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On the discriminants of transformation equations
of modular forms

by

Yoshitaka MAEDA

Introduction.

The purpose of this note is to add some supplementary results to our previous paper [3]. We have proved there that the transformation equations of certain modular forms can be expressed by special values of the zeta functions of those forms. At the symposium, we talked about this result. Here we give some results obtained after that.

Let \( f \) be a modular form on \( \Gamma_0(p) \) of weight \( k \). We assume that \( p \) is an odd prime throughout the paper. Then the transformation equation of \( f \) is defined by

\[
\phi(X; f) = \prod_{\alpha \in \Gamma_0(p) \backslash SL_2(\mathbb{Z})} (X - f|_k^\alpha) = 0,
\]

where

\[
(f|_k^\gamma)(z) = \det(\gamma)^{k/2} f((az+b)/(cz+d))(cz+d)^{-k}
\]

for

\[
\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R}) = \{ \gamma \in GL_2(\mathbb{R}) \mid \det(\gamma) > 0 \}.
\]

Obviously all coefficients \( \sigma^\mu_\nu \) of \( \phi(X; f) \) are modular forms on \( SL_2(\mathbb{Z}) \). We call that the transformation equation \( \phi(X; f) = 0 \) is Z-rational if all coefficients \( \sigma^\mu_\nu \) have Z-rational Fourier expansions as modular forms (see §1, for the Z-rationality of Fourier expansions). Then one of our results is
Theorem 1. If the transformation equation \( \phi(X;f) = 0 \) of \( f \) is \( \mathbb{Z} \)-rational and if \( p \) is an odd prime, then the discriminant \( D \) of \( \phi(X;f) \) is expressed as

\[
D = \begin{cases} 
(\frac{-1}{p^2}) \frac{(p-1)}{2} p^p \Delta^{p+1} h^2, & \text{if } f \text{ is a cusp form,} \\
\frac{(p-1)}{2} p^p \Delta^{p-1} h^2, & \text{otherwise,}
\end{cases}
\]

where \( h \) is a modular form on \( SL_2(\mathbb{Z}) \) with a \( \mathbb{Z} \)-rational Fourier expansion and \( \Delta \) is Ramanujan's function \( \exp(2\pi i z) \prod_{n=1}^{\infty} (1-\exp(2\pi i nz))^{24} \).

Especially, when \( f \) is the special cusp form discussed in [3], we even know the divisibility of the above form \( h \) by \( \Delta^{(p+1)/2} \) (Proposition 9). We will prove this theorem in \( \S 2 \).

In Proposition 4, we will also show under the assumption in Theorem 1 the following congruence relation:

\[
\phi(X;f) \equiv (X - \sigma_1)(X^p - \sigma_p) \mod p.
\]

Here both \( \sigma_1 \) and \( \sigma_p \) are certain modular forms on \( SL_2(\mathbb{Z}) \). Though this result follows from the Eicher-Shimura congruence relation [7, Theorem 7.9], we will give an elementary proof without using their result.

In [3], we considered the transformation equation \( \phi(X;f) = 0 \) for \( f = gE_{\lambda,p}^* \), where \( g \) is a cusp form on \( \Gamma_0(p) \) and \( E_{\lambda,p}^* \) is a certain Eisenstein series. In \( \S 3 \), we will show that for a certain choice of \( g \), the transformation equation \( \phi(X;gE_{\lambda,p}^*) = 0 \) of \( gE_{\lambda,p}^* \) is \( \mathbb{Z} \)-rational. In \( \S 4 \), we will give numerical examples of the transformation equations for the above \( gE_{\lambda,p}^* \) and the specialized equations at several elliptic curves (see [3, \( \S 3 \)], for the definition of the specialized equation at an elliptic curve).
§1. Congruence relation of transformation equations.

Let $p$ be an odd prime and $\Gamma_0(p)$ be a subgroup of $\text{SL}_2(\mathbb{Z})$ defined by

$$\Gamma_0(p) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \mid c \equiv 0 \mod p \}.$$ 

We denote by $M_k(\Gamma_0(p))$ the vector space of modular forms on $\Gamma_0(p)$ of weight $k$. Hence an element $f$ of $M_k(\Gamma_0(p))$ is a function on the upper half complex plane $H$ with the following three properties:

(i) $f$ is holomorphic on $H$;
(ii) $f|_{\gamma} = f$ for all $\gamma \in \Gamma_0(p)$;
(iii) $f|_{\gamma}$ has the Fourier expansion of the form: $\sum_{n=0}^{\infty} a_\gamma(n) e(nz/p)$ at $i\infty$ for all $\gamma \in \text{SL}_2(\mathbb{Z})$ ( $e(z) = \exp(2\pi iz)$ ).

Moreover, if $a_\gamma(0) = 0$ for all $\gamma \in \text{SL}_2(\mathbb{Z})$, then $f$ is called a cusp form. The subspace of $M_k(\Gamma_0(p))$ consisting of all cusp forms is denoted by $S_k(\Gamma_0(p))$.

Let $A$ be a subring of $\mathbb{C}$ and $f$ be a function on $H$ with a Fourier expansion of the form: $\sum_{n=0}^{\infty} a(n) e(nz/N)$ for some positive integer $N$. Then we say that $f$ is $A$-rational if $a(n)$ belongs to $A$ for any $n$. Let $g(z) = \sum_{n=0}^{\infty} b(n) e(nz/N)$ be another $A$-rational function and $m$ be an ideal of $A$. Then we write $f \equiv g \mod m$ if $a(n) \equiv b(n) \mod m$ for all $n$. Further, for any field-automorphism $\sigma$ of $\mathbb{C}$, we define an action of $\sigma$ on $f$ by

$$f^\sigma(z) = \sum_{n=0}^{\infty} a(n)^\sigma e(nz/N).$$

Though the following lemma is a corollary of [8, Theorem 5], we give an elementary proof.
Lemma 2. Let $f$ be an element of $M_k(\Gamma_0(p))$. Then for any field-automorphism $\sigma$ of $C$, we have that

$$(f|_k(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}))^\sigma = f^\sigma|_k(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}).$$

Proof. It is sufficient to show this for a basis of $M_k(\Gamma_0(p))$.

First let us show the relation for Eisenstein series. We know from [4, Satz 11] that the subspace of $M_k(\Gamma_0(p))$ spanned by all Eisenstein series has a basis consisting of the following functions:

$$k > 2 : \quad G_k(z) = -B_k/2k + \sum_{n=1}^{\infty} \left\{ \sum_{d|n} d^{k-1} \right\} e(nz)$$

and $G_k(pz)$

$$k = 2 : \quad E_p(z) = G_2(z) - pG_2(pz).$$

Here $B_k$ is the $k$-th Bernoulli number and

$$G_2(z) = i/(4\pi(z-\bar{z}) - 1/24 + \sum_{n=1}^{\infty} \left\{ \sum_{d|n} d \right\} e(nz).$$

Put $\tau = (\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix})$. Since $G_k(z)$ is an Eisenstein series on $SL_2(Z)$ of weight $k$, we have that $G_k|_k\tau = G_k$, and therefore, $G_k(pz)|_k\tau = p^{-k}G_k(z/p)$. Thus both $G_k|_k\tau$ and $G_k(pz)|_k\tau$ are $Q$-rational. Then the desired result follows for $G_k(z)$ and $G_k(pz)$. A similar argument shows that the lemma holds also for $E_p$. Next let us show the relation for a basis of cusp forms. The theory of primitive forms shows that a basis of $S_k(\Gamma_0(p))$ is given by $f_i(z)$, $f_i(pz)$ and $g_j(z)$. Here $f_i(z)$ and $g_j(z)$ are primitive forms of conductor 1 and $p$, respectively. Our assertion holds for such cusp forms $f_i$. In fact, it is known that $f_i^\sigma$ are again primitive forms of conductor 1. Let us write $f(z) = f_i(pz)$. Then we have that

$$(f|_k\tau)(z) = p^{-k}f_i(z/p)$$

and
Since $f^\sigma(z) = f^\sigma(pz)$, we see that
\[(f^\sigma|_{k^\tau})(z) = p^{-k}f^\sigma(z/p).\]
It follows that $(f|_{k^\tau})^\sigma = f^\sigma|_{k^\tau}$.

Now let us show the relation for primitive forms $g_j$ of conductor $p$. We know from [1, Lemma 3] that
\[g_j|_{k^\tau} = \gamma g_j\quad \text{with } \gamma = \pm 1.\]
Since $\gamma$ is expressed as
\[\gamma = -a(p)p^{1-k/2}\]
with the $p$-th Fourier coefficient $a(p)$ of $g_j$ and since $g_j^\sigma$ is again a primitive form of conductor $p$, we have that
\[g_j^\sigma|_{k^\tau} = \gamma g_j^\sigma.\]
Thus we observe that
\[g_j|_{k^\tau} = \gamma p^{-k/2}g_j(z/p)\]
and
\[g_j^\sigma|_{k^\tau} = \gamma p^{-k/2}g_j^\sigma(z/p).\]
This concludes the proof of Lemma 2.

**Lemma 3.** Let $f$ be an element of $M_k(\Gamma_0(p))$. Then the transformation equation $\phi(X; f) = 0$ of $f$ is Z-rational if and only if both $f$ and $f|_{k^\tau} = f|_{k^\tau}(0 - 1)$ are Z-rational.

Proof. Let us write the Fourier expansions of $f$ and $f|_{k^\tau}$ as
\[(1.1_a) \quad f(z) = \sum_{n=0}^\infty a(n)e(nz),\]
and
\[(1.1_b) \quad (f|_{k^\tau})(z) = \sum_{n=0}^\infty b(n)e(nz/p),\]
respectively. Further let us write the transformation equation
of \( f \) as

\[
(1.2) \quad \phi(X; f) = X^{p+1} + \sum_{\mu=1}^{p+1} (-1)^{\mu} \sigma_{\mu} X^{p+1-\mu}.
\]

First let us show that if both \( f \) and \( f|_{k^T} \) are \( \mathbb{Z} \)-rational, then all coefficients \( \sigma_{\mu} \) are \( \mathbb{Z} \)-rational. Let us define a set \( R \) by

\[
(1.3) \quad R = \{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tau_u = \begin{pmatrix} 0 & -1 \\ 1 & u \end{pmatrix} \mid u = 0, 1, 2, \ldots, p-1 \}.
\]

Then, since \( p \) is a prime, the set \( R \) gives a complete set of representatives for \( \Gamma_0(p) \backslash \text{SL}_2(\mathbb{Z}) \) (e.g., [5, Lemma 2.2]).

