学位論文

Generic bifurcations of varieties

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Dedicated to Professor Minoru Nakaoka on his 60th birthday

We study the zero point set of a parametrized smooth map germ. It is not only a natural generalization of Mather's theory of smooth map germs (cf. [11],[12]), but also it contains the bifurcation theory of stationary solutions of parametrized ordinary differential equations.

One of our main results is a classification of parametrized smooth map germs under a certain equivalence relation.

1. Introduction

Let $f : (R^n \times R^r, 0) \longrightarrow (R^p, 0)$ be a smooth map germ. For each $u \in (R^r, 0)$, we have a germ of "varieties" $f_u^{-1}(0)$ defined by $f_u = f|R^n \times u$. In this article, we shall study bifurcations of these varieties.

Let $C_0^\infty(R^n \times R^r)$ be the ring of smooth function germs $(R^n \times R^r, 0) \longrightarrow R$. This ring has a maximal ideal M_{n+r} consisting of all germs $(R^n \times R^r, 0) \longrightarrow (R, 0)$. For each smooth map germ $f : (R^n \times R^r, 0) \longrightarrow (R^p, 0)$, we define a ring homomorphism $f^* : C_0^\infty(R^p) \longrightarrow C_0^\infty(R^n \times R^r)$ by $f^*(h) = h \circ f$.

<u>DEFINITION 1.1</u>. Smooth map germs $f, g : (R^n \times R^r, 0) \longrightarrow (R^p, 0)$ are <u>P-K-equivalent</u> (respectively <u>S.P-K-equivalent</u>) if there exists a diffeomorphism germ $\Phi : (R^n \times R^r, 0) \longrightarrow (R^n \times R^r, 0)$ of the form $\Phi(x,u) = (\Phi_1(x,u), \phi(u))$ (respectively $\Phi(x,u) = (\Phi_1(x,u), u)$) such that

$\Phi^*(I(f)) = I(g)$. Here, $I(f) = f^*(M_p)C_0^\infty(R^n \times R^r)$.

For each map germ $f : (R^n \times R^r, 0) \longrightarrow (R^p, 0)$, the bifurcation map germ $\pi_f : (f^{-1}(0), 0) \longrightarrow (R^r, 0)$ is defined by $\pi_f(x,u) = u$.

DEFINITION 1.2. For two smooth map germs $f, g : (R^n \times R^r, 0) \longrightarrow (R^p, 0)$, bifurcation map germs π_f, π_g are A-equivalent if there are diffeomorphism germs $\Phi : (R^n \times R^r, 0) \longrightarrow (R^n \times R^r, 0)$ and $\phi : (R^r, 0) \longrightarrow (R^r, 0)$ such that $\Phi(f^{-1}(0)) = g^{-1}(0)$ and $\phi \circ \pi_f = \pi_g \circ \Phi$.

PROPOSITION 1.3. For two map germs $f, g : (R^n \times R^r, 0) \longrightarrow (R^p, 0)$, if f, g are P-K-equivalent, then π_f, π_g are A-equivalent.

The proof is trivial by Definition 1.1.

REMARK. The converse of Proposition 1.3 does not always hold. For example, let $f, g : (R \times R, 0) \longrightarrow (R, 0)$ be defined by $f(x,u) = x^2$ and $g(x,u) = x$. We have $\pi_f = \pi_g$ but f and g are not P-K-equivalent. In [10], Martinet showed that, provided df and dg are surjective, π_f and π_g are A-equivalent if and only if f and g are P-K-equivalent.

Besides being very attractive and geometrically interesting, the study of bifurcations of varieties arises naturally in a number of occasions such as in the study of differential equations and economics, etc. (cf. [3],[6],[8], [13],[15],[18]).

The theory of P-K-equivalence (or S.P-K-equivalence) is interpreted as a natural generalization of the A-equivalence theory for map germs :

For each smooth map germ $f : (R^n, 0) \longrightarrow (R^p, 0)$, we define $D_f : (R^n \times R^p, 0) \longrightarrow (R^p, 0)$ by $D_f(x,y) = f(x) - y$. By Proposition 1.3 and its remark, D_f and D_g are P-K-equivalent if and only if f and g are A-equivalent in the sense of Mather.

Moreover, it contains the "Rigt-equivalence" theory for function germs on singular varieties. (cf. Section 4).

In the case where $r = 1$, the P-K-equivalence relation has been studied by M. Golubitsky and D. Schaeffer ([6]). But the situation is quite different in the case where $r \geq 2$. (See Section 3, Proposition 3.14).

All map germs and diffeomorphisms considered here, are differentiable of class C^∞, unless stated otherwise.

Main results in this paper have been announced in [9].

2. Implicit function theorem for the P-K-equivalence

In this section, we prove some fundamental theorems for the P-K-equivalence theory. We now define "singularities" via the P-K-equivalence's version.

Let $f : (R^n \times R^r, 0) \longrightarrow (R^p, 0)$ be a map germ. We define

$$df_x : T_0(R^n \times R^r) \longrightarrow T_0R^p$$

by

$$df_x(v_1, \ldots, v_n, w_1, \ldots, w_r) = (\textstyle\sum_{i=1}^n v_i \frac{\partial f}{\partial x_i}^1(0), \ldots, \sum_{i=1}^n v_i \frac{\partial f}{\partial x_i}^p(0)),$$

and

$$df_u : T_0(R^n \times R^r) \longrightarrow T_0R^p$$

by

$$df_u(v_1, \ldots, v_n, w_1, \ldots, w_r) = (\textstyle\sum_{j=1}^r w_j \frac{\partial f}{\partial u_j}^1(0), \ldots, \sum_{j=1}^r w_j \frac{\partial f}{\partial u_j}^p(0)).$$

Since $\mathrm{Ker}(df_x) \supset T_0R^r$, we may define

$$df_u^K : \mathrm{Ker}(df_x) \longrightarrow \mathrm{Coker}(df_x)$$

by

$$df_u^K = \pi \circ df_u | \mathrm{Ker}(df_x),$$

where $\pi : T_0R^p \longrightarrow \mathrm{Coker}(df_x)$ is the canonical projection.

<u>DEFINITION 2.1</u>. Let $f : (R^n \times R^r, 0) \longrightarrow (R^p, 0)$ be a map germ. We say that f is <u>non-singular at 0 (in the contact sense)</u> if $\mathrm{rank}(df_x) = \min(n,p)$ and $\mathrm{rank}(df_u^K) = \min(r, p - \min(n,p))$. We say that f is <u>singular at 0 (in the contact sense)</u> if f is not non-singular at 0 (in the contact sense).

REMARK. We interpret that f is non-singular at 0 (in the contact sense) if :

a) $n \geq p$: f_0 has rank p at 0

b) $n < p$: f_0 is an immersion, and f has maximal rank at 0. Here, $f_0 = f|R^n \times 0$.

By the above interpretation, the following theorem is obvious.

THEOREM 2.2. (Implicit function theorem for the P-K-equivalence). Let $f : (R^n \times R^r, 0) \longrightarrow (R^p, 0)$ be a non-singular map germ (in the contact sense).

1) If $n \geq p$, then f is S.P-K-equivalent to the following germ : $(x_1, \ldots, x_n, u_1, \ldots, u_r) \longrightarrow (x_1, \ldots, x_p)$.

2) If $n < p$ and $r \leq p - n$, then f is P-K-equivalent to the following germ :

$$(x_1, \ldots, x_n, u_1, \ldots, u_r) \longrightarrow (x_1, \ldots, x_n, u_1, \ldots, u_r, 0, \ldots, 0).$$

3) If $n < p$ and $r > p - n$, then f is P-K-equivalent to the following germ :

$$(x_1, \ldots, x_n, u_1, \ldots, u_r) \longrightarrow (x_1, \ldots, x_n, u_1, \ldots, u_{p-n}).$$

The following proposition will be useful in Section 5.

PROPOSITION 2.3. Let $f : (R^n \times R^r, 0) \longrightarrow (R^p, 0)$ be a map germ with $\mathrm{rank}(df_x) = s$ and $\mathrm{rank}(df_u^K) = q$.

1) f is S.P-K-equivalent to a germ $f' : (R^s \times R^{n-s} \times R^r, 0) \longrightarrow (R^s \times R^{p-s}, 0)$ of the form $f'(x^1, x^2, u) = (x^1, \bar{f}(x^2, u))$, where $x^1 = (x_1, \ldots, x_s)$, $x^2 = (x_{s+1}, \ldots, x_n)$ and $u = (u_1, \ldots, u_r)$.

