$. Then H is Λ-stable if and only if for each $g \in G$ and $\lambda \in \Lambda$,

$$T^H(\lambda^{g^{-1}}|_K) = \lambda^{|H:K|}, \text{ where } K = H \cap H^g .$$

<u>Proof</u>. For each $g \in G$, we define the mapping $\phi_g : \hat{H}_p \longrightarrow \hat{H}_p$ by

$$\phi_g : \mu \longmapsto T^H(\mu^{g^{-1}}|_K) \quad \text{for all} \quad \mu \in \hat{H}_p ,$$

where $K = H \cap H^g$. Let $\Phi = R_H^G \circ T_H^G$, then ϕ_g and Φ are endomorphisms of $\hat{H}_p$. It follows from Lemma 2.1 that for all $g \in G$ and $\mu \in \hat{H}_p$,

$$(1) \quad (\Phi \circ \phi_g)(\mu) = (\phi_g \circ \Phi)(\mu) = \Phi(\mu)^{|H:K|},$$

where $K = H \cap H^g$. We shall use additive notation on the abelian group $\hat{H}_p$. Then (1) is rewritten by

$$(2) \quad \Phi\, \phi_g = \phi_g\, \Phi = |H : H \cap H^g|\, \Phi \quad \text{for all} \quad g \in G .$$

Let $G = \sum_i Hg_iH$ be a decomposition of G to (H, H)-double cosets, $K_i = H \cap H^{g_i}$, $r_i = |H : K_i|$ and $n = |G : H|$. Set $\Delta = \Phi - n\, I$ and $\delta_i = \phi_{g_i} - r_i\, I$ for each i. Since $n = \sum_i r_i$, Mackey decomposition(Proposition 2.2) yields that

$$(3) \quad \Delta = \sum_i \delta_i .$$

By (2), we have that

$$(4) \quad \Delta\, \delta_i = \delta_i\, \Delta = -\, n\, \delta_i .$$

(3) yields that $\operatorname{Ker} \Delta \geq \bigcap_i \operatorname{Ker} \delta_i$. Conversely (4) yields that $n\, \delta_i(\operatorname{Ker} \Delta) = 0$ for each i. Since n is prime to p, $\operatorname{Ker} \Delta \leq \operatorname{Ker} \delta_i$ and hence

(5) $\quad \mathrm{Ker}\ \Delta = \bigcap_i \mathrm{Ker}\ \delta_i$.

Since H is Λ-stable if and only if $\Lambda \leq \mathrm{Ker}\ \Delta$ by Lemma 2.4, the lemma follows from (5).

Remark. Our stability is an imitation of the concept of stability in homology theory of groups ([2, XII.9]). In fact, if $H \leq G$ and λ is a stable element of $H^1(H, C^*) = \hat{H}$, then by the definition, $\lambda^{g^{-1}}|_{H \cap H^g} = \lambda|_{H \cap H^g}$ for each $g \in G$, so H is $\langle\lambda\rangle$-stable in our sense.

Lemma 2.7. Let H be a subgroup of G of index prime to p and P a p-subgroup of H . Set $L = \langle x^{-1}x^G \cap H \mid x \in P\rangle$. Then the following hold :

(1) $P \cap G' \leq LH'$.

(2) $P^G \cap H \subseteq L(\mathrm{Ker}\ T_H^G)$, where $P^G = \{x^g \mid x \in P, g \in G\}$.

Proof. Set $\Lambda = (H/LH')^\wedge$. If $\lambda \in \Lambda$, $x \in P$, $g \in G$ and $r = |\langle x\rangle : \langle x\rangle \cap H^g|$, then $gx^rg^{-1} \in x^rL$, and so $\lambda(gx^rg^{-1}) = \lambda(x^r)$. By Proposition 2.2(2), we have that $T^G(\lambda)_{|P} = \lambda_{|P}^{|G:H|}$ for all $\lambda \in \Lambda$. Thus P is Λ- and $(\Lambda \cap \mathrm{Ker}\ T^G)$-stable. The lemma follows from Lemma 2.3.

3. Preparations for Section 4.

In this section, we introduce some terminology and notation, that is, subsets of Sylow type, the functor Φ_r^* on p-groups and p-singularities.

Definition 3.1. Let $H \leq G$ and S a subset of H. Assume that if S^g is contained in H for $g \in G$, then S^g is conjugate to S via an element of H (that is, $g \in N_G(S)H$). Then S is called a subset of Sylow type in H (with respect to G).

For example, if S is weakly closed in a Sylow subgroup of H, then S is of Sylow type in H. The proof of the following lemma is easy.

Lemma 3.1. Let $H \leq G$ and P a p-subgroup of H of Sylow type in H with respect to G. Then the following hold :

(1) If $H \trianglelefteq N \leq G$, then $N = N_N(P)\,H$.

(2) H contains a Sylow p-subgroup of G if and only if H contains a Sylow p-subgroup of $N_G(P)$.

(3) If $N_G(P) \leq H$, then all p-subgroup of G containing P are contained in H.

(4) Let X and Y be P-invariant subsets of H which are conjugate in G. Assume that H contains a Sylow p-subgroup of $N_G(X)$. Then there is $u \in HN_G(P)$ such that $Y = X^u$.

<u>Definition 3.2</u>. For any p-group P and $0 \leq r \leq \infty$, we set

$$\Phi_r^*(P) = \bigcap M_r'(p) \ , \ \Phi_\infty^*(P) = \bigcap M',$$

where M ranges over all subgroup of P of index at most p. Furthermore, $\Phi_1^*(P)$ is simply denoted by $\Phi^*(P)$, that is

$$\Phi^*(P) = \bigcap \Phi(M).$$

<u>Definition 3.3</u>. We say that a p-group P is <u>weakly regular</u> if P has no epimorphism onto $Z_p \wr Z_p$.

Regular p-group are clearly weakly regular. However the weakly regularity is far general than regularity. For example, extra-special 2-groups are weakly regular except for D_8 , but they are not regular. Some other examples shall be found in Section 5. One of the purpose of this paper is to prove that Wielandt theorem ([11. IV.8.1]) holds also for weakly regular Sylow p-subgroups. See Section 4.

<u>Lemma 3.2</u>. Let P be a p-group and $0 \leq r \leq \infty$. Then the following hold :

(1) $\Phi^*(P)$ is the intersection of all subgroups of P of index at most p^2. In particular, $\Phi^*(P) \geq \Phi(\Phi(P))$.

(2) If $Q \leq P$, then $\Phi_r^*(Q) \leq \Phi_r^*(P)$.

(3) If $N \trianglelefteq P$, then $\Phi_r^*(P/N) \leq \Phi_r^*(P)N/N$. If furthermore $N \leq \Phi_r^*(P)$, then $\Phi_r^*(P/N) = \Phi_r^*(P)/N$.

(4) $cl(P/\Phi^*(P)) \leq p$.

(5) $\Phi(P) \leq Z_{p-1}(P \bmod \Phi^*(P))$.

(6) $[\mho^1(P), P] \leq \Phi^*(P)$.

(7) The following conditions are equivalent :

(a) P is weakly regular.

(b) For each maximal subgroup M of P, $cl(P/\Phi(M)) < p$.

(c) $cl(P/\Phi^*(P)) < p$.

(8) When $p = 2$, the following are equivalent :

(a) P is weakly regular.

(b) All subgroups of P of index at most 4 are normal.

<u>Proof</u>. We prove only (7). The remainders can be easily proved. By the definition, (b) and (c) are equivalent. Since $cl(Z_p \wr Z_p) = p$, (b) implies (a). Assume, by the way of contradiction, that P has a maximal subgroup M with $cl(P/\Phi(M)) = p$. Then M has a maximal subgroup L which does not contain $Z_{p-1}(P \bmod \Phi(M))$. Let N be a maximal normal subgroup of P contained in L. Set $\overline{P} = P/N$. Since $N \geq \Phi(M)$, $\overline{M}$ is an elementary abelian maximal subgroup of $\overline{P}$. It follows from the choice of L that $cl(\overline{P}) \geq p$ and $|\overline{P}| \leq p^{p+1}$. Thus P/N is isomorphic to $Z_p \wr Z_p$, that is, P is not weakly regular. Hence (a) implies (b). The proof of (7) is complete.

In the remainder of this paper, we shall use the following notation : For subsets X and Y of a group, we set

$$X^Y = \{x^y \mid x \in X,\ y \in Y\}, \text{ and}$$

$$X_Y = \bigcap_{y \in Y} X^y .$$

Don't mistake the meaning for notation in other papers.

Finally we introduce another concept which shall play an important part in the next section.

<u>Definition 3.4</u>. Let H be a finite group, $S \leq H$, $\lambda \in \hat{S}_p$ and $x \in H$. We say that (S, λ, x) is a <u>weak p-singularity</u> in H provided

(a) x is a p-element,

(b) $T^H(\lambda)(x) \neq 1$, and

(c) $\langle x^p \rangle^H \cap S \subseteq \operatorname{Ker} \lambda$.

When $|\lambda| = p^d$, d is called the <u>depth</u> of the weak singularity. When $|\lambda| = p$, we say that (S, λ, x) is a (p-)<u>singularity</u>. The subgroup S, the character λ and the element x are called a (weak) <u>singular subgroup</u> (or simply a (weak) <u>singularity</u>), a <u>singular character</u> and a <u>singular element</u>, respectively.

For example, it follows from Lemma 2.1(4) that abelian p-groups have no proper weak singular subgroup.

We shall study groups with (weak) singularities in the remainder of this section.

<u>Lemma 3.3.</u> Let H be a finite group and (S, λ, x) a weak p-singularity in H of depth d. Then the following hold :

(1) If $a, b \in H$, then $(S^a, \lambda^{a^{-1}}, x^b)$ is also a weak p-singularity.

(2) $S - \operatorname{Ker} \lambda$ contains a conjugate of x.

(3) If $S \leq R \leq H$, then $(R, T^R(\lambda), x)$ is a weak p-singularity in H of depth at most d.

(4) If $S \leq L \leq H$, then there is a conjugate y of x such that (S, λ, y) is a weak p-singularity in L of depth d.

(5) If $N \trianglelefteq H$ and $N \leq \operatorname{Ker} \lambda$, then $(S/N, \lambda', xN)$ is a weak p-singularity in H of depth d, where λ' is the linear character of S/N induced by λ.

(6) If $x \in L \leq H$, then there is a conjugate R of S such that $L \cap R$ is a weak p-singularity in L of depth at most d with singular element x. Furthermore, if $H = L\,S$, then $(S \cap L, \lambda_{|S \cap L}, x)$ is a weak singularity in L.

(7) If P is a Sylow p-subgroup of S, then $(P, \lambda_{|P}, x)$ is also a weak p-singularity in H of depth d.

<u>Proof.</u> (1) Clear.

(2) This follows from Corollary 2.2.1.

(3) Set $\mu = T^R(\lambda)$. If $y \in \langle x^p \rangle^H \cap R$, then $y^R \cap S \subseteq \langle x^p \rangle^H \cap S \subseteq \operatorname{Ker} \lambda$. Thus Corollary 2.2.1 yields that $\mu(y) = T^R(\lambda)(y) = 1$, and so $\langle x^p \rangle^H \cap R \subseteq \operatorname{Ker} \mu$, whence (R, μ, x) is a weak singularity in H.