Since \( \tau_u = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \), it follows from \( (1.1_b) \) that for any integer \( u \),

\[
(1.4) \quad (f|_{k^T} \tau_u)(z) = \sum_{n=0}^{\infty} \zeta^{nu} b(n) e(nz/p),
\]

where \( \zeta = e(1/p) \). Since all Fourier coefficients \( b(n) \) belong to \( \mathbb{Z} \), the modular forms \( f|_{k^T} \tau_u \) are \( \mathbb{Z}[\zeta] \)-rational. Thus coefficients \( \sigma_{\mu} \) are \( \mathbb{Z}[\zeta] \)-rational. In fact, coefficients \( \sigma_{\mu} \) are symmetric functions in \( \{ f|_{k^{\alpha}} \}_{\alpha \in R} \). On the other hand, \( (1.4) \) shows that for any field-automorphism \( \rho \) of \( \mathbb{C} \), \( \{ (f|_{k^{\alpha}})^\rho \}_{\alpha \in R} = \{ f|_{k^{\alpha}} \}_{\alpha \in R} \).

Thus we have that \( \sigma_{\mu}^\rho = \sigma_{\mu} \). This shows that the modular forms \( \sigma_{\mu} \) are \( \mathbb{Z} \)-rational. Conversely, assume that all the coefficients \( \sigma_{\mu} \) are \( \mathbb{Z} \)-rational. Then we have that for any field-automorphism \( \rho \) of \( \mathbb{C} \),

\[
(1.5) \quad \prod_{\alpha \in R} (X - f|_{k^{\alpha}}) = \prod_{\alpha \in R} (X - (f|_{k^{\alpha}})^\rho).
\]

Especially, we have that \( f^\rho = f \) or \( f^\rho = f|_{k^T \tau_u_0} \) for some \( u_0 \).

Assume that \( f^\rho = f|_{k^T \tau_u_0} \). Since \( f|_{k^T}(1 \quad -1) = f^\rho \) and \( \tau_u_0 = \tau(1 \quad 1)u_0 \),

we see
(1.6) \( f^\rho = f|_{k^\tau} \).

Therefore we have

(1.7) \( f|_{k^\tau} u = f|_{k^\tau} = f^\rho \) for any \( u \).

Moreover Lemma 2 and (1.6) show that

(1.8) \( (f|_{k^\tau})^\rho = f^\rho|_{k^\tau} = f \).

Thus we have that

(1.9) \( (f|_{k^\tau} u)^\rho = (f|_{k^\tau})^\rho = f \).

Rewriting the equation (1.5) through (1.7) and (1.9), we obtain that

\[
(X - f)(X - f^\rho)^\rho = (X - f^\rho)(X - f)^\rho.
\]

Hence we have that for any field-automorphism \( \rho \),

(1.10) \( f^\rho = f \).

Namely, \( f \) is \( \mathbb{Q} \)-rational. On the other hand, the vector space \( M_k(\Gamma(p)) \) has a \( \mathbb{Z} \)-rational basis (e.g., \([8, (9)]\)), where \( \Gamma(p) \) is the subgroup of \( SL_2(\mathbb{Z}) \) defined by

\[
\Gamma(p) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid a \equiv d \equiv 1, \ b \equiv c \equiv 0 \mod p \}.
\]

Thus we can write \( f \) as

\[
f(z) = c \sum_{n=0}^{\infty} d(n)e(nz)
\]

with a rational number \( c \) and rational integers \( d(n) \). Let us consider the equation \( \phi(X;f) = 0 \) over the ring \( \mathbb{Q}[[q]] \) of formal power series in \( q = e(z) \). Then \( f \) is integral over \( \mathbb{Z}[[q]] \) and is contained in its quotient field. In fact, all coefficients of \( \phi(X;f) \) are \( \mathbb{Z} \)-rational from the assumption, and therefore, they belong to \( \mathbb{Z}[[q]] \) for \( q = e(z) \). Since \( \mathbb{Z} \) is principal, \( \mathbb{Z}[[q]] \)
is a normal ring (e.g., [2, VII.3 Exercise 9c]), and this shows that \( f \) is \( Z \)-rational. Lemma 2 and (1.10) show that for any field-automorphism \( \rho \),

\[
(f|_k \tau)^\rho = f|_k \tau.
\]

Therefore a similar argument as above in \( \mathbb{Z}[[q^{1/p}]] \) shows that \( f|_k \tau \) is again \( Z \)-rational. Note that \( f|_k \tau \) belongs to \( M_k(\Gamma(p)) \). This concludes the proof of Lemma 3.

**Proposition 4.** Let \( f \) be an element of \( M_k(\Gamma_0(p)) \) and let us write the transformation equation of \( f \) as

\[
\phi(X,f) = X^{p+1} + \sum_{\mu=1}^{p+1} \varepsilon_{\mu} (-1)^\mu_X^{p+1-\mu}.
\]

If \( \phi(X,f) = 0 \) is \( Z \)-rational, then we have a congruence:

\[
\phi(X,f) \equiv (X - \sigma_1)(X^p - \sigma_p) \mod p.
\]

**Proof.** Let us write \( \zeta = e(1/p) \) and \( p \) be the unique prime ideal of \( \mathbb{Z}[\zeta] \) which divides \( p \). Then we know that \( \zeta \equiv 1 \mod p \).

Since \( f|_k \tau \) is \( Z \)-rational by Lemma 3, the Fourier expansion (1.4) of \( f|_k \tau \) shows that

\[
f|_k \tau \equiv f|_k \tau \mod p
\]

for any \( \tau \). Thus we have that

\[
\phi(X,f) \equiv (X - f)(X - f|_k \tau)^P \mod p
\]

\[
\equiv (X - f)(X^p - (f|_k \tau)^P) \mod p.
\]

Especially, we have that

\[
\sigma_1 \equiv f \mod p
\]

and

\[
\sigma_p \equiv (f|_k \tau)^P \mod p.
\]
Thus we have that
\[
\phi(X;f) \equiv (X - \sigma_1) (x^p - \sigma_p) \mod p.
\]
Here all coefficients of \(\phi(X;f)\) are \(\mathbb{Z}\)-rational, and especially both \(\sigma_1\) and \(\sigma_p\) are \(\mathbb{Z}\)-rational. Therefore we have that
\[
\phi(X;f) \equiv (X - \sigma_1) (x^p - \sigma_p) \mod p.
\]
§2. Proof of Theorem 1.

By the definition of the discriminant, we have

\[ D = \prod (f|_k \alpha - f|_k \beta)^2, \]

where the product is taken over all non-ordered pairs \((\alpha, \beta)\)
with \(\alpha \neq \beta\) in the representative set \(R\) as in (1.3). Then we see that

\[ D = \prod_{u=0}^{p-1} (f - f|_k \tau_u)^2 \cdot \prod_{0 \leq u < v \leq p-1} (f|_k \tau_u - f|_k \tau_v)^2. \]

Obviously \(D\) is a modular form on \(SL_2(\mathbb{Z})\) of weight \(kp(p+1)\). Let us put

\[ \delta = \prod_{u=0}^{p-1} (f - f|_k \tau_u) \cdot \prod_{0 \leq u < v \leq p-1} (f|_k \tau_u - f|_k \tau_v). \]

Then we see \(D = \delta^2\). First let us show

Lemma 5. \(\delta\) is a modular form on \(SL_2(\mathbb{Z})\).

Proof. It is sufficient to prove that \(\delta|_m (1 \ 1) = \delta\) and

\[ \delta|_m (0 \ -1) = \delta \] for \(m = kp(p+1)/2\). Put \(\sigma = (1 \ 1)\) and \(\tau = (0 \ -1)\).

The right multiplication of \(\sigma\) on the coset space \(\Gamma_0(p) \setminus SL_2(\mathbb{Z})\)
induces a permutation on the representative set \(R\), and \((1 \ 0)
\) is transformed to \((1 \ 0), \tau_{p-1} \) to \(\tau_0\), and \(\tau_u \) to \(\tau_{u+1}\) for \(u = 0, 1, \ldots, p-2\). Thus we observe that the first factor \(\prod_{u=0}^{p-1} (f - f|_k \tau_u)\)
of \(\delta\) is invariant under \(\sigma\), and that

\[
\{ \prod_{0 \leq u < v \leq p-1} (f|_k \tau_u - f|_k \tau_v) \}|_{kp(p-1)/2}\]

\[= (-1)^{p-1} \prod_{0 \leq u < v \leq p-1} (f|_k \tau_u - f|_k \tau_v). \]

Since \(p\) is odd, this shows that \(\delta|_m \sigma = \delta\). Also, \(\tau\) induces a
permutation on \(R\), and \((1 \ 0)\) is transformed to \(\tau_0\), \(\tau_0\) to \((1 \ 0)\),

- 2-1 -
and $\tau_u$ to $\tau_v(u)$ for $u = 1, 2, \ldots, p-1$. Here $v(u)$ is an integer in
\{1, 2, \ldots, p-1\} satisfying $v(u)u \equiv -1 \pmod{p}$. Rewriting $\delta$ as
$$\Pi_{u=0}^{p-1}(f - f|_{k\tau_u}) \cdot \Pi_{v=1}^{p-1}(f|_{k\tau_v} - f|_{k\tau_v}) \cdot \Pi_{1 \leq u < v \leq p-1}(f|_{k\tau_v} - f|_{k\tau_u})$$
we observe that
$$\delta|_{m\tau} = -\Pi_{u=0}^{p-1}(f - f|_{k\tau_u}) \cdot \Pi_{v=1}^{p-1}(f|_{k\tau_0} - f|_{k\tau_v})$$
$$\cdot \Pi_{1 \leq u < v \leq p-1}(f|_{k\tau_v}(u) - f|_{k\tau_v}(v)).$$
Thus, in order to prove that $\delta|_{m\tau} = \delta$, we have to show that the
last factor $\Pi_{1 \leq u < v \leq p-1}(f|_{k\tau_v} - f|_{k\tau_u})$ of $\delta$ is alternating under
the permutation $v$; namely, it is sufficient to prove that the
permutation $\psi$ on $(\mathbb{Z}/p\mathbb{Z})^\times$ defined by $\psi(a) = -1/a$ for $a$ ($\mathbb{Z}/p\mathbb{Z})^\times$
is an odd permutation. Since $\psi^2 = \text{id.}$, we see that
$$\psi = \Pi(a,b),$$
where the product is taken over all the transpositions $(a,b)$
between $a$ and $b$ with $ab = -1$ and $a \neq b$. The number $\ell$ of the
elements in $(\mathbb{Z}/p\mathbb{Z})^\times$ with $a^2 = -1$ is 2 or 0 according as $p \equiv 1 \pmod{4}$
or not. Thus $(p-1-\ell)/2$ is odd. Since the number of the transpositions
in $\psi$ is $(p-1-\ell)/2$, $\psi$ is an odd permutation. This concludes the
proof of Lemma 5.

