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## Singular Perturbation of Domains and Semilinear Elliptic Equation

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## Contents

§	1	Introduction
§	2	Existence of Stable Solutions
§	3	Asymptotic Behavior on the Thin Part
§	4	Construction of Unstable Solution
§	5	Concluding Remarks

§1 Introduction We consider the following semilinear elliptic boundary value problem,

(1.1) 
$$\begin{pmatrix} \Delta v + f(v) = 0 & \text{in } \Omega \\ \\ \\ \\ \frac{\partial v}{\partial v} = 0 & \text{on } \partial \Omega \end{pmatrix}.$$

where  $\Omega$  is a bounded domain in  $\mathbb{R}^n$  with smooth boundary  $\partial \Omega$  and  $\nu$  denotes the unit outer normal vector on  $\partial \Omega$ .  $\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$  is the Laplace operator and f is a real valued smooth function on  $\mathbb{R}$ .

The structure of the solutions of (1.1) and their stability largely depend upon the geometrical property of the domain  $\Omega$  and we may consider that the structure usually varies continuously under the smooth deformation of  $\Omega$ . Our subject in this paper is to consider the behavior of the solutions and their structure when the domain  $\Omega$  singularly perturbs. The domain which we deal with is exhibited in Figure 1 and it is decomposed as follows  $\Omega(\zeta) = D_1 \cup D_2 \cup Q(\zeta)$  where  $D_1$  and  $D_2$  are mutually disjoint and  $Q(\zeta)$  is a moving part which approaches a segment as  $\zeta \downarrow 0$ .



- 1 -

Then can we say that the influence of  $Q(\zeta)$  over (1.1) for  $\Omega = \Omega(\zeta)$ vanishes as  $\zeta \downarrow 0$  i.e. that the structure of the solutions of (1.1) for  $\Omega = \Omega(\zeta)$  ( for small  $\zeta > 0$  ) is equivalent to that of (1.1) for  $\Omega = \Omega_0 = D_1 \cup D_2$  (Figure 2)? In fact, Vegas [22], Hale and Vegas [10] have considered (1.1) for  $f = f(\lambda, u) = \lambda u - u^p$  on the same domain as that in Figure 1 and analyzed the bifurcation phenomenon for the bifurcation parameter  $\zeta$ ( when  $\lambda > 0$  is a sufficiently small constant ). Their bifurcation diagram in the case that p is an odd natural number and the domain  $\Omega(\zeta)$  is symmetric, is in Figure 3.



## Figure 3 (Bifurcation Diagram)

In their situation, when  $\zeta$  is very small ( i.e.  $0 < \zeta < \zeta_2$  in Figure 3 ) there are exactly nine solutions and each of them takes values near one of the values  $\left\{ 0, \lambda^{1/(p-1)}, -\lambda^{1/(p-1)} \right\}$  in  $D_i$ ( i = 1, 2 ) and its behavior on  $Q(\zeta)$  is automatically determined by the behavior on  $D_1$  and  $D_2$ . Thus the structure of the solutions for  $\Omega(\zeta)$  (  $0 < \zeta < \zeta_2$  ) is equivalent to that for  $\Omega = \Omega_0$ ( non-connected open set ). Remark that (1.1) for  $\Omega = \Omega_0$  has exactly nine solutions, each of which is equal to one of the values  $\left\{ 0, \lambda^{1/(p-1)}, -\lambda^{1/(p-1)} \right\}$  in  $D_i$  for each i for sufficiently small  $\lambda > 0$ . In this case,  $\Omega(\zeta)$  can be regarded as a perturbation from  $\Omega_0$ . Nevertheless, in this paper, we conclude that it is more natural to regard  $\Omega(\zeta)$  as a perturbation rather from the set  $\Omega_* = D_1 \cup D_2 \cup L$  (exhibited in Figure 4) where  $L = \bigcap_{\substack{\zeta > 0 \\ \zeta > 0}} Q(\zeta)$ than from  $\Omega_0 = D_1 \cup D_2$  if we consider the domain perturbation up to the structure of the solutions of (1.1) for  $\Omega = \Omega(\zeta)$ .



## Figure 4

In the situation of Hale-Vegas [10] and Vegas [22], we remark that  $\frac{\partial f}{\partial u}$  is small around the solutions from the smallness of  $\lambda > 0$  and this may ensure the uniqueness of the behavior of the solution v on  $Q(\zeta)$  when v is specified to take values near  $a_i$  in  $D_i$  (i = 1, 2) where  $f(a_i) = 0$  and  $f'(a_i) < 0$ , but the solution is rather free on  $Q(\zeta)$  for general f. We consider a family of functions  $\begin{cases} v_{\zeta} \\ \zeta > 0 \end{cases}$  such that  $v_{\zeta}$  is an arbitrary solution of (1.1) for  $\Omega = \Omega(\zeta)$  and  $\lim_{\zeta \to 0} \|v_{\zeta} - a_i\|_{L^2(D_i)} = 0$  holds for i = 1, 2 where  $a_i$  is any  $\zeta \to 0$  for satisfying  $f(a_i) = 0$  and  $f'(a_i) < 0$  and we prove that for any sequence of positive values  $\{\zeta_m\}_{m=1}^{\infty}$  such that  $\lim_{m \to \infty} \zeta_m = 0$ , there exist a subsequence  $\{\varkappa_m\}_{m=1}^{\infty} \subset \{\zeta_m\}_{m=1}^{\infty}$  and a solution v of the two point boundary value problem of the ordinary differential equation (1.2),

(1.2) 
$$\begin{pmatrix} \frac{d^2 V}{dz^2} + f(V) = 0 \text{ in } L \\ V(z) = a_i \quad z \in \overline{D}_i \cap \overline{L} \quad (i = 1, 2) \end{pmatrix}$$

such that  $v_{x_m}$  is asymptotically near to V in  $Q(x_m)$  in the sense of "uniform convergence" and near to a in  $\overline{Q(\varkappa_m)} \cap \overline{D}_i$ . In this case, the stability of  $v_{\varkappa_m}$  in (1.1) for  $\Omega = \Omega(\varkappa_m)$ coincides with the stability of V in (1.2) for large m . Conversely, we take an appropriate nonlinear term f for which (1.2) has two stable solutions  $V^{(0)} < V^{(2)}$  and another unstable solution  $V^{(1)}$  between them, in the case that  $a_1 = a_2 = b_1$  and  $f(b_1) = 0$  and  $f'(b_1) < 0$ , and we construct three distinct solutions  $v_{\zeta}^{(0)} < v_{\zeta}^{(1)} < v_{\zeta}^{(2)}$  of (1.1) for  $\Omega = \Omega(\zeta)$  small  $\zeta > 0$ such that  $v_{\zeta}^{(i)}$  behaves like  $V^{(i)}$  in  $Q(\zeta)$  (i = 0,1,2) and takes values near  $b_1$  in  $D_1 \cup D_2$  and  $v_\zeta^{(0)}$  and  $v_\zeta^{(2)}$  are stable and  $v_{\zeta}^{(1)}$  is unstable for small  $\zeta > 0$  . Therefore we see that the behavior of the solution  $\, {
m v}_{\zeta} \,$  on Q( $\zeta$ ) which is almost governed by the equation (1.2) on L , plays an important role to determine the stability of  $v_{\zeta}$  even if  $\zeta > 0$  is small. From these facts, we conclude that we should regard  $\, arnothing(\zeta) \,$  as a perturbation from  $\Omega_* = D_1 \cup D_2 \cup L .$ 

The boundary value problem (1.1) is a stationary problem of the following parabolic boundary value problem,

(1.3) 
$$\begin{pmatrix} \frac{\partial u}{\partial t} = \Delta u + f(u) & \text{in } (0, \infty) \times \Omega \\ \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } (0, \infty) \times \partial \Omega \end{pmatrix}$$

Definition 0. A solution of (1.3) which is independent of the variable t is said to be an equilibrium solution.

- 4 -

We recall the definition of the stability of an equilibrium solution.

Definition 1. The equilibrium solution v of (1.3) is said to be stable if given any  $\varepsilon > 0$ , there exists a  $\delta > 0$ , such that  $\|u(t, \cdot) - v(\cdot)\|_{L^{\infty}(\Omega)} \leq \varepsilon$  ( $0 < t < \infty$ ) for any  $w \in C^{0}(\overline{\Omega})$ satisfying  $\|v - w\|_{L^{\infty}(\Omega)} \leq \delta$ , where u is a solution of (1.3) with the initial condition u(0,x) = w(x). We say that w is unstable if v is not stable. For details, see Matano [15].

It has been observed by several authors that the stability and the structure of the equilibrium solutions are closely related to the geometry of the domain  $\Omega$ . It is known that any non-constant equilibrium solution must be unstable if  $\Omega$  is a bounded convex domain in  $\mathbb{R}^n$ . (See N.Chafee [4] for n = 1 and see H.Matano [15] and Casten-Holland [3] for general n.) More generally, the same result holds in the case that  $\Omega$  is a Riemannian manifold with non-negative Ricci curvature and  $\partial\Omega$  has non-positive definite second fundamental form with respect to the unit outer normal vector  $\nu$  on  $\partial\Omega$  (S.Jimbo [11]). On the other hand, Matano [15] has constructed a non-constant stable equilibrium solution on the same type of domain as  $\Omega(\zeta)$  in Figure 1. We shall refine his result in Section 2. On the other hand, there are several results as for the reaction-diffusion system. See K.Kishimoto and H.F.Weinberger [13], H.Matano and M.Mimura [17].

- 5 -

The contents of this paper are as follows.

In Section 2, first we will set a perturbing domain  $\Omega(\zeta) = \bigcup_{i=1}^{N} D_i \cup Q(\zeta) \text{ under a rather weak condition ( so it may be a i=1)}$ very wild perturbation ) and for small  $\zeta > 0$ , we will construct a stable equilibrium solution  $v_{\zeta}$  of (1.3) for  $\Omega = \Omega(\zeta)$  which takes values near  $a_i$  in  $D_i$  ( $1 \leq i \leq N$ ) where  $a_i$  is an arbitrary zero point of f such that  $f'(a_i) < 0$ .

In Section 3, we will establish the domain  $\Omega(\zeta)$  in Figure 1 concretely ( for the delicate argument ) and analyze the behavior on  $Q(\zeta)$  of the solution of (1.1) for  $\Omega = \Omega(\zeta)$  which takes values near  $a_i$  in  $D_i$  (  $f(a_i) = 0$ ,  $f'(a_i) < 0$  ) and we prove that  $v_{\zeta}$  is asymptotically near to some solution of the ordinary differential equation (1.2) up to the stability.

In Section 4, we will choose an appropriate f (  $a_1 = a_2 = b_1$ in this case ) so that (1.2) has two stable solutions  $v^{(0)} < v^{(2)}$ and another unstable solution  $v^{(1)}$  such that  $v^{(0)} < v^{(1)} < v^{(2)}$ . For the domain  $\Omega(\zeta)$  in Section 3 and these f and  $v^{(0)}$ ,  $v^{(1)}$ and  $v^{(2)}$ , we shall construct three distinct solutions  $v^{(0)}_{\zeta}$ ,  $v^{(1)}_{\zeta}$ and  $v^{(2)}_{\zeta}$  such that  $v^{(i)}_{\zeta}$  behaves like  $v^{(i)}$  in  $Q(\zeta)$  and takes values near  $b_1$  in  $D_1 \cup D_2$  for each i (  $0 \le i \le 2$  ) and  $v^{(0)}_{\zeta}$  and  $v^{(2)}_{\zeta}$  are stable and  $v^{(1)}_{\zeta}$  is unstable for small  $\zeta > 0$ . All the functions that we consider in this paper are real valued. §2 Existence of Stable Solutions.

Let  $D_1$ ,  $D_2$ ,  $\cdots$ ,  $D_N$  be bounded domains in  $\mathbb{R}^n$  ( $n \ge 2$ ) such that each  $D_j$  has a smooth boundary  $\partial D_j$  and  $D_i \cap D_j = \emptyset$  holds for any i and j with i > j. From now on we establish the situation.

- (II-1) Let  $\{\Omega(\zeta)\}_{\zeta>0}$  be a family of bounded domains in  $\mathbb{R}^n$  which satisfies the following conditions (1) and (2);
  - (1) Each  $\Omega(\zeta)$  has a smooth boundary and  $\Omega(\zeta_1) \supset \Omega(\zeta_2) \supset \bigcup_{i=1}^{\vee} D_i$ holds for any  $\zeta_1$  and  $\zeta_2$  such that  $\zeta_1 > \zeta_2 > 0$ .
  - (2)  $\lim_{\zeta \to 0} \operatorname{Vol}(\Omega(\zeta) \bigcup_{i=1}^{N} D_i) = 0$

(II-2) Let f be a real valued smooth function on  $\mathbb{R}$  such that the set  $\Pi = \left\{ \xi \in \mathbb{R} \mid f(\xi) = 0, f'(\xi) < 0 \right\}$  is not empty.

Under the above conditions (II-1) and (II-2), we will consider the equilibrium solutions of the following semilinear diffusion equation (2.1).

(2.1) 
$$\begin{pmatrix} \frac{\partial u}{\partial t} = \Delta u + f(u) & \text{in } (0,\infty) \times \Omega(\zeta), \\\\ \frac{\partial u}{\partial \nu} = 0 & \text{on } (0,\infty) \times \partial \Omega(\zeta). \end{pmatrix}$$

We present our first result concerning the existence of stable equilibrium solution which approaches the constant function on each D, when  $\zeta \to 0$ .

Theorem 1. For any sequence of values  $\{a_i\}_{i=1}^N$  which is contained in the set  $\Pi = \{\xi \in \mathbb{R} \mid f(\xi) = 0, f'(\xi) < 0\}$  and for small  $\xi > 0$ , the boundary value problem (2.1) has at least one stable equilibrium solution  $v_{\zeta}$  which satisfies the following condition (2.2),

$$(2.2) \left\{ \begin{array}{ccc} \lim_{\zeta \to 0} & \| v_{\zeta} - a_{i} \|_{L^{2}(D_{i})} = 0 & (1 \leq i \leq N) \\ \lim_{\zeta \to 0} & v_{\zeta} = a_{i} & \text{in } C^{\infty}(\overline{D_{i}(\eta)}) & \text{for any } \eta > 0 , \\ & (1 \leq i \leq N) \end{array} \right.$$
where we have defined  $D_{i}(\eta) = \left\{ x \in D_{i} | \operatorname{dis}(x, \Omega(\eta) - D_{i}) > \eta \right\}$ 
for any  $\eta > 0$ .

Remark. Hale and Vegas [10] have proved a similar result to our Theorem 1 ( also the uniqueness ) under some asumption concerning the bound of  $\frac{\partial f}{\partial u}$  with the aid of the Implicit Function Theorem. But we do not impose any assumption concerning the bound of  $\frac{\partial f}{\partial u}$  and therefore we cannot apply the Implicit Function Theorem, because, as we will prove in Section 4, we can not expect the uniqueness of  $v_{\zeta}$  which satisfies (2.2) in general. We apply the result of Matano [15] ( Theorem 4.2 in [15] ) essentially.

For the proof of Theorem 1, we use the Poincaré type inequality.

Proposition 1. Let D be a bounded domain in  $\mathbb{R}^n$  with smooth boundary  $\partial D$ . Let  $\{\lambda_q\}_{q=1}^{\infty}$  and  $\{\psi_q\}_{q=1}^{\infty}$  be respectively the sequence of eigenvalues arranged in increasing order and the complete system of the corresponding orthonormalized eigenfunctions associated with -4 with Neumann boundary condition. Then we have the following inequality :

- 8 -

$$\frac{1}{\lambda_{q+1}} \int_{D} |\operatorname{grad} \psi|^{2} dx + \sum_{k=1}^{q} \frac{\lambda_{q+1} - \lambda_{k}}{\lambda_{q+1}} \left( \int_{D} \psi \psi_{k} dx \right)^{2}$$

$$\geq \int_{D} |\psi|^{2} dx \quad \text{for any } \psi \in \operatorname{H}^{1}(D) \text{ and natural number } q.$$

This can be easily proved by the eigenfunction-expansion and so we omit the proof.

Hereafter we denote by {  $\lambda_{i,q}$ } q=1 and {  $\psi_{i,q}$ } q=1, respectively the sequence of eigenvalues arranged in increasing order and the complete system of corresponding orthonormalized eigenfunctions associated with the operator -4 on  $D_i$  with Neumann boundary condition. Hereafter we put  $Q(\zeta) \equiv \Omega(\zeta) - \bigcup_{i=1}^{N} D_i$ .

(2.3)  $\begin{pmatrix} A(x) = a_i \text{ for any } x \in D_i & (1 \leq i \leq N) \\ a_* \leq A(x) \leq a^* \text{ for any } x \in \mathbb{R}^n \text{ and grad } A(x) \\ \text{ has compact support in } \mathbb{R}^n. \end{cases}$ 

We define for  $w \in H^{1}(\Omega(\zeta)) \cap L^{\infty}(\Omega(\zeta))$ ,

$$J_{\zeta}(w) \equiv \int_{\Omega(\zeta)} \left(\frac{1}{2} | \text{grad } w |^2 - \int_{A(x)}^{w(x)} f(\xi) d\xi \right) dx$$

and also we define for  $\zeta > 0$  and  $\delta > 0$ ,

$$E(\delta,\zeta) = \left\{ w \in C^{2}(\Omega(\zeta)) \cap C^{1}(\overline{\Omega(\zeta)}) \mid a_{*} - \delta \leq w(x) \leq a^{*} + \delta \text{ in } \Omega(\zeta) \right\}$$

 $J_{\zeta}(w) \leq J_{\zeta}(A) + \delta^{3}, \quad \| w - a_{i} \|_{L^{2}(D_{i})} \leq \delta, i = 1, 2, \cdots, N \}.$ 

To prove the existence of stable equilibrium solution of (2.1) by the aid of Theorem 4.2 in [15], we will find a positive valued function  $\delta(\zeta)$  ( $\zeta > 0$ ) which satisfies the following conditions

 $\left(\begin{array}{c} \lim_{\zeta \to 0} \delta(\zeta) = 0 \\ \zeta \to 0 \\ E(\delta, \zeta) \text{ is a positively invariant closed subset of} \\ C^1(\overline{\Omega(\zeta)}) \cap C^2(\Omega(\zeta)) \text{ under the flow defined by the} \\ equation (2.1) \text{ when } \delta \text{ belongs to the interval} \\ [\delta(\zeta), 2\delta(\zeta)] \text{ for small } \zeta > 0 . \end{array} \right)$ 

It is clear by the aid of the Comparison - Existence Theorem that if  $\delta_0 > 0$  is small so that  $f'(\xi) < 0$  for  $\xi \in [a_* - \delta_0, a_*) \cup (a^*, a^* + \delta_0]$  hold, there exists a unique classical global solution  $u_{\zeta}(t,x)$  with  $u_{\zeta}(0,x) = w(x)$  and  $a_* - \delta \leq u_{\zeta}(t,x) \leq a^* + \delta$ ,  $x \in \Omega(\zeta)$ ,  $t \geq 0$  for any  $w \in C^2(\Omega(\zeta)) \cap C^1(\overline{\Omega(\zeta)})$  such that  $a_* - \delta \leq w(x) \leq a^* + \delta$  ( $x \in \Omega(\zeta), 0 < \delta < \delta_0$ ).

Notice that  $\delta_0$  depends only on f. From now on, we will argue about the behavior of  $u_{\zeta}(t, \cdot)$  when t grows up, under the condition that the initial condition w belongs to the set  $E(\delta, \zeta)$ . Notice that  $u_{\zeta}$  also satisfies the equation given by replacing f in (2.1) by  $\tilde{f}$  which is identical to f on the interval  $[a_* - \delta_0, a^* + \delta_0]$  and has compact support in R , because the value of  $u_{\zeta}(t,x)$  always belongs to the above interval. Therefore from now up to the end of the proof of Theorem 1, we assume, without loss of generality, that f has a compact support in R .