(4) Set $\nu = T^L(\lambda)$. Since $T^H(\lambda)(x) = T^H_L(\nu)(x) \neq 1$, Corollary 2.2.1 yields that there is a conjugate y of x with $\nu(y) \neq 1$. Since $\langle x^p \rangle^H \cap S \subseteq \operatorname{Ker} \lambda$, $\langle y^p \rangle^L \cap S \subseteq \operatorname{Ker} \lambda$. Thus (S, λ, y) is a weak singularity in L.

(5) Clear.

(6) Since $(T^H(\lambda)_{|L})(x) \neq 1$, Proposition 2.2(1) yields that there is an element $h \in H$ with $T^L(\mu)(x) \neq 1$, where $\mu = \lambda^{h^{-1}}{}_{|R \cap L}$ and $R = S^h$. Since $\langle x^p \rangle^{Lh^{-1}} \cap S \cap L^{h^{-1}} \subseteq \operatorname{Ker} \lambda$, $\langle x^p \rangle^L \cap R \cap L \subseteq \operatorname{Ker} \mu$. Thus $(R \cap L, \mu, x)$ is a weak singularity in L. If $H = S\,L$, then we may take $h = 1$, and so $R = S$, $\mu = \lambda_{|S \cap L}$. Thus $(S \cap L, \lambda_{|S \cap L}, x)$ is a weak singularity in L.

(7) By Lemma 2.1(3) and (4), $T^H(\lambda_{|P}) = T^H(T^S(\lambda_{|P})) = T^H(\lambda)^{|S:P|}$. Since $|S : P|$ is relatively prime to p, we have that $T^H(\lambda_{|P})(x) \neq 1$ and $\langle x^p \rangle^H \cap P \subseteq \operatorname{Ker} \lambda_{|P}$. Thus $(P, \lambda_{|P}, x)$ is a weak singularity in H.

<u>Lemma 3.4</u>. Let $\overline{H}$ be a transitive permutation group with a stabilizer $\overline{S}$ and let C be a cyclic group of order p^d. Construct the wreath product $H = C \wr \overline{H}$ with base subgroup V ($\cong C^{|\overline{H}:\overline{S}|}$). Then $V\overline{S}$ is a weak p-singularity in H of depth d with singular element contained in V.

<u>Proof</u>. Let v_i, $1 \leq i \leq n$, be the generators of V permuted by $\bar{H}$ and assume that $\bar{S}$ fixes v_n. Set $S = V\bar{S}$ and $K = \langle v_1, \cdots, v_{n-1}\rangle \bar{S}$, so that $S = K \times \langle v_n \rangle$. Let λ be a linear character of S with kernel K. Then $|\lambda| = p^d$ and $\lambda(v_n)$ is a primitive p^d-th root of unity. For any $x \in \langle v_n \rangle$,

$$\lambda^H(x) = n - 1 + \lambda(x), \quad 1_S^H(x) = n.$$

Thus

$$(\lambda^H - 1_S^H)|_{\langle v_n \rangle} = (\lambda - 1_S)|_{\langle v_n \rangle},$$

and so

$$T^H(\lambda)|_{\langle v_n \rangle} = \lambda|_{\langle v_n \rangle}.$$

Let x be an element of $\langle v_n \rangle$ of order p. Then $T^H(\lambda)(x) = \lambda(x) \neq 1$. Thus (S, λ, x) is a weak p-singularity in H.

<u>Lemma 3.5</u>. Let P be a p-group and let (S, λ, x) be a p-singularity in P. Set $V = S_P$ and $K = \mathrm{Ker}\,\lambda$. Assume that $x \in V$. Then $P/K_P \simeq Z_p \wr (P/V)$, where the wreath product is constructed by the faithful permutation representation of P/V on the set P/S and the base subgroup is V/K_P. In particular, $m(V/K_P) = |P : S|$ and $V = \langle x^P \rangle K_P$.

<u>Proof</u>. We may assume that $K_P = 1$. Set $P^* = Z_p \wr (P/V)$ and let V^* be the base subgroup of P^*. Then the induced representation λ^P yields a monomorphism of P to P^*. We can regard $P \leq P^*$. Then $P^* = V^*P$ and $V \leq V^*$. We need to show that $V = V^*$. Since P/V acts transitively on a subset

of V^* generating V^*, $[V^*, P]$ is of index p in V^*. Since $T^P(\lambda)(x) = T^{P^*}(\lambda')(x) \neq 1$, where λ' is a linear character of V^*S which extends λ, we have that $x \notin [V^*, P]$. Since $V \trianglelefteq P^*$, we have that $V^* = \langle x^P \rangle \leq V$, as required. (The necessary facts about wreath products are found in [11, I.15 and V.18).

<u>Lemma 3.6</u>. Let P be a p-group, A a maximal subgroup of P, $\lambda \in \hat{A}$, $K = \operatorname{Ker} \lambda$ and $u \in P - A$. Then the following hold :

(1) $T^P(\lambda)_{|A} = \lambda^{\bar{u}}$, where $\bar{u} = 1 + u + \cdots + u^{p-1}$, and $T^P(\lambda)(u) = \lambda(u^p)$. If $x \in \Omega_1(A \bmod K_P)$, then $T^P(\lambda)(x) = \lambda([x, u; p-1])$.

(2) $(\Omega_1 Z_{p-1})(P \bmod K_P) \leq \operatorname{Ker} T^P(\lambda)$.

(3) (A, λ, x) is a weak singularity in P of depth d for an element x of P if and only if $\Omega_1(A/K_P) \not\leq Z_{p-1}(P/K_P)$.

(4) (A, λ, x) is a singularity in P for an element x of P if and only if $P/K_P \simeq Z_p \wr Z_p$.

<u>Proof</u>. (1) Since $\lambda^P_{|A} = \sum_i \lambda^{u^i}$ and $1^P_{A|A} = p\, 1_A$, $T^P(\lambda)_{|A} = \prod_i \lambda^{u^i} = \lambda^{\bar{u}}$. By Proposition 2.2(2), $T^P(\lambda)(u) = \lambda(u^p)$. If $x \in \Omega_1(A \bmod K_P)$, then $T^P(\lambda)(x) = \lambda(x^{\bar{u}}) = \lambda(x^{(1-u)^{p-1}}) = \lambda([x, u; p-1])$.

(2) This follows from (1).

(3) We may assume that $K_P = 1$. If (A, λ, x) is a weak singularity in P, then $x^p \in Z(P) \cap \operatorname{Ker} \lambda = 1$. By (1),

$[x, u; p-1] \neq 1$, so $x \in \Omega_1(A) - Z_{p-1}(P)$. Conversely, if $x \in \Omega_1(A) - Z_{p-1}(P)$, then $T^P(\lambda)(x) \neq 1$ by (1). Thus (A, λ, x) is a weak singularity.

(4) If $|\lambda| = p$, then A/K_P is elementary. Thus (4) follows from (3), Lemma 3.4 and 3.5.

<u>Proposition 3.7</u>. Let P be a p-group. Then the following hold :

(1) P has no proper singularity if and only if P is weakly regular.

(2) If $cl(P/\Phi_d^*(P)) < p$, then P has no proper weak singularity of depth at most d.

(3) If $Q < P$ and $\lambda \in \Omega_r(\hat{Q})$, then $(\Omega_1 Z_{p-1})(P \bmod \Phi_r^*(P)) \leq \operatorname{Ker} T^P(\lambda)$.

<u>Proof</u>. (1) If P is not weakly regular, then P has a quotient group isomorphic to $Z_p \wr Z_p$, so P has a proper singularity by Lemma 3.4. Conversely if P has a proper singularity, then by Lemma 3.3(3), P has a singularity of index p. Thus P is not weakly regular by Lemma 3.6(4).

(2) This follows from Lemma 3.3(3) and 3.6(3).

(3) Let M be a maximal subgroup of P containing Q and let $\mu = T^M(\lambda)$, then $|\mu| \leq |\lambda| \leq p^r$. Since $\Phi_r^*(P) \leq (\operatorname{Ker} \mu)_P$, it follows from Lemma 3.6(2) that

$$\begin{aligned}(\Omega_1 Z_{p-1})(P \bmod \Phi_r^*(P)) &\leq (\Omega_1 Z_{p-1})(P \bmod (\operatorname{Ker} \mu)_P) \\ &\leq \operatorname{Ker} T^P(\mu) \\ &\leq \operatorname{Ker} T^P(\lambda).\end{aligned}$$

<u>Lemma 3.8</u>. Let P be a p-group and (S, λ, x) a weak singularity in P of depth d. Then the following hold :

(1) $N_P(\mathrm{Ker}\ \lambda) = S$.

(2) Let $Q \leq P$ and assume that $[x, y; p-1] \in \Phi_d^*(Q)$ for all $y \in Q$. Then $\langle Q , x\rangle$ is contained in a conjugate of S .

(3) Let Q be a subgroup of P normalized by x. Assume that $cl(Q/\Phi_d^*(Q)) \leq p-2$. Then $Q\langle x\rangle$ is contained in a conjugate of S .

(4) $N_P(\langle x\rangle)$ is contained in a conjugate of S .

(5) If p is odd, then $|P| < |S|^2$.

(6) If $N \trianglelefteq P$ and $N \cap S \leq \mathrm{Ker}\ \lambda$, then N is contained in $(\mathrm{Ker}\ \lambda)_P$.

<u>Proof</u>. (1) Suppose false. By Lemma 3.3(4), we may assume that $\mathrm{Ker}\ \lambda \trianglelefteq P$ and $|P : S| = p$. But then Lemma 3.6(3) implies a contradiction.

(2) Suppose false. By Lemma 3.3(6), we may assume that $P = \langle Q , x\rangle$. Let M be a maximal subgroup of P containing S and let $\mu = T^M(\lambda)$. By Lemma 3.6(1), $T^P(\lambda)(x) = T^P(\mu)(x) = \mu([x, y; p-1]) \neq 1$, where $y \in Q - M$. But $[x, y; p-1]$ $\Phi_d^*(Q) \leq (M \cap Q)_d'(p) \leq \mathrm{Ker}\ \mu$, a contradiction.

(3) Since $[x, y; p-1] \in [Q , y; p-1] \leq \Phi_d^*(Q)$ for every $y \in Q$, (3) follows from (2).

(4) By Lemma 3.3(6), we may assume that $P = N_P(\langle x\rangle)$ and $(\mathrm{Ker}\ \lambda)_P = 1$. Then x is of order p, so $x \in Z(P)$.

By (2), $S = P$.

(5) We shall first show that λ^P is irreducible. We argue by induction on $|P : S|$. Let M be a maximal subgroup of P containing S. Since S is a weak singularity in M with singular character λ, λ^M is irreducible by induction. If λ^P is not irreducible, then $\lambda^P{}_{|M} = p\,\lambda^M$. Since $p \neq 2$, $T^P(\lambda)_{|M} = \det(\lambda^P{}_{|M}) = \det(p\,\lambda^M) = (\det \lambda^M)^p = T^M(\lambda)^p$. Thus $T^P(\lambda)(x) = T^M(\lambda)(x^p)$. But since $\langle x^p\rangle^M \cap S \subseteq \operatorname{Ker} \lambda$, we have that $T^M(\lambda)(x^p) = 1$ by Corollary 2.2.1. This is a contradiction. Hence λ^P is irreducible. By [9, Lemma 4.6.3], $\lambda^P(1)^2 = |P : S|^2$ divides $|P : Z(P)|$, proving (5).