We claim that
\begin{equation}
\delta = cg, \text{ where } c \text{ is a constant with } c^2 = (-1)^{(p-1)/2}p
\end{equation}
and $g$ is a $\mathbb{Z}$-rational modular form on $\text{SL}_2(\mathbb{Z})$.

In fact, both $f$ and $f|_{k\tau}$ are $\mathbb{Z}$-rational by Lemma 3. Thus,
using the Fourier expansion (1.4) of $f|_{k\tau_u}$, we can find polynomials
$\beta_n(x,y)$ in $\mathbb{Z}[x,y]$ so that
(2.2) \( f|_{k \tau_u} - f|_{k \tau_v} = (\zeta^u - \zeta^v) \sum_{n=1}^{\infty} \beta_n (\zeta^u, \zeta^v) e(nz/p) \)

for all \( \tau_u \) and \( \tau_v \). Put \( c = \prod_{0 \leq u < v < p-1} (\zeta^u - \zeta^v) \). Then we see that

\[
c^2 = (-1)^{p(p-1)/2} \prod_{0 \leq u < v < p-1} (\zeta^u - \zeta^v)(\zeta^v - \zeta^u)
= (-1)^{(p-1)/2} p^{-1} \sum_{u \neq v} (\zeta^u - \zeta^v)
= (-1)^{(p-1)/2} p^{-1} \sum_{u=0}^{p-1} (\zeta^u - \zeta^v)
= (-1)^{(p-1)/2} p^{-1}.
\]

Further put \( g = \prod_{u=0}^{p-1} (f - f|_{k \tau_u}) \cdot \prod_{0 \leq u < v < p-1} (\sum_{n=1}^{\infty} \beta_n (\zeta^u, \zeta^v) e(nz/p)) \).

Then \( \prod_{u=0}^{p-1} (f - f|_{k \tau_u}) \) is \( \mathbb{Z} \)-rational as shown below. For any field-automorphism \( \rho \) of \( \mathbb{C} \), we defined the action of \( \rho \) on a Fourier series \( \varphi = \sum_{n=0}^{\infty} c(n) e(nz/p) \) by \( \varphi^\rho = \sum_{n=0}^{\infty} c(n)^\rho e(nz/p) \). Then the Fourier expansions (1.1a) and (1.4) of \( f \) and \( f|_{k \tau_u} \) show that any automorphism \( \rho \) induces a bijection of the set \( \{ f - f|_{k \tau_u} \mid u = 0, 1, 2, \ldots, p-1 \} \). Thus \( \prod_{u=0}^{p-1} (f - f|_{k \tau_u}) \) is \( \mathbb{Q} \)-rational. Moreover, all Fourier coefficients of \( f - f|_{k \tau_u} \) belong to \( \mathbb{Z}[\zeta] \). Thus \( \prod_{u=0}^{p-1} (f - f|_{k \tau_u}) \) is \( \mathbb{Z} \)-rational. A similar argument combined with the symmetricity in \( x \) and \( y \) of the polynomial \( \beta_n(x,y) \) shows that \( \prod_{0 \leq u < v < p-1} (\sum_{n=1}^{\infty} \beta_n (\zeta^u, \zeta^v) e(nz/p)) \) is also \( \mathbb{Z} \)-rational. Therefore \( g \) is \( \mathbb{Z} \)-rational. Note that \( \delta = cg \). Then Lemma 5 shows that \( g \) is a \( \mathbb{Z} \)-rational modular form on \( \text{SL}_2(\mathbb{Z}) \). Thus (2.1) is established.

Now we prove the divisibility of \( g \) by the power of \( \Delta \) as indicated in the theorem. Let us put \( h = g/\Delta^{(p-1)/2} \). Since the Fourier expansion of \( g \) starts from \( e(\Delta^{p-1}/2 \cdot z) \), \( h \) is still a modular
form. In fact, the Fourier expansion (2.2) of $f|_{k\tau_u} - f|_{k\tau_v}$ starts from $e(z/p)$, and therefore, by counting the number of such factors in the product for $g$, we know that the Fourier expansion of $g$ starts from $e(P^{-1}z)$. Further the cusp form $\Delta$ is nowhere vanishing on $H$ and its Fourier expansion starts from $e(z)$. Thus $h$ is holomorphic on $H$ and even at $i\infty$. Since the first coefficient of the Fourier expansion of $\Delta$ is equal to 1, $h$ is again $\mathbb{Z}$-rational. Next we assume that $f$ is a cusp form. Then, since the Fourier expansion of the first factor $\prod_{u=0}^{P-1}(f - f|_{k\tau_u})$ of $g$ starts from $e(z)$, that of $g$ starts from $e(P^{-1}z)$ or higher term. A similar argument as above is still valid. This completes the proof of Theorem 1.

We remark the following direct consequence of Theorem 1, which may be well known:

**Corollary 6.** Let us consider the specialized equation of $f$ at an elliptic curve $E$ defined over $\mathbb{Q}$. For the definition of the specialized equation and the details, see [3, §3]. Under the same notation as in [3, §3], if the specialized equation $\Phi(X;f,E) = 0$ is irreducible, then the prime $p$ always ramifies in the splitting field of the equation.

Remark. The above proof shows that without assuming the $\mathbb{Z}$-rationality of transformation equation, we may express $D$ as

$$D = \Delta^{P-1}h^2$$

with a modular form $h$ on $SL_2(\mathbb{Z})$. 
§3. The transformation equation of $gE^*_\lambda, p$.

In this section, let us consider in more detail the transformation equation of $gE^*_\lambda, p$ as in [3]. Here $g$ is a cusp form in $S_\lambda(\Gamma_0(p))$ and $E^*_\lambda, p$ is the Eisenstein series defined by

$$E^*_\lambda, p(z) = \sum_{\gamma \in \Gamma_0(p)} (cz + d)^{-\lambda}, \quad (\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}),$$

where $\lambda$ is an even integer $> 2$ and $\Gamma_\infty = \{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z} \}$. The Eisenstein series $E^*_\lambda, p$ is a modular form on $\Gamma_0(p)$ of weight $\lambda$, and is expressed as follows (see [9, p.794]):

$$(3.1) \quad E^*_\lambda, p(z) = \frac{2\lambda}{(p^\lambda - 1)B_\lambda} \left( G_\lambda(z) - p^{\lambda} G_\lambda(pz) \right),$$

where $B_\lambda$ is the $\lambda$-th Bernoulli number and

$$(3.2) \quad G_\lambda(z) = -B_\lambda/2\lambda + \sum_{n=1}^{\infty} \sum_{d \mid n} d^{\lambda-1} e(nz).$$

Let us take a cusp form $\varphi(z) = \sum_{n=1}^{\infty} a(n)e(nz)$ in $S_\lambda(\Gamma_0(p))$ with the following three properties:

$$(3.3)\begin{align*}
(3.3a) \quad & \varphi \mid_{\lambda \Gamma_0(p) 0 \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}} = \gamma \varphi \quad \text{for } \gamma = \pm 1; \\
(3.3b) \quad & a(1) = 1; \\
(3.3c) \quad & \varphi \text{ is } \mathbb{Z}\text{-rational.}
\end{align*}$$

For example, any $\mathbb{Q}$-rational primitive form in $S_\lambda(\Gamma_0(p))$ satisfies these conditions. Moreover, we can construct another example of such cusp forms. We will give this example at the end of this section.

Let us put

$$(3.4) \quad g = \frac{p^{\lambda/2}}{d^{\lambda}} N_\lambda \varphi,$$
where we write

\[(3.5) \quad (p^\lambda - 1) B_{\lambda}/2\lambda = N_{\lambda}/D_{\lambda}\]

with mutually prime integers $N_{\lambda}$ and $D_{\lambda}$, and $d$ is the greatest common divisor of $p^{\lambda/2}$ and $D_{\lambda}$. Then we have

Proposition 7. The transformation equation of $gE_\lambda^*, p$ is Z-rational.

Proof. It is sufficient to show that $gE_\lambda^*, p$ and $gE_\lambda^*, p | \lambda + \lambda \tau$
are Z-rational (see Lemma 3). Here $\tau = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We see from (3.1), (3.2) and (3.5) that

\[(3.6) \quad E_{\lambda, p}(z) = \frac{D_{\lambda}}{N_{\lambda}} \left\{ \frac{N_{\lambda}}{D_{\lambda}} + \sum_{n=1}^{\infty} b(n)e(nz) \right\}\]

with $b(n) \in \mathbb{Z}$. Thus $N_{\lambda}E_{\lambda, p}$ is Z-rational, and therefore, $gE_{\lambda, p}$
is Z-rational. Since $G_{\lambda}(z)$ is a modular form in $M_\lambda(SL_2(\mathbb{Z}))$, we have that

\[(3.7) \quad G_{\lambda}(pz) \bigg| \lambda \tau = p^{-\lambda} G_{\lambda}(z/p),\]

and hence, (3.1) shows that

\[(3.8) \quad (N_{\lambda}E_{\lambda, p} \bigg| \lambda \tau)(z) = D_{\lambda} \{ G_{\lambda}(z) - G_{\lambda}(z/p) \}.\]

On the other hand, it follows from (3.3a) that

\[(3.9) \quad (\varphi \bigg| \lambda \tau)(z) = \gamma p^{-\lambda/2} \varphi(z/p).\]

Thus $gE_{\lambda, p} | \lambda + \lambda \tau$ is again Z-rational. In fact, both the modular forms $\varphi(z/p)$ and $G_{\lambda}(z) - G_{\lambda}(z/p)$ are Z-rational; therefore, the modular form:

\[(3.10) \quad (gE_{\lambda, p} \bigg| \lambda \tau)(z) = \gamma \frac{D_{\lambda}}{d} \varphi(z/p) \{ G_{\lambda}(z) - G_{\lambda}(z/p) \}\]
is \(\mathbb{Z}\)-rational, where \(k = \lambda + \mu\). This is what we wanted to show.