- 10 -

For each i ( $1 \leq i \leq N$ ), we define  $v_{i,k}^{\zeta}$  and  $u_{i,q}^{\zeta}$  as follows,

$$v_{i,k}^{\zeta}(t) = \int_{D_{i}} u_{\zeta}(t,x) \psi_{i,k}(x) dx$$
$$u_{i,q}^{\zeta}(t,x) = \sum_{k=1}^{q} v_{i,k}^{\zeta}(t) \psi_{i,k}(x)$$

and applying the inequality of Proposition 1 to  $\psi = u_{\zeta} - u_{i,k}^{\zeta}$  and

D = D<sub>i</sub> there, we have the following inequality for each i.  
(2.5) 
$$\int_{D_{i}} |\operatorname{grad} u_{\zeta}(t,x)|^{2} dx \geq \lambda_{i,q+1} \int_{D_{i}} |u_{\zeta}(t,x)-u_{i,q}^{\zeta}(t,x)|^{2} dx$$

$$+ \sum_{k=1}^{q} \lambda_{i,k} (\nu_{i,k}^{\zeta}(t))^{2}.$$

On the other hand, the following inequality (2.6) is derived from  $\frac{\partial}{\partial t} J_{\zeta}(u_{\zeta}(t, \cdot)) \leq 0.$ Notice that the second s

$$(2.6) \sum_{i=1}^{N} \int_{D_{i}} \left(\frac{1}{2} |\operatorname{grad} u_{\zeta}(t,x)|^{2} - \int_{A(x)}^{u_{\zeta}(t,x)} f(\xi) d\xi \right) dx + \int_{Q(\zeta)} \left(\frac{1}{2} |\operatorname{grad} u_{\zeta}(t,x)|^{2} - \int_{A(x)}^{u_{\zeta}(t,x)} f(\xi) d\xi \right) dx \\ \leq J_{\zeta}(w) \leq J_{\zeta}(A) + \delta^{3}.$$

By (2.5) and (2.6), we have,

$$(2.7) \frac{1}{2} \sum_{i=1}^{N} (\lambda_{i,q+1} \int_{D_{i}} |u_{\zeta} - u_{i,q}^{\zeta}|^{2} dx + \sum_{k=1}^{q} \lambda_{i,k} (\nu_{i,k}^{\zeta}(t))^{2}) - \sum_{i=1}^{N} \int_{D_{i}} \int_{A}^{u_{\zeta}} f(\xi) d\xi dx + \int_{Q(\zeta)} (\frac{1}{2} |grad u_{\zeta}|^{2} - \int_{A}^{u_{\zeta}} f(\xi) d\xi) dx \leq J_{\zeta}(w) \leq J_{\zeta}(A) + \delta^{3}.$$

Concerning the second term, we have for each i (  $1 \leq i \leq N$  ),

- 12 -

Then we have from the above,

$$(2.10) \int_{D_{i}} \int_{u_{i,q}^{\zeta}}^{u_{\zeta}} f(\xi) d\xi dx \leq c_{1}(\frac{1}{2} + \frac{1}{2\alpha}) \int_{D_{i}} |u_{\zeta} - u_{i,q}^{\zeta}|^{2} dx + \frac{1}{2\alpha} |u_{\zeta} - u_{i,q}^{\zeta}|^{2} dx + \frac{1}{2\alpha} |u_{\zeta} - u_{i,q}^{\zeta}|^{2} dx$$

From (2.7),(2.8),(2.9) and (2.10), we have the following inequality (2.11) by using  $\lambda_{i,1} = 0$ :

$$(2.11) \quad \sum_{i=1}^{N} \left( \frac{1}{2} \lambda_{i,q+1} - \frac{c_{1}}{2} - \frac{c_{1}}{2a} \right) \int_{D_{i}} | u_{\zeta} - u_{i,q}^{\zeta} |^{2} dx$$

$$+ \sum_{i=1}^{N} \sum_{k=2}^{Q} \left( \frac{1}{2} \lambda_{i,k} - \frac{c_{1}}{2} \right) \cdot \left( \nu_{i,q}^{\zeta} \right)^{2} + \sum_{i=1}^{N} \int_{D_{i}} \int_{u_{i,q}^{\zeta}}^{A} f(\xi) d\xi dx$$

$$+ \int_{Q(\zeta)} \left( \frac{1}{2} + | \operatorname{grad} u_{\zeta} |^{2} - \int_{A}^{U_{\zeta}} f(\xi) d\xi \right) dx$$

$$\leq J_{\zeta}(w) \leq J_{\zeta}(A) + \delta^{3} \quad \text{for } t \geq 0 , a > 0 , q \geq 2 .$$
Now we put  $a = \frac{1}{c_{1}} \inf_{1 \leq i \leq N} \lambda_{i,2} > 0 \text{ and fix it, so that we }$ 

$$have \quad \frac{1}{2} \lambda_{i,k} - \frac{c_{1}}{2} a \geq \frac{1}{4} - \lambda_{i,2} \quad (1 \leq i \leq N, k \geq 2).$$
Next we take q sufficiently large so that the inequality   

$$\frac{1}{2} \lambda_{i,q+1} - \frac{1}{2} c_{1} - \frac{c_{1}}{2a} \geq 1 \quad \text{holds for any } i \quad (1 \leq i \leq N)$$

and fix this natural number q .

For  $\alpha$  and q which we have determined above, the following inequality (2.12) easily follows from (2.11).

$$(2.12) \sum_{i=1}^{N} \int_{D_{i}} |u_{\zeta}(t,x) - u_{i,q}^{\zeta}(t,x)|^{2} dx + \sum_{i=1}^{N} \sum_{k=2}^{q} \frac{1}{4} \lambda_{i,2} (\nu_{i,q}^{\zeta}(t))^{2} + \sum_{i=1}^{N} \int_{D_{i}} \int_{u_{i,q}^{q}(t,x)}^{a_{i}} f(\xi) d\xi dx + \int_{Q(\zeta)} \left\{ \frac{1}{2} |\operatorname{grad} u_{\zeta}(t,x)|^{2} - \int_{A(x)}^{u_{\zeta}(t,x)} f(\xi) d\xi \right\} dx \leq J_{\zeta}(w) \leq J_{\zeta}(A) + \delta^{3} (t \geq 0, \zeta > 0, 0 < \delta < \delta_{0}, w \in E(\delta,\zeta)).$$

The inequality (2.12) is the our main tool to prove that  $u_{\zeta}(t, \cdot)$ always stay near A in L<sup>2</sup>-sense if the initial condition w is near A. In the inequality (2.12), only the third term is difficult to deal with and it may be negative if w is not near to A. From now on, we will prove that if  $\delta$  and  $\zeta$  are small, the third term of (2.12) is always nonnegative and furthermore  $|u_{i,q}^{\zeta}(t,x) - a_{i}|$ can be estimated in D<sub>i</sub> for the initial condition  $w \in E(\delta, \zeta)$ . We introduce the following function  $B_{i}(\sigma)$ .

$$B_{i}(\sigma) \equiv \int_{a_{i}+\sigma}^{a_{i}} f(\xi) d\xi \quad (1 \leq i \leq N)$$

From (II-2) and  $\{a_i\}_{i=1}^N \subset \prod$ , it is easy to see that  $B_i$  satisfies the following properties (2.13),(2.14) and (2.15).

- $(2.13) \quad B_{i}(0) = 0$
- (2.14) There exists a positive constant  $\sigma_*$  such that  $B_i(\sigma)$  is positive for any  $\sigma \in [-\sigma_*, 0) \cup (0, \sigma_*]$
- (2.15)  $B_i(\sigma)$  is a strictly convex function in  $\sigma$  on  $(-\sigma_*, \sigma_*)$ .

It is clear that  $K \equiv \min \min \{B_i(-\sigma_*), B_i(\sigma_*)\}$  is positive.  $1 \leq i \leq N$ 

If  $w \in E(\delta, \zeta)$ , we have

$$\delta^{2} \geq \|\mathbf{w} - \mathbf{a}_{i}\|_{L^{2}(D_{i})}^{2} = \{ v_{i,1}^{\zeta}(0) - \mathbf{a}_{i} \operatorname{Vol}(D_{i})^{1/2} \}^{2} + \sum_{k=2}^{\infty} (v_{i,k}^{\zeta}(0))^{2}.$$

We put 
$$c_2 \equiv \max_{1 \leq i \leq N, 1 \leq k \leq q} \| \psi_{i,k} \|_{L^{\infty}(D_i)}$$
 and  $\delta_1 \equiv \min \{\frac{\sigma_*}{1 \leq i \leq N, 4c_2 q^{1/2}, \delta_0} \}$ 

Then for any  $\delta$ ,  $\zeta$  such that  $0 < \delta < \delta_1$ ,  $\zeta > 0$ , we have,

$$(2.16) | u_{i,q}^{\zeta}(0,x) - a_{i} | \leq | \nu_{i,1}^{\zeta}(0) \psi_{i,1}(x) - a_{i} | +$$

 $\sum_{k=2}^{q} | v_{i,k}^{\zeta}(0) \psi_{i,k}(x) | \leq \{ \max_{\substack{1 \leq i \leq N, 1 \leq k \leq q \\ L^{\infty}(D_i)}} \| \psi_{i,k} \|_{L^{\infty}(D_i)} q^{1/2} \}$ 

$$\{ | r_{i,1}^{\zeta}(0) - a_{i} \operatorname{Vol}(D_{i})^{1/2} |^{2} + \sum_{k=2}^{q} (r_{i,k}^{\zeta}(0))^{2} \}^{1/2}$$

 $\leq c_2 q^{1/2} \quad \delta \leq c_2 q^{1/2} \quad \delta_1 \leq \frac{1}{4} \quad \sigma_* \quad \text{in } D_i \quad (1 \leq i \leq N).$ Here we define, for  $\delta$  and  $\zeta$ ,

 $T(\delta,\zeta) =$ 

$$\sup \left\{ \begin{array}{l} t_{*} \geq 0 \mid |\nu_{i,1}^{\zeta}(t)\psi_{i,1}(x) - a_{i}| + \sum_{k=2}^{q} \mid \nu_{k,i}^{\zeta}(t) \psi_{i,k}(x) \mid \leq \sigma_{*} \right.$$
 for any  $(t,x) \in [0,t_{*}] \times D_{i}$   $\left( 1 \leq i \leq N \right) \right\} .$  It is clear that  $T(\delta,\zeta)$  is positive if  $w \in E(\delta,\zeta)$  for  $\delta$  and

 $\zeta$  such that  $0 < \delta < \delta_1$  and  $\zeta > 0$  hold. From now on we will prove that  $T(\delta,\zeta)$  is infinity if  $\delta$  and  $\zeta$  is small.

Lemma 2.1. Let  $\delta_2 \in (0, \delta_1)$  and  $\zeta_1 > 0$  be positive constants such that the following inequality (2.17) holds for any  $(\delta, \zeta) \in (0, \delta_2] \times (0, \zeta_1].$ 

$$(2.17) \quad \delta^3 + c_3 \operatorname{Vol}(Q(\zeta)) \leq$$

 $\min_{\substack{1 \leq i \leq N \\ x \in \mathbb{R}^n}} \min \left\{ B_i(\sigma_*/8) \operatorname{Vol}(D_i), B_i(-\sigma_*/8) \operatorname{Vol}(D_i), \lambda_{i,2} \sigma_*^2/64(q-1)c_2^2 \right\}.$ where  $c_3 \equiv \sup_{x \in \mathbb{R}^n} |\operatorname{grad} A(x)|^2 + \int_{\mathbb{R}} |f(\xi)| d\xi .$ 

Then  $T(\delta,\zeta) = \infty$  for any  $(\delta,\zeta) \in (0,\delta_2] \times (0,\zeta_1]$ .

(Proof of Lemma 2.1) We assume that  $T(\delta,\zeta)$  is finite for some  $(\delta,\zeta) \in (0,\delta_2] \times (0,\zeta_1]$  and  $w \in E(\delta,\zeta)$ . If t belongs to the interval  $[0,T(\delta,\zeta)]$ , the following inequality (2.18) follows from

- 16 -

the definition of  $T(\delta,\zeta)$ ,

$$(2.18) | u_{i,q}^{\zeta}(t,x) - a_{i} | \leq | \nu_{i,1}^{\zeta}(t) \psi_{i,1}(x) - a_{i} |$$
  
+  $\sum_{k=2}^{q} | \nu_{i,k}^{\zeta}(t) \psi_{i,k}(x) | \leq \sigma_{*}$  on [0, T( $\delta,\zeta$ )] × D<sub>i</sub>.

Hence it follows from (2.13), (2.14) and (2.15)

$$\int_{D_{i}} \int_{u_{i,q}^{\zeta}(t,x)}^{a_{i}} f(\xi) d\xi dx \ge 0 \quad (0 \le t \le T(\delta,\zeta), i = 1,2,\cdots,N)$$
  
follows.

As we have the following inequality (2.19) from (2.12) and the definition of  $\rm J_{\zeta}$  and  $\rm c_3$  :

$$(2.19) \sum_{i=1}^{N} \int_{D_{i}} |u_{\zeta}(t,x) - u_{i,q}^{\zeta}(t,x)|^{2} dx + \sum_{i=1}^{N} \sum_{k=2}^{q} \lambda_{i,2} (\nu_{i,k}^{\zeta}(t))^{2} + \sum_{i=1}^{N} \int_{D_{i}} \int_{u_{i,k}^{\zeta}(t,x)}^{a_{i}} f(\xi) d\xi dx \leq c_{3} \operatorname{Vol}(Q(\zeta)) + \delta^{3},$$
we have, for any  $(\delta,\zeta) \in (0,\delta_{2}] \times (0, \zeta_{1}]$  and from (2.17), the following inequalities (2.20) and (2.21),

$$(2.20) \sum_{i=1}^{N} \sum_{k=2}^{q} \frac{1}{4} \lambda_{i,2} (\nu_{i,k}^{\zeta}(t))^{2} \leq \min_{\substack{1 \leq i \leq N \\ 1 \leq i \leq N}} \lambda_{i,2} \sigma_{*}^{2/64(q-1)c_{2}^{2}}$$

$$(2.21) 0 \leq \int_{D_{i}} \int_{u_{i,q}^{q}(t,x)}^{a_{i}} f(\xi) d\xi dx \leq \min_{\substack{1 \leq i \leq N \\ 1 \leq i \leq N}} \{B_{i}(\pm \sigma_{*}/8) \operatorname{Vol}(D_{i})\}$$

$$(0 \leq t \leq T(\delta,\zeta)).$$

By (2.20) and a little calculation, we have  $\sum_{k=2}^{q} |\nu_{i,k}^{\zeta}(t)| \leq \frac{\sigma_{\star}}{4c_{2}}$ (  $0 \leq t \leq T(\delta, \zeta)$  ). Hence we get the following inequality (2.22) : (2.22) |  $u_{i,q}^{\zeta}(t,x) - v_{i,1}^{\zeta}(t) \psi_{i,1}(x)$ |  $= \left| \sum_{k=2}^{q} \nu_{i,k}^{\zeta}(t) \psi_{i,k}(x) \right| \leq \frac{1}{4} \sigma_{\star}.$ Next that from (2.21) by the aid of  $\psi_{i,1}(x) = \text{Vol}(D_i)^{-1/2}$ , we obtain  $(2.23) \ 0 \leq \sum_{i=1}^{N} \int_{D.} B_{i}(\nu_{i,1}^{\zeta}(t) \operatorname{Vol}(D_{i})^{-1/2} - a_{i} + \Psi_{i}(t,x)) \ dx$  $\leq \min_{1 \leq i \leq N} \{ B_{i}(\pm \sigma_{*}/8) Vol(D_{i}) \}$ (  $0 \leq t \leq T(\delta,\zeta)$ ,  $0 < \zeta \leq \zeta_1$ ,  $0 < \delta \leq \delta_2$  ) where we put  $\Psi_{i}(t,x) = \sum_{k=2}^{q} \nu_{i,k}^{\zeta}(t) \psi_{i,k}(x) .$ Remark that  $\Psi_i(t,x)$  is estimated in (2.22). It follows from (2.16)  $|v_{i,1}^{\zeta}(0)\psi_{i,1}(x) - a_{i}| \leq \sigma_{\star}/4$  in  $D_{i}$   $(1 \leq i \leq N)$ . tha Now we assert the following inequality :  $(2.24) a_{i} - \sigma_{*} / 2 \leq \nu_{i,1}^{\zeta}(t) \psi_{i,1}(x) = \nu_{i,1}^{\zeta}(t) \operatorname{Vol}(D_{i})^{-1/2} \leq a_{i} + \sigma_{*} / 2$ for any  $t \in [0,T(\delta,\zeta)]$  and  $i = 1,2,\cdots,N$ . If the inequality (2.24) breaks at  $t = t' \in [0, T(\delta, \zeta)]$  for the first time for some i, then  $|\nu_{i,1}^{\zeta}(t') \operatorname{Vol}(D_i)^{-1/2} - a_i| = \frac{\sigma_*}{2}$ 

holds and  $|u_{i,q}^{\zeta}(t',x) - a_i| \ge \frac{1}{4} \sigma_*$  in  $D_i$  follows from (2.22) and we have

 $\int_{D_{i}}^{B_{i}(u_{i,q}^{\zeta}(t',x) - a_{i})} dx \ge \min \{B_{i}(\frac{1}{4}\sigma_{*}), B_{i}(-\frac{1}{4}\sigma_{*})\}.$ But this contradicts the inequality (2.23) by (2.13),(2.14) and (2.15) and the continuity. Thus we have ascertained the inequality (2.24).

Then again by (2.22) and (2.24), we have the inequality,

$$| v_{i,1}^{\zeta}(t)\psi_{i,1}(x) - a_i| + \sum_{k=2}^{q} | v_{i,k}^{\zeta}(t)\psi_{i,k}(x)| \leq \frac{3}{4} \sigma,$$
  
on  $[0,T(\delta,\zeta)] \times D_i \quad (1 \leq i \leq N).$ 

Then there exists ( by the continuity of  $u_{\zeta}(t,\cdot)$  ) T' > T( $\delta,\zeta$ ) such that

$$| v_{i,1}^{\zeta}(t) \psi_{i,1}(x) - a_i | + \sum_{k=2}^{q} | v_{i,k}^{\zeta}(t) \psi_{i,k}(x) | \leq \sigma_*$$

holds on  $[0,T'] \times D_i$  ( $1 \leq i \leq N$ ). But this is a contradiction to the definition of  $T(\delta,\zeta)$ . Consequently we conclude  $T(\delta,\zeta) = \infty$ for any  $(\delta,\zeta) \in (0,\delta_2] \times (0,\zeta_1]$ .

Lemma 2.1 is thus proved.

Therefore, from the inequality (2.19) and Lemma 2.1 we have the following estimates (2.25),(2.26) and (2.27) concerning the behavior of  $u_{\zeta}(t,\cdot)$  with intial condition  $w \in E(\delta,\zeta)$ ,

$$(2.25) \sum_{i=1}^{N} \int_{D_{i}} |u_{\zeta}(t,x) - u_{i,q}^{\zeta}(t,x)|^{2} dx \leq c_{3} \operatorname{Vol}(Q(\zeta)) + \delta^{3}$$

$$(2.26) \sum_{i=1}^{N} \sum_{k=2}^{q} \frac{1}{4} \lambda_{i,2} (\nu_{i,k}^{\zeta}(t))^{2} \leq c_{3} \operatorname{Vol}(Q(\zeta)) + \delta^{3}$$

holds and  $|u_{i,q}^{\zeta}(t',x) - a_i| \ge \frac{1}{4} \sigma_*$  in  $D_i$  follows from (2.22) and we have

$$\int_{D_{i}} B_{i}(u_{i,q}^{\zeta}(t',x) - a_{i}) dx \ge \min \{B_{i}(\frac{1}{4}\sigma_{*}), B_{i}(-\frac{1}{4}\sigma_{*})\}.$$

But this contradicts the inequality (2.23) by (2.13), (2.14) and (2.15) and the continuity. Thus we have ascertained the inequality (2.24).

Then again by (2.22) and (2.24), we have the inequality,

$$| v_{i,1}^{\zeta}(t) \psi_{i,1}(x) - a_i | + \sum_{k=2}^{q} | v_{i,k}^{\zeta}(t) \psi_{i,k}(x) | \leq \frac{3}{\sqrt{4}} \sigma,$$
  
on  $[0,T(\delta,\zeta)] \times D_i (1 \leq i \leq N).$ 

Then there exists ( by the continuity of  $u_{\zeta}(t,\cdot)$  )  $T' > T(\delta,\zeta)$  such that

$$| v_{i,1}^{\zeta}(t) \psi_{i,1}(x) - a_i | + \sum_{k=2}^{q} | v_{i,k}^{\zeta}(t) \psi_{i,k}(x) | \leq \sigma_*$$

holds on  $[0,T'] \times D_i$  ( $1 \leq i \leq N$ ). But this is a contradiction to the definition of  $T(\delta,\zeta)$ . Consequently we conclude  $T(\delta,\zeta) = \infty$ for any  $(\delta,\zeta) \in (0,\delta_2] \times (0,\zeta_1]$ .