(6) We may assume that $(\operatorname{Ker} \lambda)_P = 1$. If $N \neq 1$, then $1 \neq Z(P) \cap N \leq S \cap N \leq \operatorname{Ker} \lambda$, a contradiction.

We shall add some results about singularities.

<u>Lemma 3.9</u>. Let (S, λ, x) be a singularity in a p-group P with $K = \operatorname{Ker} \lambda$ and $|P : S| = p^n$. Set $\overline{P} = P/K_P$. Then the following hold :

(1) $\overline{S} = \overline{K} \times Z(\overline{P})$ and $|Z(\overline{P})| = p$.

(2) If $S < Q \leq P$, then Q is not weakly regular.

(3) If $S \leq R \trianglelefteq Q \leq P$, then $m(R/\Phi(R)) \geq |Q : R|$.

(4) Let $P = P_0 > P_1 > \cdots > P_n = S$ be a series of subgroups such that $|P_i : P_{i+1}| = p$ for $0 \leq i < n$. Then $\operatorname{Ker} T^{P_i}(\lambda) \not\leq P_{i+1}$ for $0 \leq i < n$. Let $a_0 \in P_0 - P_1$ and

$a_i \in \operatorname{Ker} T^{P_i}(\lambda) - P_{i+1}$ for $0 < i < n$. Define inductively elements x_i, $0 \leq i \leq n$, by the rule $x_0 = x$ and $x_{i+1} = [x_i, a_i; p-1]$. Then $T^P(\lambda)(x) = \lambda(x_n) \neq 1$.

(5) $\bar{x} \notin Z_{n(p-1)}(\bar{P})$, $cl(\bar{P}) > n(p-1)$, $|\bar{x}| \leq p^n$, $\exp(\bar{P}) \leq p^{n+1}$, $\mho^n(\bar{P}) \leq Z(\bar{P})$, $\mho^n(\Phi(\bar{P})) = \mho^{n+1}(\bar{P}) = 1$.

(6) If $K_P = 1$ and N is a cyclic normal subgroup of P, then $x^2 \in C_P(N)$ and $\langle x\rangle N/\langle x^2\rangle$ is dihadral or semidihedral.

<u>Proof</u>. (1) follows from Lemma 3.8(4). (2) follows from Lemma 3.3(4) and Proposition 3.7(1). (3) follows from Lemma 3.3(3), 3.3(4) and 3.5.

We shall prove (4). Set $\lambda_i = T^{P_i}(\lambda)$. Then P_{i+1} is a singularity in P_i with singular character λ_{i+1} by Lemma 3.3. Since $T^{P_i}(\lambda_{i+1}) = \lambda_i$, Lemma 3.6 yields that $\operatorname{Ker} \lambda_i \not\leq P_{i+1}$. We shall argue the second statement by induction on n. We may assume that $n \geq 1$. Set $Q = P_{n-1}$ and $\mu = T^Q(\lambda)$. By Lemma 3.3(3), (Q, μ, x) is also a singularity in P. Thus by induction, we have that

$$T^P(\lambda)(x) = T^P(\mu)(x) = \mu(x_{n-1}).$$

Lemma 3.6(1) implies that

$$\mu(x_{n-1}) = \begin{cases} \lambda(x_{n-1}{}^p) & \text{if } x_{n-1} \notin S, \\ \lambda([x_{n-1}, a_{n-1}; p-1]) & \text{if } x_{n-1} \in S. \end{cases}$$

Thus we may assume that $x_{n-1} \notin S$. By Lemma 3.6, $Q/K_Q \simeq Z_p \wr Z_p$, $x_{n-1}{}^p \notin K_Q$ and $a_{n-1}{}^p \in K_Q$. It follows easily that

$$x_{n-1}{}^{p} \equiv [x_{n-1}, a_{n-1}; p-1] \mod K_Q .$$

Hence $T^P(\lambda)(x) = \mu(x_{n-1}) = \lambda(x_n)$, as required.

$\overline{P}$ can be regarded as a subgroup of a Sylow p-subgroup of the symmetric group of degree p^{n+1}. So (5) follows from (4) and some properties of Sylow p-subgroups of symmetric groups. See [11. Section III.15].

We shall finally prove (6). We may assume that $P = \langle x \rangle N$. Let y be a generator of N and $z = y^{|y|/2}$. Then $Z(P) = \langle z \rangle$. By the structure of the automorphism groups of cyclic groups, we have that if x^2 does not centralize N, then x^2 is conjugate to $x^2 z$. Since $z \notin \operatorname{Ker} \lambda$, the definition of the singularities derives a contradiction. Hence $x^2 = 1$ and P is dihedral or semidihedral.

<u>Lemma 3.10</u>. Let P be a 2-group and (S, λ, x) a singularity in P. Let N be a subgroup of P normalized by x. Assume that $\exp(N/N') \leq 2^n$, $n \geq 2$, and that all subgroups of P of index at most 2^{n+1} contain N'. Then there is a conjugate T of S such that $|N : N \cap T| < 2^n$.

<u>Proof</u>. Let $K = \operatorname{Ker} \lambda$. By Lemma 3.3, we may assume that $K_P = 1$, $P = N\langle x \rangle$ and $|P : S| = 2^n$. Since $|N : N \cap K| \leq 2^{n+1}$, the hypothesis of the lemma implies that $N' \leq N \cap K \trianglelefteq P = NS$, and so we have that $N' = N \cap K = 1$. By Lemma 3.9(1), $N \cap S = Z(P)$ and $|N \cap S| = 2$. Thus $|N| = 2^{n+1}$. Let $P = P_0 > P_1 > \cdots > P_n = S$ be a series of subgroups with $|P_i : P_{i+1}| = 2$.

Since N is abelian, we have that $\mathrm{Ker}\, T^{P_i}(\lambda) \cap N \leq P_{i+1}$ for $i \geq 1$. Let $N_i = N \cap P_i$ and let $\nu \in \hat{N}$ be an extension of $\lambda|_{N \cap S}$ ($\neq 1$). Then $|N_i : N_{i+1}| = 2$. By Lemma 2.1(4) and Proposition 2.2,

$$T^{P_i}(\lambda)|_{N_i} = T^{N_i}(\lambda|_{N \cap S}) = (\nu|_{N_i})^{2^{n-i}}.$$

Since $|N_i : N \cap S| = 2^{n-i}$ and $K \cap N = 1$, this implies that N_1 is cyclic of order 2^n and $\mathrm{Ker}\, \nu \cap N_1 = 1$. Let $M = P_1$. Since $\exp(N) \leq 2^n$, $\mathrm{Ker}\, \nu \neq 1$. Let $1 \neq a \in \mathrm{Ker}\, \nu$, then $a \in N - M$. By Lemma 2.1(3) and 3.6(1), we have

$$T^P(\lambda)(x) = T^M(\lambda)([x, a]) = \nu([x, a])^{2^{n-1}}.$$

Since $a \in \Omega_1(N) \trianglelefteq P$ and $n \geq 2$, $[x, a]^{2^{n-1}} = 1$. This is a contradiction. The lemma is proved.

<u>Lemma 3.11</u>. Let P be a p-group and (S, λ, x) a singularity in P. Assume that P is the central product of two subgroups P_1 and P_2. Assume $x = x_1x_2$, where $x_i \in P_i$ for $i = 1, 2$. Then for some i, (S, λ, x_i) is a singularity in P. In particular, for such i, $(S \cap P_i, \lambda|_{S \cap P_i}, x_i)$ is a singularity in P_i and $S \geq P_j$, where $j \neq i$.

<u>Proof</u>. Since $T^P(\lambda)(x) = T^P(\lambda)(x_1) \cdot T^P(\lambda)(x) \neq 1$, we have that $T^P(\lambda)(x_i) \neq 1$ for $i = 0$ or 1. We need to show that $\langle x_i^p \rangle^P \cap S \subseteq \mathrm{Ker}\, \lambda$. By Lemma 3.8(4), we may assume that $C(x) \leq S$. Then $x_1, x_2 \in S$. Take $k \geq 1$ and $u \in P_i$ such that $x_i^{kpu} \in S$. Since $[P_1, P_2] = 1$, $x_j^{kp} = x_j^{kpu} \in \mathrm{Ker}\, \lambda$ for $j \neq i$.

Since $x = x_i x_j$ and $x^{kpu} \in \operatorname{Ker} \lambda$, we have that $x_i^{kpu} \in \operatorname{Ker} \lambda$. Hence $\langle x_i^p \rangle^P \cap S \subseteq \operatorname{Ker} \lambda$, and so (S, λ, x_i) is a singularity in P. Since $P_j \trianglelefteq P$ and $P_j \leq C(x_i)$, Lemma 3.8(4) yields that $P_j \leq S$. Thus $S = (S \cap P_i)P_j$ and $P = P_i S$. By Lemma 3.3(6), we have that $(S \cap P_i, \lambda|_{S \cap P_i}, x_i)$ is a singularity in P_i.

4. Applications to transfer theorems

In this section, we prove some transfer theorems by the applications of character-theoretic transfers. In Section 2, we saw that a character-theoretic transfer is the dual of an ordinary transfer. Thus using character-theoretic transfers instead of ordinary transfers, we can rewrite the proofs of some transfer theorems ([9, 9.4], etc.) and Wielandt method suggests generalizations of these transfer theorems. We list some well-known theorems which shall be generalized in the later.

Let P be a Sylow p-subgroup of G.

(1) If P is $Z_p \wr Z_p$ free, then $P \cap G' = P \cap N(P)'$. (Wielandt, see [11, IV.8]). A generalization shall be found in Theorem 4.2 in this paper.

(2) $P \cap G' = \langle P \cap N(P)', P \cap Q' \mid Q \in \mathrm{Syl}_p(G) \rangle$. (Grün's first theorem, see [9, Theorem 7.4.2). See Theorem 4.3.

(3) If Q is a weakly closed subgroup of P and $[x, y; p-1] = 1$ for all $x \in P$ and $y \in Q$, then $P \cap G' = P \cap N(Q)'$.(Hall-Wielandt, see [10, Theorem 14.4.2]). See Theorem 4.4.

(4) If A is a strongly closed abelian subgroup of P, then $P \cap G' = P \cap N(A)'$. (Glauberman [7, Theorem 6.1]). See Theorem 4.5.

(5) If W is a weakly closed subgroup of P, then $\Omega_1(C_P(W) \cap G') \leq N(W)'$. (Zappa ?). See Theorem 4.6.

(6) $P \cap G' = \langle x^{-1}y \mid x, y \in P, x \sim y \text{ in } G \rangle$. (Focal subgroup theorem, see [9, Theorem 7.3.4]). See Theorem 4.7 and Lemma 2.7.

(7) If $\mathfrak{F}$ is a weak conjugation family for P (see [1], [8]), then $P \cap G' = \langle P', [H, T] \mid (H, T) \in \mathfrak{F} \rangle$. (Alperin [1]). See Theorem 4.8 and 4.9.

The following proposition shall be use in the proofs of all theorems in this section except Theorem 4.6.

