Singular Perturbation of Domains and Semilinear Elliptic Equation

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Contents

§1 Introduction We consider the following semilinear elliptic boundary value problem,

$$(1.1)\quad \begin{cases} \Delta v + f(v) = 0 & \text{in } \Omega \\ \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega . \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and ν denotes the unit outer normal vector on $\partial\Omega$. $\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ is the Laplace operator and f is a real valued smooth function on $\mathbb{R}$.

The structure of the solutions of (1.1) and their stability largely depend upon the geometrical property of the domain Ω and we may consider that the structure usually varies continuously under the smooth deformation of Ω . Our subject in this paper is to consider the behavior of the solutions and their structure when the domain Ω singularly perturbs. The domain which we deal with is exhibited in Figure 1 and it is decomposed as follows $\Omega(\zeta) = D_1 \cup D_2 \cup Q(\zeta)$ where D_1 and D_2 are mutually disjoint and $Q(\zeta)$ is a moving part which approaches a segment as $\zeta \downarrow 0$. Therefore the volume of $Q(\zeta)$ decreases to zero as $\zeta \downarrow 0$.

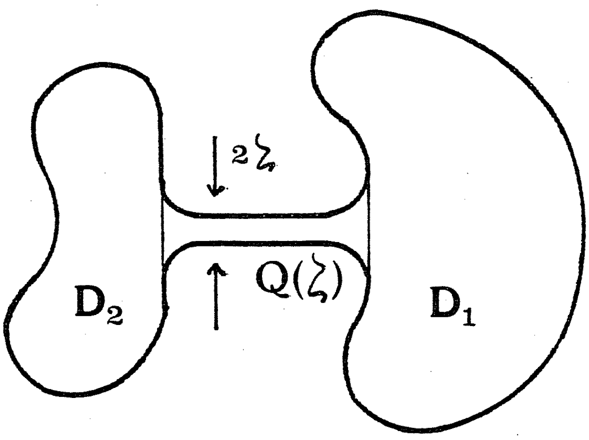


Figure 1

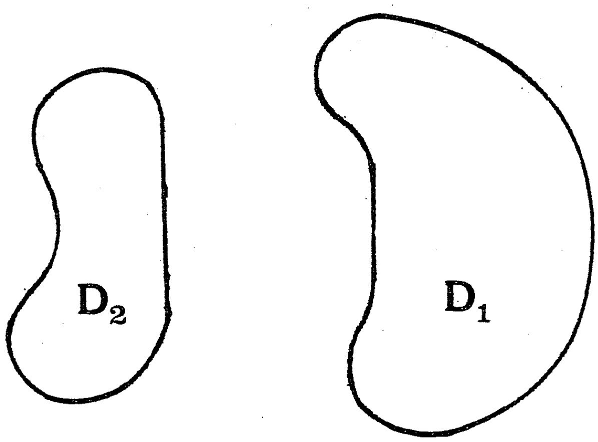


Figure 2

Then can we say that the influence of $Q(\zeta)$ over (1.1) for $\Omega = \Omega(\zeta)$ vanishes as $\zeta \downarrow 0$ i.e. that the structure of the solutions of (1.1) for $\Omega = \Omega(\zeta)$ (for small $\zeta > 0$) is equivalent to that of (1.1) for $\Omega = \Omega_0 \equiv D_1 \cup D_2$ (Figure 2)?

In fact, Vegas [22], Hale and Vegas [10] have considered (1.1) for $f = f(\lambda, u) = \lambda u - u^p$ on the same domain as that in Figure 1 and analyzed the bifurcation phenomenon for the bifurcation parameter ζ (when $\lambda > 0$ is a sufficiently small constant). Their bifurcation diagram in the case that p is an odd natural number and the domain $\Omega(\zeta)$ is symmetric, is in Figure 3.

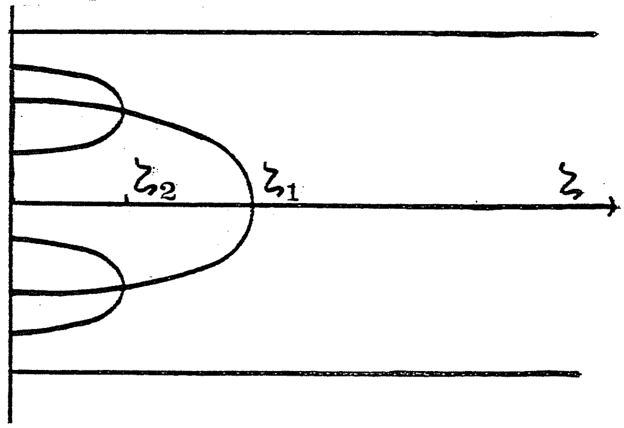


Figure 3 (Bifurcation Diagram)

In their situation, when ζ is very small (i.e. $0 < \zeta < \zeta_2$ in Figure 3) there are exactly nine solutions and each of them takes values near one of the values $\left\{ 0, \lambda^{1/(p-1)}, -\lambda^{1/(p-1)} \right\}$ in D_i ($i = 1, 2$) and its behavior on $Q(\zeta)$ is automatically determined by the behavior on D_1 and D_2 . Thus the structure of the solutions for $\Omega(\zeta)$ ($0 < \zeta < \zeta_2$) is equivalent to that for $\Omega = \Omega_0$ (non-connected open set). Remark that (1.1) for $\Omega = \Omega_0$ has exactly nine solutions, each of which is equal to one of the values $\left\{ 0, \lambda^{1/(p-1)}, -\lambda^{1/(p-1)} \right\}$ in D_i for each i for sufficiently small $\lambda > 0$. In this case, $\Omega(\zeta)$ can be regarded as a perturbation from Ω_0. Nevertheless,in this paper, we conclude that it is more natural to regard $\Omega(\zeta)$ as a perturbation rather from the set $\Omega_* = D_1 \cup D_2 \cup L$ (exhibited in Figure 4) where $L = \bigcap_{\zeta>0} Q(\zeta)$ than from $\Omega_0 = D_1 \cup D_2$ if we consider the domain perturbation up to the structure of the solutions of (1.1) for $\Omega = \Omega(\zeta)$.

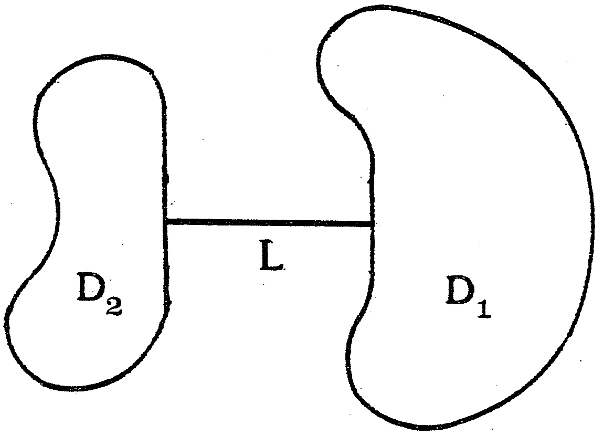


Figure 4

In the situation of Hale-Vegas [10] and Vegas [22], we remark that $\frac{\partial f}{\partial u}$ is small around the solutions from the smallness of $\lambda > 0$ and this may ensure the uniqueness of the behavior of the solution v on $Q(\zeta)$ when v is specified to take values near a_i in D_i ($i = 1, 2$) where $f(a_i) = 0$ and $f'(a_i) < 0$, but the solution is rather free on $Q(\zeta)$ for general f .

We consider a family of functions $\left\{ v_\zeta \right\}_{\zeta>0}$ such that v_ζ is an arbitrary solution of (1.1) for $\Omega = \Omega(\zeta)$ and $\lim_{\zeta\to 0} \| v_\zeta - a_i \|_{L^2(D_i)} = 0$ holds for $i = 1 , 2$ where a_i is any point satisfying $f(a_i) = 0$ and $f'(a_i) < 0$ and we prove that for any sequence of positive values $\{ \zeta_m \}_{m=1}^{\infty}$ such that $\lim_{m\to\infty} \zeta_m = 0$, there exist a subsequence $\{ \varkappa_m \}_{m=1}^{\infty} \subset \{ \zeta_m \}_{m=1}^{\infty}$ and a solution V of the two point boundary value problem of the ordinary differential equation (1.2),

$$
(1.2) \quad \begin{cases} \dfrac{d^2V}{dz^2} + f(V) = 0 \quad \text{in} \quad L \\[2ex] V(z) = a_i \qquad z \in \overline{D}_i \cap \overline{L} \qquad (i = 1 , 2) \end{cases}
$$

such that $v_{\varkappa_m}$ is asymptotically near to V in $Q(\varkappa_m)$ in the sense of "uniform convergence" and near to a_i in $\overline{Q(\varkappa_m)} \cap \overline{D}_i$. In this case, the stability of $v_{\varkappa_m}$ in (1.1) for $\Omega = \Omega(\varkappa_m)$ coincides with the stability of V in (1.2) for large m. Conversely, we take an appropriate nonlinear term f for which (1.2) has two stable solutions $V^{(0)} < V^{(2)}$ and another unstable solution $V^{(1)}$ between them, in the case that $a_1 = a_2 = b_1$ and $f(b_1) = 0$ and $f'(b_1) < 0$, and we construct three distinct solutions $v_\zeta^{(0)} < v_\zeta^{(1)} < v_\zeta^{(2)}$ of (1.1) for $\Omega = \Omega(\zeta)$ small $\zeta > 0$ such that $v_\zeta^{(i)}$ behaves like $V^{(i)}$ in $Q(\zeta)$ ($i = 0,1,2$) and takes values near b_1 in $D_1 \cup D_2$ and $v_\zeta^{(0)}$ and $v_\zeta^{(2)}$ are stable and $v_\zeta^{(1)}$ is unstable for small $\zeta > 0$. Therefore we see that the behavior of the solution v_ζ on $Q(\zeta)$ which is almost governed by the equation (1.2) on L, plays an important role to determine the stability of v_ζ even if $\zeta > 0$ is small. From these facts, we conclude that we should regard $\Omega(\zeta)$ as a perturbation from $\Omega_* = D_1 \cup D_2 \cup L$.