Lemma 2.1 is thus proved.

Therefore, from the inequality (2.19) and Lemma 2.1 we have the following estimates (2.25),(2.26) and (2.27) concerning the behavior of  $u_{\zeta}(t,\cdot)$  with intial condition  $w \in E(\delta,\zeta)$ ,

$$(2.25) \sum_{i=1}^{N} \int_{D_{i}} |u_{\zeta}(t,x) - u_{i,q}^{\zeta}(t,x)|^{2} dx \leq c_{3} \operatorname{Vol}(Q(\zeta)) + \delta^{3}$$

$$(2.26) \sum_{i=1}^{N} \sum_{k=2}^{q} \frac{1}{4} \lambda_{i,2} (\nu_{i,k}^{\zeta}(t))^{2} \leq c_{3} \operatorname{Vol}(Q(\zeta)) + \delta^{3}$$

$$(2.27) \quad 0 \leq \int_{D_{i}} \int_{u_{i,q}^{\xi}(t,x)}^{a_{i}} f(\xi) d\xi dx \leq c_{3} \operatorname{Vol}(Q(\zeta)) + \delta^{3}$$
$$(0 \leq t < \infty, 0 < \zeta < \zeta_{1}, 0 < \delta < \delta_{2}).$$

From  $\lim_{\zeta \to 0} \operatorname{Vol}(\mathbb{Q}(\zeta)) = 0$ , it is clear that there exists a strictly monotone continuous function  $\zeta(\delta)$  on some interval  $(0, \delta_3]$  $(0 < \delta_3 < \delta_2)$  with the following properties (2.28) and (2.29),  $(2.28) \quad \lim_{\delta \to 0} \zeta(\delta) = 0$  $(2.29) \quad \delta^3 + c_3 \operatorname{Vol}(\mathbb{Q}(\zeta)) \leq 0$ 

 $\min_{\substack{1 \leq i \leq N}} \min \left\{ \frac{\delta^2}{4}, \frac{\lambda_{i,2}}{64(1+c_2(q-1)^{1/2} \operatorname{Vol}(D_i)^{1/2})^2}, \operatorname{Vol}(D_i) B_i(\frac{\pm \delta}{8\operatorname{Vol}(D_i)^{1/2}}) \right\}$ 

}

for any  $\zeta \in (0, \zeta(\delta)]$ .

We define a function  $\delta(\zeta)$  to be the inverse function of the above function  $\zeta(\delta)$ . It is easy to see that  $\delta(\zeta)$  is defined on some interval  $(0, \zeta_2]$   $(0 < \zeta_2 < \zeta_1)$  and  $\lim_{\zeta \to 0} \delta(\zeta) = 0$  holds.

Lemma 2.2. The set  $E(\delta,\zeta)$  is positively invariant for any  $(\delta,\zeta) \in [\delta(\zeta),\delta_3] \times (0,\zeta_2]$ , i.e. for any  $w \in E(\delta,\zeta)$ , the solutio of (2.1)  $u_{\zeta}(t,\cdot)$  belongs to  $E(\delta,\zeta)$  for any  $t \ge 0$ .

(Proof of Lemma 2.2) For any  $w \in E(\delta,\zeta)$  ( $\delta(\zeta) \leq \delta \leq \delta_3$ ,  $0 < \zeta \leq \zeta_2$ ), we can obtain from (2.25),(2.26),(2.27),(2.28) and (2.29) the following inequalities,

Hence applying the same argument as the last part of the proof of Lemma 2.1 ( which deduced the inequality (2.24) ) to the inequality (2.32), we have the following estimate.

$$(2.35) | \mathbf{a}_{i} - \mathbf{v}_{i,1}^{\zeta}(t) | \mathbf{w}_{i,1}(x) | \leq \frac{\delta}{4 | \operatorname{Vol}(D_{i})^{1/2}} + \frac{c_{2}(q-1)^{1/2} \delta}{4 | 1 + c_{2}(q-1)^{1/2} | \operatorname{Vol}(D_{i})^{1/2} |}$$
  
on  $[0,\infty) \times D_{i}$  ( $1 \leq i \leq N$ ), and then we have,  
$$(2.36) || \mathbf{a}_{i} - \mathbf{v}_{i,1}^{\zeta}(t) | \mathbf{w}_{i,1} ||_{L^{2}(D_{i})} \leq \frac{1}{|\operatorname{Vol}(D_{i})^{1/2}} + \frac{c_{2}(q-1)^{1/2} ||_{L^{2}(D_{i})^{1/2}}}{1 + c_{2}(q-1)^{1/2} |\operatorname{Vol}(D_{i})^{1/2}}$$

Therefore, using (2.30), (2.31) and (2.32), we have

$$\| u_{\zeta}(t, \cdot) - a_{i} \|_{L^{2}(D_{i})} \leq \| u_{\zeta}(t, \cdot) - u_{i,q}^{\zeta}(t, \cdot) \|_{L^{2}(D_{i})}$$

$$+ \| \sum_{k=2}^{q} v_{i,k}^{\zeta}(t) \|_{\mu,k} \|_{L^{2}(D_{i})} + \| a_{i} - v_{i,1}^{\zeta}(t) \|_{\mu,1} \|_{L^{2}(D_{i})}$$

$$\leq \delta \qquad (t \geq 0, 1 \leq i \leq N).$$

Thus we have proved the positive invariance  $E(\delta,\zeta)$  under the conditions  $0 < \zeta \leq \zeta_2$  and  $\delta(\zeta) \leq \delta \leq \delta_3$  and we have complete the proof of Lemma 2.2.

Thus we are in the situation where we can apply Theorem 4.2 in Matano [15] to the closed subset  $E(\delta(\zeta),\zeta)$  of  $C^2(\Omega(\zeta)) \cap C^1(\overline{\Omega(\zeta)})$ because it is easy to see that  $E(\delta(\zeta),\zeta)$  has " the property (S)" in [15] for  $\zeta > 0$  (  $0 < \zeta < \zeta_2$ ). Thus we have obtained a stable equilibrium solution  $v_{\zeta}$  in  $E(\delta(\zeta),\zeta)$  for small  $\zeta > 0$ .  $(2.35) | \mathbf{a}_{i} - \mathbf{v}_{i,1}^{\zeta}(t) | \mathbf{\psi}_{i,1}(\mathbf{x}) | \leq \frac{\delta}{4 | \operatorname{Vol}(\mathsf{D}_{i})^{1/2}} + \frac{c_{2}(q-1)^{1/2} \delta}{4 | 1 + c_{2}(q-1)^{1/2} | \operatorname{Vol}(\mathsf{D}_{i})^{1/2} |}$ on  $[0,\infty) \times \mathsf{D}_{i}$  ( $1 \leq i \leq \mathsf{N}$ ), and then we have,  $(2.36) || \mathbf{a}_{i} - \mathbf{v}_{i,1}^{\zeta}(t) | \mathbf{\psi}_{i,1} ||_{L^{2}(\mathsf{D}_{i})} \leq \frac{1}{4 | \sigma | \operatorname{Vol}(\mathsf{D}_{i})^{1/2} |} \left\{ \frac{1}{|\operatorname{Vol}(\mathsf{D}_{i})^{1/2}} + \frac{c_{2}(q-1)^{1/2}}{1 + c_{2}(q-1)^{1/2} | \operatorname{Vol}(\mathsf{D}_{i})^{1/2}} \right\}$ 

Therefore, using (2.30), (2.31) and (2.32), we have

$$\| u_{\zeta}(t, \cdot) - a_{i} \|_{L^{2}(D_{i})} \leq \| u_{\zeta}(t, \cdot) - u_{i,q}^{\zeta}(t, \cdot) \|_{L^{2}(D_{i})}$$

$$+ \| \sum_{k=2}^{q} \nu_{i,k}^{\zeta}(t) \psi_{i,k} \|_{L^{2}(D_{i})} + \| a_{i} - \nu_{i,1}^{\zeta}(t) \psi_{i,1} \|_{L^{2}(D_{i})}$$

$$\leq \delta \qquad (t \geq 0, 1 \leq i \leq N).$$

Thus we have proved the positive invariance  $E(\delta,\zeta)$  under the conditions  $0 < \zeta \leq \zeta_2$  and  $\delta(\zeta) \leq \delta \leq \delta_3$  and we have completd the proof of Lemma 2.2.

Thus we are in the situation where we can apply Theorem 4.2 in Matano [15] to the closed subset  $E(\delta(\zeta),\zeta)$  of  $C^2(\Omega(\zeta)) \cap C^1(\overline{\Omega(\zeta)})$ because it is easy to see that  $E(\delta(\zeta),\zeta)$  has " the property (S)" in [15] for  $\zeta > 0$  (  $0 < \zeta < \zeta_2$ ). Thus we have obtained a stable equilibrium solution  $v_{\zeta}$  in  $E(\delta(\zeta),\zeta)$  for small  $\zeta > 0$ .

- 22 -

Next we examine the property of  $v_{\zeta}$ . For any i  $(1 \leq i \leq N)$  $v_{r}$  satisfies the following relations.

$$(2.37) \qquad \Delta v_{\zeta} + f(v_{\zeta}) = 0 \quad \text{in } D_{i}$$

$$(2.38) a_* - \delta(\zeta) \leq v_{\zeta}(x) \leq a^* + \delta(\zeta) \text{ in } D_{i}$$

(2.39) 
$$\frac{\partial v_{\zeta}}{\partial \nu} = 0$$
 on  $\partial \Omega(\zeta) \cap \partial D_{i}$ 

For any  $\eta > 0$ , applying the Schauder estimate to  $v_{\zeta}$  on the domain  $D_{i}(\eta/2)$ , we obtain the boundedness of  $\{v_{\zeta}\}_{\zeta>0}$  in  $C^{1+\beta}(D_{i}((1-(1/2)^{2})\eta))$  for some  $\beta \in (0,1)$  and also the boundedness of  $\{f(v_{\zeta})\}_{\zeta>0}$  in  $C^{1+\beta}(D_{i}((1-(1/2)^{2})\eta))$ . Again, applying the Schauder estimate to the domain  $D_{i}((1-(1/2)^{2})\eta)$ , we obtain the boundedness of  $\{v_{\zeta}\}_{\zeta>0}$  in  $C^{3+\beta}(D_{i}((1-(1/2)^{3})\eta))$ . Repeating this bootstrap argument, we obtain the boundedness of  $\{v_{\zeta}\}_{\zeta>0}$  in  $C^{\infty}(D_{i}(\eta))$ . On the other hand, we already have  $\lim_{\zeta \to 0} \|v_{\zeta} - a_{i}\|_{L^{2}(D_{i})} = 0$ , then we conclude  $\lim_{\zeta \to 0} v_{\zeta} = a_{i}$  in  $C^{\infty}(D_{i}(\eta))$  ( $1 \leq i \leq N$ ). This completes the proof of Theorem 1.

§ 3 Asymptotic Behavior on The Thin Part.

In this section we consider the behavior of the solution on the perturbation part  $Q(\zeta)$ , but the domains introduced in Section 2 can contain extremely wild perturbation because the condition (II-1) is too weak. For the sake of the delicate argument about the behavior of the solution, we establish the domain concretely which is the special case of those in Section 2.

We set the domain  $\Omega(\zeta)$  in the following form :

$$\Omega(\zeta) = D_1 \cup D_2 \cup Q(\zeta)$$

where  $D_i$  (i=1,2) and  $Q(\zeta)$  are defined in the following (III-1) and (III-2) where  $x' = (x_2, x_3, \dots, x_n) \in \mathbb{R}^{n-1}$ .

(III-1)  $\rm D_1$  and  $\rm D_2$  are bounded domains in  ${\rm I\!R}^n$  (mutually disjoint) with smooth boundary which satisfy the following conditions for some constant  $\zeta_{\star}$ .

$$\begin{split} \vec{D}_{1} \cap \{ x = (x_{1}, x') \in \mathbb{R}^{n} \mid x_{1} \leq 1, |x'| < 3\zeta_{*} \} \\ &= \{ (1, x') \in \mathbb{R}^{n} \mid |x'| < 3\zeta_{*} \} \\ &= \{ (1, x') \in \mathbb{R}^{n} \mid x_{1} \geq -1, |x'| < 3\zeta_{*} \} \\ &= \{ (-1, x') \in \mathbb{R}^{n} \mid |x'| < 3\zeta_{*} \} \\ &= \{ (-1, x') \in \mathbb{R}^{n} \mid |x'| < 3\zeta_{*} \} \\ \end{split}$$
$$(III-2) \qquad Q(\zeta) = R_{1}(\zeta) \cup R_{2}(\zeta) \cup \Gamma(\zeta) \\ R_{1}(\zeta) = \{ (x_{1}, x') \in \mathbb{R}^{n} \mid 1 -2\zeta < x_{1} \leq 1, |x'| < \zeta\rho((x_{1}-1)/\zeta) \} \end{split}$$

$$R_{2}(\zeta) = \{ (x_{1}, x') \in \mathbb{R}^{n} \mid -1 \leq x_{1} < -1 + 2\zeta, |x'| < \zeta \rho((-1 - x_{1})/\zeta) \}$$

$$\Gamma(\zeta) = \{ (x_1, x') \in \mathbb{R}^n \mid -1 + 2\zeta \leq x_1 \leq 1 - 2\zeta , |x'| < \zeta \}$$

where  $\rho \in C^{0}((-2,0]) \cap C^{\infty}((-2,0))$  is a positive valued monotone increasing function such that  $\rho(0) = 2$ ,  $\rho(s) = 1$  for  $s \in (-2,-1)$ and  $\lim_{s\uparrow=0} \frac{d^{k}\rho}{ds^{k}}(s) = +\infty$  holds for any positive integer k. We also assume that

 $(III-3) \quad \overline{\lim}_{\xi \to \infty} f(\xi) < 0 , \quad \underline{\lim}_{\xi \to -\infty} f(\xi) > 0$ 

Remark. The domain determined above satisfies (II-1) and (II-2) therefore it is a special case of that dealt in Section 2 and so we use the same notation  $\Omega(\zeta)$ .

Under the situation supported by the conditions (II-2),(III-1), (III-2) and (III-3), we analyze the asymptotic behavior of some solutions ( which will be characterized by (III-4) ) of the following semilinear elliptic boundary value problem (3.1).

(3.1) 
$$\begin{pmatrix} \Delta v + f(v) = 0 & \text{in } \Omega(\zeta), \\ \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial \Omega(\zeta). \end{pmatrix}$$

(III-4) Let  $v_{\zeta}$  be an arbitrary solution of the above (3.1) for  $\zeta$ (  $0 < \zeta < \zeta_*$  ) such that the family of the functions {  $v_{\zeta}$  } $_{0 < \zeta < \zeta_*}$  satisfies the following condition.

$$\lim_{\zeta \to 0} \| v_{\zeta} - a_{i} \|_{L^{2}(D_{i})} = 0 \quad (i = 1, 2)$$

where  $f(a_i) = 0$  and  $f'(a_i) < 0$  (i = 1, 2). (See (II-2))

- 25 -

Definition 2. Let  $\mu_1(\zeta)$  be the first eigenvalue of the following eigenvalue problem.

(3.2) 
$$\begin{pmatrix} \Delta \psi + f'(v_{\zeta})\psi + \mu \psi = 0 & \text{in } \Omega(\zeta), \\ \\ \frac{\partial \psi}{\partial \gamma} = 0 & \text{on } \partial \Omega(\zeta). \end{pmatrix}$$

Remark. It is well-known that if  $\mu_1(\zeta) > 0$  (resp.  $\mu_1(\zeta) < 0$ )  $v_{\zeta}$  is stable (resp. unstable) as an equilibrium solution of (1.1) for  $\Omega = \Omega(\zeta)$ .

Remark. The two values  $a_1$  and  $a_2$  are not necessarily mutually distinct.

We define  $M_* = \inf \{ \xi \in \mathbb{R} \mid f(\xi) = 0 \}$  and  $M^* = \sup \{ \xi \in \mathbb{R} \mid f(\xi) = 0 \}$ . It is easily seen by (II-2) and (III-3) that  $M_*$  and  $M^*$  are well defined and that (3.3)  $M_* \leq v_{\zeta}(x) \leq M^*$  for  $x \in \Omega(\zeta)$ .

Then we have the following theorem.

Theorem 2. Assume  $n \ge 3$ , then we have, for i (i = 1, 2),

$$\lim_{\zeta \to 0} \sup_{\mathbf{x} \in D_{\mathbf{i}} \cup R_{\mathbf{i}}(\zeta)} | \mathbf{v}_{\zeta}(\mathbf{x}) - \mathbf{a}_{\mathbf{i}} | = 0.$$

We prepare the ordinary differential equation which describes the asymptotic behavior of  $\,v_\zeta\,$  on Q(\zeta) when  $\,\zeta\,\downarrow\,0$  .

(3.4) 
$$\begin{pmatrix} \frac{d^2 V}{dz^2} + f(V) = 0 & \text{in} - 1 < z < 1 \\ V(1) = a_1, & V(-1) = a_2. \end{pmatrix}$$

- 26 -

Definition 3. Let  $\lambda_V$  and  $\Phi_V$  be respectively the first eigenvalue and the first eigenfunction of the following eigenvalue problem (3.5) for a solution V of (3.4).

(3.5) 
$$\begin{pmatrix} \frac{d^2 \phi}{dz^2} + f'(V(z)) \phi + \lambda \phi = 0 & \text{in} - 1 < z < 1 \\ \phi(1) = 0 , \phi(-1) = 0 . \end{pmatrix}$$

Now we present one of the main results of this paper.

Theorem 3. Assume  $n \ge 3$ , then for any sequence of positive values  $\{\zeta_m\}_{m=1}^{\infty}$  such that  $\lim_{m\to\infty} \zeta_m = 0$ , there exist a subsequence  $\{\varkappa_m\}_{m=1}^{\infty} \subset \{\zeta_m\}_{m=1}^{\infty}$  and a solution V of (3.4) with the following asymptotic property (3.6) :

(3.6) 
$$\lim_{m \to \infty} \sup_{x \in Q(x_m)} |v_x(x_1,x') - V(x_1)| = 0$$
.

Furthermore concerning the above V, if  $\lambda_{V} > 0$  (resp.  $\lambda_{V} < 0$ ), then  $\lim_{m \to \infty} \mu_{1}(x_{m}) > 0$  (resp.  $\lim_{m \to \infty} \mu_{1}(x_{m}) < 0$ ) holds.

Before starting the proof we introduce some notations.

$$p_1 = (1,0,\cdots,0) , \quad p_2 = (-1,0,\cdots,0) ,$$
  

$$\mathcal{I}_1(\eta) = \{ (x_1,x') \in \mathbb{R}^n \mid x_1 > 1 , \mid x - p_1 \mid < \eta \} ,$$
  

$$\mathcal{I}_2(\eta) = \{ (x_1,x') \in \mathbb{R}^n \mid x_1 < -1 , \mid x - p_2 \mid < \eta \} .$$

It can be easily seen by the last part of the proof of Theorem 1 and the condition (III-4) that the following convergence (3.7)follows.

(See (3.38), (3.40) and Lemma<sup>3</sup>, 2 mentioned later.)

ζ

Here concerning the convergence (3.9), we have the following estimate.

Lemma 3.1.

For the analysis of the behavior of  $v_{\zeta}$  on the small part  $\Sigma_1(\zeta)$ , we change the scale of the variable x into y around the point  $p_1$  as follows.

(3.13) 
$$\begin{pmatrix} x - p_1 = \eta_m \cdot (y - p_1) \\ U_m(y) = v_{\zeta_m}(\eta_m \cdot (y - p_1) + p_1) \end{pmatrix}$$

By (3.13), the equation (3.1) is transformed into the following equation (3.14) in some neighborhood of  $p_1$  .

$$(3.14) \left\{ \begin{array}{rrr} \Delta_{y} U_{m} + \eta_{m}^{2} f(U_{m}) = 0 & \text{in } \Sigma_{1}(3\zeta_{*}/\eta_{m}) \\ \\ \frac{\partial U_{m}}{\partial y_{1}} (0, y') = 0 & \text{for } y' \text{ such that } \frac{2\zeta_{m}}{\eta_{m}} < |y'| < \frac{3 \zeta_{*}}{\eta_{m}} \end{array} \right.$$

We put  $\gamma_{m} \equiv \max_{\substack{y_{1} \geq 1, |y-p_{1}| = 3\zeta_{*}/\eta_{m}}} |U_{m}(y) - a_{1}|$  $= \max_{\substack{x \in D_{1}, |x-p_{1}| = 3\zeta_{*}}} |v_{\zeta_{m}}(x) - a_{1}|$ Then it is easy to see  $\lim_{m \to \infty} \gamma_{m} = 0$ .