<u>Proposition 4.1</u>. Let $R \leq H \leq G$, $\lambda \in \hat{H}_p$ and x be a p-element of R. Assume the following :

(a) $\langle x^p \rangle^G \cap H \subseteq \operatorname{Ker} \lambda$, and

(b) For each $g \in G$, (R_g, λ'_g, x) is not a weak p-singularity in R, where

$$R_g = R \cap H^g \quad \text{and} \quad \lambda'_g = \lambda^{-1}_{|R_g} \cdot \lambda^{g^{-1}}_{|R_g} .$$

Then

$$T^G(\lambda)(x) = \lambda(x)^{|G:H|}.$$

<u>Proof</u>. Suppose false. Let $G = \sum_i Hg_iR$ be a decomposition of G to (H, R)-double cosets. Set $R_i = R_{g_i}$ and $\lambda'_i = \lambda'_{g_i}$. Since $T^R(\lambda_{|R_i}) = \lambda^{|R:R_i|}_{|R}$ and $\sum_i |R:R_i| = |G:H|$, Mackey

decomposition yields that

$$1 \neq \lambda(x)^{-|G:H|}T^G(\lambda)(x) = \prod_i T^R(\lambda_i')(x).$$

Thus there is $g \in G$ such that $T^R(\lambda_g')(x) \neq 1$. Let $y \in \langle x^p\rangle^R \cap R_g$. By (a), we have $\lambda(y) = \lambda(gyg^{-1}) = 1$, so $y \in \operatorname{Ker} \lambda_g'$. Hence $\langle x^p\rangle^R \cap R_g \subseteq \operatorname{Ker} \lambda_g'$. This means that (R_g, λ_g', x) is a weak p-singularity in R, contrary to (b).

<u>Corollary 4.1.1</u>. Let $P, R \leq H \leq G$, $(|G:H|, p) = 1$, P be a p-group and $\Lambda \leq \hat{H}_p$. Let x be an element of $P \cap T^G(\Lambda)^\perp - \Lambda^\perp$ of minimal order. Assume that $x \in R$ and that

$$\langle x^p\rangle^G \cap H \subseteq \Lambda^\perp \cup P^H .$$

Then there is $g \in G$ and $\lambda \in \Lambda$ such that (R_g, λ_g', x) is a weak p-singularity in R, where

$$R_g = R \cap H^g \quad \text{and} \quad \lambda_g' = \lambda^{-1}_{|R_g} \cdot \lambda^{g^{-1}}_{|R_g} .$$

<u>Proof</u>. There is $\lambda \in \Lambda$ such that $\lambda(x) \neq 1$. By Proposition 4.1, it will suffice to show that

$$\langle x^p\rangle^G \cap H \subseteq \operatorname{Ker} \lambda.$$

Let $y \in \langle x^p\rangle^G \cap P^H$, then $|y| < |x|$ and $y \in P^H \cap T^G(\Lambda)^\perp$. Thus the minimality of the order of x implies that $y \in \Lambda^\perp \leq \operatorname{Ker} \lambda$, as required.

The following theorem is another refinement of Thompson's fusion lemma than [15]. This is an easy corollary of Proposition 4.1, but we shall prove at the begining.

<u>Theorem 4.1</u>. Let P be a Sylow 2-subgroup of G and M a maximal subgroup of P. Let x be an element of $P \cap G' - M$ of minimal order. Then there is an element g of G such that $x \in M^g$ and $P \cap P^g$ is a singular subgroup in P with singular element x.

<u>Proof</u>. Let λ be a linear character of P with kernel M. For each $g \in G$, we set

$$\lambda_g = \lambda^{g^{-1}}_{|P \cap P^g} .$$

Assume that for some $g \in G - N(P)$, $T^P(\lambda_g)(x) \neq 1$. By the choise of x, we have that $\langle x^2 \rangle^P \cap P^g \subseteq P \cap M^g = \mathrm{Ker}\, \lambda_g$. Thus $(P \cap P^g, \lambda_g, x)$ is a singularity in P. Set $S = P \cap P^g$ and $K = P \cap M^g = \mathrm{Ker}\, \lambda_g$. Let R be a subgroup of P such that $|R : S| = 2$. By Lemma 3.3(4), we may assume that (S, λ_g, x) is a singularity in R. Then $R/K_R \cong D_8$, so we may assume that $x \in K$. In this case the theorem holds. Since $T^G(\lambda)(x) = 1$, by Mackey decomposition we may assume that $1 = T_P^{N(P)}(\lambda)(x)$. As $|N(P):P|$ is prime to p, there is $g \in N(P)$ such that $\lambda(gxg^{-1}) = 1$, and so $x \in M^g$. Thus the theorem holds again.

We shall first generalize Wielandt theorem.

<u>Theorem 4.2.</u> If G has a weakly regular Sylow p-subgroup P (that is, P has no epimorphism onto $Z_p \wr Z_p$), then $P \cap G' = P \cap N_G(P)'$.

<u>Proof</u>. Suppose false. Set $H = N(P)$ and $\Lambda = (H/H^pH')^\wedge$. By Proposition 2.5, P is not Λ-stable. Let x be an element of $P \cap T^G(\Lambda)^\perp - \Lambda^\perp$ of minimal order. Applying Corollary 4.1.1 to $R = P$, we have that there is g in G such that (P_g, λ'_g, x) is a singularity in P. By Proposition 3.7, P has no proper singularity, and so $g \in H$. Since $H' \leq \operatorname{Ker} \lambda$, we have $\lambda'_g = 1$. This is a contradiction.

The following result is a generalization of Grun's first theorem. To apply to Theorem 4.5, the expression is somewhat unsystematic.

<u>Theorem 4.3</u>. Let H be a subgroup of G of index prime to p and P a p-subgroup of Sylow type in H with respect to G. Let P_p be the p-th term of the lower central series of P and set

$$K = \langle \langle P_p - \Phi_r^*(P) \rangle^G \cap H \rangle [P, N(P)] H'_r(p),$$

$$L = \langle \langle x^p \rangle^G \cap H - P^H \mid x \in P \cap G'_r(p) - K \rangle .$$

Then $P \cap G'_r(p) \leq KL \leq \langle \Phi(P)^G \cap H \rangle [P, N(P)] H'_r(p)$.

<u>Proof</u>. Set $\Lambda = (H/KL)^{\wedge}$. It will suffice to show that P is Λ-stable. Suppose false and let x be an element of $P \cap T^G(\Lambda)^{\perp} - \Lambda^{\perp}$ of minimal order. Then

$$\langle x^p \rangle^G \cap H \subseteq P^H \cap L \subseteq P^H \quad .$$

Applying Corollary 4.1.1 to $R = P$, we have that (P_g, λ'_g, x) is a weak singularity in P for some $g \in G$ and $\lambda \in \Lambda$. Since

$$P_g \cap \langle P_p - \Phi^*_r(P) \rangle \leq P_g \cap \operatorname{Ker} \lambda \cap (\operatorname{Ker} \lambda)^g$$
$$\leq \operatorname{Ker} \lambda'_g ,$$

we have that $P_g = P$ by Lemma 3.8(6) and Proposition 3.7(2), and so $P \leq H^g$. Since P is of Sylow type in H, $g \in H\,N(P)$. But then $\lambda'_g = 1$, a contradiction.

<u>Corollary 4.3.1</u>. Let H, P, K and L be subgroups as in the theorem. Then the following hold :

(1) If $cl(P/\Phi^*_r(P)) < p$, then $P \cap G'_r(p) \leq [P, N(P)]LH'_r(p)$.

(2) If $P^G \cap H = P^H$, then $P \cap G'_r(p) \leq K$.

(3) If $P^G \cap H = P^H$ and $cl(P/\Phi^*_r(P)) < p$, then $P \cap G'_r(p) = [P, N(P)](P \cap H'_r(p))$.

(4) If $y \in P$, then $y^G \cap PKL \subseteq yKL$.

(5) If $P^G \cap H = P^H$, then $y^G \cap H \subseteq yK$ for each $y \in P$.

<u>Corollary 4.3.2</u>. If P is a Sylow p-subgroup of G, then

$$P \cap G' = [P, N(P)]\, \langle \langle P_p - \Phi^*(P) \rangle^G \cap P \rangle .$$

These corollaries follow directly from Theorem 4.3 and Proposition 2.5.

The following is a generalization of Hall-Wielandt theorem.

<u>Theorem 4.4</u>. Let $P \in Syl_p(G)$ and $Q \trianglelefteq P \leq H \leq G$. Assume that Q is of Sylow type in H and that $[x, y; p-1] \in \Phi^*(Q)$ for all $x \in P$, $y \in Q$. Then $P \cap G' = (P \cap H')(P \cap N(Q)')$.

<u>Proof</u>. Let $N = N(Q)$ and $\Lambda = (H/H^pH'(P \cap N'))^\wedge$. Let x be an element of $P \cap T^G(\Lambda)^\perp - \Lambda^\perp$ of minimal order. Applying Corollary 4.1.1 to $R = Q, x$, we have that there is $g \in G$ and $\lambda \in \Lambda$ such that (R_g, λ'_g, x) is a singularity in R. By Lemma 3.8(2), we have $R_g = R$. Since Q is of Sylow type in H, the element g is in HN. We may assume that $g \in N$. Take an element h of $H \cap N$ such that $hgxg^{-1}h^{-1} \in P$, then $\lambda'_g(x) = \lambda(x^{-1}hgxg^{-1}h^{-1})$. But $x^{-1}hgxg^{-1}h^{-1} \in P \cap N'$, and so $\lambda'_g(x) = 1$, a contradiction. Hence $P \cap G^pG' \leq \Lambda^\perp = H^pH'(P \cap N')$. The theorem follows from Proposition 2.5.

The following theorem is a generalization of Glauberman theorem.

<u>Theorem 4.5</u>. Let $P \in Syl_p(G)$ and $Q \trianglelefteq P \leq H \leq G$. Assume that Q is of Sylow type in H with respect to G. Set $K = \langle y^{-1}y^G \cap P \mid y \in Q \rangle$. Then $P \cap G' = K(P \cap H')(P \cap N(Q)')$.

<u>Proof</u>. Set $N = N(Q)$ and $\Lambda = (H/H^pH'K(P \cap N'))^\wedge$. Suppose $P \cap G^pG' \not\leq \Lambda^\perp$. By Lemma 2.4, there is $\lambda \in \Lambda$ such that $T^G(\lambda) = 1$ and $\lambda_{|P} \neq 1$. By Lemma 2.7, we have $Q^G \cap H \subseteq QKH' \leq (\mathrm{Ker}\ T_H^G)K \leq \mathrm{Ker}\ \lambda$. Set $R = Q\langle x\rangle$. By Corollary 4.1.1, there is $g \in G$ such that (R_g, λ_g', x) is a singularity in R. Since $R_g \cap Q = Q \cap H^g \subseteq R_g \cap \mathrm{Ker}\ \lambda \cap (\mathrm{Ker}\ \lambda)^g \leq \mathrm{Ker}\ \lambda_g'$, it follows from Lemma 3.3(2) and Lemma 3.8(6) that $R_g = R$, and so $g \in HN$. Since $P \cap N' \leq \mathrm{Ker}\ \lambda$, we have $\lambda_g'(x) = 1$. This is a contradiction. Hence $P \cap G^pG' \leq \Lambda^\perp$, so the theorem follows from Proposition 2.5.

<u>Corollary 4.5.1</u>. Let P, Q and H be the same subgroups as in the theorem. Then the following hold :

(1) $P \cap G' = (P \cap H')(P \cap N(Q)')(\langle Q^G \cap P\rangle \cap G')$.