2) f is P-K-equivalent to a germ $f' : (R^s \times R^{n-s} \times R^q \times R^{r-q}, 0) \longrightarrow (R^s \times R^q \times R^{p-(s+q)}, 0)$ of the form $f'(x^1, x^2, u^1, u^2) = (x^1, u^1 + g_1(x^2, u^1, u^2), g_2(x^2, u^1, u^2))$, where $x^1 = (x_1, \ldots, x_s)$, $x^2 = (x_{s+1}, \ldots, x_n)$, $u^1 = (u_1, \ldots, u_q)$, $u^2 = (u_{q+1}, \ldots, u_r)$ and the order of g_i is at least 2, $(i = 1,2)$.

Proof. 1) If necessary, by the coordinate change of $(R^n,0)$ and $(R^p,0)$, we may assume that

$$(*) \qquad df_x = \left(\begin{array}{c|c} E & 0 \\ \hline 0 & 0 \end{array}\right), \text{ where } E \text{ is the } s\times s\text{-unit matrix.}$$

Then $f : (R^n\times R^r,0) \longrightarrow (R^s\times R^{p-s},0)$ has components f_1 and f_2 with respect to the splitting $R^p = R^s\times R^{p-s}$. We now define a map germ $\Psi : (R^s\times R^{n-s}\times R^r,0) \longrightarrow (R^s\times R^{n-s}\times R^r,0)$ by $\Psi(x^1,x^2,u) = (f_1(x^1,x^2,u),x^2,u)$. By the assumption $(*)$, Ψ is a local diffeomorphism germ. Hence, we put $\Phi(x^1,x^2,u) = (\phi(x^1,x^2,u),x^2,u)$ as the local inverse of Ψ.

Now, we have $f\circ\Phi(x^1,x^2,u) = (x^1,f_2\circ\Phi(x^1,x^2,u))$. It is clear that $I(x^1,f_2\circ\Phi(x^1,x^2,u)) = I(x^1,f_2\circ\Phi(0,x^2,u))$. Thus the proof of 1) is completed.

2) By the statement 1), it is enough to prove the statement in the case where $\mathrm{rank}(df_x) = 0$ and $\mathrm{rank}(df_u) = q$.

If necessary, by the coordinate change of $(R^r,0)$ and $(R^p,0)$, we may assume that

$$(**) \qquad df_u = \left(\begin{array}{c|c} E & 0 \\ \hline 0 & 0 \end{array}\right), \text{ where } E \text{ is the } q\times q\text{-unit matrix.}$$

Then $f : (R^n\times R^r,0) \longrightarrow (R^q\times R^{p-q},0)$ has components f_1 and f_2. We now define a map germ $\Psi : (R^q\times R^{r-q},0) \longrightarrow (R^q\times R^{r-q},0)$ by $\Psi(u^1,u^2) = (f_1(0,u^1,u^2),u^2)$. By the assumption $(**)$, Ψ is a local diffeomorphism germ. Hence, we put $\Phi(u^1,u^2) = (\phi(u^1,u^2),u^2)$ as the inverse of Ψ. If we represent $f\circ(1_{R^n}\times\Phi)(x,u^1,u^2) = (f_1'(x,u^1,u^2),f_2'(x,u^1,u^2))$, then $f_1'(0,u^1,u^2) = u^1$. Hence, the first order term of f_1' is $u^1 = (u_1,\ldots,u_q)$.

This completes the proof. Q.E.D.

3. Finite determinacy

We now turn our attention to the question of whether a map germ is P-K-equivalent (or S.P-K-equivalent) to a polynomial.

DEFINITION 3.1. Let $f, g : (R^n \times R^r, 0) \longrightarrow (R^p, 0)$ be a map germs.

i) f, g are k-jet equivalent if $(f^* - g^*)(M_p) \subset M_{n+r}^{k+1}$.

ii) f, g are (k_1, k_2)-jet equivalent if $(f^* - g^*)(M_p) \subset (M_n^{k_1+1} + M_r^{k_2+1})C_0^\infty(R^n \times R^r)$.

These are clearly equivalence relations ; we respectively denote $j_0^k f$ and $j_0^{(k_1,k_2)} f$ of equivalence classes represented by f. If we want to find "normal forms" for P-$\mathcal{K}$-equivalence classes, the following are a rather important chain of ideas :

DEFINITION 3.2. A map germ $f : (R^n \times R^r, 0) \longrightarrow (R^p, 0)$ is k-determined (respectively (k_1, k_2)-determined) relative to $\mathcal{S}$ if every map germ $g : (R^n \times R^r, 0) \longrightarrow (R^p, 0)$ such that $j_0^k f = j_0^k g$ (respectively $j_0^{(k_1,k_2)} f = j_0^{(k_1,k_2)} g$) is $\mathcal{S}$-equivalent to f. The notion of finite determinacy is same as the usual case. Here, $\mathcal{S}$ denotes P-$\mathcal{K}$ or S.P-$\mathcal{K}$.

In order to characterize finitely determined map germs, we need some algebraic tools. Let $f : (R^n \times R^r, 0) \longrightarrow (R^p, 0)$ be a smooth map germ. The set of all smooth vector field germs along f is written by $\theta(f)$; it is a $C_0^\infty(R^n \times R^r)$-module. We also write $\theta(r) = \theta(1_R r)$.

DEFINITION 3.3. 1) $tf : \theta(\pi_n) \longrightarrow \theta(f)$ is defined by $tf(\xi) = df \circ i(\xi)$, where $\pi_n : (R^n \times R^r, 0) \longrightarrow (R^n, 0)$ is the canonical projection and $i : \theta(\pi_n) \longrightarrow \theta(n+r)$ is the canonical inclusion. This is a $C_0^\infty(R^n \times R^r)$-homomorphism.

2) $\tau f : \theta(r) \longrightarrow \theta(f)$ is defined by $\tau f(\eta) = df \circ \iota(\eta \circ \pi_r)$, where $\pi_r : (R^n \times R^r, 0) \longrightarrow (R^r, 0)$ is the canonical projection and $\iota : \theta(\pi_r) \longrightarrow \theta(n+r)$ is the canonical inclusion. This is a $C_0^\infty(R^r)$-homomorphism.

The proof of the following theorem is same as usual method. And it is a corollary of Damon's very general theorem. (cf. [2]).

THEOREM 3.4. (Characterization theorem). Let $f : (R^n \times R^r, 0) \longrightarrow (R^p, 0)$ be a map germ. Then the following are equivalent :

1) f is finitely determined relative to P-K (respectively S.P-K).

2) $\dim_R \theta(f)/tf(\theta(\pi_n)) + \tau f(\theta(r)) + f^*(M_p)\theta(f) < +\infty$ (respectively $\dim_R \theta(f)/tf(\theta(\pi_n)) + f^*(M_p)\theta(f) < +\infty$).

From now on, we make estimates of orders of variable separated determinacy. Define $J^k(n+r,p)$ (respectively $J^{(k_1,k_2)}(n+r,p)$) to be the set of k-jets $j_0^k f$ (respectively (k_1,k_2)-jets $j_0^{(k_1,k_2)} f$). By the same argument as in [12], we can show that P-K-equivalence classes and S.P-K-equivalence classes in a k-jet space are smooth submanifolds. Moreover, we can calculate their tangent spaces. Under this observation, we introduce some notations. For any map germ $f : (R^n \times R^r, 0) \longrightarrow (R^p, 0)$, we define

$T(P\text{-}K)(f) = tf(M_{n+r}\theta(\pi_n)) + \tau f(M_r\theta(r)) + f^*(M_p)\theta(f)$,

$T_e(P\text{-}K)(f) = tf(\theta(\pi_n)) + \tau f(\theta(r)) + f^*(M_p)\theta(f)$,

$T(S.P\text{-}K)(f) = tf(M_{n+r}\theta(\pi_n)) + f^*(M_p)\theta(f)$,

$T_e(S.P\text{-}K)(f) = tf(\theta(\pi_n)) + f^*(M_p)\theta(f)$.

The following proposition is the key of estimates of orders of determinacy.

PROPOSITION 3.5. Suppose that a map germ $f : (R^n \times R^r, 0) \longrightarrow (R^p, 0)$ is k-determined relative to P-K (respectively S.P-K). Let $(k_1,k_2) \in N \times N$ be such that $k_1 + k_2 \leq k$. If

(*) $(M_n^{k_1} + M_r^{k_2})\theta(g) \subset T(P\text{-}K)(g) + M_{n+r}^{k+1}\theta(g)$

(respectively

$(M_n^{k_1} + M_r^{k_2})\theta(g) \subset T(S.P\text{-}K)(g) + M_{n+r}^{k+1}\theta(g)$)

for any map germ $g : (R^n \times R^r, 0) \longrightarrow (R^p, 0)$ with the same (k_1,k_2)-jet as f. Then f is (k_1,k_2)-determined relative to

P-K (respectively S.P-K).

For the proof, we need the following two lemmas.