We have the following properties (3.15),(3.16) and (3.17) by the definition of  $\eta_{\rm m} = \eta(\epsilon_1, \zeta_{\rm m})$  and  $U_{\rm m}$ .

Here we define a comparison function  $\mbox{G}_{m}$  which will estimate  $\mbox{U}_{m}$  for large  $\mbox{y}$  .

$$G_{m}(y) = \frac{\varepsilon_{1}}{|y - p_{1}|^{n-2}} + \gamma_{m}$$

It can be easily seen by (3.16) and the assumption of  $\varepsilon_1$  that  $f(U_m(y)) < 0$  for any  $y \in (\Sigma_1(3\zeta_*/\eta_m) - \Sigma_1(1)) \cap \{ y \mid U_m(y) > a_1 \}$   $f(U_m(y)) > 0$  for any  $y \in (\Sigma_1(3\zeta_*/\eta_m) - \Sigma_1(1)) \cap \{ y \mid U_m(y) < a_1 \}$ and that  $G_m$  is a harmonic function in  $\Sigma_1(\zeta_*/\eta_m) - \Sigma_1(1)$  with the boundary condition  $\frac{\partial G_m}{\partial y_1}(0, y') = 0$  (1 <  $|y'| < \frac{3 \zeta_*}{\eta_m}$ ).

Then we can apply the standard argument similar to the Comparison Theorem to the function  $U_m - a_1$  in the domain  $\Sigma_1(3\zeta_*/\eta_m) - \Sigma_1(1)$ by using (3.14), (3.15) and the definition of  $\gamma_{\rm m}$  and we obtain the following estimate (3.18) for sufficiently large m . Recall  $\lim_{m\to\infty} \zeta_m / \eta_m = 0$ .  $(3.18) | U_{m}(y) - a_{1} | \leq G_{m}(y) \text{ for } y \in \Sigma_{1}(3\zeta_{*}/\eta_{m}) - \Sigma_{1}(1) .$ Applying the same argument as the the last part of the proof of Theorem 1 and moreover the diagonal argument to the family {  $U_m$  } m=1in (3.14) with a-priori bound (3.17) by using  $\lim \eta_{\rm m}$  = 0 and lim  $\zeta_m / \eta_m = 0$ , we can choose a subsequence  $\{ U_m \}_{j=1}^{\infty}$  such that there exists a smooth function U in  $C^{\infty}(\{(y_1,y') \in \mathbb{R}^n \mid y_1 \ge 1\} - \{p_1\})$ with the following conditions (3.19), (3.20), (3.21) and (3.22).  $(3.19) M_{*} \leq U(y) \leq M^{*} \text{ in } \{ (y_{1}, y') \mid y_{1} \geq 1 \} - \{ p_{1} \}$  $(3.20) \Delta_{v} U = 0 \quad in \{ (y_{1}, y') \in \mathbb{R}^{n} \mid y_{1} > 1 \}$  $(3.21) \quad \frac{\partial U}{\partial y_1} (1, y') = 0 \quad \text{for } y' \in \mathbb{R}^{n-1} \text{ such that } y' \neq 0$  $(3.22) \qquad \lim_{i \to \infty} U_{m_i} = U$ in  $C^{\infty}(\{ (y_1, y') \mid y_1 \ge 1, \eta \le | y - p_1 \mid \le \frac{1}{n} \})$ for any  $\eta > 0$  .

On the other hand, from the estimate (3.18), the convergence (3.22) and  $\lim_{m\to\infty} \gamma_m = 0$ , U satisfies the following estimates

Here we can define a function  $\overline{U} \in C^{\infty}(\mathbb{R}^n - \{p_1\})$  by using the Laplace equation (3.20) and the Neumann boundary condition (3.21) as follows

$$\overline{U}(y_{1},y') = \begin{pmatrix} U(y) & \text{for } y_{1} \ge 1, y \neq p_{1} \\ \\ U(2-y_{1},y') & \text{for } y_{1} < 1 \end{pmatrix}$$

By a simple calculation, we have,

$$\begin{pmatrix} \Delta_{y} U = 0 & \text{in } \mathbb{R}^{n} - \{p_{1}\} \\ \\ M_{*} \leq U(y) \leq M^{*} & \text{in } \mathbb{R}^{n} - \{p_{1}\} \end{pmatrix}$$

Therefore, applying the removable singularity theorem, we can extend  $\overline{U}$  on  $\mathbb{R}^n$  as a bounded harmonic function. We denote it also by  $\overline{U}$  .
Thus  $\overline{U}$  must be a constant function by the Harnack Theorem. But it is impossible by (3.23) and (3.25). This is a contradiction and we complete the proof of Lemma 3.1.

By Lemma 3.1, we take a constant  $\beta > 0$  such that

$$(3.26) \quad \underline{\lim}_{\zeta \to 0} \quad \frac{\zeta}{\eta(\varepsilon_0, \zeta)} > \beta > 0 , \quad 0 < \beta < 1/2 .$$

We change the variable x into y around  $p_1$  by the following,

$$(3.27) \left( \begin{array}{ccc} x & - & p_{1} & = & \zeta \cdot (y - p_{1}) \\ & & U_{\zeta}(y) & = & v_{\zeta}(\zeta(y - p_{1}) + p_{1}) \end{array} \right)$$

By (3.27), the equation (3.1) is transformed into the following equation (3.28)-(3.29)

$$(3.28) \qquad \varDelta_{y} U_{\zeta} + \zeta^{2} f(U_{\zeta}) = 0 \quad \text{in } H_{\zeta} ,$$

(3.29) 
$$\frac{\partial U_{\zeta}}{\partial \nu} = 0$$
 on  $\partial H_{\zeta} \cap \partial H$ .

Here we have put,

$$\begin{split} \mathrm{H} &= \{ (\mathrm{y}_{1},\mathrm{y}') \in \mathbb{R}^{n} \mid \mathrm{y}_{1} > 1 \} \\ & \cup \{ (\mathrm{y}_{1},\mathrm{y}') \in \mathbb{R}^{n} \mid -1 < \mathrm{y}_{1} \leq 1 , |\mathrm{y}'| < \rho(\mathrm{y}_{1}-1) \} \\ & \cup \{ (\mathrm{y}_{1},\mathrm{y}') \in \mathbb{R}^{n} \mid \mathrm{y}_{1} \leq -1 , |\mathrm{y}'| < 1 \} , \\ & \mathrm{H}_{\zeta} &= \mathrm{H} \quad \cap \ \{ (\mathrm{y}_{1},\mathrm{y}') \in \mathbb{R}^{n} \mid \mathrm{y}_{1} \leq -1 , \text{ or } | \mathrm{y} - \mathrm{p}_{1} | \leq 3\zeta_{*}/\zeta \} \\ & \mathrm{and} \quad \nu \text{ denotes the unit outer normal vector on } \partial \mathrm{H}. \end{split}$$

Here we define  $\tau_{\zeta} = \max_{\substack{y_1 \ge 1, |y-p_1| = 3\zeta_*/\zeta}} |U_{\zeta}(y) - a_1|$ =  $\max_{x \in D_1, |x-p_1| = 3\zeta_*} |v_{\zeta}(x) - a_1|$ 

It is easily seen by (3.7) that  $\lim_{\zeta \to 0} \tau_{\zeta} = 0$ . From (3.26) and the definition of  $\eta(\varepsilon_0, \zeta)$ , we have,

(3.30)  $\eta(\varepsilon_0,\zeta) < \zeta/\beta$  for sufficiently small  $\zeta > 0$  and also we have,

$$(3.31) \max_{x \in \mathcal{I}_{1}(\zeta/\beta)} |v_{\zeta}(x) - a_{1}| = \max_{y_{1} \ge 1, |y-p_{1}| \le 1/\beta} |U_{\zeta}(y) - a_{1}|$$

$$\geq \max_{x_{1} \ge 1, |x-p_{1}| = \eta(\varepsilon_{0}, \zeta)} |v_{\zeta}(x) - a_{1}| = \varepsilon_{0}$$

$$(3.32) |U_{\zeta}(y) - a_{1}| \le \varepsilon_{0}$$

$$in \{ (y_{1}, y') \in \mathbb{R}^{n} \mid y_{1} \ge 1, \frac{1}{\beta} \le |y - p_{1}| \le \frac{3 \zeta_{*}}{\zeta} \}.$$

$$(3.33) M_{*} \le U_{\zeta}(y) \le M^{*} \quad in H_{\zeta}$$

$$(3.34) |U_{\zeta}(y) - a_{1}| \le \frac{\varepsilon_{0}}{\beta^{n-2} |y - p_{1}|^{n-2}} + \tau_{\zeta}$$

$$in \quad \mathcal{I}_{1}(3\zeta_{*}/\zeta) - \mathcal{I}_{1}(1/\beta)$$

By the same argument in (3.28), (3.29), (3.33) and (3.34) as the proof of Lemma 3.1, we can choose a convergent subsequence

 $\left\{ \begin{array}{ll} U_{\zeta_{m}} \end{array} \right\}_{m=1}^{\infty} \subset \left\{ \begin{array}{ll} U_{\zeta} \end{array} \right\}_{0 < \zeta < \zeta_{\star}} & \text{such that } \lim_{m \to \infty} \zeta_{m} = 0 \text{ and a} \\ \\ \text{function } U \in C^{\infty}(\overline{H}) & \text{which satisfy the following equations} \\ \\ (3.35) \qquad \Delta_{y} U = 0 \text{ in } H \end{array}$ 

$$(3.36) \qquad \frac{\partial U}{\partial \nu} = 0 \quad \text{on} \quad \partial H$$

(3.37) 
$$\lim_{m \to \infty} U_{\zeta_m} = U \quad \text{in } C^{\infty}(\overline{H}_{\eta}) \text{ for any } \eta > 0$$

(3.38) 
$$| U(y) - a_1 | \leq \frac{c_0}{\beta^{n-2} | y - p_1 |^{n-2}}$$

in { 
$$(y_1, y') \in \mathbb{R}^n \mid y_1 \ge 1$$
 ,  $\mid y - p_1 \mid \ge 1/\beta$  }

•

$$(3.39) \qquad M_{\star} \leq U(y) \leq M^{\star} \qquad \text{in } H$$

On the other hand, from (3.31),(3.37) and the compactness of the set  $\{ (y_1, y') \in \mathbb{R}^n \mid y_1 \ge 1, \mid y - p_1 \mid \le 1/\beta \}$ , we obtain

Thus (3.38) and (3.40) imply that U is a non-constant function in H . But this is impossible from (3.35),(3.36),(3.39) and the following Lemma 3.2.

Lemma 3.2. Let  $\psi$  be a bounded function which belongs to C  $^{\infty}(\ \overline{\rm H}\ )$  and satisfies the following equations

 $(3.41) \qquad \varDelta_{y} \psi = 0 \qquad \text{in } H$ 

 $(3.42) \quad \frac{\partial \psi}{\partial \nu} = 0 \quad \text{on } \partial H$ 

$$\begin{array}{cccc} (3.43) & \lim & | \psi(y) - a | = 0 \\ y_1 \ge 1, |y| \rightarrow \infty \end{array}$$

Then  $\psi = a$  in H

( Proof of Lemma 3.2 ) We assume the contrary. Without loss of generality, we may assume

$$(3.44) \qquad \sup_{y \in H} \psi(y) = M > a$$

We choose a sequence of points  $\{r_m\}_{m=1}^{\infty} \subset H$  such that

 $\lim_{m\to\infty} \psi(r_m) = a . Using the Strong Maximum Principle, the Hopf Lemma (See [19]) and the equation (3.41)-(3.42), we can easily see that <math>\psi$  cannot attain its maximum on  $\overline{H}$ , because  $\psi$  is a non-constant function. Moreover  $\{r_m\}_{m=1}^{\infty}$  does not have an accumulation point on  $\overline{H}$  and so from (3.43), we obtain

$$\lim_{m\to\infty} r_{m,1} = -\infty$$

We assume  $r_{m,1} < 0$  for any m;

here we denoted by  $r_{m,i}$  the i-th component of the point  $r_m$ . We define a family of functions {  $\psi_m$  } $_{m=1}^{\infty}$  as follows,

$$\psi_{m}(y_{1},y') = \psi(y_{1}+r_{m,1}+2,y')$$
.

Each  $\psi_{\rm m}$  satisfies the following equations,

$$(3.45) \qquad \Delta_{\rm v} \, \psi_{\rm m} = 0 \qquad \text{in } {\rm H} \, \cap \, \{ \, {\rm y}_1 < 0 \, \}$$

(3.46) 
$$\frac{\partial \psi_{\rm m}}{\partial \nu} = 0$$
 on  $\partial H \cap \{ y_1 < 0 \}$ 

$$(3.47) \qquad \psi_{\rm m}({\rm y}) \leq {\rm M} \qquad {\rm in} \ {\rm H}$$

$$(3.48) \qquad \lim_{m \to \infty} \max_{H \cap \{y_1 = -2\}} \psi_m(y) = M$$

By the standard compactness argument cocerning the solutions of the elliptic boundary value problem and the Maximum Principle in (3.45)-(3.48), we deduce the following convergence,

$$(3.49) \lim_{m \to \infty} \psi_{m} = M \quad \text{in } C^{\infty}(\overline{H \cap \{-3 < y_{1} < -1\}})$$

On the other hand, integrating the equation (3.41) in y' on  $\{ |y'| < 1 \}$  by using the Neumann boundary condition, we have,

$$\frac{d^2}{dy_1^2} \int_{|y'|<1} \psi(y_1,y') \, dy' = 0 \quad \text{for} \quad y_1 \leq 0.$$

But the boundedness of  $\psi$  implies the boundedness of

$$\int_{|y'|<1} \psi(y_1,y') \, dy' \quad \text{in} \quad -\infty < y_1 \leq 0 .$$
  
Therefore  $\int_{|y'|<1} \psi(y_1,y') \, dy'$  is independent of  $y_1$  when  $y_1$  is negative. We denote its value by K.

Therefore we have the following equality,

$$(3.50)^{\circ} \int_{|y'| < 1} \psi_{m}(-2, y') dy' = \int_{|y'| < 1} \psi(r_{m,1}, y') dy' = K$$

We remark that the left hand side (3.50) tends to the value  $M \int_{|y'|<1} 1 \, dy' \text{ when } m \text{ tends to } \infty \text{ . Then we obtain,}$   $\int_{|y'|<1} \psi(r_{m,1},y') \, dy' = \int_{|y'|<1} M \, dy' \text{ for any } m \text{ .}$ 

- 37 -

From (3.44), the above equality implies  $\psi(\mathbf{r}_{m,1},\mathbf{y}') = \mathbf{M}$  for  $\mathbf{y}'$ such that  $|\mathbf{y}'| < 1$ . But this contradicts to the fact that  $\psi$ cannot attain its maximum on  $\overline{\mathbf{H}}$ . This completes the proof of Lemma 3.2 and also the proof of  $\lim_{\zeta \to 0} \sup_{\mathbf{x} \in \mathbf{D}_1} |\mathbf{v}_{\zeta}(\mathbf{x}) - \mathbf{a}_1| = 0$ . To prove  $\lim_{\zeta \to 0} \sup_{\mathbf{x} \in \mathbf{R}_1(\zeta)} |\mathbf{v}_{\zeta}(\mathbf{x}) - \mathbf{a}_1| = 0$ , we remember the transformation (3.27),(3.28),(3.29) and by means of a similar argument there and we get the compactness of the family {  $\mathbf{U}_{\zeta}$  } $_{0<\zeta<\zeta_*}$ in  $\mathbb{C}^{\infty}(\mathbf{H}_{\eta})$  for any  $\eta > 0$ . Let {  $\mathbf{U}_{\zeta_m}$  } $_{m=1}^{\infty}$  be any convergent subsequence of the above family such that  $\lim_{\mathbf{m} \to \infty} \zeta_{\mathbf{m}} = 0$  and there exists  $\overline{\mathbf{U}} \in \mathbb{C}^{\infty}(\overline{\mathbf{H}})$  such that  $\lim_{\mathbf{m} \to \infty} \mathbf{U}_{\zeta_m} = \overline{\mathbf{U}}$  in  $\mathbb{C}^{\infty}(\overline{\mathbf{H}}_{\eta})$ for any  $\eta > 0$ . Then  $\overline{\mathbf{U}}$  is a harmonic function in  $\mathbf{H}$ . (See (3.35) and (3.36).) But we have already proved

$$\begin{split} \lim_{\zeta \to 0} & \sup_{x \in D_1} | v_{\zeta}(x) - a_1 | = 0 & \text{which implies } \overline{U}(y_1, y') = a_1 & \text{for} \\ \text{any} & y \in H \cap \{ y_1 > 1 \}. & \text{Hence by the Unique Continuation Theorem,} \\ \text{we get } \overline{U} = a_1 & \text{in } H & \text{. Then we conclude} \end{split}$$

$$\begin{split} \lim_{\zeta \to 0} & \sup_{y \in H_{\eta}} | U_{\zeta}(y) - a_{1} | = 0 \quad \text{for any } \eta > 0 \text{ . This implies} \\ \lim_{\zeta \to 0} & \sup_{x \in R_{1}(\zeta)} | v_{\zeta}(x) - a_{1} | = 0 \text{ . Thus we complete the proof} \end{split}$$

of Theorem 2.

( Proof of the Former Half of Theorem 3 )

To analyze the asymptotic behavior of  $v_{\zeta}$  in the thin part Q( $\zeta$ ), we change the variable x into y as follows.

(3.51) 
$$\begin{pmatrix} y_1 = x_1 \\ \zeta y' = x' \\ U_{\zeta}(y) = v_{\zeta}(y_1, \zeta y') \end{pmatrix}$$

We define  $\iota(\zeta) = \sum_{i=1}^{2} \sup_{\substack{x \in R_{i}(\zeta) \\ \zeta \to 0}} |v_{\zeta}(x) - a_{i}|$  and so by Theorem 2, we have  $\lim_{\substack{x \in Q_{i}(\zeta) \\ \zeta \to 0}} \iota(\zeta) = 0$ . We put  $\omega = \max_{\substack{x \in Q_{i}(\zeta) \\ M_{*} \leq \zeta \leq M}} |f(\zeta)|$ . By (3.51), the equation (3.1) is transformed into the following

$$(3.52)$$
  $(\frac{\partial^2}{\partial d} + \frac{1}{\partial d} + \frac{n}{\partial d} + \frac{\partial^2}{\partial d})$   $U + f(U) = 0$  in

equation in the part corresponding to  $Q(\zeta)$ .

$$(3.52) \quad \left(\frac{\partial^2}{\partial y_1^2} + \frac{1}{\zeta^2} \sum_{j=2}^{n} \frac{\partial^2}{\partial y_j^2}\right) U_{\zeta} + f(U_{\zeta}) = 0 \quad \text{in } G(\zeta)$$

$$(3.53) \quad \frac{\partial U_{\zeta}}{\partial \nu} = 0 \quad \text{on} \quad \partial G \cap \{ -1 + \zeta < \not{\xi} < 1 - \zeta \}$$
  
where we have put  $G = \{ (y_1, y') \in \mathbb{R}^n \mid |y'| < 1, -\infty < y_1 < \infty \}$ 

 $G(\zeta)=G\cap\{-1+\zeta< y_1<1-\zeta\}$  and denoted by  $\nu$  the unit outer normal vector on  $\partial G$  .