The boundary value problem (1.1) is a stationary problem of the following parabolic boundary value problem,

$$
(1.3) \quad \begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + f(u) & \text{in} \quad (0,\infty) \times \Omega \\[2ex] \dfrac{\partial u}{\partial \nu} = 0 & \text{on} \quad (0,\infty) \times \partial\Omega . \end{cases}
$$

Definition 0. A solution of (1.3) which is independent of the variable t is said to be an equilibrium solution.

We recall the definition of the stability of an equilibrium solution.

Definition 1. The equilibrium solution v of (1.3) is said to be stable if given any $\varepsilon > 0$, there exists a $\delta > 0$, such that $\| u(t,\cdot) - v(\cdot) \|_{L^\infty(\Omega)} \leqq \varepsilon$ $(0 < t < \infty)$ for any $w \in C^0(\overline{\Omega})$ satisfying $\| v - w \|_{L^\infty(\Omega)} \leqq \delta$, where u is a solution of (1.3) with the intial condition $u(0,x) = w(x)$. We say that w is unstable if v is not stable.
For details, see Matano [15].

It has been observed by several authors that the stability and the structure of the equilibrium solutions are closely related to the geometry of the domain Ω. It is known that any non-constant equilibrium solution must be unstable if Ω is a bounded convex domain in $\mathbb{R}^n$. (See N.Chafee [4] for $n = 1$ and see H.Matano [15] and Casten-Holland [3] for general n.) More generally, the same result holds in the case that Ω is a Riemannian manifold with non-negative Ricci curvature and $\partial\Omega$ has non-positive definite second fundamental form with respect to the unit outer normal vector ν on $\partial\Omega$ (S.Jimbo [11]). On the other hand, Matano [15] has constructed a non-constant stable equilibrium solution on the same type of domain as $\Omega(\zeta)$ in Figure 1. We shall refine his result in Section 2. On the other hand, there are several results as for the reaction-diffusion system. See K.Kishimoto and H.F.Weinberger [13], H.Matano and M.Mimura [17].

The contents of this paper are as follows.

In Section 2, first we will set a perturbing domain $\Omega(\zeta) = \bigcup_{i=1}^{N} D_i \cup Q(\zeta)$ under a rather weak condition (so it may be a very wild perturbation) and for small $\zeta > 0$, we will construct a stable equilibrium solution v_ζ of (1.3) for $\Omega = \Omega(\zeta)$ which takes values near a_i in D_i ($1 \leqq i \leqq N$) where a_i is an arbitrary zero point of f such that $f'(a_i) < 0$.

In Section 3, we will establish the domain $\Omega(\zeta)$ in Figure 1 concretely (for the delicate argument) and analyze the behavior on $Q(\zeta)$ of the solution of (1.1) for $\Omega = \Omega(\zeta)$ which takes values near a_i in D_i ($f(a_i) = 0$, $f'(a_i) < 0$) and we prove that v_ζ is asymptotically near to some solution of the ordinary differential equation (1.2) up to the stability.

In Section 4, we will choose an appropriate f ($a_1 = a_2 = b_1$ in this case) so that (1.2) has two stable solutions $V^{(0)} < V^{(2)}$ and another unstable solution $V^{(1)}$ such that $V^{(0)} < V^{(1)} < V^{(2)}$. For the domain $\Omega(\zeta)$ in Section 3 and these f and $V^{(0)}$, $V^{(1)}$ and $V^{(2)}$, we shall construct three distinct solutions $v_\zeta^{(0)}$, $v_\zeta^{(1)}$ and $v_\zeta^{(2)}$ such that $v_\zeta^{(i)}$ behaves like $V^{(i)}$ in $Q(\zeta)$ and takes values near b_1 in $D_1 \cup D_2$ for each i ($0 \leqq i \leqq 2$) and $v_\zeta^{(0)}$ and $v_\zeta^{(2)}$ are stable and $v_\zeta^{(1)}$ is unstable for small $\zeta > 0$.
All the functions that we consider in this paper are real valued.

§2 Existence of Stable Solutions.

Let $D_1, D_2, \cdots, D_N$ be bounded domains in $\mathbb{R}^n$ ($n \geqq 2$) such that each D_j has a smooth boundary ∂D_j and $D_i \cap D_j = \emptyset$ holds for any i and j with $i > j$. From now on we establish the situation.

(II-1) Let $\{\Omega(\zeta)\}_{\zeta>0}$ be a family of bounded domains in $\mathbb{R}^n$ which satisfies the following conditions (1) and (2) ;

(1) Each $\Omega(\zeta)$ has a smooth boundary and $\Omega(\zeta_1) \supset \Omega(\zeta_2) \supset \bigcup_{i=1}^{N} D_i$ holds for any ζ_1 and ζ_2 such that $\zeta_1 > \zeta_2 > 0$.

(2) $$\lim_{\zeta \to 0} \mathrm{Vol}(\Omega(\zeta) - \bigcup_{i=1}^{N} D_i) = 0$$

(II-2) Let f be a real valued smooth function on $\mathbb{R}$ such that the set $\Pi \equiv \{ \xi \in \mathbb{R} \mid f(\xi) = 0,\ f'(\xi) < 0 \}$ is not empty.

Under the above conditions (II-1) and (II-2), we will consider the equilibrium solutions of the following semilinear diffusion equation (2.1).

$$(2.1) \quad \begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + f(u) & \text{in } (0,\infty) \times \Omega(\zeta), \\[2ex] \dfrac{\partial u}{\partial \nu} = 0 & \text{on } (0,\infty) \times \partial\Omega(\zeta). \end{cases}$$

We present our first result concerning the existence of stable equilibrium solution which approaches the constant function on each D_i when $\zeta \to 0$.

Theorem 1. For any sequence of values $\{ a_i \}_{i=1}^{N}$ which is contained in the set $\Pi = \left\{ \xi \in \mathbb{R} \mid f(\xi) = 0 , f'(\xi) < 0 \right\}$ and for small $\zeta > 0$, the boundary value problem (2.1) has at least one stable equilibrium solution v_ζ which satisfies the following condition (2.2),

$$(2.2) \quad \begin{cases} \lim_{\zeta \to 0} \| v_\zeta - a_i \|_{L^2(D_i)} = 0 \quad (1 \leqq i \leqq N) \\ \lim_{\zeta \to 0} v_\zeta = a_i \quad \text{in } C^\infty(\overline{D_i(\eta)}) \quad \text{for any } \eta > 0 , \\ \qquad\qquad (1 \leqq i \leqq N) \end{cases}$$

where we have defined $D_i(\eta) \equiv \left\{ x \in D_i \mid \operatorname{dis}(x, \Omega(\eta) - D_i) > \eta \right\}$ for any $\eta > 0$.

Remark. Hale and Vegas [10] have proved a similar result to our Theorem 1 (also the uniqueness) under some asumption concerning the bound of $\frac{\partial f}{\partial u}$ with the aid of the Implicit Function Theorem. But we do not impose any assumption concerning the bound of $\frac{\partial f}{\partial u}$ and therefore we cannot apply the Implicit Function Theorem, because, as we will prove in Section 4, we can not expect the uniqueness of v_ζ which satisfies (2.2) in general. We apply the result of Matano [15] (Theorem 4.2 in [15]) essentially.

For the proof of Theorem 1, we use the Poincaré type inequality.