(2) If $Q^G \cap H = Q^H$, then

$$P \cap G' = (P \cap H')(P \cap N(Q)')\ \langle\langle Q_p - \Phi^*(Q)\rangle^G \cap P\rangle .$$

(3) If $Q^G \cap H = Q^H$ and Q is weakly regular, then

$$P \cap G' = (P \cap H')(P \cap N(Q)').$$

<u>Proof</u>. These follow from Theorem 4.3 and 4.5.

The following theorem is a generalization of Zappa theorem. The proof does not need Proposition 4.1.

<u>Theorem 4.6</u>. Let H be a subgroup of G of index prime to p, Q be a p-subgroup of Sylow type in H with respect to G, and P be a p-subgroup of $N_H(Q)$. Assume that P is generated by $P \cap H'_r(p)[P, N(Q)]$ and elements x's such that $\langle x^p \rangle [x, y; p-1] \in \Phi^*_r(Q)$ for all $y \in Q$. Then $P \cap G'_r(p) \leq H'_r(p)[P, N(Q)]$.

<u>Proof</u>. Set $N = N(Q)$ and $\Lambda = (H/H'_r(p)(H \cap [P, N]))^\wedge$. Suppose P is not Λ-stable. Since $\Lambda^\perp \leq G'_r(p)$, there is $\lambda \in \Lambda$ such that $T^G(\lambda) = 1$ and $\lambda_{|P} \neq 1$ by Lemma 2.4. Thus there is $x \in P$ such that $\lambda(x) \neq 1$ and $\langle x^p \rangle [x, y; p-1] \in \Phi^*_r(Q)$ for all $y \in Q$. Set $R = Q\langle x \rangle$. For $g \in G$, we define R_g and λ'_g as in Proposition 4.1. By Mackey decomposition and Lemma 2.1(4), there is $g \in G$ such that $T^R(\lambda'_g)(x) \neq 1$. By Proposition 3.7(3), we have that $R_g = R$, so we may assume that $g \in N$. Since $[P, N] \cap H \leq \operatorname{Ker} \lambda$, $T^R(\lambda'_g)(x) = \lambda'_g(x) = \lambda(x^{-1}gxg^{-1}) = 1$. This is a contradiction. Hence the theorem is proved.

<u>Corollary 4.6.1</u>. Let H and Q be the same subgroups as in the theorem. Let P be a Sylow p-subgroup of $C_H(Q)$. Then $\Omega_1(P) \cap G' \leq H'[\Omega_1(P), N(Q)]$.

The following theorem is a further generalization of the focal subgroup theorem. (Compare with Lemma 2.7). This theorem shall suggests Theorem 4.8 and 4.9.

<u>Theorem 4.7</u>. Let $P \in Syl_p(G)$ and $P \leq H \leq G$. Take elements $x_1, \cdots, x_m$ of P such that

(a) $H = H^pH'\langle x_1, \cdots, x_m\rangle$, and

(b) x_k is an element of $P - H^pH'\langle x_1, \cdots, x_{k-1}\rangle$ of minimal order for each k.

Let G_k, $1 \leq k \leq m$, be the subsets of G which consist of $g \in G - H$ such that

(c) $H \cap H^g$ is a p-singularity in H with singular element x_k,

(d) $P \neq P \cap P^g \in Syl_p(H \cap H^g)$, and

(e) $O^{p'}(N(Q)) \not\leq C(Q/\Phi(Q))H$, where $Q = P \cap P^g$.

Finally, set $P_k = \langle x_k^{-1}x_k^{Hg^{-1}} \cap P \mid g \in G_k\rangle$, $1 \leq k \leq m$. Then

$$P \cap G' = P_1 \cdots P_m(P \cap H')[P, N(P)].$$

<u>Proof</u>. Set $\Lambda = (H/P_1 \cdots P_m[P, N(P)]H^pH')\hat{}$. It will suffice to show that P is Λ-stable. Suppose false. Then by Lemma 2.4, there is $\lambda \in \Lambda$ such that $T^G(\lambda) = 1$ and $\lambda_{|P} \neq 1$. Take $x = x_k$ such that $x_1, \cdots, x_{k-1} \in \operatorname{Ker} \lambda$ and $\lambda(x) \neq 1$. By (b), $\langle x^p\rangle^G \cap H \subseteq \operatorname{Ker} \lambda$. By Proposition 4.1, there is $g \in G$ such that $(H \cap H^g, \lambda'_g, x)$ is a p-singularity in H. Choose g such that $|P \cap P^g|$ is maximal. Set $Q = P \cap P^g$. If $a, b \in H$, then $(H \cap H^{agb}, \lambda'_{agb}, x)$ is also a singularity in H by Lemma 3.3(1), and so $Q \in Syl_p(H \cap H^g)$. Set $C = C(Q/\Phi(Q))$ and suppose $P^g \cap N(Q) \leq HC$. If $Q = P$, then $g \in N(P)$. But

then $T^P(\lambda'_g)(x) = \lambda'_g(x) = \lambda(x^{-1}gxg^{-1}) = 1$, a contradiction. Thus $Q < P$, so $Q < P^g \cap N(Q) \leq HC$. Thus there is $c \in C$ such that $Q < P^{gc} \cap N(Q) \leq H$, and so $|H \cap H^{gc}|_p > |H \cap H^g|_p$. Since P is a Sylow p-subgroup of H, it follows from the maximality of Q that $(H \cap H^{gc}, \lambda'_{gc}, x)$ is not singularity in H. But since $Q < H \cap H^{gc}$ and $\lambda'_{gc|Q} = \lambda'_{g|Q}$, we have a contradiction by Corollary 2.2.2 and Lemma 3.3(3), (7). Hence we have $P^g \cap N(Q) \not\leq HC$, so $g \in G_k$. By Lemma 3.3(2), $x^H \cap Q \not\leq \mathrm{Ker}\, \lambda'_g$. On the other hand, if $x^h \in Q$ for $h \in H$, then $\lambda'_g(x^h) = \lambda(x^{-1}gx^hg^{-1})$, since $g \in G_k$. This is a contradiction. The theorem is proved.

The following theorem is a generalization of Alperin's theorem [1].

<u>Theorem 4.8</u>. Let $P \in \mathrm{Syl}_p(G)$, $P \leq H \leq G$ and $\mathfrak{F}$ be a weak conjugation family for P. Take elements $x_1, \cdots, x_m$ of P such that

(a) $H = H^pH'\langle x_1, \cdots, x_m\rangle$, and

(b) x_k is an element of $P - H^pH'\langle x_1, \cdots, x_{k-1}\rangle$ of minimal order for every k.

Let $\mathfrak{F}'$ be the set of all $(F, N) \in \mathfrak{F}$ such that F contains a G-conjugate of a singular subgroup Q in H such that

(c) For some $g \in G$ and k, $Q = P \cap P^g \in \mathrm{Syl}_p(H \cap H^g)$ and $O^{p'}(N(Q)) \not\leq C(Q/\Phi(Q))H$. Furthermore, the singular element of Q is x_k.

Then $P \cap G' = (P \cap H')[P, N(P)] \langle [F, N] \mid (F, N) \in \mathcal{F}' \rangle$.

<u>Proof</u>. Let P^* be the right side of the above statement. Set $\Lambda = (H/H^pH'P^*)^\wedge$. Suppose the theorem is false. By Proposition 2.5, there is $\lambda \in \Lambda$ such that $T^G(\lambda) = 1$ and $\lambda_{|P} \neq 1$. Take $x = x_k$ such that $x_1, \cdots, x_{k-1} \in \operatorname{Ker} \lambda$ and $\lambda(x) \neq 1$. Then $x^{p^G} \cap H \subseteq \operatorname{Ker} \lambda$. By Proposition 4.1, there is $g \in G$ such that $(H \cap H^g, \lambda'_g, x)$ is a singularity in H. We choose g such that $|P \cap P^g|$ is maximal. Then $P \cap P^g \in \operatorname{Syl}_p(H \cap H^g)$. Set $Q = P \cap P^g$ and $R = Q^{g^{-1}}$. By Lemma 3.3(8), $(Q, \lambda'_{g|Q}, x)$ is also a singularity in H. Since $[P, N(P)] \leq \operatorname{Ker} \lambda$, we have $Q \neq P$. Suppose $N(Q) \cap P^g \leq C(Q/\Phi(Q))H$. Then there are $c \in C(Q/\Phi(Q))$ and $h \in H$ such that $Q^h < P^{gch} \cap N(Q)^{ch} \leq P \cap P^{gch}$. Since the element c centralizes $Q/\operatorname{Ker} \lambda'_{g|Q}$, we have that $\lambda'_{g|Q} = \lambda'_{gc|Q}$, and so $\lambda'_{g|Q}$ can be extended to $P^{h^{-1}} \cap P^{gc}$. By Corollary 2.2.2 and Lemma 3.3(3), we have a contradiction. Hence we have that $N(Q) \cap P^g \not\leq C(Q/\Phi(Q))H$, and so $O^{p'}(N(Q)) \not\leq C(Q/\Phi(Q))H$. Now, we take $(F_i, N_i) \in \mathcal{F}$ and $g_i \in N_i$, $1 \leq i \leq n$, such that

$$R^{g_1 \cdots g_i} \leq F_i, \quad i = 1, \cdots, n,$$

$$R^{g'} = Q, \quad \lambda^{g^{-1}}{}_{|Q} = \lambda^{g'^{-1}}{}_{|Q}, \text{ where } g' = g_1 \cdots g_n.$$

Then $\lambda'_{g|Q} = \lambda'_{g'|Q}$ and $Q \leq P \cap P^{g'}$. By Lemma 2.1(4), we have $Q \in \operatorname{Syl}_p(H \cap H^{g'})$, so $Q = P \cap P^{g'}$. Since $(H \cap H^{g'}, \lambda'_{g'}, x)$ is also a singularity in H, we may assume that

$g = g'$. Since Q satisfies the condition (c), $(F_i, N_i) \in \mathfrak{F}'$ for every i. We set, for $0 \leq i \leq n$,

$$g_i' = g_{i+1} \cdots g_n,\ g_n' = 1,\ \lambda_i = \lambda^{g_i'^{-1}}|_Q .$$

Since $Q \leq F_i^{g_i'} \leq P^{g_i'}$, λ_i are well-defined. We have

$$\lambda_{g|Q}' = \prod_{i=1}^{n} (\lambda_i^{-1} \lambda_{i-1}).$$

Thus $T^H(\lambda_i^{-1} \lambda_{i-1})(x) \neq 1$ for some i. Since $\langle x^p \rangle^G \cap H \subseteq \mathrm{Ker}\ \lambda$, we have that $(Q, \lambda_i^{-1} \lambda_{i-1}, x)$ is a singularity in H. By Lemma 3.3(2), $x^H \cap Q \not\subseteq \mathrm{Ker}(\lambda_i^{-1} \lambda_{i-1})$. Let $y \in x^H \cap Q$ and $z = g_i' y g_i'^{-1}$. Then $(\lambda_i^{-1}\lambda_{i-1})(y) = \lambda(z^{-1} g_i z g_i^{-1})$. Since $Q \leq F_i^{g_i'} \cap F_i^{g_{i-1}'}$, we have $z^{-1} g_i z g_i^{-1} \in [F_i, N_i] \leq \mathrm{Ker}\ \lambda$. Thus $x^H \cap Q \subseteq \mathrm{Ker}(\lambda_i^{-1}\lambda_{i-1})$. This is a contradiction. The theorem is proved.