We decompose  $U_{\zeta}$  as  $U_{\zeta} = U_{1,\zeta} + U_{2,\zeta}$  by the folloeing equations which determine  $U_{1,\zeta}$  and  $U_{2,\zeta}$  uniquely.

$$(3.54) \quad \left(\frac{\partial^2}{\partial y_1^2} + \frac{1}{\zeta^2} \sum_{j=2}^n \frac{\partial^2}{\partial y_j^2}\right) U_{1,\zeta}(y) = 0 \quad \text{in } G(\zeta)$$

(3.55) 
$$\begin{pmatrix} U_{1,\zeta}(y) = U_{\zeta}(y) & \text{on } G \cap \{ y_1 = 1 - \zeta \} \\ U_{1,\zeta}(y) = U_{\zeta}(y) & \text{on } G \cap \{ y_1 = -1 + \zeta \} \end{pmatrix}$$

$$(3.56) \quad \frac{\partial U_{1,\zeta}}{\partial \nu}(y) = 0 \quad \text{on} \quad \partial G \cap \{-1 + \zeta < y_1 < 1 - \zeta \}$$

$$(3.57) \quad U_{2,\zeta} = U_{\zeta} - U_{1,\zeta}$$

By the above definition,  ${\rm U}_{2,\,\zeta}$  automaticaly satisfies the following equation

$$(3.58) \left( \frac{\partial^2}{\partial y_1^2} + \frac{1}{\zeta^2} \sum_{j=2}^n \frac{\partial^2}{\partial y_j^2} \right) U_{2,\zeta} + f(U_{\zeta}) = 0 \text{ in } G(\zeta)$$

$$(3.59) \quad U_{2,\zeta}(1-\zeta,y') = U_{2,\zeta}(-1+\zeta,y') = 0 \quad (|y'| < 1)$$

$$(3.60) \quad \frac{\partial U_2}{\partial \nu} \zeta = 0 \quad \partial G \quad \cap \{ -1 + \zeta \leq y_1 \leq 1 - \zeta \} .$$

Hereafter we denote the differential operator by  $P_{\zeta}$  as follows

$$P_{\zeta} = \frac{\partial^2}{\partial y_1^2} + \frac{1}{\zeta^2} \sum_{j=2}^n \frac{\partial^2}{\partial y_j^2} .$$

We can deduce the following estimate by applying the comparison theorem in (3.54)-(3.55) by the aid of the definition of  $\iota(\zeta)$  .

Lemma 3.3. For any  $\zeta \in (0, \zeta_*)$ , we have,

(3.61) 
$$\sup_{y \in G(\zeta)} | U_{1,\zeta}(y) - \frac{1-\zeta-y_1}{2-2\zeta} a_1 - \frac{1-\zeta+y_1}{2-2\zeta} a_2 | \leq \iota(\zeta)$$

We define functions  $\Phi_{\pm}$  in G( $\zeta$ ) which estimate U<sub> $\zeta$ </sub> roughly.

$$\Phi_{\pm,\zeta}(y_1,y') = \frac{y_1 + 1 - \zeta}{2 - 2\zeta} a_1 + \frac{1 - \zeta - y_1}{2 - 2\zeta} a_2$$
  
$$\pm \frac{\omega}{2}(y_1 + 1 - \zeta)(1 - \zeta - y_1) \pm \iota(\zeta)$$

Lemma 3.4. For any  $\zeta \in (0, \zeta_*)$ , we have the following estimate

$$\Phi_{-,\zeta}(\mathbf{y}) \leq U_{\zeta}(\mathbf{y}) \leq \Phi_{+,\zeta}(\mathbf{y}) \text{ in } G(\zeta).$$

(Proof of Lemma 3.4) By an easy calculation, we have,

$$P_{\zeta} \quad \Phi_{\pm} = - \omega \quad \text{in } G(\zeta)$$

 $\frac{\partial \Phi_{\pm}}{\partial \nu} = 0 \quad \text{on} \quad \partial G \cap \{ -1 + \zeta < y_1 < 1 - \zeta \} \text{ and by the}$  definition of  $\iota(\zeta)$ , we also have

$$\begin{aligned} \mathbf{a}_{1} - \iota(\zeta) &= \Phi_{-,\zeta}(1-\zeta,\mathbf{y}') \leq U_{\zeta}(1-\zeta,\mathbf{y}') \leq \Phi_{+,\zeta}(1-\zeta,\mathbf{y}') = \mathbf{a}_{1} + \iota(\zeta) \\ \mathbf{a}_{2} - \iota(\zeta) &= \Phi_{-,\zeta}(-1+\zeta,\mathbf{y}') \leq U_{\zeta}(-1+\zeta,\mathbf{y}') \leq \Phi_{+,\zeta}(-1+\zeta,\mathbf{y}') = \mathbf{a}_{2} + \iota(\zeta) \\ \text{Applying the comparison theorem, we have the consequence.} \end{aligned}$$

Lemma 3.5. There exists a positive constant 
$$c_1$$
 such that  
(3.63)  $\int_{G(\zeta)} \left| \frac{\partial U_{2,\zeta}}{\partial y_1} \right|^2 dy + \frac{1}{\zeta^2} \sum_{j=2}^n \int_{G(\zeta)} \left| \frac{\partial U_{2,\zeta}}{\partial y'} \right|^2 dy$   
 $\leq c_1$  for any  $\zeta \in (0,\zeta_*)$ 

We can deduce this inequality by integrating the equation (3.58)after multiplying  $U_{2,\zeta}$  and using the boundedness (3.3) and (3.61). We define a function which bounds  $U_{2,\zeta}$  in  $G(\zeta)$ .

$$\Psi_{\zeta}(y_{1},y') = \frac{\omega}{2} (1-\zeta - y_{1})(1-\zeta + y_{1})$$

Lemma 3.6. There exists a positive constant 
$$c_2$$
 such that  
(3.64)  $| U_{2,\zeta}(y) | \leq \Psi_{\zeta}(y)$  in  $G(\zeta)$   
(3.65)  $| \frac{\partial U_{2,\zeta}}{\partial y_1}(1-\zeta,y') | \leq c_2$   
( $|y'| \leq 1$ )  
(3.66)  $| \frac{\partial U_{2,\zeta}}{\partial y_1}(-1+\zeta,y') | \leq c_2$ 

(Proof of Lemma 3.6)

 $\varPsi_{r}$  satisfies the following equations,

$$(3.67) \quad P_{\zeta} \Psi_{\zeta} + \omega = 0 \quad \text{in } G(\zeta)$$

(3.68) 
$$\frac{\partial \Psi_{\zeta}}{\partial \nu} = 0$$
 on  $\partial G \cap \{ -1 + \zeta \leq y_1 \leq 1 - \zeta \}$ 

$$(3.69) \quad \Psi_{\zeta}(-1+\zeta, y') = \Psi_{\zeta}(1-\zeta, y') = 0 \qquad |y'| < 1 .$$

Then applying the comparison theorem to (3.58)-(3.60) and (3.67)-(3.69), we see that

$$(3.70) - \Psi_{\zeta}(\mathbf{y}) \leq U_{2,\zeta}(\mathbf{y}) \leq \Psi_{\zeta}(\mathbf{y}) \quad \text{in } G(\zeta) .$$

Then taking account of the boundary condition (3.59) and (3.69) we have,

$$\left| \begin{array}{c} \frac{\partial U_{2,\zeta}}{\partial y_{1}}(1-\zeta,y') \right| \leq \left| \begin{array}{c} \frac{\partial \Psi_{\zeta}}{\partial y_{1}}(1-\zeta,y') \right| = \omega(1-\zeta) \leq \omega \end{array} \right|$$

By the same argument, we have  $|\frac{\partial U_{2,\zeta}}{\partial y_1}(-1+\zeta,y')| \leq \omega$ . Thus we conclude the result.

Lemma 3.7. For any  $\delta \in (0,1)$ , there exists a constant  $c_{3,\delta} > 0$  such that  $(3.71) | \frac{\partial U_{\zeta}}{\partial y_1}(y) | \leq c_{3,\delta}$  in  $G(\delta) (0 < \zeta \leq \delta/2)$   $(3.72) | \frac{\partial U_{1,\zeta}}{\partial y_1}(y) | \leq c_{3,\delta}$  in  $G(\delta) (0 < \zeta \leq \delta/2)$   $(3.73) | \frac{\partial U_{2,\zeta}}{\partial y_1}(y) | \leq c_{3,\delta}$  in  $G(\delta) (0 < \zeta \leq \delta/2)$ (Proof of Lemma 3.7) We will prove (3.71). For any  $y_{\star} \in [0, 1-\delta]$ , we define a function  $W_1$  which is defined on  $G \cap \{2y_{\star}-1+\zeta \leq y_1 \leq y_{\star}\}$  and satisfies the following equations  $(3.74) \quad \frac{\partial W_1}{\partial y_1}(y_{\star},y') = \frac{\partial U_{\zeta}}{\partial y_1}(y_{\star},y') \text{ for } |y'| < 1$  $(3.75) P_{\zeta} W_1 + \frac{1}{2} (f(U_{\zeta}(y)) - f(U_{\zeta}(2y_{\star}-y_1,y')) = 0$ 

in  $G \cap \{ 2y_{*} - 1 + \zeta < y_{1} < y_{*} \}$ 

(3.76) 
$$\frac{\partial W_1}{\partial \nu} = 0$$
 on  $\partial G \cap \{ 2y_* - 1 + \zeta \leq y_1 \leq y_* \}$ 

$$(3.77) \quad W_1(y_*,y') = 0 \quad \text{for } |y'| < 1.$$

We define a comparison function  $\theta_1$  as follows

$$\Theta_{1}(y_{1}, y') = \frac{\omega}{2}(y_{*} - y_{1}) \cdot (y_{1} - 2y_{*} + 1 - \zeta) + \frac{\overline{M}}{1 - y_{*} - \zeta}(y_{*} - y_{1})$$

- 43 -

where we have put  $\overline{M} = \max(|M_*|, |M^*|)$ .

This satisfies the following equations

(3.78) 
$$P_{\zeta} \Theta_1 + \omega = 0$$
 in  $G \cap \{ 2y_* - 1 + \zeta < y_1 < y_* \}$ 

(3.79) 
$$\frac{\partial \Theta_1}{\partial \nu} = 0$$
 on  $\partial G \cap \{ 2y_* - 1 + \zeta \leq y_1 \leq y_* \}$ 

(3.80) 
$$\Theta_1(2y_*-1+\zeta,y') = \overline{M}$$
 for  $|y'| < 1$ 

(3.81) 
$$\theta_1(y_*,y') = 0$$
 for  $|y'| < 1$ .

Applying the comparison theorem to (3.74)-(3.76) and (3.78) - (3.80)(Notice  $P_{\zeta}(\Theta_1 - W_1)(y) \leq 0$ .), we obtain

$$- \theta_1(y) \leq W_1(y) \leq \theta_1(y) \text{ in } G \cap \{ 2y_* - 1 + \zeta \leq y_1 \leq y_* \}.$$
  
Taking notice of the boundary condition (3.77) and (3.81), we deduce

from (3.82) by (3.74) that

$$(3.83) \mid \frac{\partial U_{\zeta}}{\partial y_{1}}(y_{*},y') \mid = \mid \frac{\partial W_{1}}{\partial y_{1}}(y_{*},y') \mid \leq \mid \frac{\partial \Theta_{1}}{\partial y_{1}}(y_{*},y') \mid$$
$$= \frac{\omega}{2} (1 - y_{*} - \zeta) + \frac{\overline{M}}{1 - y_{*} - \zeta} \leq \frac{\omega}{2} + \frac{2 \overline{M}}{\delta}$$
for any  $\zeta \in (0, \delta/2]$ .

The above estimate holds uniformly in  $y_* \in [0, 1-\delta]$ . For the case that  $y_* \in [-1+\delta,0]$ , the proof is the same as the above case. On the other hand, we can prove (3.72) and (3.73) by the completely same argument ( reflection technique ) as (3.71). Lemma 3.8. For any  $\delta \in (0,\zeta_*)$ , there exists a positive constant  $c_{4,\delta}$  such that

$$(3.84) \qquad \begin{array}{c} n \\ \Sigma \\ j=2 \end{array} \mid \frac{\partial U_{\zeta}}{\partial y_{j}}(y) \mid^{2} \leq c_{4,\delta} \zeta^{4} \\ \\ \text{on } \partial G \cap \{ -1+\delta \leq y_{1} \leq 1-\delta \} \text{ for any } \zeta \in (0,\delta/2] . \end{array}$$

(Proof of Lemma 3.8) For the sake of constructing a comparison function, we take a function  $h \in C^{\infty}([0,\infty))$  which satisfies

(i) 
$$h(0) = 0$$
,  $h(1) = 1$   
(ii)  $\frac{d^k h}{d\xi^k}(0) = 0$  for any natural number  $k$ .  
 $\frac{dh}{d\xi}(\xi) > 0$  for any  $\xi \in (0,1)$ .

Take an arbitrary hyperplane  $\pi$  in  $\mathbb{R}^n$  which contains the y<sub>1</sub>-axis. By an appropriate orthogonal transformation of coordinate in  $(y_2, \dots, y_n)$ , we can assume that  $\pi$  is expressed by the equation  $y_2 = 0$  without loss of generality. Remark that the equation (3.52) is invariant under the above transformation. Now we define a domain  $G_+(\zeta)$  and a function  $W_2(y)$  in  $G_+(\zeta)$ 

as follows.

(3.85) 
$$G_{+}(\zeta) = G(\zeta) \cap \{ y_1 > 0 \}$$

(3.86) 
$$W_2(y) = \frac{1}{2} (U_{\zeta}(y_1, y_2, \dots, y_n) - U_{\zeta}(y_1, -y_2, y_3, \dots, y_n))$$

It is easily seen that  $W_2$  satisfies the following equations

$$(3.87) \quad \frac{\partial W_2}{\partial y_2}(y) = \frac{\partial U_{\zeta}}{\partial y_2}(y) \quad \text{on} \quad \pi \cap \partial G_{+}(\zeta)$$

$$(3.88) P_{\zeta} W_{2} + \frac{1}{2} (f(U_{\zeta}) - f(U_{\zeta}(y_{1}, -y_{2}, y_{3}, \cdots, y_{n})) = 0 \text{ in } G_{+}(\zeta)$$

$$(3.89) W_{2}(y) = 0 \text{ on } \pi \cap \partial G_{+}(\zeta)$$

(3.90) 
$$\frac{\partial W_2}{\partial \nu}(y) = 0$$
 on  $\partial G_+(\zeta) \cap \partial G(\zeta)$ 

We put a comparison function  $\theta_2(y)$  as follows

$$(3.91) \quad \Theta_{2}(\mathbf{y}) = \begin{pmatrix} e(\delta)\zeta^{2}\mathbf{y}_{2}(3-\mathbf{y}_{2}) + \overline{M} h(\frac{\mathbf{y}_{1}-1+\delta}{\delta-\zeta}) & (\mathbf{y}_{1} > 1-\delta) \\ e(\delta)\zeta^{2}\mathbf{y}_{2}(3-\mathbf{y}_{2}) & (-1+\delta \leq \mathbf{y}_{1} \leq 1-\delta) \\ e(\delta)\zeta^{2}\mathbf{y}_{2}(3-\mathbf{y}_{2}) + \overline{M} h(\frac{-\mathbf{y}_{1}-1+\delta}{\delta-\zeta}) & (\mathbf{y}_{1} < -1+\delta) \end{pmatrix}$$

where  $e(\delta) = 1 + \omega + \frac{2 \overline{M}}{\delta^2} \sup_{\xi \in [0,1]} |h''(\xi)|$ .

By a simple calculation, we obtain

$$(3.92) P_{\zeta} \Theta_{2}(\mathbf{y}) = \begin{pmatrix} -2 e(\delta) + \frac{\overline{M}}{(\delta-\zeta)^{2}} h''(\frac{\mathbf{y}_{1}-1+\delta}{\delta-\zeta}) & (\mathbf{y}_{1} > 1-\delta) \\ -2 e(\delta) & (-1+\delta \leq \mathbf{y}_{1} \leq 1-\delta) \\ -2 e(\delta) + \frac{\overline{M}}{(\delta-\zeta)^{2}} h''(\frac{-\mathbf{y}_{1}-1+\delta}{\delta-\zeta}) & (\mathbf{y}_{1} < -1+\delta) \end{pmatrix}$$

- 46 -

$$(3.93) \quad \frac{\partial \theta_2}{\partial r} (y) > 0 \quad \text{on} \quad \partial G_+(\zeta) \cap \partial G$$

$$(3.94) \quad \theta_2(y) \ge \overline{M} \quad \text{on} (\partial G_+(\zeta) \cap \{y_1=1-\zeta\}) \cup (\partial G_+(\zeta) \cap \{y_1=-1+\zeta\})$$

$$(3.95) \quad \theta_2(y) = 0 \quad \text{on} \quad \partial G_+(\delta) \cap \pi$$
By using  $0 < \zeta \le \delta/2$  and the definition of  $e(\delta)$ , we obtain from
$$(3.88)-(3.90) \text{ and } (3.92)-(3.94) \text{ that}$$

$$(3.96) \quad P_{\zeta}(\theta_2 - W_2)(y) < 0 \quad \text{in} \quad G_+(\zeta)$$

$$(3.97) \quad \frac{\partial}{\partial r} (\theta_2 - W_2) > 0 \quad \text{on} \quad \partial G_+(\zeta) \cap \partial G$$

$$(3.98) \quad \theta_2(y) - W_2(y) \ge 0 \quad \text{on} \quad \partial G_+(\zeta) \cap \pi$$

$$(3.99) \quad \theta_2(y) - W_2(y) \ge 0 \quad \text{on} \quad \partial G_+(\zeta) \cap \pi$$

$$(3.100) \quad \theta_2(y) - W_2(y) = 0 \quad \text{on} \quad \partial G_+(\delta) \cap \pi$$
According the Nextern Detection  $d = (2, 0\zeta) / (2, 00)$ 

Applying the Maximum Principle to (3.96)-(3.99), we obtain,

 $\Theta_2(y) - W_2(y) \ge 0$  in  $G_+(\zeta)$ .

By a similar argument with respect to  $-\Theta_2$  and  $W_2$  in  $G_+(\zeta)$ , we have  $-\Theta_2(y) \leq W_2(y)$  in  $G_+(\zeta)$ . Then we conclude that (3.101)  $|W_2(y)| \leq \Theta_2(y)$  in  $G_+(\zeta)$ . Thus by the inequality (3.101) with the boundary condition (3.100),

we have  $\left|\frac{\partial W_2}{\partial y_2}(y)\right| \leq \frac{\partial \Theta_2}{\partial y_2}(y)$  on  $\partial G_+(\delta) \cap \pi$ .

- 47 -

Therefore we have from (3.87) that

$$(3.101) \quad \left| \begin{array}{c} \frac{\partial U_{\zeta}}{\partial y_{2}}(y) \right|_{\partial G_{+}(\delta) \cap \pi} \\ \left| \begin{array}{c} \leq \end{array} \right| \begin{array}{c} \frac{\partial \Theta_{2}}{\partial y_{2}}(y) \\ \frac{\partial \Theta_{+}(\delta) \cap \pi}{\partial y_{2}} \\ \left| \begin{array}{c} \partial \Theta_{+}(\delta) \cap \pi \end{array} \right| \\ = 3 \quad e(\delta) \quad \zeta^{2} \\ \end{array}$$

Then we have 
$$\left| \frac{\partial U_{\zeta}}{\partial y_{2}}(y) \right| \partial G_{+}(\delta) \cap \pi \cap \partial G \right| \leq 3 e(\delta) \zeta^{2}$$
.

On the other hand we have the Neumann boundary condition

$$\frac{\partial U_{\zeta}}{\partial \nu}(y) = 0$$
 on  $\partial G_{+} \cap \pi \cap \partial G$ .