Proposition 8. Let us write the transformation equation of \(g_{E, \lambda, p}\) as

\[
\phi(x; g_{E, \lambda, p}) = x^{p+1} + \sum_{\mu=1}^{p+1} (-1)^\mu \sigma_{\mu} x^{p+1-\mu}.
\]

Then the modular form \(\sigma_p\) has a Fourier expansion of the form:

\[
\sigma_p(z) = -\gamma \left(\frac{D_{\lambda}}{d}\right) P_e(2z) + \sum_{n=3}^{\infty} c(n) e(nz)
\]

with rational integers \(c(n)\).

Proof. Since \(\sigma_p\) is the \(p\)-th elementary symmetric function in \(\{g_{E, \lambda}\}_{\lambda \in \mathbb{R}}\), we have by (1.3) that

\[
\sigma_p = g_{E, \lambda, p} \left\{ \sum_{\mu=0}^{p-1} (g_{E, \lambda, p} | k^{\tau_{v}}) \right\} + \prod_{u=0}^{p-1} (g_{E, \lambda, p} | k^{\tau_{u}}).
\]

Here \(\tau_{u} = (0, -1)\). Since \(\tau_{u} = \tau(1, 0)\), it follows from (3.10) that

\[
(3.12) \quad (g_{E, \lambda, p} | k^{\tau_{u}})(z) = \gamma \left(\frac{D_{\lambda}}{d}\right) \phi \left(\frac{(z+u)/p}{\left(G_{\lambda}(z) - G_{\lambda}(z+u/p)\right)}\right).
\]

Thus we can find polynomials \(b_n(x)\) in \(\mathbb{Z}[x]\) so that (3.12) is rewritten as

\[
(3.13) \quad (g_{E, \lambda, p} | k^{\tau_{u}})(z) = -\gamma \left(\frac{D_{\lambda}}{d}\right) \sum_{n=2}^{\infty} b_n(\zeta^u) e(nz/p)
\]

for all \(\tau_{u}\). Here \(\zeta = e(1/p)\). Especially we have the second polynomial \(b_2(x) = x^2\). Therefore we have that

\[
(3.14) \quad \prod_{u=0}^{p-1} (g_{E, \lambda, p} | k^{\tau_{u}})(z) = -\gamma \left(\frac{D_{\lambda}}{d}\right) \sum_{n=2}^{\infty} w(n) e(nz/p)
\]

with rational integers \(w(n)\). Note \(w(2p) = 1\). On the other hand, since the Fourier expansion of \(g_{E, \lambda, p}\) starts from \(e(z)\), (3.13) shows that

\[
(3.15) \quad g_{E, \lambda, p} \left\{ \prod_{\mu=0}^{p-1} \prod_{v=0}^{p-1} (g_{E, \lambda, p} | k^{\tau_{v}}) \right\} = \sum_{n=3}^{\infty} 2w'(n) e(nz/p)
\]
with rational integers \( w^*(n) \). Thus, considering \( w^*(2p) = 1 \) and
\[ 3p - 2 > 2p, \]
we see that the Fourier expansion of \( \sigma_p \) starts from
\[ e(2z) \]
with the coefficient \( -\gamma \left( \frac{D}{d} \right)^p \). Moreover, it follows from
Proposition 7 that \( \sigma_p \) is \( \mathbb{Z} \)-rational modular form. This concludes
the proof of Proposition 8.

Remark. Our modification of \( \varphi \) as in (3.4) is best possible.

Analyzing carefully the above proof of Proposition 7, one sees
that if both \( c \varphi_{\lambda, p}^* \) and \( c \varphi_{\lambda, p}^* |_{k_T} \) are \( \mathbb{Z} \)-rational for a constant \( c \),
then \( c \) is a rational integer and a multiple of \( p^{2N_A/d} \).

Proposition 9. The discriminant \( D \) of \( \psi(X; g_{\lambda, p}^*) \) is expressed
as
\[ D = (-1)^{(p-1)/2} p \Delta^2 (p+1) h^2, \]
where \( h \) is a \( \mathbb{Z} \)-rational modular form on \( \text{SL}_2(\mathbb{Z}) \).

Proof. The Fourier expansion (3.13) of the modular form
\[ g_{\lambda, p}^* |_{k_T} \]
shows that both the Fourier expansions of the modular forms \( g - g_{\lambda, p}^* |_{k_T} \) and \( g_{\lambda, p}^* |_{k_T} - g_{\lambda, p}^* |_{k_T} \)
start from \( e(2z/p) \) for any \( T \) and \( T' \). Thus that of the modular form
\[ \prod_{u=0}^{p-1} (g_{\lambda, p}^* |_{k_T} - g_{\lambda, p}^* |_{k_T}) \]
starts from \( e((p+1)z) \). Then a similar argument as in the proof of
Theorem 1 shows our assertion.

Now let us give examples of the cusp forms \( \varphi \) satisfying
the conditions (3.3 \( a, b, c \) ) when there exists a primitive form
of conductor \( p \). Let us take a primitive form \( f \) in \( S_2(\Gamma_0(p)) \)
of conductor \( p \) and write the Fourier expansion of \( f \) as
\[ f(z) = \sum_{n=1}^{\infty} b(n) e(nz). \]
(For the primitiveness of cusp forms, see, for example, [9, p789].) We denote by \( M \) the module generated over \( \mathbb{Z} \) by all \( b(n) \) in \( \mathbb{C} \) and by \( K \) the field generated by \( M \). For any isomorphism \( \sigma \) of \( K \) into \( \mathbb{C} \), we define the conjugate \( f^\sigma \) of \( f \) by

\[
f^\sigma(z) = \sum_{n=1}^{\infty} b(n)^\sigma e(nz).
\]

As is well known, \( f^\sigma \) is again a primitive form in \( S_\kappa(\Gamma_0(p)) \).

We define a cusp form \( \text{Tr}(af) \) in \( S_\kappa(\Gamma_0(p)) \) for any \( a \) in \( K \) by

\[
\text{Tr}(af) = \sum \sigma f^\sigma,
\]

where \( \sigma \) runs over all isomorphisms of \( K \) into \( \mathbb{C} \). Since \( f \) is primitive, it follows from [1, Lemma 3] that

\[
\text{(3.16a)} \quad f|_\kappa(\begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix}) = \gamma f, \quad \gamma = \pm 1.
\]

Moreover we have that for any conjugate \( f^\sigma \),

\[
\text{(3.16b)} \quad f^\sigma|_\kappa(\begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix}) = \gamma f^\sigma
\]

for the above \( \gamma \), because \( \gamma \) is expressed as

\[
\text{(3.17)} \quad \gamma = -b(p)p^{1-\kappa/2}.
\]

Thus we see that for any \( a \) in \( K \),

\[
\text{(3.18)} \quad \text{Tr}(af)|_\kappa(\begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix}) = \gamma \text{Tr}(af).
\]

We see easily that

\[
\text{(3.19)} \quad \text{Tr}(af) \text{ is } \mathbb{Z}\text{-rational if and only if } a \text{ belongs to } D = \{ \beta \in K \mid \text{Tr}_{K/Q}(\beta x) \in \mathbb{Z} \text{ for all } x \in \mathcal{M} \}.
\]

Proposition 10. Let us put

\[
U = \{ a \in D \mid \text{Tr}(af) \text{ has a Fourier expansion of the form: } e(z) + \sum_{n=2}^{\infty} c(n)e(nz) \text{ with rational integers } c(n) \},
\]

and
\[ V = \{ \alpha \in D \mid \text{Tr}_{K/Q}(\alpha) = 0 \}. \]

Then we have

1. \( U \) is not empty;
2. \( V \) is isomorphic to \( \mathbb{Z}^{d-1} \) for \( d = [K:Q] \);
3. \( U = \alpha_0 + V \) for any element \( \alpha_0 \) of \( U \).

Proof. Let \( \alpha \) be an element of \( D \). Since \( b(1) = 1 \) and
\[ \text{Tr}(\alpha f) = \sum_{n=1}^{\infty} \text{Tr}_{K/Q}(\alpha b(n)) e(nz), \]
we see that
\[ (3.20) \quad \alpha \text{ belongs to } U \text{ if and only if } \text{Tr}_{K/Q}^a = 1. \]

Since \( M \) generates \( K \) and since \( M \) is a \( \mathbb{Z} \)-free module, \( M \) is isomorphic to \( \mathbb{Z}^d \). Let \( \{ \omega_i \}_{i=1}^d \) be a \( \mathbb{Z} \)-basis of \( M \) and \( \{ \eta_i \}_{i=1}^d \) be the dual basis of \( \{ \omega_i \} \) with respect to \( \text{Tr}_{K/Q} \); hence we have
\[ (3.21) \quad \text{Tr}_{K/Q}(\omega_i \eta_j) = \delta_{ij}. \]

Then we know that
\[ (3.22) \quad D = \sum_{i=1}^{d} \mathbb{Z} \eta_i \quad \text{(direct sum)}. \]

Since \( 1 (=b(1)) \) belongs to \( M \), we may write as
\[ 1 = \sum_{i=1}^{d} m_i \omega_i \]
for some rational integers \( m_i \). Then (3.21) shows that
\[ m_i = \text{Tr}_{K/Q}^a \eta_i \]
for any \( i \). Namely, we have that
\[ (3.23) \quad 1 = \sum_{i=1}^{d} (\text{Tr}_{K/Q}^a \eta_i) \omega_i. \]

Let \( c \) be the greatest common divisor of \( \{ \text{Tr}_{K/Q}^a \eta_i \}_{i=1}^d \). Since \( f \) is primitive, all Fourier coefficients \( b(n) \) of \( f \), and therefore, all \( \omega_i \) are algebraic integers. This combined with (3.23) shows that \( c \) is equal to 1. Thus considering (3.22), we know that
\[ (3.24) \quad \text{Tr}_{K/Q}^D = \mathbb{Z}. \]
Especially there exists an element $\alpha_0$ of $D$ such that $\text{Tr}(\alpha_0f)$ belongs to $U$. Since $V \otimes_2 Q$ is isomorphic to $Q^{d-1}$, we see the assertion (2). The third assertion is clear from (3.20).