Now, we shall prepare for Theorem 4.9. Let $1 \neq P \in \mathrm{Syl}_p(G)$. We say that G is p-<u>isolated</u> (or G has a p-<u>strongly embedded subgroup</u>) provided

$$G \neq \langle N(A) \mid 1 \neq A \leq P \rangle .$$

If G is a 2-isolated simple group of 2-rank at least 2, then G is isomorphic to $L_2(q)$, $Sz(q)$, or $U_3(q)$, $q = 2^m > 2$ (Bender's theorem). These simple groups are called <u>Bender groups</u>.

Let B be a Bender group. By Richen and Steinberg theorems ([12]), the following hold :

(a) B has a unique 2-block of defect 0;

(b) If N is an irreducible $GF(2^\infty)B$-module, then $\dim N \leq |B|_2$.

Let $\overline{M}$ be an irreducible $GF(2^\infty)B$-module belonging to the unique 2-block of defect 0. Then there is a $GF(2)B$-module M such that $\overline{M} = GF(2^\infty) \otimes M$. The module M is called a <u>Steinberg module</u> of B . Then $\dim M = |B|_2$. Let <u>St(B)</u> be the semidirect product of the Steinberg module M by the Bender group B .

The Steinberg module is constructed as follows. Let T be a Sylow 2-subgroup of the Bender group B and $q^* = |T|$. It is wellknown that B has a doubly transitive permutation representation on the set $\Omega = \{0, 1, \cdots, q^*\}$. Let V be a vector space over $GF(2)$ with basis $\{v_i \mid i \in \Omega\}$. Then B acts on V by $v_i b = v_{ib}$ for $i \in \Omega$, $b \in B$, so we obtain a $GF(2)B$-module of degree $q^* + 1$. Set $M = [V, B] = \langle v_0 + v_i \mid 1 \leq i \leq q^* \rangle$. It is easy to prove that M is an absolutely irreducible $GF(2)B$-module, so M is the Steinberg module of B. The Steinberg modules of a Bender group are isomorphic.

In order to prove Theorem 4.9, we shall prepare two lemmas.

Lemma 4.1. Let B be a Bender group with Sylow 2-subgroup T of order q^* and M an irreducible $GF(2)B$-module. Assume that $[M, T; q^*-1] \neq 0$. Then the following hold :

(1) M is isomorphic to the Steinberg module of B ;

(2) The Sylow 2-subgroup $M \cdot T$ of the semidirect product $M \cdot B$ is isomorphic to $Z_2 \wr T$, where the wreath product is constructed by the regular permutation of T.

(3) $H^1(B, M) = 0$, $H^2(B, M) = 0$.

Proof. As $GF(2^\infty) \otimes M$ is complete reducible, (1) follows from Richen and Steinberg theorems. (2) is clear by the above construction of the Steinberg module. By (2), we see that $H^1(T, M) = 0$ and $H^2(T, M) = 0$. Hence Gaschutz theorem ([10], Section 15.8) yields (3). The lemma was proved.

Lemma 4.2. Let G be a 2-constrained group such that $O(G) = 1$ and $O^{2'}(G) = G$. Let S be a Sylow 2-subgroup of G , $F = O_2(G)$ and $\overline{G} = G/\Phi(F)$. Assume that $m(S/F) > 1$ and that

$$[F, G] \neq \langle [F, N(T)] \mid F < T \leq S \rangle .$$

Then G/F is 2-isolated and there are $U, V \triangleleft \overline{G}$ such that $\overline{F} = U \times V$, $V \leq C_{\overline{G}}(O_{22'}(\overline{G}))$ and

$$(G/U)/O(G/U) \simeq St(G/O_{22'}(G)).$$

<u>Proof</u>. We may assume that F is elementary. Set $H = \langle N(T) \mid F < T \leq S \rangle$. Then $[F, H] \neq [F, G]$, so $H < G$. This means that G/F is 2-isolated. As $m(S/F) > 1$, H contains $O_{22'}(G)$. If Q is a Hall 2'-subgroup of $O_{22'}(G)$, then $G = N(Q)F$ and $F = [F, Q] \times C_F(Q)$, so we may assume that $Q = 1$. Then $B = G/F$ is isomorphic to a Bender group. If F involves a chief G-factor isomorphic to the Steinberg module, then the lemma follows from Lemma 4.1. We may assume that G is the semidirect product $F \cdot B$ of B with kernel F. Let E be a minimal normal subgroup of G contained in $[F, G]$. Since $Z(G) \cap G' \leq S'$ ([7], Proposition 4.4), we see that if $E = [E, H]$ or $E \leq Z(G)$, then G/E satisfies the hypothesis of the lemma, so the lemma holds by the induction argument. Thus we may assume that $[E, H] < E$ and further that $E = F$. Then F is an irreducible B-module and $F \cap G' = F \not\leq H'$. Since B is a TI-group, it follows from Corollary 4.1.1 that F is a singular subgroup in S. By Lemma 3.5, we have that $\dim F \geq |S/F|$, and so F is isomorphic to the Steinberg module, as required.

In the following theorem, we find an especial family which controls the transfer. A part of the conclusion (a) – (f) are satisfied by Goldschmidt's families [8], too, and the rest gives the subset of his family which controls transfers. The proof was suggested by the idea of Masahiko Miyamoto which simplified the ordinary proof.

<u>Theorem 4.9</u>. Let $P \in Syl_p(G)$ and $P \leq H \leq G$. Take elements $x_1, \cdots, x_m$ in $P - H^pH'$ such that $H = H^pH'\langle x_1, \cdots, x_m\rangle$ and x_k is of minimal order in $P - H^pH'\langle x_1, \cdots x_{k-1}\rangle$ for every k. Let $\mathfrak{F}$ be the set of paires (F, N), where $F < P$ and $F \leq N \leq N(F)$, satisfying the following conditions :

(a) $N_P(F)$ is a Sylow p-subgroup of $N(F)$;

(b) $N(F)/F$ is p-isolated;

(c) F is a Sylow p-subgroup of $O_{p'p}(N(F))$;

(d) For any $x \in N_P(F) - F$, $N = \langle x^N, N_P(F)\rangle$;

(e) If $F_2 \trianglelefteq F_1 \leq F$, $N = N_N(F_1/F_2)F$ and $C_{N_P(F)}(F_1/F_2) \not\leq F$, then $N = C_N(F_1/F_2)N_P(F)$;

(f) $N(F)$ is p-constrained;

(g) F contains a conjugate of a singular subgroup Q in H with singular element x_k for some k. Furthermore, $P \neq Q = P \cap P^g \in Syl_p(H \cap H^g)$ for some $g \in G$ and $O^{p'}(N(Q)) \not\leq C(Q/\Phi(Q))H$;

(h) $N_P(F)$ has a normal subgroup K such that $\Phi(F) \leq K \leq F$ and $N_P(F)/K \cong Z_p \wr (N_P(F)/F)$, where the wreath product is constructed by the regular permutation of $N_P(F)/F$ and the base subgroup is F/K. In particular, $m(F/K) \geq |N_P(F):F|$. Furthermore, if $m(N_P(F)/F) > 1$, then $[F, O_{pp'}(N(F))] \leq K$;

(i) If $p = 2$ and $m(N_P(F)/F) > 1$, then there are $U, V \trianglelefteq \overline{N} = N/\Phi(F)$ such that $\overline{F} = U \times V$, $[V, O_{22'}(\overline{N})] = 1$ and $(\overline{N}/U)/O(\overline{N}/U)$ is isomorphic to $St(N/O_{22'}(N))$.

Define the equivalence relation $\sim$ on $\mathfrak{F}$ by $(F_1, N_1) \sim (F_2, N_2)$ if and only if F_1 and F_2 are conjugate in G. Let $\mathfrak{F}_0$ be the complete set of representatives of equivalence classes of $\mathfrak{F}$. Then

$$P \cap G' = (P \cap H')[P, N(P)] \langle [F, N] \mid (F, N) \in \mathfrak{F}_0 \rangle .$$

<u>Proof</u>. Let P^* be the right side of the statement. Suppose $P \cap G' < P^*$. Take a subgroup F of P such that

(1) $[F, N(F)] \not\leq P^*$ and F satisfies the condition (g) in this theorem;

(2) $|F|$ is maximal subject to (1);

(3) $|N_P(F)|$ is maximal subject to (1) and (2).

Since there exists a conjugation family by Alperin theorem, the existence of F follows from Theorem 4.8. Next, take a subgroup N of $N(F)$ such that

(4) $N_P(F) \leq N$ and $[F, N] \not\leq P^*$;

(5) $|N|$ is minimal subject to (4).

Clearly, the subgroup N exists.

We shall show that $(F, N) \in \mathfrak{F}$. As $[P, N(P)] \leq P^*$, we have $F < P$. By Alperin theorem, if $g \in N(F)$, then there are $E_i \leq P$ and $g_i \in N(E_i)$, $1 \leq i \leq n$, such that $g = g_1 \cdots g_n$ and for each i, $N_P(E_i) \in \mathrm{Syl}_p(N(E_i))$ and $F^{g_1 \cdots g_n} \leq E_i$. By induction argument on n, we have

$$[F, g] \leq [F, g_1][F^{g_1}, g_2] \cdots [F^{g_1 \cdots g_{n-1}}, g_n]$$
$$\leq [E_i, N(E_i)] \mid 1 \leq i \leq n \quad .$$

Thus the maximalities of $|F|$ and $|N_P(F)|$ yield that $N_P(F)$ $Syl_p(N(F))$, and so (F, N) satisfies the condition (a). By the maximality of $|F|$, we have that $N(F) \neq \langle N(F) \cap N(E) \mid F < E \leq N_P(F)\rangle$. This means that $N(F)/F$ is p-isolated, and so (F, N) satisfies (b). If E is a Sylow p-subgroup of $O_{p'p}(N(F))$ then $N(F) = O_{p'}(N(F))(N(F) \cap N(E))$. Thus $[F, N(F)] = [E, N(E)]$. By the maximality of $|F|$, we have $E = F$, so (F, N) satisfies (c). Let x be an element of $N_P(F) - F$. Set $L = \langle x^N, F\rangle$ and $E = L \cap P$. As $L \trianglelefteq N$, we have that $E \in Syl_p(L)$, so $N = N_N(E)L$. Since $[F, N] \leq [F, L]\,[E, N(E)]$ and $F < E$, we have that $[E, N(E)] \leq P^*$, and so the maximality of $|F|$ yields that $[F, L] \not\leq P^*$. By the minimality of $|N|$, we have $N = N_P(F)L$, and so (F, N) satisfies (d). Assume $F_2 \trianglelefteq F_1 \leq F$, $N = N_N(F_1/F_2)F$ and $x \in C_{N_P(F)}(F_1/F_2) - F$. Then $\langle x^N, F\rangle \leq C_N(F_1/F_2)F \trianglelefteq N$. Thus (d) yields that $N = C_N(F_1/F_2)N_P(F)$, so (F, N) satisfies (e). If $C_P(F) \not\leq F$, then (e) yields that $N = C_N(F)N_P(F)$, so $[F, N] \leq P'$, a contradiction. Thus $C_P(F) \leq F$. Thus $N(F)$ is p-constrained, so (F, N) satisfies (f). The condition (g) is already satisfied. Set $M = \langle N_N(T) \mid F < T \leq N_P(F)\rangle$. Then by Tate theorem, we have $F \cap N^pN' \neq F \cap M^pM'$. (Notice that $F \cap N^pN' = [F, N]\Phi(N_P(F))$). By Corollary 4.1.1, there is $g \in N - M$ such that $N_P(F) \cap N_P(F)^g = F$ is a singular

subgroup in $N_P(F)$. By Lemma 3.5, the first statement of (h) follows. Assume that $m(N_P(F)/F) > 1$. Let Q be a Hall p'-subgroup of $O_{pp'}(N)$, so that $N = N_N(Q)F$. Set $\overline{N} = N/\Phi(F)$. Then $\overline{F} = [\overline{F}, \overline{Q}] \times C_{\overline{F}}(\overline{Q})$. As $Q \leq M$, we have $C_{\overline{F}}(\overline{Q}) \not\leq \overline{M}'$. Thus by Proposition 4.1 and Lemma 3.5, we can take the subgroup K as in (h) in this case. Thus (F, N) satisfies (h). Finally, it follows from Lemma 4.2 that (F, N) satisfies (i). Hence we proved that $(F, N) \in \mathfrak{F}$.