LEMMA 3.6. Let f, g have the same (k_1,k_2)-jet. Then

1) $f^*(\phi) - g^*(\phi) \in (M_n^{k_1+1} + M_r^{k_2+1})C_0^\infty(R^n \times R^r)$ for any $\phi \in C_0^\infty(R^p)$,

2) $tf(\xi) - tg(\xi) \in R(M_n^{k_1} + M_r^{k_2})\theta(f)$ for any $\xi \in R\theta(\pi_n)$ and for any subring $R \subset C_0^\infty(R^n \times R^r)$,

3) $\tau f(\eta) - \tau g(\eta) \in M_r^{k_2}\theta(f)$ for any $\eta \in \theta(r)$.

Proof. The statement 1) is trivial by the definition of the (k_1,k_2)-jet equivalence. Let $(x_1,\ldots,x_n,u_1,\ldots,u_r)$ be a system of local coordinate at 0. Then

$$tf(\frac{\partial}{\partial x_i}) - tg(\frac{\partial}{\partial x_i}) = \sum_{j=1}^p \frac{\partial(f_j - g_j)}{\partial x_i} \cdot \frac{\partial}{\partial y_j} \circ f \quad (i = 1,\ldots,n).$$

By the definition,

$$\frac{\partial(f_j - g_j)}{\partial x_i} \in (M_n^{k_1} + M_r^{k_2+1})\theta(f) \subset (M_n^{k_1} + M_r^{k_2})\theta(f) \text{ for any } i.$$

Since any element $\xi \in R\theta(\pi_n)$ is an $RC_0^\infty(R^n \times R^r)$-linear combination of the $\frac{\partial}{\partial x_i}$'s, 2) follows. The proof of 3) is same as that of 2). Q.E.D.

LEMMA 3.7. Let G be a Lie group acting on a manifold M : and let V be a connected submanifold of M. Then V is contained in a single orbit if and only if

a) $TV_x \subset T(Gx)_x$ for any $x \in V$

b) $\dim T(Gx)_x$ is constant for every $x \in V$.

Here, Gx is the orbit through x in M.

Proof. See (3.1) of [12].

Proof of Proposition 3.5. The idea is apply Lemma 3.7 with $M = J^k(n+r,p)$, $G = (P\text{-}K)^k$ (respectively $(S.P\text{-}K)^k$) and $V = \{j_0^k g \in J^k(n+r,p) \mid j_0^{(k_1,k_2)}g = j_0^{(k_1,k_2)}f\}$, where $(P\text{-}K)^k$ (respectively $(S.P\text{-}K)^k$) is the Lie group of k-jets of P-K-equivalences (respectively S.P-K-equivalences).

Since $(M_n^{k_1+1} + M_r^{k_2+1})\theta(g) \subset (M_n^{k_1} + M_r^{k_2})\theta(g)$, the condition (*) implies that $TV_w \subset T((P-K)^k w)_w$ (respectively $TV_w \subset T((S.P-K)^k w)_w$) for every $w = j_0^k g \in V$.

Moreover, we consider the following exact sequence :

$$0 \to \mathrm{Ker}(\pi) \to \frac{T(P-K)(g)+M_{n+r}^{k+1}\theta(g)}{M_{n+r}^{k+1}\theta(g)} \xrightarrow{\pi} \frac{T(P-K)(g)+(M_n^{k_1}+M_r^{k_2})\theta(g)}{(M_n^{k_1}+M_r^{k_2})\theta(g)} \to 0$$

for any $j_0^k g \in V$, where $\pi : \theta(g)/M_{n+r}^{k+1}\theta(g) \to \theta(g)/(M_n^{k_1}+M_r^{k_2})\theta(g)$ is the canonical epimorphism. Since

$$\mathrm{Ker}(\pi) = \frac{(T(P-K)(g)+M_{n+r}^{k+1}\theta(g)) \cap (M_n^{k_1}+M_r^{k_2})\theta(g)}{M_{n+r}^{k+1}\theta(g)},$$

it has a constant dimension by the condition (*) and Lemma 3.6. Hence,

$$\dim_R \frac{T(P-K)(g)+M_{n+r}^{k+1}\theta(g)}{M_{n+r}^{k+1}\theta(g)}$$

is constant for every $j_0^k g \in V$. This implies the condition b) in Lemma 3.7. This completes the proof. Q.E.D.

The following simple and algebraic lemma is very useful.

LEMMA 3.8. *Let R be a commutative ring. Let E be an R-module, and $Z \subset R$ be a subring (not necessarily with unit) such that $Z^k E = 0$ (for some integer k). If $E = ZE$, then $E = 0$.*

Proof. If $E = ZE$, then $ZE = Z^2E = \cdots = Z^kE$. So, since $Z^kE = 0$, $E = 0$. Q.E.D.

We now have the following estimate.

THEOREM 3.9. *Let $f : (R^n \times R^r, 0) \longrightarrow (R^p, 0)$ be a map germ.*

i) *Suppose f is ℓ-determined relative to P-K for sufficiently large ℓ. Let D be a $C_0^\infty(R^n \times R^r)$-submodule of $\theta(f)$.*

If $D \subset T_e(P\text{-}K)(f) + (M_n^s 1 + M_r^s 2)\theta(f)$
and
$(M_n^s 1 + M_r^s 2)\theta(f) \subset T(S.P\text{-}K)(f) + M_r D + M_{n+r}(M_n^s 1 + M_r^s 2)\theta(f)$.
Then f *is* (s_1,s_2)*-determined relative to* $P\text{-}K$.

ii) *Suppose* $r = 1$ *and* f *is* ℓ*-determined relative to* $S.P\text{-}K$ *for sufficiently large* ℓ.

If
$(M_n^s 1 + M_1^s 2)\theta(f) \subset T(S.P\text{-}K)(f) + M_{n+1}(M_n^s 1 + M_1^s 2)\theta(f)$.
Then f *is* (s_1,s_2)*-determined relative to* $S.P\text{-}K$.

Proof. i) Suppose g has the same (s_1,s_2)-jet as f. It follows from Lemma 3.6 (with $R = M_{n+r}$) that

(1) $D \subset T_e(P\text{-}K)(g) + (M_n^s 1 + M_r^s 2)\theta(g)$

and

(2) $(M_n^s 1 + M_r^s 2)\theta(g) \subset T(S.P\text{-}K)(g) + M_r D + M_{n+r}(M_n^s 1 + M_r^s 2)\theta(g)$.

Let E be the $C_0^\infty(R^n \times R^r)$-module

$$\frac{T(S.P\text{-}K)(g) + M_r D + (M_n^s 1 + M_r^s 2)\theta(g)}{T(S.P\text{-}K)(g) + M_r D + M_{n+r}^k(M_n^s 1 + M_r^s 2)\theta(g)},$$

where $k + s \geq \ell + 1$ and $s = \min(s_1,s_2)$. Then, (2) implies $ZE = E$ (with $Z = M_{n+r}$). By Lemma 3.8, $E = 0$. Hence,
$(M_n^s 1 + M_r^s 2)\theta(g) \subset T(S.P\text{-}K)(g) + M_r D + M_{n+r}^k(M_n^s 1 + M_r^s 2)\theta(g)$.
Substituting for D,

(3) $(M_n^s 1 + M_r^s 2)\theta(g) \subset T(P\text{-}K)(g) + M_r(M_n^s 1 + M_r^s 2)\theta(g) + M_{n+r}^k(M_n^s 1 + M_r^s 2)\theta(g)$.

Let E' be the $C_0^\infty(R^r)$-module

$$\frac{T(P\text{-}K)(g) + (M_n^s 1 + M_r^s 2)\theta(g)}{T(P\text{-}K)(g) + M_{n+r}^k(M_n^s 1 + M_r^s 2)\theta(g)}.$$

Then, (3) implies $Z'E' = E'$ (with $Z' = M_r$). Since $M_r^k \subset M_{n+r}^k$, $Z'^k E' = 0$. By Lemma 3.8, $E' = 0$. Hence, we have
$(M_n^s 1 + M_r^s 2)\theta(g) \subset T(P\text{-}K)(g) + M_{n+r}^k(M_n^s 1 + M_r^s 2)\theta(g)$. We now have the following inclusions :
$M_{n+r}^k(M_n^s 1 + M_r^s 2)\theta(g) \subset M_{n+r}^k M_{n+r}^s \theta(g) \subset M_{n+r}^{\ell+1}\theta(g)$.

It follows that

$$(M_n^s 1 + M_r^s 2)\theta(g) \subset T(P\text{-}K)(g) + M_{n+r}^{\ell+1}\theta(g).$$

Since this holds for all g with the same (s_1,s_2)-jet as f, it follows by Proposition 3.5 that f is (s_1,s_2)-determined relative to P-K.

The proof of ii) is same as (easier than) that of the case i). This completes the proof. Q.E.D.

REMARK. In ([6], Theorem 2.8), there is the following estimate : If $M^s_{n+1}\theta(f) \subset T(S.P\text{-}K)(f)$, then f is s-determined relative to P-K. The statement ii) of the above theorem fills up their estimate.