Now let us put $\mathcal{V} = \text{Tr}(af)$ for any element $\alpha$ of $U$. Then (3.18), (3.19) and (3.20) show that $\mathcal{V}$ satisfies the conditions (3.3a,b,c).

In this section, we are going to give several numerical examples of the transformation equations \( \phi(x; g_{E^*}^{*,\lambda,p}) = 0 \) and the specialized equations \( \phi(x; g_{E^*}^{*,\lambda,p'},E) = 0 \) at various elliptic curves \( E \) defined over \( \mathbb{Q} \). See [3, §3] for the definition of the specialized equation at an elliptic curve. For simplicity, we consider only the case \( \dim S_2(\Gamma_0(p)) = 1 \). Thus we may take as \( \varphi \) in (3.4) with (3.3 \( a, b, c \) ) the unique primitive form in \( S_2(\Gamma_0(p)) \). Let us modify \( \varphi \) as in (3.4) and write the modified modular form as \( g \).

Let us explain how to read the table given below by taking the following case I as an example. This case is the restatement of the example given in our previous paper [3, §5]. We will add several new examples here. We use the same notation in §3 and write simply \( G, H, \) and \( D \) for \( 12g_2, 216g_3, \) and \( \Delta \), respectively. Here \( g_2 \) (resp., \( g_3 \)) is the Eisenstein series in \( M_4(SL_2(\mathbb{Z})) \) (resp., \( M_6(SL_2(\mathbb{Z})) \)) whose constant term of the Fourier expansion is equal to \( 1/12 \) (resp., \( 1/216 \)). Thus they are \( \mathbb{Q} \)-rational.

Case I. \( p = 5, \ i = 4, \ \lambda = 4, \ k = 8. \)
\[
g = -5 \cdot 13 \varphi.
\]
\[
\begin{array}{c}
x^6 & 1 \\
x^5 & 0 \\
x^4 & -25GD \\
x^3 & -1440D^2 \\
x^2 & 155G^2D^2 \\
x & GH^2D^2 + 18096GD^3 \\
1 & 65H^2D^3 + 538240D^4
\end{array}
\]
The above table can be read that the transformation equation 
\( \phi(X; gE_{4,5}) \) is given by the polynomial

\[
(4.1) \quad x^6 - 25GDX^4 - 1440D^2x^3 + 155G^2Dx^2 + (GH^2D^2 + 18096GDG^3)x \\
+ (65H^2D^3 + 538240D^4).
\]

Thus, for example, the monomial \(-25GD\) given at the right-hand side of \(X^4\) is the isobaric polynomial of the coefficient of \(X^4\).

\[
\begin{align*}
\text{Tr}(X) &= 0 \\
\text{Tr}(X^2) &= 50GD = 2^{-30} \cdot 3 \cdot 141 \cdot \frac{220}{36 \cdot 7^2 \cdot 11 \cdot 13} f_{16} \\
\text{Tr}(X^3) &= 4320D^2 = 2^{-46} \cdot 3 \cdot 221 \Sigma (\frac{229}{3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 114169}) f_{24} \\
\text{Tr}(X^4) &= 630G^2D^2 = 2^{-62} \cdot 3 \cdot 301 \\
&\quad \cdot \Sigma (\frac{234}{3 \cdot 1 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 40}) f_{32} \\
\text{Tr}(X^5) &= -5GH^2 + 89520GD^3 \\
&= 2^{-78} \cdot 3 \cdot 381 \Sigma (-\frac{243}{3 \cdot 1 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 40}) f_{40} \sigma \\
N(\alpha-353792) &= 2^{18} \cdot 3 \cdot 7 \cdot 11 \cdot 13 \cdot 31 \cdot 17 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \psi^{(1)}(\alpha) \\
\text{Tr}(X^6) &= 7610H^2D^3 + 16815360D^4 \\
&= 2^{-94} \cdot 3 \cdot 461 \Sigma (\frac{256}{3 \cdot 1 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \psi^{(1)}(\alpha) f_{48} \psi^{(2)}(\alpha)) \\
N(5117\alpha + 17457217536) &= -2^{37} \cdot 3^8 \cdot 11 \cdot 383 \cdot 3129512851870124265857
Here $\text{Tr}(X^\mu)$ indicates the $\mu$-th power sum $\text{Tr}(gE_{4,5}^\mu)$ of all the roots of the transformation equation given in (4.1), and the corresponding isobaric polynomial is given at the right-hand side of $\text{Tr}(X^\mu)$. As we have seen in [3], the power sum $\text{Tr}(gE_{4,5}^\mu)$ can be expressed as

$$\text{Tr}(gE_{4,5}^\mu) = 2^{-2(\mu-1)} \cdot 3 \cdot (8\mu-2)! \sum_f \mathcal{P}(\pi, \mu) \frac{D(8\mu-1, f, gE_{4,5}^\mu, \mu-1)}{\pi^{8\mu-1} \langle f, f \rangle}.$$ 

(See [3, Theorem] for the notation.) After the isobaric polynomial in the table, we have given this expression of power sum. (This expression is not given in [3, §5].) Thus, for example, in the expression corresponding to $\text{Tr}(X^5)$, the value $2^{43} (\alpha - 3537792)/\{3^{18} \cdot 7 \cdot 5 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \psi'(\alpha)\}$ gives the special value $D(39, f_{40}, g_{40}^\mu, 4)$. Here $f_{40}$ indicates a primitive form in $S_{40}(\text{SL}_2(\mathbb{Z}))$; $\alpha$ is a generator of the field $K(f_{40})$ generated over $\mathbb{Q}$ by all Fourier coefficients of $f_{40}$; $\psi$ indicates the characteristic polynomial of $\alpha$ and $\psi'(x) = d\psi/dx$. Further, in the above expression of $\text{Tr}(X^5)$, the summation is over all isomorphisms $\sigma$ of $K(f_{40})$ into $\mathbb{C}$. Note that in the limit of the calculation we have done, all the primitive forms in $S_\infty(\text{SL}_2(\mathbb{Z}))$ are conjugate under the automorphisms of $\mathbb{C}$. We denote by $N(\gamma)$ the norm of an algebraic number $\gamma$, for example, $N(\alpha - 3537792)$ indicates the norm of the number $\alpha - 3537792$. If the factors in the listed numbers are less than $10^{10}$, then they are primes; otherwise, we do not know whether they are prime or not.

Let us now list the characteristic polynomials $\psi(x)$ and their discriminants $D(\psi)$ of a generator $\alpha$ of the fields $K(f_m)$:
<table>
<thead>
<tr>
<th>m</th>
<th>$\psi(x)$ and $D(\psi)$</th>
</tr>
</thead>
</table>
| 40 | $\psi(x) = x^3 - 548856x^2 - 810051757056x + 213542160549543936$  
$D(\psi) = 2^{26} \cdot 3^{12} \cdot 5^2 \cdot 7^2 \cdot 13^2 \cdot 73 \cdot 59077 \cdot 92419245301$ |
| 48 | $\psi(x) = x^4 - 5785560x^3 - 467142374034432x^2 + 1426830562183253852160x$  
$+ 3297913828840214320807673856$  
$D(\psi) = 2^{70} \cdot 3^{22} \cdot 5^6 \cdot 7^6 \cdot 31 \cdot 383^2 \cdot 10210753616344141199245524873423941499439$ |
| 50 | $\psi(x) = x^3 + 24225168x^2 - 566746931810304x - 13634883228742736412672$  
$D(\psi) = 2^{32} \cdot 3^{12} \cdot 5^4 \cdot 7^4 \cdot 12284628694131742619401$ |
| 60 | $\psi(x) = x^5 + 449691864x^4 - 2209450184054433792x^3$  
$- 736010060393513697870348288x^2$  
$+ 810634763334812972416233648439689216x$  
$+ 263222216157060824115203098902237248565018624$  
$D(\psi) = 2^{148} \cdot 3^{38} \cdot 5^8 \cdot 7^8 \cdot 17^4 \cdot 23 \cdot 1019$  
$\cdot 6519163204712103878902727074480503094855670432357132070088$  
$298897328058850220694574301717487795597^*$ |

* This number $65191\ldots 5597$ is a number of 97-figures.

Case II. $p = 5$, $\ell = 4$, $\lambda = 6$, $k = 10$.  
$g = 5^2 \cdot 31^\varphi$

| $x^6$ | 1 |
| $x^5$ | 0 |
| $x^4$ | $-145G^2D$ |
| $x^3$ | $587520HD^2$ |
| $x^2$ | $3635GH^2D^2 - 377403840GD^3$ |
| $x$ | $G^2H^3D^2 + 6290064G^2HD^3$ |
| 1 | $-775H^4D^3 - 7058849600H^2D^4$ |
\[ \text{Tr}(X) = 0 \]

\[ \text{Tr}(X^2) = 290G^2_D = 2^{-38} \cdot 3 \cdot 18! \cdot \frac{2^{23} \cdot 29}{3 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17} \cdot f_{20} \]

\[ \text{Tr}(X^3) = -1762560H_D^2 = 2^{-58} \cdot 3 \cdot 28! \cdot \left( \frac{2^{44} (131\alpha + 1196402688)}{3 \cdot 17 \cdot 5 \cdot 7 \cdot 11^3 \cdot 13 \cdot 17^2 \cdot 19^2 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 47 \cdot \psi'(\alpha)} \right) \cdot f_{30} \cdot \sigma \]

\[ N(131\alpha + 1196402688) = -2^{27} \cdot 3 \cdot 7^2 \cdot 3833 \cdot 32619042931 \]

\[ \text{Tr}(X^4) = 27510G_H^2D^2 + 1582277760G_D^3 \]

\[ = 2^{-78} \cdot 3 \cdot 38! \cdot \left( \frac{2^{44} (131\alpha + 1196402688)}{3 \cdot 17 \cdot 5 \cdot 7 \cdot 11^3 \cdot 13 \cdot 17^2 \cdot 19^2 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 47 \cdot \psi'(\alpha)} \right) \cdot f_{40} \cdot \sigma \]