Proposition 1. Let D be a bounded domain in $\mathbb{R}^n$ with smooth boundary ∂D. Let $\{ \lambda_q \}_{q=1}^{\infty}$ and $\{ \psi_q \}_{q=1}^{\infty}$ be respectively the sequence of eigenvalues arranged in increasing order and the complete system of the corresponding orthonormalized eigenfunctions associated with $-\Delta$ with Neumann boundary condition. Then we have the following inequality :

$$\frac{1}{\lambda_{q+1}}\int_D |\text{grad}\ \psi|^2\,dx + \sum_{k=1}^{q}\frac{\lambda_{q+1}-\lambda_k}{\lambda_{q+1}}\left(\int_D \psi\,\psi_k\,dx\right)^2$$

$$\geqq \int_D |\,\psi\,|^2\,dx \quad \text{for any} \quad \psi \in H^1(D) \quad \text{and natural number} \quad q\,.$$

This can be easily proved by the eigenfunction-expansion and so we omit the proof.

Hereafter we denote by $\{\lambda_{i,q}\}_{q=1}^{\infty}$ and $\{\psi_{i,q}\}_{q=1}^{\infty}$, respectively the sequence of eigenvalues arranged in increasing order and the complete system of corresponding orthonormalized eigenfunctions associated with the operator $-\Delta$ on D_i with Neumann boundary condition. Hereafter we put $Q(\zeta) \equiv \Omega(\zeta) - \bigcup_{i=1}^{N} D_i$.

(Proof of Theorem 1) We put $a^* \equiv \max_{1\leqq i\leqq N} a_i$ and $a_* \equiv \min_{1\leqq i\leqq N} a_i$. Let $A(x) \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ satisfy the following conditions :

$$(2.3)\quad \begin{cases} A(x) = a_i \quad \text{for any} \quad x \in D_i \quad (1 \leqq i \leqq N) \\ a_* \leqq A(x) \leqq a^* \quad \text{for any} \quad x \in \mathbb{R}^n \quad \text{and} \quad \text{grad}\,A(x) \\ \text{has compact support in } \mathbb{R}^n. \end{cases}$$

We define for $w \in H^1(\Omega(\zeta)) \cap L^\infty(\Omega(\zeta))$,

$$J_\zeta(w) \equiv \int_{\Omega(\zeta)}\left(\frac{1}{2}|\,\text{grad}\ w\,|^2 - \int_{A(x)}^{w(x)} f(\xi)\,d\xi\right)dx$$

and also we define for $\zeta > 0$ and $\delta > 0$,

$$E(\delta,\zeta) \equiv \Big\{ w \in C^2(\Omega(\zeta)) \cap C^1(\overline{\Omega(\zeta)}) \;\Big|\; a_* - \delta \leqq w(x) \leqq a^* + \delta \quad \text{in } \Omega(\zeta)$$

$$J_\zeta(w) \leqq J_\zeta(A) + \delta^3, \quad \|\,w - a_i\,\|_{L^2(D_i)} \leqq \delta, \ i = 1,2,\cdots,N \Big\}.$$

To prove the existence of stable equilibrium solution of (2.1) by the aid of Theorem 4.2 in [15], we will find a positive valued function $\delta(\zeta)$ ($\zeta > 0$) which satisfies the following conditions

$$
(2.4)\quad
\begin{cases}
\lim_{\zeta\to 0} \delta(\zeta) = 0 \\
E(\delta,\zeta) \text{ is a positively invariant closed subset of} \\
C^1(\overline{\Omega(\zeta)}) \cap C^2(\Omega(\zeta)) \text{ under the flow defined by the} \\
\text{equation (2.1) when } \delta \text{ belongs to the interval} \\
[\delta(\zeta), 2\delta(\zeta)] \text{ for small } \zeta > 0 .
\end{cases}
$$

It is clear by the aid of the Comparison - Existence Theorem that if $\delta_0 > 0$ is small so that $f'(\xi) < 0$ for $\xi \in [a_*-\delta_0, a_*) \cup (a^*, a^*+\delta_0]$ hold, there exists a unique classical global solution $u_\zeta(t,x)$ with $u_\zeta(0,x) = w(x)$ and $a_* - \delta \leqq u_\zeta(t,x) \leqq a^* + \delta$, $x \in \Omega(\zeta)$, $t \geqq 0$ for any $w \in C^2(\Omega(\zeta)) \cap C^1(\overline{\Omega(\zeta)})$ such that

$a_* - \delta \leqq w(x) \leqq a^* + \delta$ ($x \in \Omega(\zeta)$, $0 < \delta < \delta_0$).

Notice that δ_0 depends only on f . From now on, we will argue about the behavior of $u_\zeta(t,\cdot)$ when t grows up, under the condition that the initial condition w belongs to the set $E(\delta,\zeta)$. Notice that u_ζ also satisfies the equation given by replacing f in (2.1) by $\bar{f}$ which is identical to f on the interval $[a_*- \delta_0, a^*+ \delta_0]$ and has compact support in $\mathbb{R}$, because the value of $u_\zeta(t,x)$ always belongs to the above interval. Therefore from now up to the end of the proof of Theorem 1, we assume, without loss of generality, that f has a compact support in $\mathbb{R}$.

For each i ($1 \leqq i \leqq N$), we define $\nu^{\zeta}_{i,k}$ and $u^{\zeta}_{i,q}$ as follows,

$$\nu^{\zeta}_{i,k}(t) \equiv \int_{D_i} u_{\zeta}(t,x)\ \psi_{i,k}(x)\ dx$$

$$u^{\zeta}_{i,q}(t,x) \equiv \sum_{k=1}^{q} \nu^{\zeta}_{i,k}(t)\ \psi_{i,k}(x)$$

and applying the inequality of Proposition 1 to $\psi = u_{\zeta} - u^{\zeta}_{i,k}$ and

$D = D_i$ there, we have the following inequality for each i .

$$(2.5) \qquad \int_{D_i} |\ \text{grad}\ u_{\zeta}(t,x)\ |^2 dx \geqq \lambda_{i,q+1} \int_{D_i} |u_{\zeta}(t,x) - u^{\zeta}_{i,q}(t,x)|^2\ dx + \sum_{k=1}^{q} \lambda_{i,k}\ (\ \nu^{\zeta}_{i,k}(t))^2\ .$$

On the other hand, the following inequality (2.6) is derived from

$$\frac{\partial}{\partial t}\ J_{\zeta}(\ u_{\zeta}(t,\cdot)) \leqq 0\ .$$

$$(2.6) \qquad \sum_{i=1}^{N} \int_{D_i} (\frac{1}{2} |\text{grad}\ u_{\zeta}(t,x)|^2 - \int_{A(x)}^{u_{\zeta}(t,x)} f(\xi)\ d\xi\ )\ dx\ + \int_{Q(\zeta)} (\frac{1}{2}\ |\ \text{grad}\ u_{\zeta}(t,x)|^2 - \int_{A(x)}^{u_{\zeta}(t,x)} f(\xi)\ d\xi\ )\ dx \leqq J_{\zeta}(w) \leqq J_{\zeta}(A) + \delta^3\ .$$

By (2.5) and (2.6), we have,

$$(2.7) \quad \frac{1}{2} \sum_{i=1}^{N} (\ \lambda_{i,q+1} \int_{D_i} |\ u_{\zeta} - u^{\zeta}_{i,q}\ |^2 dx + \sum_{k=1}^{q} \lambda_{i,k}\ (\ \nu^{\zeta}_{i,k}(t)\ )^2\ ) - \sum_{i=1}^{N} \int_{D_i} \int_{A}^{u_{\zeta}} f(\xi)\ d\xi\ dx\ + \int_{Q(\zeta)} (\frac{1}{2} |\text{grad}\ u_{\zeta}|^2 - \int_{A}^{u_{\zeta}} f(\xi)\ d\xi\ )\ dx \leqq J_{\zeta}(w) \leqq J_{\zeta}(A) + \delta^3\ .$$