We have many other estimates as corollaries of the above theorem. Since the estimate of ii) in the above theorem is very nice, we will only consider the case where $r \geq 2$.

COROLLARY 3.10. *If*

(1) $(M^{k_1}_n + M^{k_2-1}_r)\theta(f) \subset T_e(P\text{-}K)(f) + (M^{k_1+\ell}_n + M^{k_2}_r)\theta(f)$

and

(2) $M^{\ell}_n\theta(f) \subset T_e(S.P\text{-}K)(f) + M_r\theta(f) + (M^{k_1+\ell}_n + M^{k_2}_r)\theta(f)$,

then f *is* $(k_1 + \ell, k_2)$*-determined relative to P-K*.

Proof. Multiply (2) through by $M^{k_1}_n$, to obtain
$M^{k_1+\ell}_n\theta(f) \subset tf(M^{k_1}_n\theta(\pi_n)) + M_rM^{k_1}_n\theta(f) + f^*(M_p)M^{k_1}_n\theta(f) + M^{k_1}_n(M^{k_1+\ell}_n + M^{k_2}_r)\theta(f)$.
We always have the inclusion : $M^{k_2}_r\theta(f) \subset tf(M_{n+r}\theta(\pi_n)) + M_rM^{k_2-1}_r\theta(f) + f^*(M_p)\theta(f)$.

By (3), (4) and the inclusion

$$M^{k_1}_n(M^{k_1+\ell}_n + M^{k_2}_r)\theta(f) \subset M_{n+r}(M^{k_1+\ell}_n + M^{k_2}_r)\theta(f),$$

we have $(M^{k_1+\ell}_n + M^{k_2}_r)\theta(f) \subset T(S.P\text{-}K)(f) + M_r(M^{k_1}_n + M^{k_2-1}_r)\theta(f) + M_{n+r}(M^{k_1+\ell}_n + M^{k_2}_r)\theta(f)$.

The result follows, from this and (1), by Theorem 3.9.

This completes the proof. Q.E.D.

COROLLARY 3.11. *If* $M^k_{n+r}\theta(f) \subset T_e(P\text{-}K)(f)$, *then* f *is* $(s(k,r),k+1)$*-determined relative to* $P\text{-}K$. *Here*,

$$s(k,r) = \begin{cases} k(r+1) & \text{if } \operatorname{rank}(df_u) = p, \\ k(r+2) & \text{otherwise}. \end{cases}$$

Proof. By the hypothesis,

$$E = \frac{T_e(P\text{-}K)(f) + M_r T_e(P\text{-}K)(f)}{T_e(S.P\text{-}K)(f) + M_r T_e(P\text{-}K)(f) + (M^k_{n+r})^{r+1} T_e(P\text{-}K)(f)}$$

has an R_k-module structure, where R_k is a subring of $C^\infty_0(R^n \times R^r)$ generated by R and M^k_{n+r}. Then,

$$\dim_R E \le \dim_R \frac{\tau f(\theta(r))}{\tau f(M_r\theta(r))} = r.$$

By Corollary (1.6) in [11] applied to R_k-module E, it follows that

$$(M^k_{n+r})^r T_e(P\text{-}K)(f) \subset T_e(S.P\text{-}K)(f) + M_r T_e(P\text{-}K)(f).$$

In the case where $\operatorname{rank}(df_u) = p$, $e_i = (0,\ldots,1,\ldots,0)$ $(i = 1,2,\ldots,p)$ are elements of $T_e(P\text{-}K)(f)$. Then

$$(M^k_{n+r})^r \theta(f) \subset (M^k_{n+r})^r T_e(P\text{-}K)(f).$$

In other cases, since $M^k_{n+r}\theta(f) \subset T_e(P\text{-}K)(f)$,

$$M^{k(r+1)}_{n+r}\theta(f) \subset M^k_{n+r}(M^k_{n+r})^r\theta(f) \subset (M^k_{n+r})^r T_e(P\text{-}K)(f).$$

In either cases, we have inclusions

$$M^{s(k,r)-k}_{n+r}\theta(f) \subset (M^k_{n+r})^r T_e(P\text{-}K)(f) \subset T_e(S.P\text{-}K)(f) + M_r\theta(f).$$

The result follows, from this and Corollary 3.10. Q.E.D.

COROLLARY 3.12. *If* $\theta(f) = T_e(P\text{-}K)(f)$ *then* f *is* $(r+1,1)$*-determined relative to* $P\text{-}K$.

Proof. In this situation, by the same argument as Corollary 3.11, we have

$$M^r_{n+r}\theta(f) \subset T_e(S.P\text{-}K)(f) + M_r\theta(f).$$

Hence, by Corollary 3.10, the result follows. Q.E.D.

REMARK. The estimate in Corollary 3.12 is a generalization of Mather's estimate for stable map germs. (cf. Proposition 3.5 in [12]).

We now consider the finite determinacy relative to S.P-K.

DEFINITION 3.13. For any map germ $f : (R^n \times R^r, 0) \longrightarrow (R^p, 0)$, we let

$\Sigma^*(f) = \{(x,u) \in (R^n \times R^r, 0) \mid \mathrm{rank}(df_x(x,u)) < p$ and $f(x,u) = 0\}$.

We say that $\Sigma^*(f)$ is a bifurcation set of f with respect to S.P-K. This notion is defined also on the complex field C.

For any C^ω map germ $f : (R^n \times R^r, 0) \longrightarrow (R^p, 0)$, we denote the complexification of f by $f_C : (C^n \times C^r, 0) \longrightarrow (C^p, 0)$.

By the same method as the usual case, we have the following proposition which indicates the difference of situations in cases of $r = 1$ and $r \geq 2$. (cf. Wall [16]).

PROPOSITION 3.14. Let $f : (R^n \times R^r, 0) \longrightarrow (R^p, 0)$ be a smooth map germ with $r \geq 2$. The following are equivalent :

1) f is finitely determined relative to S.P-K.
2) f is S.P-K-equivalent to the following germ :
$(x_1, \ldots, x_n, u_1, \ldots, u_r) \longrightarrow (x_1, \ldots, x_p)$.
3) $\Sigma^*(f_C) \subset \{0\}$, if f has the complexification f_C.

REMARKS. i) By the above proposition, Golubitsky and Schaeffer's estimate (Theorem 2.8 in [6]) is not useful in the case where $r \geq 2$.

ii) The above proposition is related to the well known remark : the "Right-equivalence" theory for map germs $(R^n, 0) \longrightarrow (R^p, 0)$ $p \geq 2$ is "almost impossible". The reason, why the above situation happens, is the S.P-K-equivalence theory for $f : (R^n \times R^r, 0) \longrightarrow (R^p, 0)$ implies the Right-equivalence theory for $\pi_f : (f^{-1}(0), 0) \longrightarrow (R^r, 0)$.

EXAMPLE 3.15. Consider a map germ $f(x,u) = x^k \pm u^s x^m$ $(k \geq 1 ; s, m \geq 0)$. One calculates $T(S.P\text{-}K)(f) = \langle kx^k \pm mu^s x^m, kx^{k-1}u \pm mu^{s+1}x^{m-1}, x^k \pm u^s x^m \rangle_{C_0^\infty(R\times R)}$.

In the cases

$m \geq 2$: $u^k \notin T(S.P\text{-}K)(f)$ for any $k \in N$, so f is not finitely determined relative to S.P-K.

$m = 1$; $k = 1$: $x, u \in T(S.P\text{-}K)(f)$, so by Theorem 3.9 ii), f is (1,1)-determined relative to S.P-K.

$m = 1$; $k \geq 2$: $x^k, u^{2s} \in T(S.P\text{-}K)(f)$, so by Theorem 3.9 ii), f is $(k,2s)$-determined relative to S.P-K.

$m = 0$: $x^k, u^s \in T(S.P\text{-}K)(f)$, so by Theorem 3.9 ii), f is (k,s)-determined relative to S.P-K.

EXAMPLE 3.16. Consider map germs 1) $f(x,y,u) = \pm u + x^3 - xy^2$, 2) $f(x,y,u) = \pm u + x^3 + y^3$, 3) $f(x,y,u) = \pm u \pm (x^2y + y^4)$. By the same calculation as that of Example 3.15, 1) and 2) are (3,1)-determined relative to S.P-K and 3) is (4,1)-determined relative to S.P-K.

EXAMPLE 3.17. Consider map germs 1) $f(x,y,u) = (\pm u^s + xy, x^2 \pm y^2)$, 2) $f(x,y,u) = (\pm xy, \pm u^s + x^2 \pm y^2)$, 3) $f(x,y,u) = (\pm xy, \pm u^s + x^2 \pm y^3)$. By the same argument as that of Example 3.15, 1) and 2) are $(2,s)$-determined relative to S.P-K and 3) is $(3,s)$-determined relative to S.P-K.