Then by considering the arbitrariness of  $\pi$  ( containing y<sub>1</sub>-axis ) and the uniformness of the above argument in taking the hyperplane  $\pi$ , we conclude that

$$\sum_{j=2}^{n} \left| \frac{\partial U_{\zeta}}{\partial y_{j}}(y) \right|_{\partial G \cap \partial G_{+}}(\delta) \right| \leq 3 (n-1) e(\delta) \zeta^{2} (0 < \zeta \leq \delta/2) .$$
We complete the proof of Lemma 3.8 by putting  $c_{4,\delta} = (3(n-1)e(\delta))^{2}$ .  
By the aid of Lemma 3.3 - Lemma 3.8, we will obtain a convergent subsequence of  $\{U_{\zeta_{m}}\}_{m=1}^{\infty}$ . From Lemma 3.4 and  $\lim_{\zeta \to 0} \iota(\zeta) = 0$ , it is easy to see that for  $\varepsilon > 0$ , there exists a constant  $\overline{\zeta} = \overline{\zeta}(\varepsilon)$  and  $\overline{\delta} = \overline{\delta}(\varepsilon)$  such that  $\overline{\zeta}(\varepsilon)$  and  $\overline{\delta}(\varepsilon)$  depend monotonously on  $\varepsilon$  and  $\lim_{\varepsilon \to 0} \overline{\delta}(\varepsilon) = 0$ ,  $\lim_{\varepsilon \to 0} \overline{\zeta}(\varepsilon) = 0$  and such that  $\varepsilon = 0$   $(3.102) \sup_{0 < \zeta \leq \overline{\zeta}} \{\sup_{1-2\overline{\delta} \leq y_{1} \leq 1-\zeta} | U_{\zeta}(y) - a_{1} | + \sup_{-1+\zeta \leq y_{1} \leq -1+2\overline{\delta}} | U_{\zeta}(y) - a_{2} | \}$ 

On the other hand we deal with the convergence on the domain  $G(2\overline{\delta}(\epsilon))$ . From Lemma 3.5,  $\{U_2,\zeta_m\}_{m=1}^{\infty}$  is bounded in the Sobolev space  $H^1(G(\overline{\delta}(\epsilon)))$  and it is compact in  $H^{1/2}(G(2\overline{\delta}(\epsilon)))$  by the Imbedding Theorem. Moreover  $\{U_2,\zeta_m \mid \partial G(2\overline{\delta}) \cap \partial G\}_{m=1}^{\infty}$  is compact in  $L^2(\partial G(2\overline{\delta}) \cap \partial G)$  by the Trace Theorem. (Taylor [21] Chapter I)

Now take a sequence of positive values {  $\epsilon_k$  }  $_{k=1}^{\infty}$  such that  $\epsilon_1 > \epsilon_2 > \cdots > \epsilon_k > \epsilon_{k+1} > \cdots > 0$  and  $\lim_{k \to \infty} \epsilon_k = 0$ . By the above compactness argument for  $\epsilon = \epsilon_1$ , we have a subsequence {  $\zeta_m^{(1)}$  }  $_{m=1}^{\infty} \subset (0, \overline{\zeta}(\epsilon_1))$  such that {  $U_{2,\zeta_m^{(1)}}$  }  $_{m=1}^{\infty}$ is convergent in  $H^{1/2}(G(2\overline{\delta}(\epsilon_1)))$  and also in  $L^2(\partial G(2\overline{\delta}(\epsilon_1)) \cap \partial G)$ and its limit function is independent of y' by Lemma 3.5. Therefore by Lemma 3.3, {  $U_{\zeta_m^{(1)}}$  }  $_{m=1}^{\infty}$  is convergent in  $H^{1/2}(G(2\overline{\delta}(\epsilon_1)))$  and also in  $L^2(\partial G(2\overline{\delta}(\epsilon_1)) \cap \partial G)$ ( Recall  $U_{\zeta_m^{(1)}} = U_{1,\zeta_m^{(1)}} + U_{2,\zeta_m^{(1)}}$ ). Then there exists a function  $U^{(1)} \in H^{1/2}(G(2\overline{\delta}(\epsilon_1)))$  which is independent of y', such that

$$\lim_{m \to \infty} U_{\chi_m^{(1)}} = U^{(1)} \quad \text{in } H^{1/2}(G(2\overline{\delta}(\varepsilon_1))) .$$

Again applying the same argument to the sequence  $\left\{ \begin{array}{c} U \\ \zeta_{m}^{(1)} \end{array} \right\}_{m=1}^{\infty}$  for  $\varepsilon = \varepsilon_{2}$ , we get a subsequence  $\left\{ \zeta_{m}^{(2)} \right\}_{m=1}^{\infty} \subset \left\{ \zeta_{m}^{(1)} \right\}_{m=1}^{\infty} \cap (0, \overline{\zeta}(\varepsilon_{2}))$  and a function  $U^{(2)} \in H^{1/2}(G(2\overline{\delta}(\varepsilon_{2})))$  which is independent of y', such that

$$\lim_{m \to \infty} U_{\zeta_m}^{(2)} = U^{(2)} \text{ in } H^{1/2}(G(2\overline{\delta}(\varepsilon_2)))$$
$$U^{(2)}|G(2\overline{\delta}(\varepsilon_2)) = U^{(1)}$$

Repeating this process inductively , we obtain a sequence of subsequences of {  ${\zeta_{\rm m}}}_{\rm m=1}^\infty$  such that

$$\{\zeta_{\mathfrak{m}}\}_{\mathfrak{m}=1}^{\infty} \supset \{\zeta_{\mathfrak{m}}^{(1)}\}_{\mathfrak{m}=1}^{\infty} \supset \{\zeta_{\mathfrak{m}}^{(2)}\}_{\mathfrak{m}=1}^{\infty} \supset \cdots \supset \{\zeta_{\mathfrak{m}}^{(q)}\}_{\mathfrak{m}=1}^{\infty} \supset \cdots$$

and a function V which is independent of y' such that  $\lim_{m\to\infty} U_{\zeta_{m}^{(q)}} = V \text{ in } H^{1/2}(G(2\overline{\delta}(\varepsilon_{q}))) \text{ for any natural number } q.$ Determine the subsequence  $\{z_{m}\}_{m=1}^{\infty} \subset \{\zeta_{m}\}_{m=1}^{\infty}$  by  $z_{m} = \zeta_{m}^{(m)}$   $(m \ge 1)$ . From the way of the construction  $\{\zeta_{m}^{(q)}\}_{m=1}^{\infty}$ , we have  $(3.105) \lim_{m\to\infty} U_{z_{m}} = V \text{ in } H^{1/2}(G(2\overline{\delta}(\varepsilon_{q}))) \quad (q \ge 1)$   $(3.106) \lim_{m\to\infty} U_{z_{m}} = V \text{ in } L^{2}(\partial G(2\overline{\delta}(\varepsilon_{q})) \cap \partial G) \quad (q \ge 1)$  $(3.107)_{1-2\overline{\delta}(\varepsilon_{q}) \le y_{1} \le 1-z_{m}} | U_{z_{m}}(y)-a_{1}| + \sup_{-1+z_{m}} \sup_{\le y_{1} \le -1+2\overline{\delta}(\varepsilon_{q})} | U_{z_{m}}(y)-a_{2}|$ 

$$\leq \epsilon_q \quad (m \geq q \geq 1)$$

From now on, we will investigate the uniform convergence of  $\{ \begin{array}{c} U_{\varkappa_{m}} \\ m=1 \end{array}$ . By Lemma 3.7 and Lemma 3.8 and the Ascoli-Arzera Theorem,  $\{ \begin{array}{c} U_{\varkappa_{m}} \\ \partial G(2\overline{\delta}(\varepsilon_{q})) \cap \partial G \end{array}\}_{m=1}^{\infty}$  is compact in  $C^{O}(\partial G(2\overline{\delta}(\varepsilon_{q})) \cap \partial G)$ for any natural number q. On the other hand we already have (3.106). Then we conclude that V is continuous in the interval (-1,1) and

$$(3.108) \lim_{m \to \infty} \sup_{y \in \partial G(2\overline{\delta}(\varepsilon_{g})) \cap \partial G} | U_{\varkappa}(y_{1},y') - V(y_{1}) | = 0$$

for any integer  $q \ge 1$  .

Then, let m tend to  $\infty$  in (3.107) and we have

$$(3.109) \sup_{1-2\overline{\delta}(\varepsilon_{q}) \leq y_{1} < 1} | v(y_{1})-a_{1}| + \sup_{-1 < y_{1} \leq -1+2\overline{\delta}(\varepsilon_{q})} | v(y_{1})-a_{2}| \leq \varepsilon_{q}$$

This concludes that V is continuous on [-1,1] and (3.110) V(1) =  $a_1$ , V(-1) =  $a_2$ . Therefore from (3.107),(108),(3.109) and (3.110), we have,

$$\begin{array}{cccc} \overline{\lim} & \sup & | U_{\chi}(y_1, y') - V(y_1) | \\ m \rightarrow \infty & y \in \partial G \cap \partial G(x_m) & m \end{array}$$

 $\leq \overline{\lim_{m \to \infty}} \qquad \sup_{\mathbf{y} \in \partial G(2\delta(\varepsilon_q)) \cap \partial G} \mid U_{\varkappa_m}(\mathbf{y}) - V(\mathbf{y}_1) \mid$ 

+ 
$$\overline{\lim}_{m \to \infty}$$
 sup | ( $U_{\varkappa}(y) - a_1$ ) + ( $a_1 - V(y_1)$ ) |  
 $m \to \infty$  1-2 $\overline{\delta}(\varepsilon_q) \leq y_1 \leq 1 - \varkappa_m$ ,  $|y'| = 1$  m

Again by (3.107) and (3.109) we conclude that

$$\lim_{m \to \infty} \sup_{\mathbf{y} \in \partial \mathbf{G}(\mathbf{x}_m)} | \mathbf{U}_{\mathbf{x}_m}(\mathbf{y}_1, \mathbf{y}') - \mathbf{V}(\mathbf{y}_1) | = 0$$

From the equation (3.52) and (3.53), we have

$$\left(\frac{\partial^2}{\partial y_1^2} + \frac{1}{\kappa_m^2} \int_{j=2}^n \frac{\partial^2}{\partial y_j^2}\right) U_{\kappa_m} + f(U_{\kappa_m}) = 0 \quad \text{in } G(\kappa_m)$$

$$\frac{\partial \mathcal{O}_{\varkappa_{\mathrm{m}}}}{\partial \nu}(\mathbf{y}) = 0 \qquad \text{on} \quad \partial \mathbf{G}(\varkappa_{\mathrm{m}}) \cap \partial \mathbf{G}$$

Take any  $\phi \in C_0^{\infty}((-1,1))$  and integrate the above equation in  $G(\varkappa_m)$  after multiplying  $\phi(y_1,y') = \phi(y_1)$ . Then we have for sufficiently large m so that  $\operatorname{supp} \phi \subset (-1+\varkappa_m, 1-\varkappa_m)$ ,

$$\int_{G(\varkappa_{m})} U_{\varkappa_{m}}(y) P_{\varkappa_{m}} \phi dy + \int_{G(\varkappa_{m})} \phi f(U_{\varkappa_{m}}) dy = 0$$

(Remark that  $P_{\varkappa_{m}} \phi(y) = \frac{\partial^{2} \phi}{\partial y_{1}^{2}}(y_{1})$ )

Let m tend to  $\infty$  and we get by (3.105) that

$$\int_{|y'| \leq 1} dy' \int_{-1}^{1} (V(y_1) \frac{\partial^2}{\partial y_1^2} \phi(y_1) + \phi(y_1) f(V(y_1)) ) dy_1 = 0$$

By the arbitrariness of  $\phi$  , we have

$$\frac{d^2}{dy_1^2} V(y_1) + f(V(y_1)) = 0 \text{ in } (-1,1) .$$

Lemma 3.9.

$$\lim_{m \to \infty} \sup_{\mathbf{y} \in G(\mathbf{x}_m)} | U_{\mathbf{x}_m}(\mathbf{y}_1, \mathbf{y}') - V(\mathbf{y}_1) | = 0$$

(Proof of Lemma 3.9) We define a comparison function  $\Theta_{\pm,m}$  by

$$\Theta_{\pm,m}(\mathbf{y}) = \mathbf{V}(\mathbf{y}_1) \pm \frac{\omega}{n-1} (1 - |\mathbf{y}'|^2) \mathbf{z}_m^2$$
$$\pm \sup_{\mathbf{y} \in \partial \mathbf{G}(\mathbf{z}_m)} |\mathbf{U}_{\mathbf{z}_m}(\mathbf{y}_1, \mathbf{y}') - \mathbf{V}(\mathbf{y}_1)|$$

$$\begin{split} \theta_{\pm,m} & \text{satisfy the following equations by (3.112)} \\ P_{\varkappa_{m}} \left( \theta_{\pm,m} - U_{\varkappa_{m}} \right) = -f(V) = 2 \omega + f(U_{\varkappa_{m}}) \stackrel{\leq}{=} 0 & \text{in } G(\varkappa_{m}) \\ \theta_{\pm,m}(y) - U_{\varkappa_{m}}(y) \stackrel{\geq}{=} 0 & \text{on } \partial G(\varkappa_{m}) \end{split}$$

Then applying the Maximum Principle, we have

$$\begin{array}{rcl} \Theta_{\pm,m}(\mathbf{y}) & - & U_{\varkappa_{m}}(\mathbf{y}) \stackrel{\geq}{\leq} & 0 & \text{ in } & \mathbf{G}(\varkappa_{m}) \\ \\ \text{or } & \Theta_{-,m}(\mathbf{y}) & \leq & U_{\varkappa_{m}}(\mathbf{y}) \leq & \Theta_{+,m}(\mathbf{y}) & \text{ in } & \mathbf{G}(\varkappa_{m}) \end{array}.$$

By the definition of  $\,\,\theta_{\pm\,,\,\mathrm{m}}^{\phantom{1}}$  , we conclude that

$$\lim_{m \to \infty} \sup_{\mathbf{y} \in G(\mathbf{x}_m)} | U_{\mathbf{x}_m}(\mathbf{y}) - V(\mathbf{y}_1) | = 0$$

and complete the proof of Lemma 3.9 .

Expressing the equality in Lemma 3.9 in the original variable x, we complete the proof of the former assertion of Theorem 3.

( Proof of the Latter Half of Theorem 3 )

(1) The case  $\lambda_{V} < 0$ .

We will prove that the first eigenvalue  $\mu_1(\varkappa_m)$  in (3.2) for  $\nu_{\varkappa_m}$  is bounded from below by a negative constant for sufficiently large m.

It is well-known that

(3.113) 
$$\mu_{1}(\varkappa_{m}) = \inf_{\substack{\psi \in H^{1}(\Omega(\varkappa_{m}))}} \frac{\int_{\Omega(\varkappa_{m})} (|\nabla \psi|^{2} - f'(\nabla_{\varkappa_{m}})\psi^{2}) dx}{\int_{\Omega(\varkappa_{m})} |\psi|^{2} dx}$$

Here we define a function

$$\psi_{\mathbf{m}}(\mathbf{x}_{1},\mathbf{x}') = \begin{pmatrix} 0 & \mathbf{x} \in D_{1} \cup D_{2} \cup R_{1}(\mathbf{x}_{\mathbf{m}}) \cup R_{2}(\mathbf{x}_{\mathbf{m}}) \\ \\ \Phi_{\mathbf{V}}(\mathbf{x}_{1}) - \Phi_{\mathbf{V}}(1-2\mathbf{x}_{\mathbf{m}}) & \mathbf{x} \in \Gamma(\mathbf{x}_{\mathbf{m}}) \end{pmatrix}$$

Remark that  $\Phi_{V}(z) = \Phi_{V}(-z)$  on (-1,1) and  $\psi_{m} \in H^{1}(\Omega(x_{m}))$ . To estimate  $\mu_{1}(x_{m})$  from above by using (3.113), we calculate (3.114)  $\int_{\Omega(x_{m})} (|\nabla \psi_{m}||^{2} - f'(v_{x_{m}}) \psi_{m}^{2}) dx$   $= \int_{|x'| \leq x_{m}} dx' \int_{-1+2x_{m}}^{1-2x_{m}} (|\frac{\partial \Phi_{V}}{\partial x_{1}}|^{2} - f'(v_{x_{m}}) |\Phi_{V}(x_{1}) - \Phi_{V}(1-2x_{m})|^{2}) dx_{1}$  $= -\int \int dx' dx_{1} \{\frac{d^{2} \Phi_{V}}{dx_{1}^{2}} + f'(v_{x_{m}}) (\Phi_{V}(x_{1}) - \Phi_{V}(1-2x_{m})) \} (\Phi_{V}(x_{1}) - \Phi_{V}(1-2x_{m})) \}$ 

$$= \int_{|\mathbf{x}'| \leq \mathbf{x}_{m}} \left\{ \int_{-1+2\mathbf{x}_{m}}^{1-2\mathbf{x}_{m}} \{ \lambda_{V} + f'(V(\mathbf{x}_{1})) - f'(v_{\mathbf{x}_{m}}(\mathbf{x}_{1},\mathbf{x}')) \} \Phi_{V}(\mathbf{x}_{1})^{2} d\mathbf{x}_{1} \right. \\ + \int_{-1+2\mathbf{x}_{m}}^{1-2\mathbf{x}_{m}} \{ -\lambda_{V} \Phi_{V}(\mathbf{x}_{1}) + (2f'(v_{\mathbf{x}_{m}}) - f'(V)) \Phi_{V}(\mathbf{x}_{1}) - f'(v_{\mathbf{x}_{m}}) \Phi_{V}(1-2\mathbf{x}_{m}) \} \times \\ \Phi_{V}(1-2\mathbf{x}_{m}) d\mathbf{x}_{1} \right\} d\mathbf{x}'$$

Using the former assertion of Theorem 3 which we have already proved we have  $|f'(V) - f'(v_{\varkappa_m})| \leq -\lambda_V/4$  in  $\Gamma(\varkappa_m)$  for sufficiently large m. On the other hand,  $\lim_{m\to\infty} \Phi_V(1-2\varkappa_m) = 0$  holds from the boundary condition  $\Phi_V(1) = 0$  and then we have the inequality,  $|\int_{-1+2\varkappa_m}^{1-2\varkappa_m} \{-\lambda_V \Phi_V(\varkappa_1) + (2f'(v_{\varkappa_m}) - f'(V)) \Phi_V(1-2\varkappa_m) - f'(v_{\varkappa_m}) \Phi_V(1-2\varkappa_m)\}$  $\times \Phi_V(1-2\varkappa_m) dx_1| \leq -\frac{\lambda_V}{4} \int_{-1+2\varkappa_m}^{1-2\varkappa_m} \Phi_V(\varkappa_1)^2 dx_1$  for large m. Then we have

$$\int_{\Omega(\varkappa_{m})} (|\nabla \psi_{m}|^{2} - f'(v_{\varkappa_{m}}) |\psi_{m}|^{2}) dx \leq \frac{\lambda_{V}}{2} \int \int_{\Gamma(\varkappa_{m})} \Phi_{V}(x_{1})^{2} dx_{1} dx'$$
for large m.

On the other hand one can easily check that

$$\begin{split} &\int_{\Omega(\varkappa_{\mathrm{m}})} \psi_{\mathrm{m}}(\mathbf{x})^{2} \mathrm{d} \mathbf{x} \leq 2 \int \int_{\Gamma(\varkappa_{\mathrm{m}})} \Phi_{\mathrm{V}}(\mathbf{x}_{1})^{2} \mathrm{d} \mathbf{x}_{1} \mathrm{d} \mathbf{x}' \text{ for large } \mathbf{m} \; . \end{split}$$
Then we conclude that  $\mu_{1}(\varkappa_{\mathrm{m}}) \leq \lambda_{\mathrm{V}}/4$  for sufficiently large  $\mathbf{m}$ .
This concludes the result the case (1).

(2) The case  $\lambda_V > 0$ .

From now on we will prove that  $\mu_1(\varkappa_m)$  is bounded from below by a positive constant for sufficiently large m. To prove by the

contradiction we assume that there exists a subsequence  $\left\{ \begin{array}{c} m(j) \end{array} \right\}_{j=1}^{\infty}$  such that

Let  $\psi_j$  be the corresponding eigenfunction of (3.2) to the eigenvalue  $\mu_1(\varkappa_{m(j)})$  such that

$$(3.115) \|\psi_{j}\|_{L^{2}(\Omega(\varkappa_{m(j)}))} = 1 \quad (j \ge 1).$$

Lemma 3.10. Under the condition (\*),

$$\lim_{j\to\infty} \psi_j = 0 \quad \text{in} \quad C^{\infty}((\overline{D_1 - \mathcal{I}_1(\eta)}) \cup (\overline{D_2 - \mathcal{I}_2(\eta)})) \quad \text{for any } \eta > 0.$$

(Proof of Lemma 3.10) Applying the bootstrap argument by the a-priori estimate in S.Agmon, A.Douglas and L.Nirenberg [1], we see

(3.116) { 
$$\psi_j$$
 }  $\overset{\infty}{j=1}$  is compact in  $C^{\infty}((\overline{D_1 - \mathcal{I}_1(\eta)}) \cup (\overline{D_2 - \mathcal{I}_2(\eta)}))$   
for any  $\eta > 0$ .

On the other hand, we take two functions  $\phi_1$ ,  $\phi_2 \in C^{\infty}(\mathbb{R}^n)$  such that

$$\phi_1(\mathbf{x}) = 1 \text{ in } D_1, \quad \phi_1(\mathbf{x}) = 0 \text{ in } D_2, \quad \phi_2(\mathbf{x}) = 0 \text{ in } D_1, \\ \phi_2(\mathbf{x}) = 1 \text{ in } D_2, \quad \text{supp } \phi_1 \cap \text{ supp } \phi_2 = \emptyset.$$

We put , for i = 1, 2 and  $j = 1, 2, 3, \cdots$  ,

$$\theta_{j}^{(i)} = \| (\varDelta + f'(v_{\varkappa_{m(j)}})) \phi_{i} - f'(a_{i}) \phi_{i} \|_{L^{2}(\Omega(\varkappa_{m(j)}))} / \| \phi_{i} \|_{L^{2}(\Omega(\varkappa_{m(j)}))}$$

and we can easily check that  $\lim_{j\to\infty} \theta_j^{(i)} = 0$  ( i = 1, 2 ) by

Theorem 2 and a simple calculation.