Concerning the second term, we have for each i ($1 \leqq i \leqq N$),

$$(2.8) \quad -\int_{D_i}\int_A^{u_\zeta} f(\xi)\,d\xi\,dx = \int_{D_i}\int_{u^\zeta_{i,q}}^{A} f(\xi)\,d\xi\,dx - \int_{D_i}\int_{u^\zeta_{i,q}}^{u_\zeta} f(\xi)\,d\xi\,dx$$

$$(2.9) \quad \int_{D_i}\int_{u^\zeta_{i,q}}^{u_\zeta} f(\xi)\,d\xi\,dx = \int_{D_i}\int_{u^\zeta_{i,q}}^{u_\zeta} (\, f(\xi) - f(u^\zeta_{i,q}\,)\,)\,d\xi\,dx$$

$$+ \int_{D_i} f(u^\zeta_{i,q}(t,x))\,(\,u_\zeta(t,x) - u^\zeta_{i,q}(t,x)\,)\,d\xi\,dx$$

$$\leqq \int_{D_i}\int_{u_\zeta}^{u^\zeta_{i,q}} \sup_{0<\mu<1} |f'(u^\zeta_{i,q} + \mu(u_\zeta - u^\zeta_{q,i}))|\cdot(u^\zeta_{i,q} - \xi)\,d\xi\,dx \; +$$

$$\int_{D_i}(u_\zeta - u^\zeta_{i,q})\{f(\nu^\zeta_{i,1}\psi_{i,1}) + \int_0^1 f'(\nu^\zeta_{i,1}\psi_{i,1} + \mu\sum_{k=2}^{q}\nu^\zeta_{i,k}\psi_{i,k})\sum_{k=2}^{q}\nu^\zeta_{i,k}\psi_{i,k}d\mu\}dx$$

$$\leqq \frac{1}{2}c_1\int_{D_i}|u_\zeta(t,x) - u^\zeta_{i,q}(t,x)|^2\,dx$$

$$+ \quad c_1\int_{D_i}|u_\zeta(t,x) - u^\zeta_{i,q}(t,x)|\cdot|\sum_{k=2}^{q}\nu^\zeta_{i,k}(t)\psi_{i,k}(x)|\,dx\,,$$

where $c_1 \equiv \sup_{\xi \in \mathbb{R}} |f'(\xi)|$. In the above we have used $\int_{D_i}(u_\zeta(t,x) - u^\zeta_{i,q}(t,x))f(\nu^\zeta_{i,1}(t)\psi_{i,1}(x))\,dx = 0$ which follows from the orthogonality relation of the eigenfunctions and the fact that $\psi_{i,1}$ is a constant function in D_i .

Then we have from the above,

$$(2.10)\quad \int_{D_i} \int_{u^\zeta_{i,q}}^{u_\zeta} f(\xi)\, d\xi\, dx \leqq c_1\left(\frac{1}{2} + \frac{1}{2\alpha}\right) \int_{D_i} | u_\zeta - u^\zeta_{i,q}|^2\, dx$$

$$+ \frac{1}{2} c_1 \alpha \sum_{k=2}^{q} (\nu^\zeta_{i,q})^2 \qquad (\alpha > 0).$$

From (2.7),(2.8),(2.9) and (2.10), we have the following inequality (2.11) by using $\lambda_{i,1} = 0$:

$$(2.11)\quad \sum_{i=1}^{N} \left(\frac{1}{2} \lambda_{i,q+1} - \frac{c_1}{2} - \frac{c_1}{2\alpha}\right) \int_{D_i} | u_\zeta - u^\zeta_{i,q} |^2\, dx$$

$$+ \sum_{i=1}^{N} \sum_{k=2}^{q} \left(\frac{1}{2} \lambda_{i,k} - \frac{c_1\,\alpha}{2}\right) \cdot (\nu^\zeta_{i,q})^2 + \sum_{i=1}^{N} \int_{D_i} \int_{u^\zeta_{i,q}}^{A} f(\xi)\, d\xi\, dx$$

$$+ \int_{Q(\zeta)} \left(\frac{1}{2} | \text{grad } u_\zeta |^2 - \int_{A}^{u_\zeta} f(\xi)\, d\xi \right) dx$$

$$\leqq J_\zeta(w) \leqq J_\zeta(A) + \delta^3 \qquad \text{for } t \geqq 0 , \alpha > 0 , q \geqq 2 .$$

Now we put $\alpha \equiv \frac{1}{c_1} \inf_{1\leqq i\leqq N} \lambda_{i,2} > 0$ and fix it, so that we

have $\frac{1}{2} \lambda_{i,k} - \frac{c_1}{2} \alpha \geqq \frac{1}{4} \lambda_{i,2}$ $(1 \leqq i \leqq N , k \geqq 2)$.

Next we take q sufficiently large so that the inequality

$\frac{1}{2} \lambda_{i,q+1} - \frac{1}{2} c_1 - \frac{c_1}{2\alpha} \geqq 1$ holds for any i $(1 \leqq i \leqq N)$

and fix this natural number q .

For α and q which we have determined above, the following inequality (2.12) easily follows from (2.11).

$$(2.12) \sum_{i=1}^{N} \int_{D_i} |u_\zeta(t,x) - u_{i,q}^\zeta(t,x)|^2 \, dx + \sum_{i=1}^{N} \sum_{k=2}^{q} \frac{1}{4} \lambda_{i,2}(\nu_{i,q}^\zeta(t))^2$$

$$+ \sum_{i=1}^{N} \int_{D_i} \int_{u_{i,q}^\zeta(t,x)}^{a_i} f(\xi) \, d\xi \, dx$$

$$+ \int_{Q(\zeta)} \{ \frac{1}{2} |\text{grad } u_\zeta(t,x)|^2 - \int_{A(x)}^{u_\zeta(t,x)} f(\xi) \, d\xi \} \, dx$$

$$\leqq J_\zeta(w) \leqq J_\zeta(A) + \delta^3$$

$$(t \geqq 0 , \zeta > 0 , 0 < \delta < \delta_0 , w \in E(\delta,\zeta)).$$

The inequality (2.12) is the our main tool to prove that $u_\zeta(t,\cdot)$ always stay near A in L^2-sense if the initial condition w is near A . In the inequality (2.12), only the third term is difficult to deal with and it may be negative if w is not near to A . From now on, we will prove that if δ and ζ are small, the third term of (2.12) is always nonnegative and furthermore $|u_{i,q}^\zeta(t,x) - a_i|$ can be estimated in D_i for the initial condition $w \in E(\delta,\zeta)$.
We introduce the following function $B_i(\sigma)$.

$$B_i(\sigma) \equiv \int_{a_i+\sigma}^{a_i} f(\xi) \, d\xi \quad (1 \leqq i \leqq N)$$

From (II-2) and $\{a_i\}_{i=1}^N \subset \prod$, it is easy to see that B_i satisfies the following properties (2.13),(2.14) and (2.15).

(2.13) $B_i(0) = 0$

(2.14) There exists a positive constant σ_* such that $B_i(\sigma)$ is positive for any $\sigma \in [-\sigma_*, 0) \cup (0, \sigma_*]$

(2.15) $B_i(\sigma)$ is a strictly convex function in σ on $(-\sigma_*, \sigma_*)$.

It is clear that $K \equiv \min_{1 \leqq i \leqq N} \min \{ B_i(-\sigma_*), B_i(\sigma_*)\}$ is positive.

If $w \in E(\delta, \zeta)$, we have

$$\delta^2 \geqq \| w - a_i \|^2_{L^2(D_i)} = \{ \nu^{\zeta}_{i,1}(0) - a_i \, \mathrm{Vol}(D_i)^{1/2} \}^2$$

$$+ \sum_{k=2}^{\infty} (\nu^{\zeta}_{i,k}(0))^2 .$$

We put $c_2 \equiv \max_{1 \leqq i \leqq N, 1 \leqq k \leqq q} \| \psi_{i,k} \|_{L^\infty(D_i)}$ and $\delta_1 \equiv \min_{1 \leqq i \leqq N} \{ \frac{\sigma_*}{4 c_2 \, q^{1/2}}, \delta_0 \}$

Then for any δ, ζ such that $0 < \delta < \delta_1$, $\zeta > 0$, we have,

$$(2.16) \quad | u^{\zeta}_{i,q}(0,x) - a_i | \leqq | \nu^{\zeta}_{i,1}(0) \, \psi_{i,1}(x) - a_i | +$$

$$\sum_{k=2}^{q} | \nu^{\zeta}_{i,k}(0) \, \psi_{i,k}(x) | \leqq \{ \max_{1 \leqq i \leqq N, 1 \leqq k \leqq q} \| \psi_{i,k} \|_{L^\infty(D_i)} \, q^{1/2} \}$$

$$\times \{ | \nu^{\zeta}_{i,1}(0) - a_i \, \mathrm{Vol}(D_i)^{1/2} |^2 + \sum_{k=2}^{q} (\nu^{\zeta}_{i,k}(0))^2 \}^{1/2}$$

$$\leqq c_2 q^{1/2} \delta \leqq c_2 q^{1/2} \delta_1 \leqq \frac{1}{4} \sigma_* \text{ in } D_i \quad (1 \leqq i \leqq N).$$

Here we define, for δ and ζ,

$T(\delta,\zeta) \equiv$

$$\sup \Big\{ t_* \geqq 0 \mid |\nu^{\zeta}_{i,1}(t)\psi_{i,1}(x) - a_i| + \sum_{k=2}^{q} |\nu^{\zeta}_{k,i}(t)\psi_{i,k}(x)| \leqq \sigma_*$$

$$\text{for any } (t,x) \in [0,t_*] \times D_i \quad (1 \leqq i \leqq N) \Big\}.$$

It is clear that $T(\delta,\zeta)$ is positive if $w \in E(\delta,\zeta)$ for δ and ζ such that $0 < \delta < \delta_1$ and $\zeta > 0$ hold. From now on we will prove that $T(\delta,\zeta)$ is infinity if δ and ζ is small.