Take $(F_0, N_0) \in \mathfrak{F}_0$ such that F_0 and F are conjugate in G. We will show that $[F, N] \leq P^*$. By Alperin theorem, we may assume that there are $E \leq P$ and $g \in N(E)$ such that $\langle F, F_0 \rangle \leq E$ and $F = F_0{}^g$. But then $[F, N(F)] \leq [F_0, N(F_0)]$ $[E, g] \leq P^*[E, N(E)]$, so by the maximality of $|F|$, we have $E = F = F_0$. This is a contradiction. The proof of the theorem is completed.

<u>Remark</u>. The above proof is essentiall same as the method by M. Miyamoto. A complete set of representatives of ~-equivalence classes of the family of the pairs satisfying the conditions (a) to (e) (respectively, (a) to (f)) is a conjugation family (respectively, a weak conjugation family). This is proved by the similar way as [8].

5. Examples

In this section, we give some examples and outlines of their proofs.

Example 5.1.(Compare with [5]). Suppose G has a Sylow p-subgroup P of maximal class and p is odd. Then the following hold :

(1) If $P \not\cong Z_p \wr Z_p$, then $P \cap G' = P \cap N(P)'$.

(2) If $P \cong Z_p \wr Z_p$, then $P \cap G' = P \cap N(J)'$, where J is a unique elementary abelian subgroup of P of rank p.

Proof. When $P \cong Z_p \wr Z_p$, the statement follows from Theorem 4.4. When $P \not\cong Z_p \wr Z_p$, show that P is weakly regular and apply Theorem 4.2.

Example 5.2. (See Thompson [5, Theorem 10.1]). Let p and q be distinct primes, p odd. Assume that all p-local subgroups of G are p-constrained and that $SCN_3(p)$ is not empty. Let P be a Sylow p-subgroup of G and Q a maximal element of $\mathcal{M}_G(P; q)$. Then $P \cap G' = P \cap N(Q)'$.

Proof. By Thompson transitivity theorem ([9, Theorem 8.5.3]), an element A of $SCN_3(P)$ is of Sylow type in $N(Q)$ with respect to G and $N(A) = (N(A) \cap N(Q))O_{p'}(N(A))$. Thus we have $P \cap N(A)' \leq P \cap N(Q)'$. The statement follows from Theorem 4.4.

Example 5.3. Let $P \in Syl_p(G)$, $p \geq 5$, and $N(P) \leq M \leq G$. Assume that $m_p(M \cap M^g) \leq 2$ or $|P \cap M^g|^2 \leq |P|$ for every $g \in G - M$. Then $P \cap G' = P \cap M'$.

Proof. Suppose false. By Theorem 4.7, there is $g \in G$ such that $Q = P \cap P^g$ is a singularity in M and $N(Q) \not\leq M$. If $m(Q) \leq 2$, then $|Q/\Phi(Q)| \leq p^3$ by Blackburn theorem ([11, Satz III.12.4]). Thus Lemma 3.8(5) and 3.9(3) yield that $Q = P$, a contradiction.

Remark. K. Harada conjectured that if P is a p-group of rank m, then $|P/\Phi(P)| \leq p^{2m}$. If the conjecture was correct, we can replace the condition $m_p(M \cap M^g) \leq 2$ with $m_p(M \cap M^g) \leq (p - 1)/2$. His conjecture will suggest the refinement of some results in this paper, for example Theorem 4.9.

Example 5.4. (See [9, Theorem 6.5.2]). Let G be a p-solvable group with abelian Hall p'-subgroup and let $S \in Syl_p(G)$. Then the following hold :

(1) If S is weakly regular, then G is of p-length 1.

(2) If $N \trianglelefteq S$ and $cl(N/\Phi^*(N)) \leq p-2$, then $N \leq O_{p'p}(G)$.

Proof. We may assume that $O_{p'}(G) = 1$ and $O^{p'}(G) = G$. Then $N(S) = S$, so (1) follows from Theorem 4.2. In case (2), we may further assume that $S = O_p(G)N$. Then N is of Sylow type in S. By Theorem 4.4, G has a normal p-complement, proving (2).

Example 5.5. Let G be a p,q-group with weakly regular Sylow p-subgroup. Assume that $G = O_{pqp}(G)$, $O_q(G) = 1$ and $O^q(G) = G$. Then Sylow q-subgroups of G are special.

Proof. This is an easy corollary of 5.4. Apply Thompson's lemma ([11, Satz III.13.6]).

Example 5.6. If G has a Sylow 2-subgroup S of class 2, then $S \cap G' = \langle [F, N(F)] \mid F \leq S, |S:F| \leq 2 \rangle$.

Proof. Use Lemma 3.9(5) and Theorem 4.7.

Example 5.7. Let $S \in Syl_2(G)$ and A an abelian subgroup of S of exponent at most 4. Assume that $|A^g \cap N(A)| \leq |A|/4$ for every $g \in G - N(A)$. Then $S \cap G' = S \cap N(A)'$.

Proof. Apply Lemma 3.10 and Theorem 4.7.

Example 5.8. (Compare with Hall-Wielandt theorem, [10, 14.4]). Let W be a weakly closed subgroup of a Sylow p-subgroup P of G. Assume $cl(W/\Phi^*(W)) \leq p-2$. Then $P \cap G' = P \cap N(W)'$.

Proof. Apply Theorem 4.4 (or Lemma 3.8(3) and Theorem 4.7).

Example 5.9. If S is a Sylow p-subgroup of $GL(n, p^m)$, $p \geq 5$, then S is weakly regular. In particular, if G has a Sylow p-subgroup S^* isomorphic to S, then $S^* \cap G' = S^* \cap N(S^*)'$.

Proof. Since $\Phi^*(S) \geq \Phi(\Phi(S)) = [S, S, S, S]$, S is weakly regular. The remainder follows from Theorem 4.1.

<u>Remark</u>. It is conjectured that almost all unipotent groups of simple groups of Lie type are weakly regular except for some cases : $L_n(2)$, $U_n(2)$, $Sp(2n, 2)$, $Sz(8)$, etc.

Let $\sum$ be the root system of type A_n with fundamental roots α_i , $1 \leq i \leq n$, and Dynkin diagram :

$$\underset{\alpha_1}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\underset{\alpha_2}{\circ}\!\!-\!\!\!-\!\!\!-\ \cdots\cdots\ -\!\!\!-\!\!\!-\!\!\underset{\alpha_{n-1}}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\underset{\alpha_n}{\circ}$$

Then $\sum^+ = \{\alpha_i + \cdots + \alpha_j \mid 1 \leq i \leq j \leq n\}$. Let $U_\alpha = \langle x_\alpha \rangle$ be the root subgroup of $GL(n+1, 2)$ for $\alpha \in \sum$ and let $U = \langle U_\alpha \mid \alpha \in \sum^+ \rangle$. Then U is a Sylow 2-subgroup of $GL(n+1, 2)$. We have that for $\alpha, \beta \in \sum^+$,

$$[x_\alpha , x_\beta] = \begin{cases} x_{\alpha+\beta} & \text{if } \alpha + \beta \in \sum^+, \\ 1 & \text{if } \alpha + \beta \notin \sum^+. \end{cases}$$

For each i, we set $U^i = \langle U_\alpha \mid \alpha \in \sum^+, \alpha \neq \alpha_i \rangle$.

Now, it is wellknown that the family $\{(U^i, N(U^i)) \mid i = 1, 2, \cdots, n\}$ in $L_{n+1}(2)$ is a conjugation family for U . How about all groups with Sylow 2-subgroup U ? This is probably a difficult problem. But it is very easy to find a family which controls the transfer.

<u>Example 5.10</u>. If G has a Sylow 2-subgroup U of typ $L_{n+1}(2)$, $n \neq 3$, then

$$U \cap G' = \langle [U^i, N_G(U^i)] \mid i = 1, 2, \cdots n \rangle .$$

<u>Proof</u>. Set $x_i = x_{\alpha_i}$, $1 \leq i \leq n$. Let F be a proper subgroup of U such that $N(F)/F$ is 2-isolated, $N_U(F) \in Syl_2(N(F))$ and F contains a conjugate of a singular subgroup S with singular element x_i for some i. We claim $S \geq U^{i-1} \cap U^{i+1} < U$. By Lemma 3.8(4), we may assume that $S \geq C_U(x_i)$. We can easily find an elementary abelian subgroup E of U such that

$$C_U(x_i)[E, C_U(x_i)] = U^{i-1} \cap U^{i+1}.$$

Thus by Lemma 3.10, we have $S \geq U^{i-1} \cap U^{i+1}$. Without proof, we refer to the fact that U and $\langle U^{i-1} \cap U^{i+1}, x_{i-1}x_{i+1} \rangle$ have no nontrivial automorphisms of odd order. Since F has a nontrivial automorphism of odd order, we havethat $F = U^{i-1}$ or U^{i+1}. The statement follows from Theorem 4.9.

<u>Example 5.11</u>. If G has a Sylow 2-subgroup of type A_8 and $O^2(G) = G$, then there is a 2-local subgroup N such that either

(1) $N/O(N) \simeq St(L_2(4))$, or

(2) $N/O(N)$ is isomorphic to the semidirect product of Z_2^3 by $L_3(2)$.