Therefore the eigenvalue problem (3.2) for  $\zeta = \varkappa_{m(j)}$  has eigenvalues  $\mu^{(1)}(j)$  and  $\mu^{(2)}(j)$  for large j such that

$$\mu^{(i)}(j) \in [-f'(a_{i}) - \theta_{j}^{(i)^{1/2}}, -f'(a_{i}) + \theta_{j}^{(i)^{1/2}}] = I_{j}^{(i)}$$

$$\| P_{I_{j}^{(i)}} \phi_{i} - \phi_{i} \|_{L^{2}(\Omega(\varkappa_{m(j)}))} / \| \phi_{i} \|_{L^{2}(\Omega(\varkappa_{m(j)}))} \leq \theta_{j}^{(i)^{1/2}}$$

for i = 1, 2 and large  $j \ge 1$ , where  $\Pr_{\substack{i \ j \ j}}$  is the eigenprojection ( associated with the self-adjoint operator  $-A - f'(v_{\varkappa_{m(j)}})$  ), onto the subspace of  $L^2(\varrho(\varkappa_{m(j)}))$  corresponding to the the interval  $I_j^{(i)}$ . We have  $\mu_1(\varkappa_{m(j)}) \notin \bigcup_{i=1}^2 [-f'(a_i) - \theta_j^{(i)}, -f'(a_i) + \theta_j^{(i)}]$  for large j by (\*) and then  $(\psi_j, \Pr_{\substack{i \ j \ j}} e_{j}^{(i)}) = 0$  for large j and i = 1,2. Therefore we have for i = 1,2,  $|(\psi_j, \phi_i)_{L^2(\varrho(\varkappa_{m(j)}))}| / ||\phi_i||_{L^2(\varrho(\varkappa_{m(j)}))} \leq \theta_j^{(i)}|^{1/2}$  for large j and so we can easily deduce  $\lim_{j \to \infty} \int_{D_i} \psi_j dx = 0$  ( i = 1,2 ). Remark that  $\psi_j(x) > 0$  in  $\overline{\varrho(\varkappa_{m(j)})}$  and we have that  $\lim_{j \to \infty} \psi_j(x) = 0$  for a.e.  $x \in D_1 \cup D_2$ . By the compactness (3.116), we conclude the result of Lemma 3.10.

By using Lemma 3.10, we can choose a monotone sequence of positive values {  $\ell_j$  } such that

$$(3.117) \begin{cases} \lim_{j \to \infty} \ell_{j} = 0, \quad \ell_{j} > \varkappa_{m(j)} \\ \lim_{j \to \infty} K(j) = 0 \\ \lim_{j \to \infty} K(j) = 0 \\ \text{where } K(j) = \sup_{x \in (D_{1} - \Sigma_{1}(2\ell_{j})) \cup (D_{2} - \Sigma_{2}(2\ell_{j}))} |\psi_{j}(x)| > 0. \end{cases}$$

Here we define two sets,

$$\begin{split} \mathbf{S}_{j} &= (\ \mathbf{Q}(\mathbf{x}_{m(j)}) \cup \ \mathbf{\Sigma}_{1}(2t_{j}) \cup \ \mathbf{\Sigma}_{2}(2t_{j}) \ ) \\ & \cap \quad \{ \ |\mathbf{x}_{1}| < 1 + (\ (2t_{j})^{2} - (\mathbf{x}_{m(j)})^{2} \ )^{1/2} \ \} \\ \mathbf{T}_{j} &= \{(\mathbf{x}_{1}, \mathbf{x}') \in \mathbb{R}^{n} | \ |\mathbf{x}'| < \mathbf{x}_{m(j)}, \ |\mathbf{x}_{1}| \leq 1 + ((2t_{j})^{2} - (\mathbf{x}_{m(j)})^{2} \ )^{1/2} \ \} \end{split}$$

Now we decompose eigenfunction 
$$\psi_{j}$$
 uniquely as follows  
 $\psi_{j}(x) = \psi_{j}^{(1)} + \psi_{j}^{(2)}$  in  $S_{j}$ , by the following equations,  
(3.118)  
 $\begin{pmatrix} \Delta \ \psi_{j}^{(1)} = 0 & \text{in } S_{j} \\ \psi_{j}^{(1)}(x) = \psi_{j}(x) & \text{on } \partial S_{j} - \partial \Omega(z_{m(j)}) \\ \frac{\partial \psi_{j}^{(1)}}{\partial \nu}(x) = 0 & \text{on } \partial S_{j} \cap \partial \Omega(z_{m(j)}) \end{pmatrix}$   
(3.119)  $\psi_{j}^{(2)}(x) = \psi_{j}(x) - \psi_{j}^{(1)}(x) & \text{in } S_{j}$ 

Apply the maximum principle to (3.118), we obtain the inequality, (3.120)  $0 < \psi_j^{(1)}(x) \leq K_j$  in  $S_j$ .

Now we calculate as follows. (3.121)  $\mu_1(\varkappa_{m(j)}) = \int_{\Omega(\varkappa_{m(j)})} (|\nabla \psi_j|^2 - f'(v_{\varkappa_{m(j)}}) |\psi_j^2|) dx$   $= \int_{\Omega(\varkappa_{m(j)}) - S_j} (|\nabla \psi_j|^2 - f'(v_{\varkappa_{m(j)}}) |\psi_j^2|) dx$   $+ \int_{S_j} |\nabla \psi_j^{(1)}|^2 dx$  $+ \int_{S_j} (|\nabla \psi_j^{(2)}|^2 - f'(v_{\varkappa_{m(j)}}) |\psi_j^{(2)}|^2) dx$ 

$$- \int_{S_{j}} f'(v_{\varkappa_{m}(j)}) (2 \psi_{j}^{(2)} - \psi_{j}^{(1)}) \psi_{j}^{(1)} dx$$
  
=  $B_{1}(j) + B_{2}(j) + B_{3}(j) + B_{4}(j)$ 

We have used  $\int_{S_j} \nabla \psi_j^{(1)} \nabla \psi_j^{(2)} dx = 0 \text{ in the above.}$ By Theorem 2,  $-f'(v_{\varkappa_m(j)}) \ge \beta_* / 2 \text{ in } \Omega(\varkappa_m(j)) - S_j \text{ for } j$ , where  $\beta_* = \min(-f'(a_1), -f'(a_2))$ . Then we have ,

$$(3.122) \quad B_{1}(j) \geq \min(1,\beta_{*}/2) \quad \left( \|\psi_{j}\|_{L^{2}(\Omega(\varkappa_{m(j)})-S_{j})} \right)^{2} \text{ for large } j$$

By (3.117) and the boundedness of  $\|\psi_j^{(2)}\|_{L^2(S_j)}$  ( j = 1,2,3,...),

$$\begin{array}{ccc} (3.123) & \lim_{j \to \infty} B_{4}(j) = 0 \\ \end{array}$$

Hereafter we estimate  $B_{3}(j)$  from below.

$$B_{3}(j) = \int_{T_{j}} (|\nabla \psi_{j}^{(2)}|^{2} - f'(v_{\varkappa_{m(j)}}) ||\psi_{j}^{(2)}|^{2}) dx$$
  
+ 
$$\int_{S_{j}^{-T_{j}}} (|\nabla \psi_{j}^{(2)}|^{2} - f'(v_{\varkappa_{m(j)}}) ||\psi_{j}^{(2)}|^{2}) dx$$

Again from Theorem 2, the second term of  $B_{3}(j)$ 

$$\geq \min (1, \beta_{*}/2) ( \|\psi_{j}^{(2)}\|_{H^{1}(S_{j}-T_{j})})^{2}$$

To estimate the first term of  $B_3(j)$ , we change the variable x into y in  $T_j$  as follows.

$$\begin{cases} x_{1} = (1 + \sigma_{j}) y_{1} \\ x' = y' \\ z_{j}(y_{1}, y') = \psi_{j}^{(2)}(\sigma_{j}y_{1}, y') \text{ for } |y_{1}| \leq 1 , |y'| < x_{m(j)} \\ \text{where } \sigma_{j} = 1 + ((2\ell_{j})^{2} - (x_{m(j)})^{2})^{1/2} . \\ \text{Remark that } \lim_{j \to \infty} \sigma_{j} = 1 , z_{j}(y) = 0 \text{ for } y_{1} = \pm 1 , |y'| < x_{m(j)}. \\ \text{Then we estimate as follows,} \\ \int_{T_{j}} (|\nabla \psi_{j}^{(2)}|^{2} - f'(v_{x_{m(j)}})^{||\psi_{j}^{(2)}|^{2}}|^{2}) dx \\ \geq \int_{|y'| \leq x_{m(j)}} dy' \int_{-1}^{1} dy_{1} \{ \frac{1}{\sigma_{j}^{2}} | \frac{\partial z_{j}}{\partial y_{1}}(y_{1}, y')|^{2} \\ - f'(v_{x_{m(j)}}(\sigma_{j}y_{1}, y')) z_{j}^{2} \} \sigma_{j} \\ = \frac{1}{\sigma_{j}} \int_{|y'| \leq x_{m(j)}} dy' \int_{-1}^{1} (|\frac{\partial z_{j}(y_{1}, y')}{\partial y_{1}}|^{2} - f'(v(y_{1}))|z_{j}(y_{1}, y')|^{2}) dy_{1} \\ + \int_{|y'| \leq x_{m(j)}} \int_{-1}^{1} (\frac{1}{\sigma_{j}^{2}} f'(v(y_{1})) - f'(v_{x_{m(j)}}(\sigma_{j}y_{1}, y'))) z_{j}^{2} \sigma_{j} dy_{1} \\ \geq \frac{1}{\sigma_{j}} \int_{|y'| \leq x_{m(j)}} x_{y} \int_{-1}^{1} z_{j}(y)^{2} dy_{1} - \int_{|y'| \leq x_{m(j)}} dy' \int_{-1}^{1} z_{j}(y)^{2} dy_{1} \\ \geq \frac{1}{\sigma_{j}} \int_{|y'| \leq x_{m(j)}} |y_{1}| < 1 + \frac{1}{\sigma_{j}^{2}} f'(v(y_{1})) - f'(v_{x_{m(j)}}(\sigma_{j}y_{1}, y'))| \\ = \frac{\lambda_{y}}{\sigma_{j}^{2}} (|\psi_{j}^{(2)}||_{L^{2}(T_{j})})^{2} - (|\psi_{j}^{(2)}||_{L^{2}(T_{j})})^{2} - \frac{1}{\sigma_{j}} + \sup | ... |.$$

By the first half of Theorem 3 and  $\lim_{j\to\infty} \sigma_j = 0$ , the second term of the above line is minor to the first for large j. Then we have the following inequality (3.124) for large j.

Therefore we have completed the proof of Theorem 3.

§ 4 Construction of Unstable Solution.

In this section, we will consider the equation (3.1) on the domain  $\Omega(\zeta)$  established in Section 3 where we choose f in (3.1) as the one we will establish below. We will construct a family of solutions {  $v_{\zeta}$  }<sub> $\zeta>0$ </sub> in (III-4) where  $v_{\zeta}$  is an unstable solution of (3.1) under the condition  $a_1 = a_2 = b_1$  for small  $\zeta > 0$ .

We determine the nonlinear term f in the following form. (4.1)  $f(\xi) = \vartheta g(\xi)$  ( $\vartheta > 0$ ) where  $g \in C^{\infty}(\mathbb{R})$  satisfies the following conditions (IV-1)-(IV-2) and the parameter  $\vartheta$  will be chosen later.

(IV-1) There exist three points  $b_1 < b_2 < b_3$  such that

$$\begin{split} g(b_{i}) &= 0 \ (1 \leq i \leq 3), \ g'(b_{1}) < 0, \ g'(b_{3}) < 0 \\ g(\xi) > 0 \ in \ (-\infty, b_{1}) \ \cup \ (b_{2}, b_{3}) \\ g(\xi) < 0 \ in \ (b_{1}, b_{2}) \ \cup \ (b_{3}, \infty). \end{split}$$

$$(IV-2) \qquad \int_{b_1}^{b_3} g(\xi) d\xi > 0$$

From (IV-1)-(IV-2), there exists a unique  $d \in (b_2, b_3)$  such that  $\int_{b_4}^{d} g(\xi) d\xi = 0$ .

Above all things we seek for the solutions of the following two point boundary value problem of the ordinary differential equation (4.2) up to their linearized stability where the nonlinear term f is that in (4.1).

(4.2) 
$$\begin{pmatrix} \frac{d^2 V}{dz^2} + f(V) = 0 & \text{in} - 1 < z < 1 \\ V(1) = b_1, & V(-1) = b_1 \end{pmatrix}$$

- 63 -

Proposition 2. There exists a positive values  $\vartheta_0$  such that for any  $\vartheta \ge \vartheta_0$ , (4.2) has exactly three solutions

$$V^{(0)}(z) (= b_1) < V^{(1)}(z) < V^{(2)}(z) (-1 < z < 1)$$

with the following stability properties,

$$\lambda_{V(0)} > 0$$
 ,  $\lambda_{V(1)} < 0$  ,  $\lambda_{V(2)} > 0$  .

( See Definition 3 in Section 3 as for  $\lambda_{V(0)}$  ,  $\lambda_{V(1)}$  ,  $\lambda_{V(2)}$  . )



Figure 5

(Proof of Proposition 2) To construct nontrivial solutions, we must search for the value  $\xi \in (d,b_3)$  which satisfies the following equation.

(4.3) 
$$\int_{b_1}^{\xi} (2\int_{\sigma}^{\xi} f(\rho)d\rho)^{-1/2}d\sigma = 1$$
 (d <  $\xi$  < b<sub>3</sub>) (See Maginu [14].)

To examine the left hand side as a function of  $\xi$ , we define  $s(\xi)$  which is defined in  $(d,b_3)$  as follows.

$$s(\xi) = \int_{b_1}^{\xi} \left( 2 \int_{\sigma}^{\xi} g(\rho) d\rho \right)^{-1/2} d\sigma$$

 $s(\xi)$  is well-defined by (IV-1) and (IV-2) and moreover we have the following properties concerning  $s(\xi)$  .

Lemma 4.1.  $s(\xi)$  is a positively valued differentiable function on  $(d,b_3)$  with the following asymptotic conditions,

$$\begin{pmatrix} \lim_{\xi \uparrow b_3} \frac{s(\xi)}{\left(-1/g'(b_3)\right)^{1/2} \log \frac{1}{b_3 - \xi}} = 1 \\ \lim_{\xi \downarrow d} \frac{s(\xi)}{\left(-1/4g'(b_1)\right)^{1/2} \log \frac{1}{\xi - d}} = 1 \\ \lim_{\xi \uparrow b_3} \frac{d}{d\xi} s(\xi) = +\infty, \quad \lim_{\xi \downarrow d} \frac{d}{d\xi} s(\xi) = -\infty$$

(Proof of Lemma 4.1) First we deal with the case that  $\xi$  is near  $b_3$  i.e. d' < (d' +  $b_3$ )/2  $\leq \xi < b_3$  where d' is a point in  $(b_2, b_3)$  which will be determined later.

$$(4.4) \ \mathrm{s}(\xi) = \int_{b_1}^{d'} \left( 2 \int_{\sigma}^{\xi} \mathrm{g}(\rho) \ \mathrm{d}\rho \right)^{-1/2} \mathrm{d}\sigma + \int_{d'}^{\xi} \left( 2 \int_{\sigma}^{\xi} \mathrm{g}(\rho) \ \mathrm{d}\rho \right)^{-1/2} \mathrm{d}\sigma$$

It is easily seen that the first term belongs to  $C^{\infty}([(d'+b_3)/2,b_3])$ then the second term is essential to the asymptotic behavior of  $s(\xi)$  when  $\xi \uparrow b_3$ . Expand  $g(\rho)$  around  $\rho = b_3$  as follows,

$$g(\rho) = g'(b_3)(\rho - b_3) + r_1(\rho)(\rho - b_3)^2 = g_1(\rho) + g_2(\rho)$$
.

By the simple calculation, we have,

$$(4.5) \int_{d}^{\xi} \left(2\int_{\sigma}^{\xi} g_{1}(\rho) d\rho\right)^{-1/2} d\sigma = \int_{d}^{\xi} \left(2\int_{\sigma}^{\xi} g'(b_{3})(\rho - b_{3}) d\rho\right)^{-1/2} d\sigma$$
$$= \frac{1}{\left(-g'(b_{3})\right)^{1/2}} \log \frac{b_{3}^{-d'} + \left((d' - b_{3})^{2} - (\xi - b_{3})^{2}\right)^{1/2}}{b_{3}^{-\xi}}$$

$$(4.6) \quad \left| \begin{array}{c} \frac{\int_{\sigma}^{\xi} g_{2}(\rho) d\rho}{\int_{\sigma}^{\xi} g_{1}(\rho) d\rho} \right| = \left| \begin{array}{c} \frac{\int_{\sigma}^{\xi} r_{1}(\rho) (\rho - b_{3})^{2} d\rho}{\frac{1}{2} g'(b_{3}) ((\sigma - b_{3})^{2} - (\xi - b_{3})^{2})} \right| \\ \leq \frac{2 r_{*}}{-3g'(b_{3})} (2b_{3}^{-} \sigma - \xi) \leq \frac{4 r_{*}}{-3g'(b_{3})} (b_{3}^{-} d') \end{array} \right|$$

where  $r_* = \max_{\substack{b_2 \leq \rho \leq b_3}} |r_1(\rho)|$ ,  $d' < (d' + b_3)/2 \leq \xi < b_3$ .

By the power series expansion, we have,

 $(1 + Y)^{-1/2} = \sum_{j=0}^{\infty} c_j Y^j$  for |Y| < 1 (radius of convergence) where  $c_j = (-\frac{1}{2})(-\frac{1}{2}-1)(-\frac{1}{2}-2)\cdots(-\frac{1}{2}-(j-1)) / j!$ . Then by using the above expansion with the estimate (4.6), we have

the following expansion .

$$(4.7) \left( 2 \int_{\sigma}^{\xi} g(\rho) d\rho \right)^{-1/2} = \left( 2 \int_{\sigma}^{\xi} g_{1}(\rho) d\rho \right)^{-1/2} \\ \times \sum_{j=0}^{\infty} c_{j} \left( \frac{\int_{\sigma}^{\xi} g_{2}(\rho) d\rho}{\int_{\sigma}^{\xi} g_{1}(\rho) d\rho} \right)^{j} \text{ for } d' < (d'+b_{3})/2 \leq \xi < b_{3}.$$

For any  $\varepsilon > 0$ , take d' near to  $b_3$  and fix it, so that we have by the estimate (4.6), the following estimate (4.8)

(4.8) 
$$\left| 1 - \sum_{j=0}^{\infty} c_j \left( \frac{\int_{\sigma}^{\xi} g_2(\rho) d\rho}{\int_{\sigma}^{\xi} g_1(\rho) d\rho} \right)^j \right| \leq \varepsilon$$

for  $\xi \in [(d'+b_3)/2, b_3)$  and  $\sigma \in [d', \xi]$ . Integrating (4.7) with  $\sigma$  from d' to  $\xi$ , we have

$$(4.9) \quad 1 - \varepsilon \leq \frac{\int_{d}^{\xi} \left(2 \int_{\sigma}^{\xi} g(\rho) d\rho\right)^{-1/2} d\sigma}{\int_{d}^{\xi} \left(2 \int_{\sigma}^{\xi} g_{1}(\rho) d\rho\right)^{-1/2} d\sigma} \leq 1 + \varepsilon$$

for  $\xi \in [(d'+b_3)/2,b_3)$ .