Lemma 2.1. Let $\delta_2 \in (0,\delta_1)$ and $\zeta_1 > 0$ be positive constants such that the following inequality (2.17) holds for any $(\delta,\zeta) \in (0, \delta_2] \times (0, \zeta_1]$.

(2.17) $\delta^3 + c_3 \mathrm{Vol}(Q(\zeta)) \leqq$

$$\min_{1\leqq i\leqq N} \min \Big\{ B_i(\sigma_*/8)\mathrm{Vol}(D_i),\ B_i(-\sigma_*/8)\mathrm{Vol}(D_i),\ \lambda_{i,2}\, \sigma_*^2/64(q-1)c_2^2 \Big\}.$$

where $c_3 \equiv \sup_{x \in \mathbb{R}^n} |\mathrm{grad}\, A(x)|^2 + \int_{\mathbb{R}} |f(\xi)|\, d\xi$.

Then $T(\delta,\zeta) = \infty$ for any $(\delta,\zeta) \in (0,\delta_2] \times (0,\zeta_1]$.

(Proof of Lemma 2.1) We assume that $T(\delta,\zeta)$ is finite for some $(\delta,\zeta) \in (0,\delta_2] \times (0,\zeta_1]$ and $w \in E(\delta,\zeta)$. If t belongs to the interval $[0,T(\delta,\zeta)]$, the following inequality (2.18) follows from the definition of $T(\delta,\zeta)$,

$$(2.18)\quad | u^{\zeta}_{i,q}(t,x) - a_i | \leqq | \nu^{\zeta}_{i,1}(t)\, \psi_{i,1}(x) - a_i |$$

$$+ \sum_{k=2}^{q} | \nu^{\zeta}_{i,k}(t)\, \psi_{i,k}(x) | \leqq \sigma_* \quad \text{on} \quad [\, 0,\ T(\delta,\zeta)] \times D_i \ .$$

Hence it follows from (2.13),(2.14) and (2.15)

$$\int_{D_i} \int_{u^{\zeta}_{i,q}(t,x)}^{a_i} f(\xi)\ d\xi\ dx \geqq 0 \quad (\ 0 \leqq t \leqq T(\delta,\zeta),\ i = 1,2,\cdots,N\ )$$

follows.

As we have the following inequality (2.19) from (2.12) and the definition of J_ζ and c_3 :

$$(2.19)\quad \sum_{i=1}^{N} \int_{D_i} | u_\zeta(t,x) - u^{\zeta}_{i,q}(t,x) |^2\ dx + \sum_{i=1}^{N} \sum_{k=2}^{q} \lambda_{i,2} (\ \nu^{\zeta}_{i,k}(t)\ )^2$$

$$+ \sum_{i=1}^{N} \int_{D_i} \int_{u^{\zeta}_{i,k}(t,x)}^{a_i} f(\xi)\ d\xi\ dx \leqq c_3\ \mathrm{Vol}(Q(\zeta)) + \delta^3 \ ,$$

we have, for any $(\delta,\zeta) \in (0,\delta_2] \times (0,\ \zeta_1]$ and from (2.17), the following inequalities (2.20) and (2.21),

$$(2.20)\quad \sum_{i=1}^{N} \sum_{k=2}^{q} \frac{1}{4} \lambda_{i,2} (\ \nu^{\zeta}_{i,k}(t)\ )^2 \leqq \min_{1 \leqq i \leqq N} \lambda_{i,2}\ \sigma_*^{\ 2}/64(q-1)c_2^{\ 2}$$

$$(2.21)\quad 0 \leqq \int_{D_i} \int_{u^{\zeta}_{i,q}(t,x)}^{a_i} f(\xi)\ d\xi\ dx \leqq \min_{1 \leqq i \leqq N} \{\ B_i(\pm\ \sigma_*/8)\mathrm{Vol}(D_i)\ \}$$

$(\ 0 \leqq t \leqq T(\delta,\zeta)\ )$.

By (2.20) and a little calculation, we have $\sum_{k=2}^{q} |\nu_{i,k}^{\zeta}(t)| \leqq \frac{\sigma_*}{4c_2}$ ($0 \leqq t \leqq T(\delta,\zeta)$). Hence we get the following inequality (2.22) :

$$(2.22) \quad | u_{i,q}^{\zeta}(t,x) - \nu_{i,1}^{\zeta}(t)\, \psi_{i,1}(x)|$$

$$= | \sum_{k=2}^{q} \nu_{i,k}^{\zeta}(t)\, \psi_{i,k}(x) | \leqq \frac{1}{4} \sigma_* .$$

Next that from (2.21) by the aid of $\psi_{i,1}(x) = \mathrm{Vol}(D_i)^{-1/2}$, we obtain

$$(2.23) \quad 0 \leqq \sum_{i=1}^{N} \int_{D_i} B_i(\nu_{i,1}^{\zeta}(t)\mathrm{Vol}(D_i)^{-1/2} - a_i + \Psi_i(t,x))\, dx$$

$$\leqq \min_{1 \leqq i \leqq N} \{ B_i(\pm\sigma_*/8)\, \mathrm{Vol}(D_i) \}$$

($0 \leqq t \leqq T(\delta,\zeta)$, $0 < \zeta \leqq \zeta_1$, $0 < \delta \leqq \delta_2$) where we put

$$\Psi_i(t,x) = \sum_{k=2}^{q} \nu_{i,k}^{\zeta}(t)\, \psi_{i,k}(x) .$$

Remark that $\Psi_i(t,x)$ is estimated in (2.22). It follows from (2.16) tha $|\nu_{i,1}^{\zeta}(0)\psi_{i,1}(x) - a_i| \leqq \sigma_* /4$ in D_i ($1 \leqq i \leqq N$).

Now we assert the following inequality :

$$(2.24) \quad a_i - \sigma_* /2 \leqq \nu_{i,1}^{\zeta}(t)\psi_{i,1}(x) = \nu_{i,1}^{\zeta}(t)\mathrm{Vol}(D_i)^{-1/2} \leqq a_i + \sigma_* /2$$

for any $t \in [0,T(\delta,\zeta)]$ and $i = 1,2,\cdots,N$.

If the inequality (2.24) breaks at $t = t' \in [0,T(\delta,\zeta)]$ for the first time for some i , then $|\nu_{i,1}^{\zeta}(t')\mathrm{Vol}(D_i)^{-1/2} - a_i| = \frac{\sigma_*}{2}$

holds and $|u^{\zeta}_{i,q}(t',x) - a_i| \geqq \frac{1}{4}\sigma_*$ in D_i follows from (2.22) and we have

$$\int_{D_i} B_i(u^{\zeta}_{i,q}(t',x) - a_i)\,dx \geqq \min\{B_i(\tfrac{1}{4}\sigma_*), B_i(-\tfrac{1}{4}\sigma_*)\}.$$

But this contradicts the inequality (2.23) by (2.13),(2.14) and (2.15) and the continuity. Thus we have ascertained the inequality (2.24).

Then again by (2.22) and (2.24), we have the inequality,

$$|\nu^{\zeta}_{i,1}(t)\psi_{i,1}(x) - a_i| + \sum_{k=2}^{q} |\nu^{\zeta}_{i,k}(t)\,\psi_{i,k}(x)| \leqq \frac{3}{4}\sigma_*$$

on $[0,T(\delta,\zeta)] \times D_i$ ($1 \leqq i \leqq N$).

Then there exists (by the continuity of $u_\zeta(t,\cdot)$) $T' > T(\delta,\zeta)$ such that

$$|\nu^{\zeta}_{i,1}(t)\,\psi_{i,1}(x) - a_i| + \sum_{k=2}^{q} |\nu^{\zeta}_{i,k}(t)\,\psi_{i,k}(x)| \leqq \sigma_*$$

holds on $[0,T'] \times D_i$ ($1 \leqq i \leqq N$). But this is a contradiction to the definition of $T(\delta,\zeta)$. Consequently we conclude $T(\delta,\zeta) = \infty$ for any $(\delta,\zeta) \in (0,\delta_2] \times (0,\zeta_1]$.

Lemma 2.1 is thus proved.

Therefore,from the inequality (2.19) and Lemma 2.1 we have the following estimates (2.25),(2.26) and (2.27) concerning the behavior of $u_\zeta(t,\cdot)$ with intial condition $w \in E(\delta,\zeta)$,

$$(2.25)\quad \sum_{i=1}^{N} \int_{D_i} |u_\zeta(t,x) - u^{\zeta}_{i,q}(t,x)|^2\,dx \leqq c_3\,\mathrm{Vol}(Q(\zeta)) + \delta^3$$

$$(2.26)\quad \sum_{i=1}^{N} \sum_{k=2}^{q} \frac{1}{4}\lambda_{i,2}\,(\nu^{\zeta}_{i,k}(t))^2 \leqq c_3\,\mathrm{Vol}(Q(\zeta)) + \delta^3$$

holds and $|u^{\zeta}_{i,q}(t',x) - a_i| \geqq \frac{1}{4}\sigma_*$ in D_i follows from (2.22) and we have

$$\int_{D_i} B_i(u^{\zeta}_{i,q}(t',x) - a_i)\, dx \geqq \min\{B_i(\tfrac{1}{4}\sigma_*), B_i(-\tfrac{1}{4}\sigma_*)\}.$$

But this contradicts the inequality (2.23) by (2.13),(2.14) and (2.15) and the continuity. Thus we have ascertained the inequality (2.24).