EXAMPLE 3.18. Let $f : (R^n\times R^r,0) \longrightarrow (R,0)$ be a map germ given by

$$f(x_1,\dots,x_n,u_1,\dots,u_r) = \pm x_1^2 \pm \cdots \pm x_n^2 + g(u_1,\dots,u_r)$$

such that

$$M_r^k \subset M_r\langle \frac{\partial g}{\partial u_1},\dots,\frac{\partial g}{\partial u_r}\rangle_{C_0^\infty(R^r)}. \qquad (*)$$

One calculates

$T_e(P\text{-}K)(f) = \langle x_1,\dots,x_n f(x_1,\dots,x_n,u_1,\dots,u_r)\rangle_{C_0^\infty(R^n\times R^r)} + \langle \frac{\partial g}{\partial u_1},\dots,\frac{\partial g}{\partial u_r}\rangle_{C_0^\infty(R^r)}$, so that $C_0^\infty(R^n\times R^r)$-module

$D = \langle \frac{\partial g}{\partial u_1},\dots,\frac{\partial g}{\partial u_r}\rangle_{C_0^\infty(R^n\times R^r)}$ is contained in $T_e(P\text{-}K)(f)$.

By (*), $(M_n^2 + M_r^k)\theta(f) \subset T(S.P\text{-}K)(f) + M_r D$, so, by Theorem 3.9 i), f is (2,k)-determined relative to P-K.

4. Versal deformations

In this section, we consider deformations (unfoldings) of map germs. Notions of the deformation of a map germ and its versality with respect to P-K (respectively S.P-K) are analogous to those of in [5]. In which, we denote S as P-K or S.P-K. The versality theorem is the following theorem.

THEOREM 4.1. *Let* $F : (R^n \times R^r \times R^s, 0) \longrightarrow (R^p, 0)$ *be an s-parameter deformation of* $f : (R^n \times R^r, 0) \longrightarrow (R^p, 0)$: *A necessary and sufficient condition for F to be S-versal is that it should be infinitesimally S-versal. Here, we say that F is infinitesimally S-versal if* $\theta(f) = T_e(S)(f) + V_F$, *where* $V_F = \langle \frac{\partial F}{\partial v_1}|R^n \times R^r \times 0, \ldots, \frac{\partial F}{\partial v_s}|R^n \times R^r \times 0 \rangle_R$.

The proof is the same as that of in [7]. Moreover, it is a corollary of Damon's more general versality theorem in [2]. But we remark that there are some applications of the above theorem.

A) For any composed map germ $(R^n, 0) \xrightarrow{f} (R^p, 0) \xrightarrow{g} (R^q, 0)$, we define $D_{(f,g)} : (R^n \times R^p \times R^q, 0) \longrightarrow (R^p \times R^q, 0)$ by $D_{(f,g)}(x,y,z) = (f(x) - y, g(y) - z)$ and $d_{(f,g)} : (R^n \times R^p, 0) \longrightarrow (R^p \times R^q, 0)$ by $d_{(f,g)}(x,y) = (f(x) - y, g(y))$. We now have the following theorem as an application of Theorem 4.1.

THEOREM 4.2. *For a composed map germ* $(R^n, 0) \xrightarrow{f} (R^p, 0) \xrightarrow{g} (R^q, 0)$, *the following are equivalent* :

1) $(R^n, 0) \xrightarrow{f} (R^p, 0) \xrightarrow{g} (R^q, 0)$ *is stable as a composed map germ.*

2) $D_{(f,g)}$ *is a P-K-versal deformation of* $d_{(f,g)}$.

The proof is trivial by arguments in [4] and [13].

B) All arguments in this article remain valid for holomorphic map germs on the complex number field C.

Let $\begin{array}{ccc}(C^n,0) & \xrightarrow[f_i]{} & (C,0)\\ g_i\downarrow & & \\ (C^p,0) & & \end{array}$ $(i = 1,2)$ be holomorphic map germs.

We say that $f_1 : (g_1^{-1}(0),0) \longrightarrow (C,0)$ and $f_2 : (g_2^{-1}(0),0) \longrightarrow (C,0)$ are <u>right-equivalent</u> if there is biholomorphic map germ $\phi : (C^n,0) \longrightarrow (C^n,0)$ such that $\phi^*(I(g_1)) = I(g_2)$ and $f_1\circ\phi|g_1^{-1}(0) = f_2|g_2^{-1}(0)$. This equivalence relation is a natural equivalence relation for holomorphic function germs on varieties. We can treat this situation as follows : For each map germs $\begin{array}{ccc}(C^n,0) & \xrightarrow{f} & (C,0)\\ g\downarrow & & \\ (C^p,0) & & \end{array}$

we define $K_{(f,g)} : (C^n\times C,0) \longrightarrow (C^p\times C,0)$ by $K_{(f,g)}(x,y) = (g(x),f(x) - y)$. It is clear that $K_{(f_1,g_1)}$ and $K_{(f_2,g_2)}$ are S.P-$\mathcal{K}$-equivalent if and only if $f_1 : (g_1^{-1}(0),0) \longrightarrow (C,0)$ and $f_2 : (g_2^{-1}(0),0) \longrightarrow (C,0)$ are right-equivalent. By an <u>s-parameter deformation of a system</u> $\begin{array}{ccc}(C^n,0) & \xrightarrow{f} & (C,0)\\ g\downarrow & & \\ (C^p,0) & & \end{array}$

we mean a system of holomorphic germs $\begin{array}{ccc}(C^n\times C^s,0) & \xrightarrow{F} & (C,0)\\ G\downarrow & & \\ (C^p,0) & & \end{array}$

such that $(F,G)|C^n\times 0 = (f,g)$. Then, we have the following versality theorem.

<u>THEOREM 4.3</u>. <u>Let</u> $\begin{array}{ccc}(C^n\times C^s,0) & \xrightarrow{F} & (C,0)\\ G\downarrow & & \\ (C^p,0) & & \end{array}$ <u>be an s-paremeter deformation of a holomorphic germs</u> $\begin{array}{ccc}(C^n,0) & \xrightarrow{f} & (C,0)\\ g\downarrow & & \\ (C^p,0) & & \end{array}$

<u>If</u>

$\theta(g)\oplus\mathcal{O}_n = (tg(\theta(n))+g^*(M_p)\theta(g))\oplus(\langle\frac{\partial f}{\partial x_1},\ldots,\frac{\partial f}{\partial x_n},g_1,\ldots,g_p\rangle\mathcal{O}_n + V_G\oplus V_F$, <u>then</u> $F|G^{-1}(0)$ <u>is a right-versal deformation of</u> $f|g^{-1}(0)$. <u>Here</u>, $\mathcal{O}_n$ <u>is the local ring of holomorphic function germs on</u> C^n <u>at</u> 0.

Proof. Define $G_f : (C^n,0) \longrightarrow (C^n \times C,0)$ by $G_f(x) = (x,f(x))$. Then, we have a surjective map

$(G_f)^* : \theta(K_{(f,g)}) \longrightarrow \theta(g) \oplus \mathcal{O}_n$ defined by $(G_f)^*(\xi,\eta) = (\xi,\eta \circ f)$. Since $\mathrm{Ker}(G_f)^* = \langle f(x)-y \rangle_{\mathcal{O}_{n+1}}$, the result follows. Q.E.D.

5. Classifications

In this section, we will classify parametrized map germs under some conditions. At first, we must argue relations of P-K-equivalences and K-equivalences. The K-equivalence theory has been developed by Mather ([11], [12]) and Martinet [10]. Notions and results for the K-equivalence theory in these articles will be employed. For each map germ $f : (R^n \times R^r,0) \longrightarrow (R^p,0)$, we define $\iota^* : \theta(f) \longrightarrow \theta(f_0)$ by $\iota^*(\eta) = \eta \circ \iota$, where $\iota : (R^n,0) \longrightarrow (R^n \times R^r,0)$ is the canonical inclusion and $f_0 = f|R^n \times 0$. This map ι^* is a surjective R-linear map whose kernel is $M_r\theta(f)$.

LEMMA 5.1. *The map* $\iota^* : \theta(f) \longrightarrow \theta(f_0)$ *induces an R-isomorphism*

$\tilde{\iota}^* : \theta(f)/(T_e(S.P\text{-}K)(f) + M_r\theta(f)) \cong \theta(f_0)/T_e(K)(f_0)$,

where $T_e(K)(f_0) = tf_0(\theta(n)) + f_0^*(M_p)\theta(f_0)$.

The proof is trivial by definitions of $T_e(S.P\text{-}K)(f)$ and $T_e(K)(f_0)$.

DEFINITION 5.2. For each germ $f : (R^n \times R^r,0) \longrightarrow (R^p,0)$, we define P-$K$-cod$(f) = \dim_R(\theta(f)/T_e(P\text{-}K)(f))$, S.P-$K$-cod$(f) = \dim_R(\theta(f)/T_e(S.P\text{-}K)(f))$, K-cod$(f_0) = \dim_R(\theta(f_0)/T_e(K)(f_0))$. We respectively call these a *P-K-codimension of* f, an *S.P-K-codimension of* f and a *K-codimension of* f_0.