\[ N(131\alpha + 1196402688) = -2^{27} \cdot 3 \cdot 7^2 \cdot 3833 \cdot 32619042931 \]

\[ \text{Tr}(X^5) = -5G^2H^3D^2 - 457402320G^2HD^3 \]

\[ = 2^{-98} \cdot 3 \cdot 48! \cdot \left( \frac{-2^{52} (\alpha - 8757800448)}{3 \cdot 23 \cdot 5 \cdot 9 \cdot 7^6 \cdot 11^4 \cdot 13 \cdot 17^2 \cdot 19^2 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 47 \cdot \psi'(\alpha)} \right) \cdot f_{50} \cdot \sigma \]

\[ N(\alpha - 8757800448) = 2^{21} \cdot 3 \cdot 7^2 \cdot 19 \cdot 73 \cdot 4235321855794559 \]

\[ \text{Tr}(X^6) = 2939450H^4D^3 + 1421841072000H^2D^4 + 585580127846400D^5 \]

\[ = 2^{-118} \cdot 3 \cdot 58! \cdot \left( \frac{-2^{58} (168536131\alpha^2 + 4799563646147788\alpha + 9993503564022187290525696)}{3 \cdot 27 \cdot 5 \cdot 11 \cdot 7 \cdot 11^5 \cdot 13 \cdot 17^3 \cdot 19^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 47 \cdot 53 \cdot \psi'(\alpha)} \right) \cdot f_{60} \cdot \sigma \]

\[ N(168536131\alpha^2 + 4799563646147788\alpha + 9993503564022187290525696) = -2^{105} \cdot 3 \cdot 23 \cdot 5^3 \cdot 11^2 \cdot 13 \cdot 17^4 \cdot 1165842531588761730920594563390304066538263826553725077932 \cdot 25580762957007062391 \]
Case III. \( p = 5, \ l = 4, \ \lambda = 8, \ k = 12 \).
\[
g = -5 \cdot 13 \cdot 313^q
\]

\[
\begin{align*}
X^6 & \quad 1 \\
X^5 & \quad 60480D \\
X^4 & \quad -625H^2D + 1301832000D^2 \\
X^3 & \quad 117115760H^2D^2 + 11768083937280D^3 \\
X^2 & \quad 69755H^4D^2 + 1728323786880H^2D^3 + 39309437117214720D^4 \\
X & \quad H^6D^2 + 17889611952H^4D^3 - 18249030627747840H^2D^4 \\
 & \quad + 15417626668505432064D^5 \\
l & \quad 20345H^6D^3 + 335091233981440H^4D^4 + 6660452326511923200H^2D^5 \\
 & \quad + 32175921734973802414080D^6
\end{align*}
\]

\[
\begin{align*}
\text{Tr}(X) & \quad -60480D \\
\text{Tr}(X^2) & \quad 1250H^2D + 1054166400D^2 \\
\text{Tr}(X^3) & \quad -464741280H^2D^2 - 20325436323840D^3 \\
\text{Tr}(X^4) & \quad 502230H^4D^2 + 27308861671680H^2D^3 + 411430804078878720D^4 \\
\text{Tr}(X^5) & \quad -5H^6D^2 - 552459647760H^4D^3 - 1001721601502668800H^2D^4 \\
 & \quad -8528203665906974392320D^5 \\
\text{Tr}(X^6) & \quad 226940810H^6D^3 + 101092055900113920H^4D^4 \\
 & \quad + 30124477620177181286400H^2D^5 + 1786641266617848672068567040D^6
\end{align*}
\]
Case IV. $P = 5$, $i = 4$, $\lambda = 10$, $k = 14$.

$g = 5^2 \cdot 71 \cdot 521 \phi$

\[
\begin{align*}
X^6 & = 1 \\
X^5 & = 0 \\
X^4 & = -2545GH^2D - 604109741760GD^2 \\
X^3 & = 25344112320H^3D^2 - 211931520573911040HD^3 \\
X^2 & = 1207235G^2H^4D^2 - 27110066987928960G^2H^2D^3 \\
& \quad - 18395423999571176837120G^2D^4 \\
X & = GH^7D^2 + 2243273283920GH^5D^3 + 5557901355458375149568GH^3D^4 \\
& \quad - 306714023877649287994543424GD^5 \\
1 & = -924775H^8D^3 - 21779093073266168000H^6D^4 \\
& \quad - 13775337937065083878647200H^4D^5 \\
& \quad - 1335397897742946615034459270400H^2D^6
\end{align*}
\]

\[
\begin{align*}
\text{Tr}(X) & = 0 \\
\text{Tr}(X^2) & = 5090G^2H^2D + 1208219483520GD^2 \\
\text{Tr}(X^3) & = -76032336960H^3D^2 + 635794561721733120HD^3 \\
\text{Tr}(X^4) & = 8125110G^2H^4D^2 + 1145901051228532640G^2H^2D^3 \\
& \quad + 803470856176952483143680G^2D^4 \\
\text{Tr}(X^5) & = -5G^7H^2D^2 - 434675195691600G^5H^3D^3 \\
& \quad - 101645803821847030179840G^3D^4 \\
& \quad + 641683050942935873278036869120GD^5 \\
\text{Tr}(X^6) & = 14539127450H^8D^3 + 2490718883181800323200H^6D^4 \\
& \quad + 72718216860527846662995763200H^4D^5 \\
& \quad + 642540701843479691260435943482982400H^2D^6 \\
& \quad + 877146390169927704752272689056106137600D^7
\end{align*}
\]
Case V. \( p = 5, \, \ell = 4, \, \lambda = 12, \, k = 16 \).

\[
g = - 5.31.601.691\mathfrak{g}
\]

\begin{align*}
x^6 & 1 \\
x^5 & 81829440G'^D \\
x^4 & -10225G^2H^2D + 1216590866568000G^2D^2 \\
x^3 & 2910441252960H^4D^2 + 612747015833407661760H^2D^3 \\
 & - 32282327635049729294991360D^4 \\
x^2 & 20065555G^6H^2 + 204260280738724356320GH^4D^3 \\
 & + 10256756271487171426170408960G^2D^4 \\
 & + 92754860107460880754044689448960GD^5 \\
x^1 & G^2H^8D^2 + 30194230743474480G^2H^6D^3 \\
 & - 161607524021417153841347584G^2H^4D^4 \\
 & + 4211742212334091386209897324806144G^2H^2D^5 \\
 & - 13223615489524979651841803555733751808G^2D^6 \\
é & 64370105H^{10}D^3 + 1860464795874207408499840H^8D^4 \\
 & + 1455861561962972042529821882490880H^6D^5 \\
 & + 272324675659205212055849466146099036160H^4D^6 \\
 & + 108890178080782704778692703693672305131520H^2D^7 \\
 & + 871861028965226012849339863959645515656724480D^8
\end{align*}
\[
\text{Tr}(X) = -81829440GD
\]
\[
\text{Tr}(X^2) = 20450G^2H^2D + 426287517577600G^2D^2
\]
\[
\text{Tr}(X^3) = -11241441830880H^4D^2 - 267658181885197261025280H^2D^3
\]
\[
- 35390154259649542464929720D^4
\]
\[
\text{Tr}(X^4) = 128839830G^6H^6D^2 + 359708164048445224320G^4H^4D^3
\]
\[
+ 17176524173998325938032762101760G^2H^2D^4
\]
\[
+ 1534860088734279858716479124930560GD^5
\]
\[
\text{Tr}(X^5) = -5G^2H^8D^2 - 33433437445696400G^2H^6D^3
\]
\[
- 6232562017950017936662318080G^2H^4D^4
\]
\[
- 1105217254226789312600343970428511518720G^2H^2D^5
\]
\[
- .704474107875704575852741574977576721448960G^2D^6
\]
\[
\text{Tr}(X^6) = 907155508010H^{10}D^3 + 50642550664820076143642880H^8D^4
\]
\[
+ 84100815151218630048984550520668160H^6D^5
\]
\[
+ 711391952787509952337600204957843023859495973120H^4D^6
\]
\[
+ 152077104881047937683933892892491027290971061616640H^2D^7
\]
\[
+ 55882397614885896504263747102166330237811042088386560D^8
\]

Case VI. \( P = 5, \quad \lambda = 6, \quad \lambda = 4, \quad k = 10. \)

\[ g = -5^2 \cdot 13^3. \]