Using (4.4), (4.5) and (4.9), we have the following estimate (4.10)

$$(4.10) \quad 1 - \varepsilon \leq \frac{\lim_{\xi \uparrow b_{3}} \frac{s(\xi)}{(-g'(b_{3}))^{-1/2} \log \frac{1}{b_{3}} - \xi}}{\left( -g'(b_{3}) \right)^{-1/2} \log \frac{1}{b_{3}} - \xi} \leq 1 + \varepsilon$$

for any  $\varepsilon > 0$ . Then we have  $\lim_{\xi \uparrow b_3} \frac{s(\xi)}{(-g'(b_3))^{-1/2} \log \frac{1}{b_3 - \xi}} = 1$ . Hereafter we take d' near  $b_3$  and fix it so that  $B(\xi,\sigma) \ge 1/2$  for  $\xi \in [(d'+b_3)/2,b_3)$  and  $\sigma \in [d',\xi]$ . We put  $B(\xi,\sigma) = \left(1 + \frac{\int_{\sigma}^{\xi} g_2(\rho) d\rho}{\int_{\sigma}^{\xi} g_1(\rho) d\rho}\right)^{-1/2}$   $F(\xi,\sigma) = \left(2 \int_{\sigma}^{\xi} g_1(\rho) d\rho\right)^{1/2}$ and then, the second term of  $(4.4) = \int_{d}^{\xi} \frac{1}{F(\xi,\sigma)} B(\xi,\sigma) d\sigma$   $= \int_{0}^{\xi-d'} \frac{1}{F(\xi,\xi-\eta)} B(\xi,\xi-\eta) d\eta = I(\xi)$ . Then we have the following.  $(4.11) \quad \frac{d}{d\xi} I(\xi) = \frac{1}{F(\xi,d')} B(\xi,d') + \int_{0}^{\xi-d'} \frac{\partial}{\partial\xi} \frac{1}{F(\xi,\xi-\eta)} B(\xi,\xi-\eta) d\eta$  $+ \int_{0}^{\xi-d'} \frac{1}{F(\xi,\xi-\eta)} \frac{\partial}{\partial\xi} B(\xi,\xi-\eta) d\eta$ 

- 67 -

$$\geq \frac{1}{2} \frac{1}{F(\xi, d')} + \frac{1}{2} \int_{0}^{\xi-d'} \frac{\partial}{\partial \xi} \frac{1}{F(\xi, \xi-\eta)} d\eta$$

$$+ \int_{0}^{\xi-d'} \frac{1}{F(\xi, \xi-\eta)} \frac{\partial}{\partial \xi} B(\xi, \xi-\eta) d\eta$$
Here we have used that  $\frac{\partial}{\partial \xi} \frac{1}{F(\xi, \xi-\eta)} \geq 0$ .

On the other hand, one can easily check that

$$\begin{array}{l} \frac{\mathrm{d}}{\mathrm{d}\xi} \left(\begin{array}{c} \int_{\xi-\eta}^{\xi} \mathrm{g}_{2}(\rho) \mathrm{d}\rho \\ \int_{\xi-\eta}^{\xi} \mathrm{g}_{1}(\rho) \mathrm{d}\rho \end{array}\right) \quad \text{is bounded for } (\mathrm{d}'+\mathrm{b}_{3})/2 \leq \xi < \mathrm{b}_{3} \quad \text{and} \\ 0 < \eta \leq \xi - \mathrm{d}' \quad \text{and also is } \frac{\partial}{\partial\xi} \mathrm{B}(\xi,\xi-\eta) \ , \ \text{then we have some} \\ \text{constant } \mathrm{M} \quad \text{such that } \left|\begin{array}{c} \frac{\partial}{\partial\xi} \mathrm{B}(\xi,\xi-\eta) \\ \frac{\partial}{\partial\xi} \mathrm{B}(\xi,\xi-\eta) \end{array}\right| \leq \mathrm{M} \\ \text{for } \xi \in \left[(\mathrm{d}'+\mathrm{b}_{3})/2,\mathrm{b}_{3}\right] \ \text{and } 0 < \eta \leq \xi - \mathrm{d}' \ . \ \text{Therefore we have} \\ (4.12) \quad \frac{\partial}{\partial\xi} \mathrm{I}(\xi) \geq -\frac{1}{2} \quad \frac{\partial}{\partial\xi} - \int_{0}^{\xi-\mathrm{d}'} \frac{1}{\mathrm{F}(\xi,\xi-\eta)} \mathrm{d}\eta - \mathrm{M} \int_{0}^{\xi-\mathrm{d}'} \frac{1}{\mathrm{F}(\xi,\xi-\eta)} \mathrm{d}\eta \\ = \quad \frac{1}{2} \quad \frac{\partial}{\partial\xi} \int_{\mathrm{d}'}^{\xi} \frac{1}{\mathrm{F}(\xi,\sigma)} \mathrm{d}\sigma - \mathrm{M} \int_{\mathrm{d}'}^{\xi} \frac{1}{\mathrm{F}(\xi,\sigma)} \mathrm{d}\sigma \\ = \quad \frac{1}{2} \quad (-\mathrm{g}'(\mathrm{b}_{3}))^{-1/2} \left( \frac{1}{\mathrm{b}_{3} - \xi} + \frac{(\mathrm{b}_{3} - \xi) \cdot ((\mathrm{d}'-\mathrm{b}_{3})^{2} - (\mathrm{b}_{3} - \xi)^{2})^{-1/2}}{\mathrm{b}_{3} - \epsilon} \right) \\ - \quad \frac{\mathrm{M}}{2} \quad (-\mathrm{g}'(\mathrm{b}_{3}))^{-1/2} \log \left( \frac{\mathrm{b}_{3} - \mathrm{d}' + ((\mathrm{d}'-\mathrm{b}_{3})^{2} - (\mathrm{b}_{3} - \xi)^{2})^{1/2}}{\mathrm{b}_{3} - \xi} \right) \\ \text{Then we conclude by } (4.4) \ \text{that} \end{array}$$
In the case that  $\xi$  approaches d, we can deal as the same procedure as above by the following decomposition.

$$s(\xi) = \int_{b_1}^{e_1} \left( 2 \int_{\sigma}^{b_1} g(\rho) d\rho + 2 \int_{d}^{\xi} g(\rho) d\rho \right)^{-1/2} d\sigma$$

$$+ \int_{e_{1}}^{d} \left(2 \int_{\sigma}^{b_{1}} g(\rho) d\rho + 2 \int_{d}^{\xi} g(\rho) d\rho \right)^{-1/2} d\sigma + \int_{d}^{\xi} \left(2 \int_{\sigma}^{\xi} g(\rho) d\rho \right)^{-1/2} d\sigma$$

Then we omit the proof of this case and we complete the proof of Lemma 4.1.

By Lemma 4.1, the equation (4.3) which is rewritten as follows

 $s(\xi) = \vartheta^{1/2}$ 

has exactly two solutions  $\xi_1 < \xi_2$  in the interval  $(d,b_3)$  by taking the parameter  $\vartheta > 0$  adequately large and at the same time (4.13) s' $(\xi_1) < 0$  and s' $(\xi_2) > 0$  hold. We fix this  $\vartheta$  and also  $f(\xi) = \vartheta g(\xi)$ . Therefore corresponding to  $\xi_1$  and  $\xi_2$ , we obtain two solutions  $V^{(1)}$  and  $V^{(2)}$  of (4.2) for  $f(\xi) = \vartheta g(\xi)$  determined above and one can easily check that  $b_1 < V^{(1)}(z) < V^{(2)}(z) < b_3$  in -1 < z < 1. By the aid of the almost same method as in K.Maginu [14], we can investigate the signature of the linearized first eigenvalues  $\lambda_{V^{(0)}}$  $\lambda_{V^{(1)}}$  and  $\lambda_{V^{(2)}}$  (See (3.5) for definition ) by (4.13) and we conclude that  $\lambda_{V^{(0)}} > 0$ ,  $\lambda_{V^{(1)}} < 0$  and  $\lambda_{V^{(2)}} > 0$  where  $V^{(0)}(z) = b_1$ . We complete the proof of Proposition 2.

Hereafter first we will construct a family of solutions of (3.1)  $\{v_{\zeta}^{(2)}\}_{0<\zeta<\zeta_{\star}}$  such that  $v_{\zeta}^{(2)}$  behaves like  $V^{(2)}$  in Q( $\zeta$ ) and takes values near  $b_1$  in  $D_1 \cup D_2$  and moreover  $\mu_1(v_{\zeta}^{(2)}) > 0$  holds for small  $\zeta > 0$ . Here we denoted by  $\mu_1(v_{\zeta}^{(2)})$  the first eigenvalue of the eigenvalue problem (3.5) for the family {  $v_{\zeta}^{(2)}$ }<sub>0<\zeta<\zeta\_{\*}</sub>. We set the function  $\Psi_*(x_1) = \Phi_{V(2)}(x_1) + \rho_*$  where  $\rho_* > 0$  is a small constant such that  $\lambda_{V(2)} \phi_{V(2)}(x_1) - \rho_* f'(V^{(2)}(x_1)) > 0$  for any  $x_1 \in [-1,1]$ . (Recall that  $V^{(2)}(-1) = V^{(2)}(1) = b_1$  and f'(b<sub>1</sub>) < 0 . ) Now we define a function  $W_{\zeta}(x)$  which is defined in  $\Omega(\zeta)$  as follows  $W_{\mathcal{E}}(\mathbf{x}_1,\mathbf{x}') \equiv$  $b_{1} + \frac{\zeta}{2} \left( \frac{dV^{(2)}}{dx_{1}} (1 - 2\zeta) - \delta_{*}(\zeta) \frac{d\Psi_{*}}{dx_{1}} (1 - 2\zeta) \right)$ for  $x \in D_1^{\cup} D_2^{\cup} (R_1(\zeta) \cap \{x_1 \ge 1-\zeta\}) \cup (R_2(\zeta) \cap \{x_1 \le -1+\zeta\})$  $b_{1} - \frac{1}{2\zeta} \left( \frac{dV^{(2)}}{dx_{1}} (1-2\zeta) - \delta_{*}(\zeta) \frac{d\Psi_{*}}{dx_{1}} (1-2\zeta) \right) \cdot (x_{1}-1+2\zeta) \cdot (x_{1}-1)$ for  $x \in R_1(\zeta) \cap \{ 1-2\zeta \le x_1 < 1-\zeta \}$  $\left\{ \begin{array}{l} \mathbb{V}^{(2)}(\mathbf{x}_{1}) - \delta_{*}(\zeta) \ \Psi_{*}(\mathbf{x}_{1}) & \text{for } \mathbf{x} \in \Gamma(\zeta) \\ \\ \mathbb{W}_{\zeta}(-\mathbf{x}_{1},\mathbf{x}') & \text{for } \mathbf{x} \in \mathbb{R}_{2}(\zeta) \cap \{ -1+\zeta < \mathbf{x}_{1} \leq -1 + 2\zeta \} \end{array} \right.$ where we have put  $\delta_{*}(\zeta) = (V^{(2)}(1-2\zeta) - b_{1}) / \Psi_{*}(1-2\zeta)$ . It is easily seen that  $\delta_*(\zeta) > 0$  and  $\lim_{\zeta \to 0} \delta_*(\zeta) = 0$ .

Lemma 4.2.  $W_{\zeta} \in C^{1}(\overline{\Omega(\zeta)})$  and we have, for small  $\zeta > 0$ ,  $\Delta W_{\zeta} + f(W_{\zeta}) > 0$ in  $\Omega(\zeta) - R_{1}(\zeta) \cap \{x_{1}=1-2\zeta \text{ or } 1-\zeta\} - R_{2}(\zeta) \cap \{x_{1}=-1+2\zeta \text{ or } -1+\zeta\}$  $\frac{\partial W_{\zeta}}{\partial \nu} = 0$  on  $\partial \Omega(\zeta)$ .

(Proof of Lemma 4.2) One can check  $W_{\zeta} \in C^{1}(\overline{\Omega(\zeta)})$  by a simple calculation. In  $D_{1} \cup D_{2} \cup (R_{1}(\zeta) \cap \{x_{1} > 1-\zeta\}) \cup (R_{2}(\zeta) \cap \{x_{1} < -1+\zeta\})$  $dV^{(2)}$ 

$$\begin{split} & \Delta W_{\zeta} = 0 \quad \text{and} \quad W_{\zeta}(\mathbf{x}) \leq b_{1} \quad \text{for small } \zeta > 0 \quad \text{from} \quad \frac{\mathrm{dv}}{\mathrm{dx}_{1}}(1) < 0 \\ & \text{and} \quad \lim_{\zeta \to 0} \delta_{\star}(\zeta) = 0 \quad \text{Then by (IV-1), we obtain the inequality.} \\ & \text{In} \quad R_{1}(\zeta) \cup \{1-2\zeta < \mathbf{x}_{1} < 1-\zeta \} \quad , \quad \Delta W_{\zeta} = \frac{-1}{2\zeta} \left(\frac{\mathrm{dV}^{(2)}}{\mathrm{dx}_{1}}(1-2\zeta) - \delta_{\star}(\zeta)\frac{\mathrm{d\Psi}_{\star}}{\mathrm{dx}_{1}}(1-2\zeta)\right) \end{split}$$

> 0 and  $W_{\zeta}(x) \leq b_1$  for small  $\zeta > 0$ . Therefore we have the inequality by the same way above also in  $R_2(\zeta) \cup \{-1+\zeta < x_1 < -1+2\zeta\}$ .

In 
$$\Gamma(\zeta)$$
, we calculate as follows,

$$\Delta W_{\zeta} + f(W_{\zeta}) = \frac{\partial^2}{\partial x_1^2} (V^{(2)} x_1) - \delta_*(\zeta) \Psi_*(x_1))$$
  
+  $f(V^{(2)}) - \delta_*(\zeta) \Psi_* f'(V^{(2)}) + \delta_*(\zeta)^2 \Psi_*^2 \varepsilon_{\zeta}$ 

(where 
$$\mathcal{Z}_{\zeta}(\mathbf{x}_1) = \int_0^1 (1-\tau) f''(\mathbf{v}^{(2)}(\mathbf{x}_1) - \tau \delta_{\star}(\zeta) \Psi_{\star}(\mathbf{x}_1)) d\tau$$
)

$$= \frac{d^{2}V^{(2)}}{dx_{1}^{2}} + f(V^{(2)}) - \delta_{*}(\zeta)\left(\frac{d^{2}\Psi_{*}}{dx_{1}^{2}} + f'(V^{(2)})\right) + \delta_{*}(\zeta)^{2}\Psi_{*}^{2}\Xi_{\zeta}$$

$$= \delta_{*}(\zeta) \left\{ \left(\lambda_{V^{(2)}V^{(2)}(x_{1})} - \rho_{*}f'(V^{(2)}(x_{1}))\right) + \delta_{*}(\zeta)\Xi_{\zeta}(x_{1}) \right\} > 0$$

holds in  $\Gamma(\zeta)$  for sufficiently small  $\zeta > 0$  by  $\lim_{\zeta \to 0} \delta_{*}(\zeta) = 0$ . Therefore we have completed the proof of Lemma 4.2.

By Lemma 4.2,  $W_{\zeta}$  is a "weak lower solution " in the sense of D.H.Sattinger [20] for small  $\zeta > 0$  and we have the following comparison property by the argument used in Section 2 and the comparison theorem.

The set  $E_1(\zeta) = \{ \psi \in C^1(\overline{\Omega(\zeta)}) \cap C^2(\Omega(\zeta)) | \psi(x) \ge W_{\zeta}(x) \text{ in } \Omega(\zeta) \}$ is a positively invariant set under the flow defined by the evolution equation (1.2) for  $\Omega(\zeta)$  and f in this section and also is the set  $E_*(\zeta) = E(\delta,\zeta) \cap E_1(\zeta)$  ( $\delta \in [\delta(\zeta), 2\delta(\zeta)]$ ) for sufficiently small  $\zeta > 0$ , where  $E(\delta,\zeta)$  and  $\delta(\zeta)$  are the ones constructed in Section 2. Therefore applying again Theorem 4.2 in [15], we have at least one stable equilibrium solution in  $E_*(\zeta)$ for small  $\zeta > 0$ . Moreover we have the following.

Lemma 4.3. For small  $\zeta > 0$ , there exists exactly one solution  $v_{\zeta}^{(2)}$  of (3.1) in  $E_{*}(\zeta)$  and the linealized first eigenvalue  $\mu_{1}(v_{\zeta}^{(2)})$  is bounded from below by a positive constant.

Here we put  $v_{\zeta}^{(0)}(x) = b_1$  in  $\Omega(\zeta)$  which is a stable solution.

One can easily check that  $v_{\zeta}^{(0)}(x) < v_{\zeta}^{(2)}(x)$  in  $\Omega(\zeta)$  by the Strong Maximum Principle and by Theorem 4.4 in Matano [15], we obtain another solution  $v_{\zeta}^{(1)}$  between the above two solutions.  $(v_{\zeta}^{(0)}(x) < v_{\zeta}^{(1)}(x) < v_{\zeta}^{(2)}(x)$  for  $x \in \Omega(\zeta)$ )  $\lim_{\zeta \to 0} \mu_1(v_{\zeta}^{(2)}) > 0$  implies that  $v_{\zeta}^{(2)}$  is locally unique for small

- 72 -

 $\zeta > 0$  and then  $v_{\zeta}^{(1)}$  must be asymptotically near to  $V^{(1)}$  on  $Q(\zeta)$  by Theorem 3 and Proposition 2. Therefore we have obtained the following result by Theorem 3.

Theorem 4. There exist three distinct solutions ( for small  $\zeta$  )  $v_{\zeta}^{(0)} < v_{\zeta}^{(1)} < v_{\zeta}^{(2)}$  of (3.1) where  $f = \vartheta g$  (  $\vartheta \ge \vartheta_0$  ) and the solutions satisfy the following asymptotic conditions.

$$\begin{split} \lim_{\zeta \to 0} \sup_{x \in D_1 \cup D_2} | v_{\zeta}^{(i)}(x) - b_1 | &= 0 \quad (i = 0, 1, 2) \\ \lim_{\zeta \to 0} \sup_{x \in Q(\zeta)} | v_{\zeta}^{(i)}(x_1, x') - V^{(i)}(x_1) | &= 0 \quad (i = 1, 2) \\ \frac{\lim_{\zeta \to 0} \mu_1(v_{\zeta}^{(0)}) > 0 \quad , \quad \frac{\lim_{\zeta \to 0} \mu_1(v_{\zeta}^{(1)}) < 0 \quad , \quad \frac{\lim_{\zeta \to 0} \mu_1(v_{\zeta}^{(2)}) > 0 \\ \text{here we denoted by} \quad \mu_1(v_{\zeta}^{(1)}) \quad \text{the first eigenvalue of the} \end{split}$$

eigenvalue problem (3.5) for the family {  $v_{\zeta}^{(1)}$ }  $0 < \zeta < \zeta_{*}$ .

whe

## §5 Concluding Remarks.

In Section 4, by choosing an appropriate nonlinear term f in the situation of Section 3 plus a condition  $a_1 = a_2 = b_1$ , we have constructed three distinct solutions  $v_{\xi}^{(0)} < v_{\xi}^{(1)} < v_{\xi}^{(2)}$  of (3.1) for small  $\zeta > 0$  such that  $v_{\xi}^{(1)}$  takes values near  $b_1$  in  $D_1 \cup D_2$  and behaves like  $V^{(1)}$  in  $Q(\zeta)$  for small  $\zeta > 0$  (i = 0,1,2) and  $\mu_1(v_{\xi}^{(0)}) \ge c$ ,  $\mu_1(v_{\xi}^{(1)}) \le -c$  and  $\mu_1(v_{\xi}^{(2)}) \ge c$  hold for small  $\zeta > 0$  and a positive constant c which is independent of  $\zeta$ . All  $v_{\xi}^{(1)}$  (i = 0,1,2) take almost same values near  $b_1$  in  $D_1 \cup D_2$  (f'( $b_1$ ) < 0) but  $v_{\xi}^{(1)}$  is unstable while  $v_{\xi}^{(0)}$  and  $v_{\xi}^{(2)}$  are stable for small  $\zeta > 0$ . This phenomenon is owing to the fact that the asymptotic behavior of  $v_{\xi}^{(1)}$  on  $Q(\zeta)$  corresponds to  $V^{(1)}$  which is an unstable solution of the ordinary differential equation (3.4) on L while those of  $v_{\xi}^{(0)}$  and  $v_{\xi}^{(2)}$  correspond to the stable ones  $V^{(0)}$  and  $V^{(2)}$ . See Figure 6.

From these results, we see that the dependence of the stability of the solution, upon the moving part  $Q(\zeta)$ , does not vanish when  $\zeta \rightarrow 0$ . Moreover the behavior of the solution on  $Q(\zeta)$  plays an important role to determine the stability and on the other hand, it is described as the solution of the ordinary differential equation (3.4) on L in this case.

Therefore we conclude that it natural to regard  $\Omega(\zeta)$  as a perturbation from  $\Omega_* = D_1 \cup D_2 \cup L$  (See Figure 4 in Section 1) if we cosider the behavior of the structure of the solutions of (3.1).

- 74 -



Figure 6

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