REMARKS. 1) S-codimensions are S-invariant for S = P-K, S.P-K or K.

2) By Lemma 5.1,

$$K\text{-cod}(f_0) = \dim_R(\theta(f)/T_e(S.P\text{-}K)(f) + M_r\theta(f)).$$

We need the following lemma.

LEMMA 5.3. *Let* $f : (R^n \times R^r, 0) \longrightarrow (R^p, 0)$ *be a germ*

i) $K\text{-cod}(f_0) \leq P\text{-}K\text{-cod}(f) + r$.

ii) *If* $\text{rank}(df_x) = s$ *and* $\text{rank}(df_u^K) = q$, *then*

$$P\text{-}K\text{-cod}(f) \geq p - (s + q).$$

Proof. i) $K\text{-cod}(f_0) = \dim_R(\theta(f)/T_e(S.P\text{-}K)(f) + M_r\theta(f))$
$\leq \dim_R(\theta(f)/T_e(S.P\text{-}K)(f) + M_r\tau f(\theta(r))$

$$= P\text{-}K\text{-cod}(f) + \dim_R \frac{T_e(P\text{-}K)(f)}{T_e(S.P\text{-}K)(f) + \tau f(M_r\theta(f))}$$

$\leq P\text{-}K\text{-cod}(f) + r$.

ii) By 1) of Proposition 2.3, we may assume that f is a germ $(R^s \times R^{n-s} \times R^r, 0) \longrightarrow (R^s \times R^{p-s}, 0)$ of the form $f(x^1, x^2, u) = (x^1, \bar{f}(x^2, u))$. We define an R-epimorphism $\Pi : \theta(f) \longrightarrow \theta(\bar{f})$ by $\Pi(\xi) = d\pi \circ \xi \circ i$ where $\pi : (R^s \times R^{p-s}, 0) \longrightarrow (R^{p-s}, 0)$ is the canonical projection and $i : (R^{n-s} \times R^r, 0) \longrightarrow (R^s \times R^{n-s} \times R^r, 0)$ is the canonical inclusion. Then, we have $\Pi(T_e(P\text{-}K)(f)) = T_e(P\text{-}K)(\bar{f})$. Hence, we may assume that $\text{rank}(df_x) = 0$.

By 2) of Proposition 2.3, f is a germ $(R^n \times R^q \times R^{r-q}, 0) \longrightarrow (R^q \times R^{p-q}, 0)$ of the form $f(x, u^1, u^2) = (u^1 + g_1(x, u^1, u^2), g_2(x, u^1, u^2))$, in where orders of g_i are at least 2 $(i = 1,2)$. Then, we have $tf(\theta(\pi_n)) \subset M_{n+r}\theta(f)$, $\tau f(\theta(r)) \subset \langle e_1, \ldots, e_q, *, \ldots, * \rangle_{C_0^\infty(R^n)}$ and $f^*(M_p) \subset M_{n+r}$.

Hence, $P\text{-}K\text{-cod}(f) \geq p - q$. Q.E.D.

The following theorem is the classification theorem for germs of zero P-K-codimension.

THEOREM 5.4. *Let* $f : (R^n \times R^r, 0) \longrightarrow (R^p, 0)$ *be a germ of* $P\text{-}K\text{-cod}(f) = 0$, $\text{rank}(df_x) = s$ *and* $\text{rank}(df_u^K) = q$. *Then*, $p = s + q$ *and there is a polynomial map germ* $g : (R^{n-s}, 0) \longrightarrow (R^q, 0)$ *of degree at most* $r + 1$ *with* $\text{rank}(dg) = 0$ *and* $K\text{-cod}(g) \leq r$ *such that* f *is P-K-equivalent*

a germ $f' : (R^s \times R^{n-s} \times R^q \times R^{r-q}, 0) \longrightarrow (R^s \times R^q, 0)$ *of the form* $f'(x^1, x^2, u^1, u_{q+1}, \ldots, u_r) = (x^1, u^1 + g(x^2) + \sum_{i=1}^{r-q} u_{q+i}\xi_i(x_2))$ *where* $\xi_1, \ldots, \xi_{r-q}$ *are generator of the R-vector space* $M_{n-s}\theta(g)/T_e(K)(g)$.

REMARK. In the above theorem, g is the K-finite map in the sense of Mather. Hence, Mather's classifications of K-versal deformations are applicable for the case of zero P-K-codimension.

LEMMA 5.5. *Let* $f : (R^n \times R^r, 0) \longrightarrow (R^p, 0)$ *be a map germ. Then the following are equivalent* :

(1) $\theta(f) = T_e(P\text{-}K)(f)$.

(2) $\theta(f_0) = T_e(K)(f_0) + V_f$.

Proof. Let $\iota : (R^n, 0) \longrightarrow (R^n \times R^r, 0)$ be the canonical inclusion. Since $\iota^*(tf(\theta(\pi_n)) = tf_0(\theta(n))$, $\iota^*(f^*(M_p)\theta(f)) = f_0^*(M_p)\theta(f_0)$ and $\iota^*(\tau f(\theta(r)) = V_f$, the proof for (1) $\Rightarrow$ (2) is trivial. By Lemma 5.1, (2) is equivalent to

$$(*) \quad \theta(f) = T_e(S.P\text{-}K)(f) + M_r\theta(f) + \langle \frac{\partial f}{\partial u_1}, \ldots, \frac{\partial f}{\partial u_r} \rangle R.$$

Hence, we have $\theta(f) = T_e(P\text{-}K)(f) + M_r\theta(f)$.
By the Malgrange preparation theorem, $\theta(f) = T_e(P\text{-}K)(f)$.
This completes the proof. Q.E.D.

This lemma indicates the fact that P-K-cod(f) = 0 if and only if f is an r-parameter K-versal deformation of f_0.

Proof of Theorem 5.4. Since P-K-cod(f) $\geq p - (s + q)$, we have $p = s + q$. By 2) of Proposition 2.3, we may assume that f is a map germ $(R^s \times R^{n-s} \times R^q \times R^{r-q}, 0) \longrightarrow (R^s \times R^q, 0)$ of the form $f(x^1, x^2, u^1, u^2) = (x^1, u^1 + \bar{f}(x^2, u^1, u^2))$.

Let $\Phi : \theta(f) \longrightarrow \theta(f_1)$ be the R-epimorphism induced by the canonical projection $(R^s \times R^q, 0) \longrightarrow (R^q, 0)$, where $f_1(x^2, u^1, u^2) = u^1 + \bar{f}(x^2, u^1, u^2)$. Since $\Phi(T_e(P\text{-}K)(f)) = T_e(P\text{-}K)(f_1)$, we also have P-$K$-cod$(f_1) = 0$. We define $g : (R^{n-s}, 0) \longrightarrow (R^q, 0)$ by $g(x^2) = f_1(x^2, 0, 0)$. By i) of Lemma 5.3, K-cod(g) $\leq$ P-K-cod$(f_1) + r = r$.

Then, there are vector field germs $\xi_1, \ldots, \xi_{r-q} \in \theta(g)$ such that these generate $M_{n-s}\theta(g)/T_e(K)(g)$ over R. If we define a map germ $f' : (R^s \times R^{n-s} \times R^q \times R^{r-q}, 0) \longrightarrow (R^s \times R^q, 0)$ by $f'(x^1, x^2, u^1, u_{q+1}, \ldots, u_r) = (x^1, u^1 + g(x^2) + \sum_{i=1}^{r-q} u_{q+i}\xi_i(x^2))$, then $f'_0 = f_0$. By the uniqueness of K-versal deformations, f and f' are P-K-equivalent. By Corollary 3.12, we may choose g as a polynomial of degree at most r + 1.

This completes the proof. Q.E.D.

Throughout remainder of this section, we will partially classify smooth germs of low P-K-codimensions.

<u>DEFINITION 5.6</u>. Let $f : (R^n \times R^r, 0) \longrightarrow (R^p, 0)$ be a map germ. We say that f has a Σ_s^k<u>-type at 0 if</u>

$\mathrm{rank}(df_x) = \min(n,p) - k$ and

$\mathrm{rank}(df_u^K) = \min(r, p - \mathrm{rank}(df_x)) - s$.

<u>LEMMA 5.7</u>. <u>Let</u> $f : (R^n \times R, 0) \longrightarrow (R^p, 0)$ $(n \geq p)$ <u>be a map germ with</u> P-K-cod(f) $\leq k$. <u>Then</u> f <u>has the</u> Σ_0^i<u>-type</u> $(0 \leq i \leq k + 1)$ <u>or the</u> Σ_1^j<u>-type</u> $(1 \leq j \leq k)$.