\[
\begin{align*}
X^6 & = 1 \\
X^5 & = 0 \\
X^4 & = -55G^2D \\
X^3 & = -40140HD^2 \\
X^2 & = 595G^2H^2D^2 - 7266240GD^3 \\
X & = -G^2H^3D^2 + 121104GD^2D^3 \\
l & = -325H^4D^3 - 2691200H^2D^4
\end{align*}
\]
\[ \text{Tr}(X) = 0 \]
\[ \text{Tr}(X^2) = 110G^2D = 2^{-38} \cdot 3 \cdot 18! \cdot \frac{23}{3 \cdot 5^2 \cdot 7^2 \cdot 13 \cdot 17} \cdot f_{20} \]
\[ \text{Tr}(X^3) = 123120HD^2 = 2^{-58} \cdot 3 \cdot 28! \cdot \left( \frac{31}{3 \cdot 11 \cdot 5^2 \cdot 7^4 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 13 \cdot 17 \cdot 23 \cdot 17 \cdot 23} \right) \cdot \frac{31}{51349} \cdot f_{30} \]
\[ \text{Tr}(X^4) = 4470G^2H^2D^2 + 39519360G^2D^3 \]
\[ = 2^{-78} \cdot 3 \cdot 38! \cdot \left( \frac{244}{3 \cdot 5^2 \cdot 7^5 \cdot 11^3 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot 17 \cdot 23 \cdot 19 \cdot 3 \cdot 19 \cdot 3 \cdot 17 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 47} \right) \cdot f_{40} \]
\[ N(149\alpha + 142350336) = -2^{25} \cdot 3 \cdot 5 \cdot 7 \cdot 17 \cdot 23 \cdot 19 \cdot 3 \cdot 19 \cdot 3 \cdot 17 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 47 \cdot 137 \cdot 41 \cdot 43 \cdot 47 \cdot 53 \]
\[ \text{Tr}(X^5) = 5G^2H^3D^2 + 10680480G^2HD^3 \]
\[ = 2^{-98} \cdot 3 \cdot 48! \]
\[ \cdot \left( \frac{252}{3 \cdot 5^2 \cdot 7 \cdot 11^3 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 43 \cdot 47} \right) \cdot f_{50} \]
\[ N(\alpha - 180741120) = 2^{22} \cdot 3 \cdot 7 \cdot 13 \cdot 1223 \cdot 1891930277 \]
\[ \text{Tr}(X^6) = 204350H^4D^3 + 8391590400H^2D^4 + 5137086873600D^5 \]
\[ = 2^{-118} \cdot 3 \cdot 58! \]
\[ \cdot \left( \frac{268}{3 \cdot 27 \cdot 5^1 \cdot 11 \cdot 7^9 \cdot 11^5 \cdot 13^4 \cdot 17^3 \cdot 19^3 \cdot 23 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 43 \cdot 47 \cdot 53} \right) \cdot f_{60} \]
\[ N(6637423\alpha^2 + 2121380494196736\alpha - 67543443341033481437184) = 2^{107} \cdot 3 \cdot 5 \cdot 7 \cdot 17 \cdot 19 \cdot 3 \cdot 19 \cdot 3 \cdot 17 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 47 \cdot 53 \]
\[ = 2^{107} \cdot 3 \cdot 5 \cdot 7 \cdot 17 \cdot 19 \cdot 3 \cdot 19 \cdot 3 \cdot 17 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 47 \cdot 53 \cdot 181 \cdot 233 \]
\[ = 2^{107} \cdot 3 \cdot 5 \cdot 7 \cdot 17 \cdot 19 \cdot 3 \cdot 19 \cdot 3 \cdot 17 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 47 \cdot 53 \cdot 181 \cdot 233 \]
\[ = 2^{107} \cdot 3 \cdot 5 \cdot 7 \cdot 17 \cdot 19 \cdot 3 \cdot 19 \cdot 3 \cdot 17 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 47 \cdot 53 \cdot 181 \cdot 233 \]
\[ = 2^{107} \cdot 3 \cdot 5 \cdot 7 \cdot 17 \cdot 19 \cdot 3 \cdot 19 \cdot 3 \cdot 17 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 47 \cdot 53 \cdot 181 \cdot 233 \]
\[ = 2^{107} \cdot 3 \cdot 5 \cdot 7 \cdot 17 \cdot 19 \cdot 3 \cdot 19 \cdot 3 \cdot 17 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 47 \cdot 53 \cdot 181 \cdot 233 \]
Case VII. \( p = 5, \, \lambda = 6, \, \kappa = 6, \, k = 12. \)
\[
g = 5^3 \cdot 31\varphi
\]
\[
x^6 = 1
\]
\[
x^5 = -10800D
\]
\[
x^4 = -175H^2D + 39994560D^2
\]
\[
x^3 = -954000H^2D^2 - 58506624000D^3
\]
\[
x^2 = 4595H^4D^2 + 5976610560H^2D^3 + 29346922598400D^4
\]
\[
x = -H^6D^2 - 26913792H^4D^3 + 1347784593408H^2D^4
\]
\[
x = 3875H^6D^3 + 35294248000H^4D^4
\]
\[
\text{Tr}(X) = 10800D
\]
\[
\text{Tr}(X^2) = 350H^2D + 56650880D^2
\]
\[
\text{Tr}(X^3) = 8532000H^2D^2 + 139408128000D^3
\]
\[
\text{Tr}(X^4) = 42870H^4D^2 + 70958165760H^2D^3 + 554255811993600D^4
\]
\[
\text{Tr}(X^5) = 5H^6D^2 + 2374938960H^4D^3 + 433666967592960H^2D^4
\]
\[
\quad + 2237768521113600000D^5
\]
\[
\text{Tr}(X^6) = 5935550H^6D^3 + 42310659964800H^4D^4 + 2350955736245043200H^2D^5
\]
\[
\quad + 9081391088816750592000D^6
\]
Case VIII. \( P = 11, \ k = 2, \ \lambda = 4, \ k = 6. \)

\[ g = -11.61 \varphi \]

\[ X_1 \]

\begin{align*}
X_{12} &= 1 \\
X_{11} &= 0 \\
X_{10} &= 11088D \\
X_9 &= -9075HD \\
X_8 &= -5962H^2D + 24952224D^2 \\
X_7 &= -77H^3D - 67215456HD^2 \\
X_6 &= 37678773H^2D^2 + 25829299584D^3 \\
X_5 &= -17237915H^3D^2 - 119108926464HD^3 \\
X_4 &= 2011493H^4D^2 + 104087609758H^2D^3 - 68766745458048D^4 \\
X_3 &= -55913H^5D^2 - 44757025102H^3D^3 + 98746847977536HD^4 \\
X_2 &= 440H^6D^2 + 6582378638H^4D^3 - 52178539740844H^2D^4 \\
&\quad + 2113825578398464D^5 \\
X &= -H^7D^2 - 308633685H^5D^3 + 6499878090033H^3D^4 \\
&\quad - 20914887319687488HD^5 \\
l &= -671H^6D^3 - 207290985242H^4D^4 + 1480882485474007H^2D^5 \\
&\quad - 3777866437306791104D^6
\end{align*}
\[ \begin{align*}
\text{Tr}(X) &= 0 \\
\text{Tr}(X^2) &= -22176D \\
\text{Tr}(X^3) &= 27225HD \\
\text{Tr}(X^4) &= 23848H^2D + 146078592D^2 \\
\text{Tr}(X^5) &= 385H^3D - 167040720HD^2 \\
\text{Tr}(X^6) &= -375645699H^2D^2 - 1221354706176D^3 \\
\text{Tr}(X^7) &= 493425009H^3D^2 + 1841678960352HD^3 \\
\text{Tr}(X^8) &= 131680032H^4D^2 - 4757929285648H^2D^3 + 11020319744130048D^4 \\
\text{Tr}(X^9) &= 4634885H^5D^2 - 9276821678520H^3D^3 - 21750837501488832HD^4 \\
\text{Tr}(X^{10}) &= 25245H^6D^2 + 340701804850H^4D^3 - 45226189141798280H^2D^4 \\
&\quad \quad + 97227250729594799616D^5 \\
\text{Tr}(X^{11}) &= 11H^7D^2 + 4400455461072H^5D^3 + 127109577108789495H^3D^4 \\
&\quad \quad + 239140646004793752384HD^5 \\
\text{Tr}(X^{12}) &= 825050231500H^6D^3 - 19665598331764275H^4D^4 \\
&\quad \quad + 393248139239476286700H^2D^5 + 845180451592627987085568D^6
\end{align*} \]
In what follows, we are going to give the specialized equations
\( \Phi(X; g_{\mathcal{E}}^+_{\lambda, p}, E) = 0 \) at several elliptic curves \( E \) defined over \( \mathbb{Q} \). Again let us explain how to read the table given below for specialized equations. We first list the curves where we specialize the transformation equations in Case I-V:

- **Case A:** \( y^2 = 4x^3 - 2 \cdot 3^{-1}x^{-1}.19 \)  
  (11A);
- **Case B:** \( y^2 = 4x^3 - 2^2x + 1 \)  
  (37A);
- **Case C:** \( y^2 = 4x^3 - 2^3.3^{-1}.5x + 3^{-3}.251 \)  
  (37B);
- **Case D:** \( y^2 = 4x^3 + 2^3.3x - 2^3 \)  
  (27A);
- **Case E:** \( y^2 = 4x^3 + 1 \)  
  (27A).

The curve in Case A is isogeneous to the modular curve \( X_0(11)/\mathbb{Q} \) (\( = \mathbb{H}/\Gamma_0(11) \)). This curve is referred in [10] as 11A. The example of Case A is the restatement of [3, §5]. The curves in Case B and Case C correspond the distinct non-isogeneous factors of the jacobian variety of \( X_0(37)/\mathbb{Q} \). The curve in Case D is found in Serre [6, 5.9.2], which has potential everywhere good reduction. The curve in Case E has complex multiplication under \( \mathbb{Q}(\sqrt{-3}) \). In the following table, we list the specialized equations of the transformation equations already listed above at these elliptic curves. In Case A, as is well known, all the specialized equations of level 5 are reducible; so, we here list only one of them which corresponds to that in Case I. All the factors of the equations listed below are irreducible over \( \mathbb{Q} \).
Case A. \( G = 2^4 \) \( H = -2^3 \cdot 19 \) \( D = -11 \)

(I) \[ X^6 + 4400X^4 - 174240X^3 + 4801280X^2 - 340643072X + 5881529280 = (X-22)(X^5 + 22X^4 + 4884X^3 - 66792X^2 + 3331856X - 267342240) \]

Case B. \( G = 2^4 \cdot 3 \) \( H = -2^3 \cdot 33 \) \( D = 37 \)

(I) \[ X^6 - 44000X^4 + 1971360X^3 + 488897280X^2 + 47063460096X + 1162360730560 = (X-22)(X^5 + 22X^4 + 4884X^3 - 66792X^2 + 3331856X - 267342240) \]

Discriminant
\[ = 2^{36} \cdot 3^{12} \cdot 5^6 \cdot 11^6 \cdot 37^{12} \cdot 42044237^2 \]

Constant term = \( 2^6 \cdot 5 \cdot 37^3 \cdot 71711 \)

(II) \[ X^6 - 12360960X^4 - 175732014080X^3 - 906454164234240X^2 - 158592818333712384X - 617317300619300044800 \]

Discriminant
\[ = 2^{90} \cdot 3^{30} \cdot 5^5 \cdot 37^{12} \cdot 43^3 \cdot 17515886745480535148167^2 \]

Constant term = \( -2^{19} \cdot 3^6 \cdot 5^2 \cdot 37^3 \cdot 1275457 \)

(III) \[ X^6 + 2237760X^5 + 1781129088000X^4 + 603569053249044480X^3 + 77756911326531793975680X^2 + 524619092816465160434614272X + 105470303081456206598924845089920 \]

Discriminant
\[ = 2^{156} \cdot 3^{60} \cdot 5^5 \cdot 37^{12} \cdot 5387^2 \cdot 11719^2 \cdot 1913209809413235735059612153^2 \]