<u>Proof</u>. Let P be a Sylow 2-subgroup of G, so that P is isomorphic to $Z_2 \wr Z_2^2$. Let A be a unique elementary abelian subgroup of P of order 16. It follows easily that $|N(P):C(P)P| = 1$ or 3. Assume first that $N(P) > C(P)P$. Then $P = A(P \cap N(P)')$. By Theorem 4.7, we have that $P \leq N(A)'$. Let

M be a maximal subgroup of P containing A. If $N(M)$ has no normal 2-complement, then all elements of $A^{\#}$ are conjugate each other in $N(A)$. Thus $2^2 \cdot 3^2 \cdot 5 \mid |N(A)|$. By Sylow theorem, we have a contradiction. Hence $N(A)/A$ is 2-isolated, and so (1) holds. Next assume that $N(P)$ has a normal 2-complement. Let E be an extra special subgroup of P of order 32. Since $N(A)/A$ has a normal 2-complement and a proper singular subgroup of which singular element is an involution of $T - A$ is E, it follows from Theorem 4.7 that $N(E)$ has no normal 2-complement and $P = (P \cap N(E)')A$. In particular, $E \leq N(E)'$. Since $N(A)/A$ has a normal 2-complement, there are an element u of $N(A) - C(A)$ of odd order and an element e of $T - A$ such that $[u, e] = 1$. Since all involutions of $T - A$ are contained in E, we have $e \in E$. Set $B = C_A(e)\langle e\rangle$. Then $B \leq E$, $B \leq T$ and $u \in N(B) - C(B)$. By an easy caluculation, we see that $N(E)/C(E)E$ is isomorphic to S_3 and any element of $N(E) - C(E)E$ of odd order normalizes B. Hence $N(E)$ has no subgroup of index 2. From this, (2) follows, as required.

<u>Example 5.12</u>. If G has a Sylow 2-subgroup of type $L_5(2)$ and $O^2(G) = G$, then $N(E)/O(N(E))E \cong GL(3, 2)$, where E is an extraspecial group of order 2^7.

<u>Proof</u>. Let P be a Sylow 2-subgroup of G containing E. Then E is the unique subgroup of T isomorphic to $D_8 * D_8 * D_8$.

We shall use the notation defined above Example 5.10. Then P is generated by the subgroups $[U^i, N(U^i)]$, $i = 1, 2, 3, 4$. Suppose $[U^2, N(U^2)] \leq P'$. The subgroup $U^1 \cap U^3 \cap U^4$ contains an elementary abelian subgroup A of order 2^6. Since P contains only two elementary abelian subgroup of order 2^6, they are weakly closed in P. Thus we have $P \leq N(A)'$. But as $P/A \cong Z_2 \times D_8$, we have a contradiction. Hence $[U^2, N(U^2)] \not\leq P'$. Similarly, $[U^3, N(U^3)] \not\leq P'$. Since $E \leq U^2 \cap U^3$ and E is weakly closed, we have that $\langle N(U^2), N(U^3) \rangle = N \leq N(E)$. Investigating the structure of the automorphism group of E, we have that $N/O(N)E$ is isomorphic to $GL(3, 2)$ and $N(E) = O(N(E))N$. Thus $N(E)/O(N(E))E$ is isomorphic to $GL(3, 2)$.

<u>Remark</u>. For the above group G, we have $P \leq N(E)'$. But without the assumption $O^2(G) = G$, the conclusion $P \cap G' = P \cap N(E)'$ does not hold in general.

<u>Example 5.13</u>. Let P be a Sylow 2-subgroup of G and E a weakly closed extraspecial subgroup of P. Set $N = N(E)$. Then the following hold :

(1) If $|E| \geq 32$, then $E \cap G' = E \cap N'$. If E is quaternion, then $E \cap G^2G' = E \cap N^2N'$;

(2) If E is not dihedral and E is strongly closed, then $P \cap G' = P \cap N'$;

(3) If $|E| \geq 128$ and $|E^g \cap P| < |E|/2$ for every $g \in G - N$, then $P \cap G' = P \cap N'$.

Proof. The following properties about E are easily proved :

(a) If $E \not\cong D_8$, then E is weakly regular;

(b) If $|E| \geq 32$, then $\Phi^*_\infty(E) = \Phi(E)$;

(c) If $|E| \geq 128$, then all subgroups of E of index 8 are normal.

Now, (1) follows from (a), (b) and Theorem 4.6. (2) follows from (a) and Corollary 4.5.1(3) If S is a singular subgroup in P and $|E| \geq 128$, then $|E : E \cap S| \leq 2$ by (c) and Lemma 3.10. Thus (3) follows from Theorem 4.7.

Example 5.14.(Chabot [3]). Let P be a Sylow 2-subgroup of G. Assume that the commtator group P' is cyclic and $O^2(G) = G$. Let Q be a Hall 2'-subgroup of $N(P)$ and set $C = C_P(Q)$, $D = [P, Q]$. Then D is weakly closed in P and $P \leq N(D)'$. Furthermore, one of the following holds :

(1) $P \leq N(P)'$ and $cl(P) \leq 2$;

(2) $P = CD$, $[C, D] = 1$, $cl(D) \leq 2$, C is generalized quaternion and $|C \cap D| \leq 2$;

(3) $P = C \times D$, C is dihedral, semidihedral or wreathed.

<u>Proof</u>. We need two results about groups whose Sylow 2-subgroups have cyclic commtator groups. We will omit the proofs, as they do't relate to the purpose of this paper.

(*) Let P be a 2-group with cyclic commutator group and let Q be a 2'-group of automorphisms of P. Set $C = C_P(Q)$ and $D = [P, Q]$. Then $P = CD$, $[C, D] = 1$ and $D' \leq Z(P)$.

(**) Let X be a 2-constrained group such that $O(X) = 1$ and $O^{2'}(X) = X$. Assume that a Sylow 2-subgroup P has the cyclic commutator group and that X is not 2-closed. Then $X/O_2(X) \cong S_3$ and $P \cap O^2(X)$ is quaternion or homocyclic of rank 2.

Now, we begin the proof of the example. We shall argue by induction on the order of G. Suppose $D = [P, Q]$ is not weakly closed in P. Then by Alperin's theorem [1], there is a subgroup F of P containing D such that $N(F) \not\leq N(D)$. Suppose F is maximal with this property. Then we have that $N = O^{2'}(N(F)) \not\leq N(D)$, $N(F)$ is 2-constrained and F is a Sylow 2-subgroup of $O_{2'2}(N(F))$. Set $\overline{N} = N/O(N)$. By (**), we have $\overline{N}/O_2(\overline{N}) \cong S_3$. Thus $[\overline{Q}, \overline{N}] \leq \overline{F}$, and so $\overline{FQ} \triangleleft \overline{N}$. This implies that $\overline{D} = [\overline{F}, \overline{Q}] \triangleleft \overline{N}$, a contradiction. We proved

(1) D is weakly closed in P.

Let $N = N(D)$ and x be an element of $P - N^2N'$ of minimal order. Then by Theorem 4.7, there is $g \in G - N$ such that $P \cap P^g$ is a singular subgroup in N with singular element x. Since P is the central product of C and D, it follows from Lemma 3.11 that $P \cap P^g$ contains D, contrary to (1). Hence we have that $N(D)$ has no subgroup of index 2. Thus we may assume that

(2) $N(P)$ has a normal 2-complement.

By Corollary 4.6.1, we have that $\Omega_1(Z(P)) \leq N(P)' \cap P = P'$. Since P' is cyclic, we have that

(3) $Z(P)$ is cyclic.

By (3), (*), (**) and Example 5.6, we see that if $cl(P) \leq 2$, then P is dihedral of order 8. Thus we may assume that

(4) $cl(P) \geq 3$.

Let F be a subgroup of P such that $N = O^{2'}(N(F))$ is 2-constrained and has no normal 2-complement. Then $N/O_{2'2}(N) \cong S_3$ by (**). Let R be a Hall 2'-subgroup of N. Suppose $|[F, R]| \geq 16$. Then $[F, R]$ is homocyclic of rank 2, $F = C_F(R) \times [F, R]$ and $\Omega_1(P') \leq [F, R]$. The subgroup F is a unique maximal abelian subgroup of $N_P(F)$, so $F = [F, R] \lhd P$ and P is wreathed. In this case, the desired conclusion holds.

Thus we may assume that $|[F, R]| \leq 8$ for every F and R as the above. Then by Alperin's theorem (or Theorem 4.9), P is generated by such $[F, R]$, so we have

(5) $P/Z = \Omega_1(P/Z)$, where $Z = \Omega_1(P')$. In particular, $P' = \Phi(P)$.

Let $x_1, \cdots, x_m$ be the generators of P such that $x_k^2 \in \Omega_1(P')$. If possible, we take the element x_k of order 2. For each k, there is $g \in G - N(P)$ such that $S = P \cap P^g$ is a singular subgroup in P with singular element x_k. We will show that each x_k inverts P'. Let λ be the singular character of S with kernel K, $x = x_k \overline{P} = P/K_P$. Then (S, λ, x) is a singularity in P. Since $x \in SP' < P$ and P' is cyclic, Lemma 3.5 yields that $M = SP'$ is of index 2 in P. Set $\mu = T^M(\lambda)$. Then for $u \in P - M$, $1 \neq T^P(\lambda)(x) = T_M^P(\mu)(x) = \mu([x, u])$. By Corollary 2.2.2, we have that $1 \neq \lambda([x, u]^{|M:S|})$. Since $\overline{S_P}$ is elementary, $|\overline{S} \cap \overline{P}'| = 2$. Since $|[\overline{x}, \overline{u}]| \leq 2|\overline{M} : \overline{S}| = |\overline{P}'|$, we have $P' = \langle [x, u] \rangle$. From $x^2 \in Z(P)$, we conclude that x inverts P'. We proved that

(6) $|P : C_P(P')| = 2$ and any involution of $P - \Phi(P)$ inverts P'.

By (6), we have $\Omega_1(C_P(P')) \leq \Phi(P) = P'$, so $C_P(P')$ is cyclic. Thus $|P : P'| = 4$, and hence P is dihedral, semidihedral or generalized quaternion. The proof is complete.

6. Concluding remarks.

In his paper [1, Section 6], Alperin expected further relations between fusion and transfer. This paper will give the answer to a part of his question.

For $H \leq G$, we have $T_H^G(1_H) = 1_G$, so the character-theoretic transfer T_H^G does not yields the information about $\det(1_H^G)$. Since the values of $\det 1_H^G$ is ± 1, the consideration of $\det 1_H^G$ is valuable in the study of 2-fusion. In fact, there are some results about transfer and fusion which can be never given by the ordinary or character-theoretic transfers. (See [15]).

Some problems about transfer are still left. Let consider about Theorem 4.9. We wish to add further restrictions to the family $\mathfrak{F}$ which controls the transfer. Can we add the condition that F is a singularity ? When G is simple, can we say that $|N_P(F):F| \leq 8$? What can we say about the structure of F ? For example, if M is a minimal normal subgroup of N covering V in (i), then M is abelian or there is $K < N$ such that M/K is extraspecial of negative type or positive type according as $N/O_{22'}(N)$ is isomorphic to $L_2(4)$ or not.

It is possible to prove Theorem 4.2 by ordinary transfers or the method in [15] (without even them, if $p = 2$). But

such proofs are nothing but the rewriting of the proof in this paper.

Let consider the classification problem of simple groups with a given Sylow 2-subgroup. First, The consideration of transfer is not useful to characterize simple groups with weakly regular Sylow 2-subgroups (Theorem 4.2). But it is useful to characterize sporadic simple groups and simple groups of Lie type over a small fields of characteristic 2. The largeness of the order of a 2-group is not a big impediment to determine the singularities. For example, When G has a Sylow 2-subgroup of type C_1, C_2, $M(22)$, or Rd and $O^2(G) = G$, it is not so difficult to prove that G has a 2-local subgroup N such that $N/O_{2'2}(N)$ is isomorphic to $L_3(2)$, $Sp(6, 2)$, M_{22}, or $L_3(2)$, respectively. Finally, the consideration of transfer is important to characterize simple groups of Lie type of odd characteristic. But for the present, Theorem 4.9 is not so convenient.

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