<u>Proof</u>. By i) of Lemma 5.3, $k \geq p - (s + q)$, where $\mathrm{rank}(df_x) = s$ and $\mathrm{rank}(df_u^K) = q$. Since $r = 1$, $q = 0$ or 1. If $q = 0$, then $p - s \leq k$. If $q = 1$, then $p - s \leq k + 1$.

This completes the proof. Q.E.D.

For classifications, we need classifications of germs with respect to K-equivalences.

<u>DEFINITION 5.8</u>. Let $f : (R^n, 0) \longrightarrow (R^p, 0)$ be a map germ with Σ_i-type and k-determined relative to K. We say that f is of <u>discrete algebra type</u> if the orbit $K^k(j_0^k f)$ in $J^k(n,p)$ has a neighbourhood which contains only finitely many K-class orbits of type Σ_i. Here, f has a Σ_i-type if $\dim(\ker(df)) = i$.

REMARKS. i) Let $f : (R^n,0) \longrightarrow (R^p,0)$ be an A-stable map germ. If (n,p) is contained in the "nice range" in the sense of Mather, f is of discrete algebra type. (cf. [1]).

ii) All stable map germs of discrete algebra types are classified by Damon ([1]).

PROPOSITION 5.9. (Damon and Wasserman).

1) *Let $f : (R^k,0) \longrightarrow (R^k,0)$ be a smooth germ of discrete algebra type with K-cod$(f) \leq 5$ and $k \leq 5$. Then f is K-equivalent to one of germs in Table I-(1).*

2) *Let $f : (R^n,0) \longrightarrow (R,0)$ be a function germ with K-cod$(f) \leq 5$. Then f is K-equivalent to one of germs in Table I-(2).*

Proof. Since the K-equivalence relation is an unfolding invariant, the notion of discrete algebra type is also an unfolding invariant. Hence, the classification is reduced to Damon [1] and Wasserman [17]. Q.E.D.

Let $f : (R^n \times R^r,0) \longrightarrow (R^p,0)$ be a map germ with P-K-cod$(f) \leq s$. By Lemma 5.3, K-cod$(f_0) \leq s + r$. Let $\xi_1(x),\ldots, \xi_\ell(x)$ be generators of the R-vector space $\theta(f_0)/T_e(K)(f_0)$ (where $\ell \leq s + r$), then

$$F(x,v_1,\ldots,v_\ell) = f_0(x) + v_1\xi_1(x) + \cdots + v_\ell\xi_\ell(x)$$

is a K-versal deformation of f_0, where $x = (x_1,\ldots,x_n)$. (See Theorem 4.4 in [10]). By the definition of the K-versal deformation of f_0, there exists a map germ $\phi : (R^n,0) \longrightarrow (R^\ell,0)$ such htat ϕ^*F is P-K-equivalent to f, where $\phi^*F(x,u) = F(x,\phi(u))$. Hence, we will classify map germs of the form ϕ^*F.

LEMMA 5.10. *Let $F : (R^n \times R^\ell,0) \longrightarrow (R^p,0)$ be a K-versal deformation of f_0 (i.e. $F(x,v) = f_0(x) + v_1\xi_1(x) + \cdots + v_\ell\xi_\ell(x)$).*

1) *Let ϕ, $\psi : (R^r,0) \longrightarrow (R^\ell,0)$ be map germs. If ϕ and ψ are right equivalent (i.e. there exists a diffeomorphism germ $h : (R^r,0) \longrightarrow (R^r,0)$ such that*

$\phi = \psi \circ h$), then ϕ^*F and ψ^*F are S.P-K-equivalent.

2) Suppose that $r = 1$. Then ϕ^*F has a Σ_0^i-type if and only if $\frac{d\phi}{du}|_{u=0} \notin \mathrm{Ker}(dF_v)$.

Proof. The statement 1) is trivial.

2) By Proposition 2.3, we may suppose that $\mathrm{rank}(df_x) = 0$. Hence, ϕ^*F has Σ_0^i-type if and only if $\mathrm{rank}(d(\phi^*F)_u) = 1$. By the chain rule, $d(\phi^*F)_u = \sum_{i=1}^{\ell} \frac{\partial F}{\partial v_i}(0) \cdot \frac{d\phi}{du} i(0)$. Q.E.D.

Our classifications are the following theorems.

THEOREM 5.11. a) Let $f : (R^n \times R, 0) \longrightarrow (R^n, 0)$ be a map germ with P-K-cod(f) ≤ 3. Then f is P-K-equivalent to one of germs in Table II-(1).

b) If f is of discrete algebra type and P-K-cod(f) ≤ 4. Then f is P-K-equivalent to one of germs in Table II-(1) or II-(2).

THEOREM 5.12. (1-parameter bifurcations of hypersurfaces). Let $f : (R^n \times R, 0) \longrightarrow (R, 0)$ be a function germ with P-K-cod(f) ≤ 4. Then f is P-K-equivalent to one of germs in Table III.

Proof of Theorem 5.11. a) By Proposition 2.3, Lemma 5.7 and Proposition 5.9, we may consider germs in the following list :

(1) $\phi_0(u) + \phi_1(u)x + \cdots + \phi_{k-2}(u)x^{k-2} \pm x^k$,

(2) $(\phi_0(u) + xy, \phi_1(u) + \phi_2(u)x + \phi_3(u)y + x^2 \pm y^2)$,

where $2 \leq k \leq 5$ and ϕ_i are polynomial function germs.

(1). Let $s = \min\{\mathrm{order}(\phi_i) \mid i = 0, \ldots, k-2\}$.

Case $(1)_0$. If $s = \mathrm{order}(\phi_0)$, then f is P-K-equivalent to $\pm u^s + \phi_1(u)x + \cdots + \phi_{k-2}(u)x^{k-2} \pm x^k$. By Example 3.15, $\pm u^s \pm x^k$ is (k,s)-determined relative to P-K. Hence, f is P-K-equivalent to

$$\pm u^s + a_1 u^s x + \cdots + a_{k-2} u^s x^{k-2} \pm x^k.$$

Since $\pm 1 + g(x) = \pm 1 + a_1 x + \cdots + a_{k-2} x^{k-2}$ is an unit elemnt, f is P-K-equivalent to $\pm u^s \pm (1/(1+g(x)))x^k$.

Because $\pm u^s \pm x^k$ is (k,s)-determined relative to P-K, $\pm u^s \pm (1/(1+g(x)))x^k$ is P-K-equivalent to $\pm u^s \pm x^k$.

In this case, one easily computes that P-K-cod(f) = $s(k-1)-1$, so that, (k,s) must be $(2,1)$, $(2,2)$, $(2,3)$, $(2,4)$, $(3,1)$, $(3,2)$, $(4,1)$ or $(5,1)$.

Case $(1)_j$. If $s = \mathrm{order}(\phi_j)$ where $j \geq 1$, then f is P-K-equivalent to a germ of the form

$$\pm u^s(x^j + \psi_0(u) + \cdots + \psi_{k-2}(u)x^{k-2}) \pm x^k.$$

If $\mathrm{order}(\psi_0(u)) = \ell$, then f is $(k,s+\ell)$-determined relative to P-K, so that, it is P-K-equivalent to a germ of the form

$$\pm u^s x^j + a_0 u^{s+\ell} + \psi_1'(u)x + \cdots + \psi_{k-2}'(u)x^{k-2} \pm x^k,$$

where $\mathrm{degree}(\psi_i'(u)) \leq s+\ell$. We can easily show that P-K-cod(f) $\geq s(k-1)$, so that, (k,s) must be $(3,1)$, $(3,2)$ or $(4,1)$. In the case where $k = 3$, j must be 1, hence $f = \pm u^s x + a_0 u^{s+\ell} \pm x^3$ where $s = 1$ or 2. In this case, if $s = 2$, then P-K-cod(f) = 5. If $s = 1$, then f is P-K-equivalent to the germ $x^3 \pm ux$. In other cases, by the same method of the above, we can show that f is P-K-equivalent to the germ $x^4 \pm ux$.

(2). Let $s = \min\{\mathrm{order}(\phi_i) \mid i = 1, 2 \text{ or } 3\}$.

Case$(2)_0$. If $s = \mathrm{order}(\phi_0)$, then f is P-K-equivalent to $(\pm u^s + xy, \phi_1(u) + \phi_2(u)x + \phi_3(u)y + x^2 \pm y^2)$.

By Example 3.17, $(\pm u^s + xy, x^2 \pm y^2)$ is $(2,s)$-determined relative to P-K. Thus, f is P-K-equivalent to

$$(\pm u^s + xy, a_1 u^s + a_2 u^s x + a_3 u^s y + x^2 \pm y^2).$$

Suppose that $s = 1$.

If $a_1 = 0$, then P-K-cod(f) ≥ 5 except for the case of $(a_2,a_3) = (0,0)$. Hence, f is P-K-equivalent to $(u \pm xy, x^2 \pm y^2)$ in this case.