Then again by (2.22) and (2.24), we have the inequality,

$$|\nu^{\zeta}_{i,1}(t)\psi_{i,1}(x) - a_i| + \sum_{k=2}^{q} |\nu^{\zeta}_{i,k}(t)\,\psi_{i,k}(x)| \leqq \frac{3}{4}\sigma_*$$

on $[0,T(\delta,\zeta)] \times D_i$ $(1 \leqq i \leqq N)$.

Then there exists (by the continuity of $u_{\zeta}(t,\cdot)$) $T' > T(\delta,\zeta)$ such that

$$|\nu^{\zeta}_{i,1}(t)\,\psi_{i,1}(x) - a_i| + \sum_{k=2}^{q} |\nu^{\zeta}_{i,k}(t)\,\psi_{i,k}(x)| \leqq \sigma_*$$

holds on $[0,T'] \times D_i$ $(1 \leqq i \leqq N)$. But this is a contradiction to the definition of $T(\delta,\zeta)$. Consequently we conclude $T(\delta,\zeta) = \infty$ for any $(\delta,\zeta) \in (0,\delta_2] \times (0,\zeta_1]$.

Lemma 2.1 is thus proved.

Therefore,from the inequality (2.19) and Lemma 2.1 we have the following estimates (2.25),(2.26) and (2.27) concerning the behavior of $u_{\zeta}(t,\cdot)$ with intial condition $w \in E(\delta,\zeta)$,

$$(2.25) \quad \sum_{i=1}^{N} \int_{D_i} |u_{\zeta}(t,x) - u^{\zeta}_{i,q}(t,x)|^2\, dx \leqq c_3\, \mathrm{Vol}(Q(\zeta)) + \delta^3$$

$$(2.26) \quad \sum_{i=1}^{N} \sum_{k=2}^{q} \frac{1}{4}\lambda_{i,2}\,(\nu^{\zeta}_{i,k}(t))^2 \leqq c_3\, \mathrm{Vol}(Q(\zeta)) + \delta^3$$

$$(2.27)\quad 0 \leqq \int_{D_i} \int_{u^{\zeta}_{i,q}(t,x)}^{a_i} f(\xi)\, d\xi\, dx \leqq c_3 \operatorname{Vol}(Q(\zeta)) + \delta^3$$

$$(0 \leqq t < \infty ,\ 0 < \zeta < \zeta_1 ,\ 0 < \delta < \delta_2).$$

From $\lim_{\zeta\to 0} \operatorname{Vol}(Q(\zeta)) = 0$, it is clear that there exists a strictly monotone continuous function $\zeta(\delta)$ on some interval $(0,\delta_3]$ $(0 < \delta_3 < \delta_2)$ with the following properties (2.28) and (2.29),

$$(2.28)\quad \lim_{\delta\to 0} \zeta(\delta) = 0$$

$$(2.29)\quad \delta^3 + c_3 \operatorname{Vol}(Q(\zeta)) \leqq$$

$$\min_{1\leqq i\leqq N} \min \{\frac{\delta^2}{4}, \frac{\lambda_{i,2} \quad \delta^2}{64(1+c_2(q-1)^{1/2}\operatorname{Vol}(D_i)^{1/2})^2}, \operatorname{Vol}(D_i)B_i(\frac{\pm\delta}{8\operatorname{Vol}(D_i)^{1/2}}) \}$$

for any $\zeta \in (0,\zeta(\delta)]$.

We define a function $\delta(\zeta)$ to be the inverse function of the above function $\zeta(\delta)$. It is easy to see that $\delta(\zeta)$ is defined on some interval $(0, \zeta_2]$ $(0 < \zeta_2 < \zeta_1)$ and $\lim_{\zeta\to 0} \delta(\zeta) = 0$ holds.

Lemma 2.2. The set $E(\delta,\zeta)$ is positively invariant for any $(\delta,\zeta) \in [\delta(\zeta),\delta_3] \times (0,\zeta_2]$, i.e. for any $w \in E(\delta,\zeta)$, the solution of (2.1) $u_{\zeta}(t,\cdot)$ belongs to $E(\delta,\zeta)$ for any $t \geqq 0$.

(Proof of Lemma 2.2) For any $w \in E(\delta,\zeta)$ $(\delta(\zeta) \leqq \delta \leqq \delta_3$, $0 < \zeta \leqq \zeta_2)$, we can obtain from (2.25),(2.26),(2.27),(2.28) and (2.29) the following inequalities,

$$(2.30) \quad \sum_{i=1}^{N} \int_{D_i} | u_\zeta(t,x) - u_{i,q}^{\zeta}(t,x)|^2 dx \leqq \left(\frac{\delta}{2} \right)^2$$

$$(2.31) \quad \sum_{i=1}^{N} \sum_{k=2}^{q} \frac{1}{4} \lambda_{i,2} (\nu_{i,k}^{\zeta}(t))^2$$

$$\leqq \min_{1 \leqq i \leqq N} \frac{\lambda_{i,2}}{64} \left\{ \frac{\delta}{1 + c_2 (q-1)^{1/2} \mathrm{Vol}(D_i)^{1/2}} \right\}^2$$

$$(2.32) \quad 0 \leqq \int_{D_i} \int_{u_{i,q}^{\zeta}(t,x)}^{a_i} f(\xi)\, d\xi\, dx \leqq \mathrm{Vol}(D_i) \min_{1 \leqq i \leqq N} \left\{ B_i\left(\frac{\pm \sigma}{8\mathrm{Vol}(D_i)^{1/2}}\right)\right\}$$

$$(0 < \zeta \leqq \zeta_2 , \ \delta(\zeta) \leqq \delta \leqq \delta_3 , \ t \geqq 0).$$

Here we have, from (2.31), the following (2.33) and (2.34),

$$(2.33) \quad \left\| \sum_{k=2}^{q} \nu_{i,k}^{\zeta}(t) \psi_{i,k} \right\|_{L^2(D_i)} \leqq \frac{\delta}{4 (1 + c_2(q-1)^{1/2} \mathrm{Vol}(D_i)^{1/2})}$$

$$(2.34) \quad \left\| \sum_{k=2}^{q} \nu_{i,k}^{\zeta}(t) \psi_{i,k} \right\|_{L^\infty(D_i)} \leqq c_2 (q-1)^{1/2} \left\{ \sum_{k=2}^{q} (\nu_{i,k}^{\zeta}(t))^2 \right\}^{1/2}$$

$$\leqq \frac{c_2(q - 1)^{1/2} \delta}{4(1 + c_2(q-1)^{1/2}\mathrm{Vol}(D_i)^{1/2})} .$$

Hence applying the same argument as the last part of the proof of Lemma 2.1 (which deduced the inequality (2.24)) to the inequality (2.32), we have the following estimate.

$$(2.35)\quad | a_i - \nu^{\zeta}_{i,1}(t)\ \psi_{i,1}(x)| \leqq$$

$$\frac{\delta}{4\ \mathrm{Vol}(D_i)^{1/2}} + \frac{c_2(q-1)^{1/2}\delta}{4\{1 + c_2(q-1)^{1/2}\ \mathrm{Vol}(D_i)^{1/2}\}}$$

on $[0,\infty) \times D_i$ $(1 \leqq i \leqq N)$, and then we have,

$$(2.36)\quad \| a_i - \nu^{\zeta}_{i,1}(t)\psi_{i,1} \|_{L^2(D_i)} \leqq$$

$$\frac{1}{4}\sigma\ \mathrm{Vol}(D_i)^{1/2} \times \left\{ \frac{1}{\mathrm{Vol}(D_i)^{1/2}} + \frac{c_2(q-1)^{1/2}}{1 + c_2(q-1)^{1/2}\mathrm{Vol}(D_i)^{1/2}} \right\}$$

Therefore, using (2.30),(2.31) and (2.32), we have

$$\| u_\zeta(t,\cdot) - a_i \|_{L^2(D_i)} \leqq \| u_\zeta(t,\cdot) - u^{\zeta}_{i,q}(t,\cdot) \|_{L^2(D_i)}$$

$$+ \| \sum_{k=2}^{q} \nu^{\zeta}_{i,k}(t)\ \psi_{i,k} \|_{L^2(D_i)} + \| a_i - \nu^{\zeta}_{i,1}(t)\ \psi_{i,1} \|_{L^2(D_i)}$$

$$\leqq \delta \qquad (t \geqq 0,\ 1 \leqq i \leqq N).$$

Thus we have proved the positive invariance $E(\delta,\zeta)$ under the conditions $0 < \zeta \leqq \zeta_2$ and $\delta(\zeta) \leqq \delta \leqq \delta_3$ and we have completd the proof of Lemma 2.2 .