Constant term = \( 2^{33} \cdot 3^{12} \cdot 5 \cdot 37^3 \cdot 71711 \cdot 1272109 \)

(IV) \[ X^6 - 39697470231920640X^4 + 2318403282667096971018240X^3 - 7957164953643574388800162365440X^2 + 215475800430255113967103637484510117888X - 180650394914609884769327204746079333724979200 \]

Discriminant
\[ = 2^{156} \cdot 3^{90} \cdot 5^5 \cdot 11^2 \cdot 37^{12} \times \left( 16926206015272284315501000005983156558085509852481903028 \right)^2 \]

Constant term = \( -2^{30} \cdot 3^{18} \cdot 5^2 \cdot 37^3 \cdot 2777 \cdot 6469 \cdot 19089662430217 \)
(V) \( X^6 + 145329085440X^5 + 3837341672479781683200X^4 \\
- 46021426623559234641189289328640X^3 \\
+ 351785812651605713387572533809327772794880X^2 \\
- 46790723656122484150478959948110814885763660906496X \\
+ 5075737063108711398438669685930545872239299744266893393920 \\
\text{Discriminant} \\
= 2^{210} \cdot 3^{102} \cdot 5^5 \cdot 37^{12} \cdot 73^2 \cdot 39521^2 \\
\times \left( 16388239180126122004055543622661872689329826018532358 \right)^2 \\
\text{Constant term} = 2^{43} \cdot 3^{18} \cdot 5 \cdot 37^3 \cdot 661 \cdot 8897132982043042382280208129 \\
\text{Case C. } G = 2^5 \cdot 5 \quad H = -2^5 \cdot 251 \quad D = 37 \\
(I) \ X^6 - 148000X^4 - 1971360X^3 + 5432192000X^2 + 1029841968640X + 14284097373120 \\
\text{Discriminant} \\
= 2^{36} \cdot 5^5 \cdot 37^{12} \cdot 97^2 \cdot 251^4 \cdot 158512865466953^2 \\
\text{Constant term} = 2^6 \cdot 3 \cdot 5 \cdot 37^3 \cdot 293749 \\
(II) \ X^6 - 137344000X^4 - 1615064279040X^3 + 151709417062400X^2 \\
- 16661863907206758400X - 53980077857153227161600 \\
\text{Discriminant} \\
= 2^{96} \cdot 5^{11} \cdot 37^{12} \cdot 251^2 \cdot 84649^2 \cdot 80524545706391590943442857^2 \\
\text{Constant term} = -2^{18} \cdot 3 \cdot 5^2 \cdot 37^3 \cdot 103 \cdot 251^2 \cdot 8353 \\
(III) \ X^6 + 2237760X^5 + 1688966528000X^4 + 1242544486072320000X^3 \\
+ 428210896271006105600000X^2 - 12210194436023481050071040000X \\
+ 12154857571766922351262826496000000 \\
\text{Discriminant} \\
= 2^{186} \cdot 5^{35} \cdot 37^{12} \cdot 149^2 \cdot 251^4 \cdot 613^2 \cdot 7681^2 \cdot 85999^2 \cdot 2828711435333527499047^2 \\
\text{Constant term} = 2^{30} \cdot 3 \cdot 5^6 \cdot 11 \cdot 37^3 \cdot 433421914559869
(IV) \( x^6 - 132384946524160000x^4 + 21274901477944622530560000x^3 \\
- 1024232100465770191290105856000000x^2 \\
- 666638187777448119334115989258240000000x \\
- 171765970451673771504422151468636281241600000000 \\
Discriminant \\
\[ = 2^{156} \cdot 5^{41} \cdot 13^{12} \cdot 232^2 \cdot 231^2 \]
\[ \times (210413886361011152471697101677904852736456762343701556447)^2 \]
Constant term \( = -2^{30} \cdot 3^7 \cdot 37^3 \cdot 231^2 \cdot 42776313398349053608831 \)

(V) \( x^6 + 484430284800x^5 + 42637091095065067520000x^4 \\
+ 1191013536577507913643327488000000x^3 \\
+ 13457249056881351216480342923175526400000000x^2 \\
+ 28017443414101677967629098569361263396126720000000000x \\
+ 17931826045430048857989553905829178499337154671411200000000000 \\
Discriminant \\
\[ = 2^{216} \cdot 5^{55} \cdot 37^{12} \cdot 231^4 \cdot 3001^2 \]
\[ \times (595692948969915419253568974827809306186927780234914489024)^2 \]
Constant term \( = 2^{42} \cdot 3^7 \cdot 37^3 \cdot 809 \)
\[ \times 679234474035518752311178057851449 \]

Case D. \( G = -2^5 \cdot 3^2 \quad H = 2^6 \cdot 3^3 \quad D = -2^6 \cdot 3^5 \)

(I) \( x^6 - 111974400X^4 - 348285175760X^3 + 5109490051329280X^2 + 19395514284707414016X \\
+ 30756189785160164188160 \\
Discriminant \\
\[ = 2^{156} \cdot 3^{102} \cdot 5^5 \cdot 523^2 \cdot 1993^2 \]
Constant term \( = 2^{30} \cdot 3^{20} \cdot 5^3 \cdot 31 \cdot 53 \)

(II) \( x^6 + 187042037760X^4 + 245549406344970240X^3 - 409599982526419477463940X^2 \\
- 3391011510639691674610040832X \\
- 1232983430314568952093894456115200 \\

\[ - 4 - 17 - \]
Discriminant

\[ \text{Discriminant} = 2^{228} \cdot 3^{132} \cdot 5^{13} \cdot 17 \cdot 6 \cdot 60 \cdot 19 \cdot 5 \cdot 8 \cdot 38 \cdot 65 \cdot 3 \cdot 13 \cdot 69 \cdot 87 \cdot 2 \]

Constant term = \(-2^{45} \cdot 3^{26} \cdot 5^2 \cdot 551461 \)

(III) \[ X^6 - 940584960X^5 + 3148962351020280800X^4 - 4418083087421892617338880X^3 \\
+ 2280132588424701078050853649121280X^2 \\
- 17214744129018591819208640165526254911488X \\
+ 437329739457071001758251942193692492633441566720 \]

Discriminant

\[ \text{Discriminant} = 2^{306} \cdot 3^{162} \cdot 5^{13} \cdot 28572 \cdot 128103862043615822809259620839427^2 \]

Constant term = \(2^{30} \cdot 3^{25} \cdot 5 \cdot 31 \cdot 53 \cdot 509 \cdot 4895575 \)

(IV) \[ X^6 + 42080459238584604426240X^4 + 1377552015614216371873334710763520X^3 \\
- 8922152257725558133720282371231512044503040X^2 \\
- 139350800506019023284957313437543882607023335021057024X \\
- 55300532147549500076664990648195586731052493633265100180684800 \]

Discriminant

\[ \text{Discriminant} = 2^{368} \cdot 3^{192} \cdot 5^{13} \cdot 47^2 \cdot 109^2 \\
\times 4427561206428995873630120430172655669459200701218607287629^2 \]

Constant term = \(-2^{72} \cdot 3^{38} \cdot 5^2 \cdot 107 \cdot 3240694984266366049 \)

(V) \[ X^6 + 366512097853440X^5 + 24406304373574174539723571200X^4 \\
- 1957290889419754283886842288845509560893440X^3 \\
+ 23787048189350037674267146417212680864094364958279598080X^2 \\
- 164676730985625649174057125121435419400135894794978454628675502473216X \\
+ 2946351257120748673032865078237641507167920249441162884821958006433022156472320 \]

Discriminant

\[ \text{Discriminant} = 2^{438} \cdot 3^{222} \cdot 5 \\
\times \left( \frac{17055481350680990459643415602664324740824343896556675510}{8629380090934997315026253707} \right)^2 \]

Constant term = \(2^{87} \cdot 3^{44} \cdot 5 \cdot 19 \cdot 9787 \cdot 20795362588083644126474341 \)
Case E. \( G = 0 \) \( H = -2^3 \cdot 3^3 \) \( D = -3^3 \)

(I) \( x^6 - 1049760x^3 + 226351350720 \)
Discriminant = \( 2^{36} \cdot 3^{66} \cdot 5^5 \cdot 11^6 \cdot 17^6 \)
Constant term = \( 2^6 \cdot 3^{12} \cdot 5 \cdot 11^3 \)

(II) \( x^6 - 92513249280x^3 - 174990344338597478400 \)
Discriminant = \( 2^{96} \cdot 3^{96} \cdot 5^{13} \cdot 7^6 \cdot 41^6 \cdot 61^6 \)
Constant term = \( -2^{18} \cdot 3^{18} \cdot 5^2 \cdot 41^3 \)

(III) \( x^6 - 1632960x^5 + 949822848000x^4 - 22764789669888000x^3 + 19303585597263052800000x^2 - 674475678084121598361600000x + 839433158209838194989465600000 = (x^2 - 544320x + 20323353600)^3 \)
Discriminant of the irreducible factor = \( 2^{18} \cdot 3^8 \cdot 5^3 \)
Constant term of the irreducible factor = \( 2^{10} \cdot 3^8 \cdot 5^2 \cdot 11^2 \)

(IV) \( x^6 - 9012189878816256000000x^3 - 19837121300256932128568336568320000000000000000 = (x^2 - 544320x + 20323353600)^3 \)
Discriminant = \( 2^{156} \cdot 3^{156} \cdot 5^{56} \cdot 17^6 \cdot 23^6 \cdot 59^6 \cdot 71^6 \cdot 701576 \)
Constant term = \( -2^{30} \cdot 3^{30} \cdot 5^{13} \cdot 59^3 \cdot 71^3 \)

(V) \( x^6 - 227831878268154706479022080000000x^3 + 4206638859544244047942013830217159698852872192000000000000 = (x^2 - 544320x + 20323353600)^3 \)
Discriminant
\[
= 2^{216} \cdot 3^{186} \cdot 5^{55} \cdot 7^6 \cdot 13^6 \cdot 173^6 \cdot 521^6 \cdot 4519^6 \\
\times 2070362216376807869728367^2
\]
Constant term = \( 2^{42} \cdot 3^{36} \cdot 5^{11} \cdot 521^3 \cdot 4519^3 \)
References


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