If $a_1 \neq 0$, then P-K-cod(f) ≥ 5 except for the case of $a_1 = \pm 1$. Hence, f is P-K-equivalent to

$$(u \pm xy, \pm u + a_2 ux + a_3 uy + x^2 \pm y^2).$$

We can easily show that P-K-cod(f) ≥ 5 except for the case of $(a_2,a_3) = (0,0)$, so that, f is P-K-equivalent to

$$(u \pm xy, x^2 \pm y^2 \pm xy).$$

In the case where $s \geq 2$, one easily computes that $P\text{-}K\text{-cod}(f) \geq 5$.

Case $(2)_1$. If $s = \text{order}(\phi_1)$, f is P-K-equivalent to $(\phi_0(u) + xy, \pm u^s + \phi_2(u)x + \phi_3(u)y + x^2 \pm y^2)$.

By Example 3.17, $(xy, \pm u^s + x^2 \pm y^2)$ is $(2,s)$-determined relative to P-K, so that, f is P-K-equivalent to

$$(a_0u^s + xy, \pm u^s + a_2u^sx + a_3u^sy + x^2 \pm y^2).$$

By the same calculation as in Case $(2)_0$, f is P-K-equivalent to $(\pm u + xy, x^2 \pm y^2)$, $(xy, \pm u + x^2 \pm y^2)$ or $(u \pm xy, x^2 \pm y^2 \pm xy)$.

Case $(2)_2$. If $s = \text{order}(\phi_2)$, then f is P-K-equivalent to $(\phi_0(u) + xy, \phi_1(u) \pm u^sx + \phi_3(u)y + x^2 \pm y^2)$.

Suppose that $\text{order}(\phi_1) = \ell$. By Example 3.17, f is P-K-equivalent to a germ of the form

$$(\psi_0(u) + xy, au^\ell \pm u^sx + \psi_3(u)y + x^2 \pm y^2),$$

where $\psi_0(u)$ and $\psi_3(u)$ are polynomials of order $\geq s$ and degree $\leq \ell$.

In this case, one easily computes that $P\text{-}K\text{-cod}(f) \geq 5$.

The case of $(2)_3$ is in the same situation as in the case of $(2)_2$. This completes the proof of the statement a).

b) In this case, by the same calculation as in the case of a), f is P-K-equivalent to one of germs in Table II-(2). This completes the proof. Q.E.D.

The proof of Theorem 5.12 is the same as that of Theorem 5.11. We respectively use Proposition 5.9 2), Example 3.15 and Example 3.16 instead of Proposition 5.9 1) and Example 3.17.

Table I-(1)

K-cod(f)	germ
ℓ	$(x_1,\ldots,x_{k-1},x_k^{\ell+1})$
4	$(x_1,\ldots,x_{k-2},x_{k-1}x_k,x_{k-1}^2\pm x_k^2)$
5	$(x_1,\ldots,x_{k-2},x_{k-1}x_k,x_{k-1}^2\pm x_k^3)$

Table I-(2)

K-cod(f)	germ
ℓ	$x_1^{\ell+1}+Q(x_2,\ldots,x_n)$
4	$x_1^3+x_2^3+Q(x_3,\ldots,x_n)$ $x_1^3-x_1x_2^2+Q(x_3,\ldots,x_n)$
5	$x_1^2x_2+x_2^4+Q(x_3,\ldots,x_n)$

Here, $Q(x_i,\ldots,x_n) = \pm x_i^2\pm x_{i+1}^2\pm\cdots\pm x_n^2$.

Table II-(1)

P-K-cod(f)	germ
0	$(x_1,\ldots,x_{n-1},u\pm x_n^2)$
1	$(x_1,\ldots,x_{n-1},u+x_n^3)$ $(x_1,\ldots,x_{n-1},\pm u^2\pm x_n^2)$
2	$(x_1,\ldots,x_{n-1},u\pm x_n^4)$ $(x_1,\ldots,x_{n-1},x_n^3\pm ux_n)$ $(x_1,\ldots,x_{n-1},\pm u^2+x_n^3)$ $(x_1,\ldots,x_{n-1},u^3\pm x_n^2)$
3	$(x_1,\ldots,x_{n-1},u+x_n^5)$ $(x_1,\ldots,x_{n-1},\pm u^2\pm x_n^4)$ $(x_1,\ldots,x_{n-1},u^3+x_n^3)$ $(x_1,\ldots,x_{n-1},x_n^4\pm ux_n)$ $(x_1,\ldots,x_{n-2},u\pm x_{n-1}x_n,x_{n-1}^2\pm x_n^2)$ $(x_1,\ldots,x_{n-2},\pm x_{n-1}x_n,u\pm x_{n-1}^2\pm x_n^2)$ $(x_1,\ldots,x_{n-2},x_{n-1}^2\pm x_n^2\pm x_{n-1}x_n,u\pm x_{n-1}^2\pm x_n^2)$

Table II-(2)

P-K-cod(f)	germ
4	$(x_1,\dots,x_{n-1},u\pm x_n^6)$
	$(x_1,\dots,x_{n-1},x_n^5\pm ux_n)$
	$(x_1,\dots,x_{n-2},\pm x_{n-1}x_n,u\pm x_{n-1}^2\pm x_n^3)$

Table III

P-K-cod(f)	germ
0	$u+Q(x_1,\dots,x_n)$
1	$u+x_1^3+Q(x_2,\dots,x_n)$
	$\pm u^2+Q(x_1,\dots,x_n)$
2	$u\pm x_1^4+Q(x_2,\dots,x_n)$
	$\pm u^2+x_1^3+Q(x_2,\dots,x_n)$
	$u^3+Q(x_1,\dots,x_n)$
	$x_1^3\pm ux_1+Q(x_2,\dots,x_n)$
3	$u+x_1^5+Q(x_2,\dots,x_n)$
	$u\pm(x_1^3+x_2^3)+Q(x_3,\dots,x_n)$
	$u\pm(x_1^3-x_1x_2^2)+Q(x_3,\dots,x_n)$
	$u^4+Q(x_1,\dots,x_n)$
	$x_1^4\pm ux_1+Q(x_2,\dots,x_n)$
4	$u\pm x_1^6+Q(x_2,\dots,x_n)$
	$u\pm(x_1^2x_2+x_2^4)+Q(x_3,\dots,x_n)$
	$u^5+Q(x_1,\dots,x_n)$
	$x_1^5\pm ux_1+Q(x_2,\dots,x_n)$

Here, $Q(x_i,\dots,x_n) = \pm x_i^2\pm\cdots\pm x_n^2$

In the above lists, P-K-versal deformations can be obtained by Theorem 4.1, but we don't have the space where these lists are displayed.

By Example 3.18, we have the following proposition.

PROPOSITION 5.13. *Let* $f : (R\times R^2,0) \longrightarrow (R,0)$ *be a map germ with* $K\text{-cod}(f_0) = 1$ *and* $P\text{-}K\text{-cod}(f) \leq 5$. *Then* f *is P-K-equivalent to one of the following germs* :

P-K-cod(f)	germ	P-K-versal deformation
0	$\pm x^2+u_1$	$\pm x^2+u_1$
1	$\pm x^2\pm u_1^2\pm u_2^2$	$\pm x^2\pm u_1^2\pm u_2^2+v_1$
2	$\pm x^2\pm u_1^2+u_2^3$	$\pm x^2\pm u_1^2+u_2^3+v_1+v_2u_2$
3	$\pm x^2\pm u_1^2\pm u_2^4$	$\pm x^2\pm u_1^2\pm u_2^4+v_1+v_2u_2+v_3u_2^2$
4	$\pm x^2+u_1^3-u_1u_2^2$	$\pm x^2+u_1^3-u_1u_2^2+v_1+v_2u_1+v_3u_1+v_4(u_1^2+u_2^2)$
	$\pm x^2+u_1^3+u_2^3$	$\pm x^2+u_1^3+u_2^3+v_1+v_2u_1+v_3u_2+v_4u_1u_2$
	$\pm x^2\pm u_1^2+u_2^5$	$\pm x^2\pm u_1^2+u_2^5+v_1+v_2u_2+v_3u_2^2+v_4u_2^3$
5	$\pm x^2\pm(u_1^2u_2+u_2^4)$	$\pm x^2\pm(u_1^2u_2+u_2^4)+v_1+v_2u_1+v_3u_2+v_4u_1^2$ $+v_5u_2^2$
	$\pm x^2\pm u_1^2\pm u_2^6$	$\pm x^2\pm u_1^2\pm u_2^6+v_1+v_2u_2+v_3u_2^2+v_4u_2^3+v_5u_2^4$

REMARK. In [15], T. Poston points out that the germ $x^2 + u_1^2 + u_2^3$ describes the phenomena of babbles broken off and dying.

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