Thus we are in the situation where we can apply Theorem 4.2 in Matano [15] to the closed subset $E(\delta(\zeta),\zeta)$ of $C^2(\Omega(\zeta)) \cap C^1(\overline{\Omega(\zeta)})$ because it is easy to see that $E(\delta(\zeta),\zeta)$ has " the property (S)" in [15] for $\zeta > 0$ $(0 < \zeta < \zeta_2)$. Thus we have obtained a stable equilibrium solution v_ζ in $E(\delta(\zeta),\zeta)$ for small $\zeta > 0$.

(2.35) $| a_i - \nu^{\zeta}_{i,1}(t)\ \psi_{i,1}(x)| \leqq$

$$\frac{\delta}{4\ \mathrm{Vol}(D_i)^{1/2}} + \frac{c_2(q-1)^{1/2}\ \delta}{4\{ 1 + c_2(q-1)^{1/2}\ \mathrm{Vol}(D_i)^{1/2} \}}$$

on $[0,\infty) \times D_i$ $(1 \leqq i \leqq N)$, and then we have,

(2.36) $\| a_i - \nu^{\zeta}_{i,1}(t)\psi_{i,1} \|_{L^2(D_i)} \leqq$

$$\frac{1}{4}\ \sigma\ \mathrm{Vol}(D_i)^{1/2} \times \left\{ \frac{1}{\mathrm{Vol}(D_i)^{1/2}} + \frac{c_2(q-1)^{1/2}}{1 + c_2(q-1)^{1/2}\mathrm{Vol}(D_i)^{1/2}} \right\}$$

Therefore, using (2.30),(2.31) and (2.32), we have

$$\| u_\zeta(t,\cdot) - a_i \|_{L^2(D_i)} \leqq \| u_\zeta(t,\cdot) - u^{\zeta}_{i,q}(t,\cdot) \|_{L^2(D_i)}$$

$$+ \| \sum_{k=2}^{q} \nu^{\zeta}_{i,k}(t)\ \psi_{i,k} \|_{L^2(D_i)} + \| a_i - \nu^{\zeta}_{i,1}(t)\ \psi_{i,1} \|_{L^2(D_i)}$$

$$\leqq \delta \qquad (t \geqq 0\ ,\ 1 \leqq i \leqq N).$$

Thus we have proved the positive invariance $E(\delta,\zeta)$ under the conditions $0 < \zeta \leqq \zeta_2$ and $\delta(\zeta) \leqq \delta \leqq \delta_3$ and we have completd the proof of Lemma 2.2 .

Thus we are in the situation where we can apply Theorem 4.2 in Matano [15] to the closed subset $E(\delta(\zeta),\zeta)$ of $C^2(\Omega(\zeta)) \cap C^1(\overline{\Omega(\zeta)})$ because it is easy to see that $E(\delta(\zeta),\zeta)$ has " the property (S)" in [15] for $\zeta > 0$ $(0 < \zeta < \zeta_2)$. Thus we have obtained a stable equilibrium solution v_ζ in $E(\delta(\zeta),\zeta)$ for small $\zeta > 0$.

Next we examine the property of v_ζ . For any i ($1 \leqq i \leqq N$) v_ζ satisfies the following relations.

$$\Delta v_\zeta + f(v_\zeta) = 0 \quad \text{in } D_i \tag{2.37}$$

$$a_* - \delta(\zeta) \leqq v_\zeta(x) \leqq a^* + \delta(\zeta) \quad \text{in } D_i \tag{2.38}$$

$$\frac{\partial v_\zeta}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega(\zeta) \cap \partial D_i \tag{2.39}$$

For any $\eta > 0$, applying the Schauder estimate to v_ζ on the domain $D_i(\eta/2)$, we obtain the boundedness of $\{ v_\zeta \}_{\zeta>0}$ in $C^{1+\beta}(D_i((1-(1/2)^2)\eta))$ for some $\beta \in (0,1)$ and also the boundedness of $\{f(v_\zeta)\}_{\zeta>0}$ in $C^{1+\beta}(D_i((1-(1/2)^2)\eta))$. Again,applying the Schauder estimate to the domain $D_i((1-(1/2)^2)\eta)$, we obtain the boundedness of $\{ v_\zeta \}_{\zeta>0}$ in $C^{3+\beta}(D_i((1-(1/2)^3)\eta))$. Repeating this bootstrap argument, we obtain the boundedness of $\{ v_\zeta \}_{\zeta>0}$ in $C^\infty(D_i(\eta))$ and also the compactness in $C^\infty(D_i(\eta))$. On the other hand, we already have $\lim_{\zeta\to0} \|v_\zeta - a_i\|_{L^2(D_i)} = 0$, then we conclude $\lim_{\zeta\to0} v_\zeta = a_i$ in $C^\infty(D_i(\eta))$ ($1 \leqq i \leqq N$). This completes the proof of Theorem 1.

§ 3 Asymptotic Behavior on The Thin Part.

In this section we consider the behavior of the solution on the perturbation part $Q(\zeta)$, but the domains introduced in Section 2 can contain extremely wild perturbation because the condition (II-1) is too weak. For the sake of the delicate argument about the behavior of the solution, we establish the domain concretely which is the special case of those in Section 2.

We set the domain $\Omega(\zeta)$ in the following form :

$$\Omega(\zeta) = D_1 \cup D_2 \cup Q(\zeta)$$

where D_i (i=1,2) and $Q(\zeta)$ are defined in the following (III-1) and (III-2) where $x' = (x_2, x_3, \cdots, x_n) \in \mathbb{R}^{n-1}$.

(III-1) D_1 and D_2 are bounded domains in $\mathbb{R}^n$ (mutually disjoint) with smooth boundary which satisfy the following conditions for some constant ζ_* .

$$\bar{D}_1 \cap \{ x = (x_1, x') \in \mathbb{R}^n \mid x_1 \leqq 1 , |x'| < 3\zeta_* \}$$
$$= \{ (1, x') \in \mathbb{R}^n \mid |x'| < 3\zeta_* \}$$

$$\bar{D}_2 \cap \{ x = (x_1, x') \in \mathbb{R}^n \mid x_1 \geqq -1 , |x'| < 3\zeta_* \}$$
$$= \{ (-1, x') \in \mathbb{R}^n \mid |x'| < 3\zeta_* \}$$

(III-2) $\quad Q(\zeta) = R_1(\zeta) \cup R_2(\zeta) \cup \Gamma(\zeta)$

$$R_1(\zeta) = \{ (x_1, x') \in \mathbb{R}^n \mid 1 - 2\zeta < x_1 \leqq 1, |x'| < \zeta\rho((x_1-1)/\zeta) \}$$

$$R_2(\zeta) = \{ (x_1, x') \in \mathbb{R}^n \mid -1 \leqq x_1 < -1 + 2\zeta, |x'| < \zeta\rho((-1-x_1)/\zeta) \}$$

$$\Gamma(\zeta) = \{ (x_1, x') \in \mathbb{R}^n \mid -1+2\zeta \leqq x_1 \leqq 1-2\zeta , |x'| < \zeta \}$$

where $\rho \in C^0((-2,0]) \cap C^\infty((-2,0))$ is a positive valued monotone increasing function such that $\rho(0) = 2$, $\rho(s) = 1$ for $s \in (-2,-1)$ and $\lim_{s\uparrow -0} \frac{d^k\rho}{ds^k}(s) = +\infty$ holds for any positive integer k .

We also assume that

(III-3) $\overline{\lim_{\xi\to\infty}} f(\xi) < 0$, $\underline{\lim_{\xi\to-\infty}} f(\xi) > 0$

Remark. The domain determined above satisfies (II-1) and (II-2) therefore it is a special case of that dealt in Section 2 and so we use the same notation $\Omega(\zeta)$.

Under the situation supported by the conditions (II-2),(III-1), (III-2) and (III-3), we analyze the asymptotic behavior of some solutions (which will be characterized by (III-4)) of the following semilinear elliptic boundary value problem (3.1).

$$
(3.1) \quad \begin{cases} \Delta v + f(v) = 0 & \text{in } \Omega(\zeta) , \\ \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega(\zeta) . \end{cases}
$$

(III-4) Let v_ζ be an arbitrary solution of the above (3.1) for ζ ($0 < \zeta < \zeta_*$) such that the family of the functions $\{ v_\zeta \}_{0<\zeta<\zeta_*}$ satisfies the following condition.

$$
\lim_{\zeta\to 0} \| v_\zeta - a_i \|_{L^2(D_i)} = 0 \qquad (i = 1 , 2)
$$

where $f(a_i) = 0$ and $f'(a_i) < 0$ ($i = 1, 2$). (See (II-2))

Definition 2. Let $\mu_1(\zeta)$ be the first eigenvalue of the following eigenvalue problem.

$$(3.2)\quad \begin{cases} \Delta\psi + f'(v_\zeta)\psi + \mu\psi = 0 & \text{in } \Omega(\zeta), \\ \dfrac{\partial\psi}{\partial\nu} = 0 & \text{on } \partial\Omega(\zeta). \end{cases}$$

Remark. It is well-known that if $\mu_1(\zeta) > 0$ (resp. $\mu_1(\zeta) < 0$) v_ζ is stable (resp. unstable) as an equilibrium solution of (1.1) for $\Omega = \Omega(\zeta)$.

Remark. The two values a_1 and a_2 are not necessarily mutually distinct.

We define $M_* = \inf\{\xi \in \mathbb{R} \mid f(\xi) = 0\}$ and $M^* = \sup\{\xi \in \mathbb{R} \mid f(\xi) = 0\}$. It is easily seen by (II-2) and (III-3) that M_* and M^* are well defined and that

$$(3.3)\quad M_* \leqq v_\zeta(x) \leqq M^* \quad \text{for } x \in \Omega(\zeta).$$

Then we have the following theorem.

Theorem 2. Assume $n \geqq 3$, then we have, for i ($i = 1, 2$),

$$\lim_{\zeta\to 0} \sup_{x \in D_i \cup R_i(\zeta)} | v_\zeta(x) - a_i | = 0 .$$

We prepare the ordinary differential equation which describes the asymptotic behavior of v_ζ on $Q(\zeta)$ when $\zeta \downarrow 0$.

$$(3.4)\quad \begin{cases} \dfrac{d^2V}{dz^2} + f(V) = 0 & \text{in } -1 < z < 1, \\ V(1) = a_1, \quad V(-1) = a_2 . \end{cases}$$

Definition 3. Let λ_V and Φ_V be respectively the first eigenvalue and the first eigenfunction of the following eigenvalue problem (3.5) for a solution V of (3.4).

$$(3.5)\quad \begin{cases} \dfrac{d^2\Phi}{dz^2} + f'(V(z))\,\Phi + \lambda\,\Phi = 0 & \text{in } -1 < z < 1 , \\ \Phi(1) = 0 , \ \Phi(-1) = 0 . \end{cases}$$

Now we present one of the main results of this paper.

Theorem 3. Assume $n \geqq 3$, then for any sequence of positive values $\{ \zeta_m \}_{m=1}^{\infty}$ such that $\lim_{m\to\infty} \zeta_m = 0$, there exist a subsequence $\{ \varkappa_m \}_{m=1}^{\infty} \subset \{ \zeta_m \}_{m=1}^{\infty}$ and a solution V of (3.4) with the following asymptotic property (3.6) :

$$(3.6)\quad \lim_{m\to\infty} \sup_{x \in Q(\varkappa_m)} | v_{\varkappa_m}(x_1,x') - V(x_1) | = 0 .$$

Furthermore concerning the above V , if $\lambda_V > 0$ (resp. $\lambda_V < 0$), then $\underline{\lim}_{m\to\infty} \mu_1(\varkappa_m) > 0$ (resp. $\overline{\lim}_{m\to\infty} \mu_1(\varkappa_m) < 0$) holds.

Before starting the proof we introduce some notations.

$$p_1 = (1,0,\cdots,0) , \quad p_2 = (-1,0,\cdots,0) ,$$

$$\Sigma_1(\eta) = \{ (x_1,x') \in \mathbb{R}^n \mid x_1 > 1 , \ | x - p_1| < \eta \} ,$$

$$\Sigma_2(\eta) = \{ (x_1,x') \in \mathbb{R}^n \mid x_1 < -1 , \ | x - p_2 | < \eta \} .$$

It can be easily seen by the last part of the proof of Theorem 1 and the condition (III-4) that the following convergence (3.7) follows.

$$(3.7) \qquad \lim_{\zeta\to 0} v_\zeta = a_i \qquad \text{in } C^\infty(\overline{D_i - \Sigma_i(\eta)})$$

for any small positive constant η ($i = 1 , 2$).

(Proof of Theorem 2) First we will prove

$$(3.8) \qquad \lim_{\zeta\to 0} \sup_{x \in D_1} | v_\zeta(x) - a_1 | = 0 .$$

We define for $\varepsilon > 0$ and $0 < \zeta < \zeta_*$,

$$K(\varepsilon,\zeta) = \left\{ x \in D_1 \;\middle|\; | v_\zeta(x) - a_1 | \geqq \varepsilon \right\}$$

$$\eta(\varepsilon,\zeta) = \inf \{ \eta > 0 \mid \Sigma_1(\eta) \supset K(\varepsilon,\zeta) \}$$

Then it follows from (3.7) that

$$(3.9) \qquad \lim_{\zeta\to 0} \eta(\varepsilon,\zeta) = 0 \quad \text{for any } \varepsilon > 0 .$$

It is easily seen that (3.8) is equivalent to the following fact (3.10).

(3.10) For any $\varepsilon > 0$, there exists $\zeta_0 = \zeta_0(\varepsilon)$ such that $\eta(\varepsilon,\zeta) = 0$ for any ζ such that $0 < \zeta < \zeta_0$.

Assume that (3.10) does not hold in spite of (3.9), that is

(3.11) there exists $\varepsilon_0 > 0$ such that $\eta(\varepsilon_0,\zeta) > 0$ for any ζ such that $0 < \zeta < \zeta_*$.

We shall show that this assumption yields a contradiction.
(See (3.38),(3.40) and Lemma 3.2 mentioned later.)

Here concerning the convergence (3.9), we have the following estimate.

Lemma 3.1.

(3.12) $$\varliminf_{\zeta\to 0} \frac{\zeta}{\eta(\varepsilon,\zeta)} > 0 \quad \text{for any } \varepsilon \text{ such that } 0 < \varepsilon < \varepsilon_0 .$$

(Proof of Lemma 3.1) If we assume the contrary, there exist ε_1 ($0 < \varepsilon_1 < \varepsilon_0$) and a sequence of positive values $\{ \zeta_m \}_{m=1}^{\infty}$ such that $\lim_{m\to\infty} \zeta_m = 0$ and $\lim_{m\to\infty} \frac{\zeta_m}{\eta(\varepsilon_1,\zeta_m)} = 0$. This last limitation also holds if ε_1 is replaced by a positive constant which is smaller than ε_1 . Therefore we assume without loss of generality that ε_1 is sufficiently small so that $f'(\xi) < 0$ holds for any $\xi \in (a_1-\varepsilon_1, a_1+\varepsilon_1)$. We denote $\eta(\varepsilon_1,\zeta_m)$ by η_m for simplicity hereafter.

For the analysis of the behavior of v_ζ on the small part $\Sigma_1(\zeta)$, we change the scale of the variable x into y around the point p_1 as follows.

(3.13) $$\begin{cases} x - p_1 = \eta_m\cdot(y - p_1) \\ U_m(y) = v_{\zeta_m}(\eta_m\cdot(y-p_1)+p_1) \end{cases}$$

By (3.13), the equation (3.1) is transformed into the following equation (3.14) in some neighborhood of p_1 .

(3.14) $$\begin{cases} \Delta_y U_m + \eta_m^2 f(U_m) = 0 \quad \text{in } \Sigma_1(3\zeta_* /\eta_m) \\ \dfrac{\partial U_m}{\partial y_1}(0,y') = 0 \quad \text{for } y' \text{ such that } \dfrac{2\zeta_m}{\eta_m} < |y'| < \dfrac{3\zeta_*}{\eta_m} \end{cases}$$

We put $\gamma_m \equiv \max_{y_1 \geqq 1, |y-p_1|=3\zeta_*/\eta_m} | U_m(y) - a_1 |$

$$= \max_{x \in D_1, |x-p_1| = 3\zeta_*} | v_{\zeta_m}(x) - a_1 |$$

Then it is easy to see $\lim_{m\to\infty} \gamma_m = 0$.

We have the following properties (3.15),(3.16) and (3.17) by the definition of $\eta_m = \eta(\varepsilon_1, \zeta_m)$ and U_m .

(3.15) $$\max_{y_1 \geqq 1, |y-p_1|=1} | U_m(y) - a_1 | = \max_{x \in D_1, |x-p_1|=\eta_m} | v_{\zeta_m}(x) - a_1 |$$

$$= \varepsilon_1$$

(3.16) $$| U_m(y) - a_1 | \leqq \varepsilon_1 \quad \text{in} \quad \overline{\Sigma_1(3\zeta_*/\eta_m) - \Sigma_1(1)}$$

(3.17) $$M_* \leqq U_m(y) \leqq M^* \quad \text{in} \quad \Sigma_1(3\zeta_*/\eta_m)$$

Here we define a comparison function G_m which will estimate U_m for large y .

$$G_m(y) = \frac{\varepsilon_1}{| y - p_1 |^{n-2}} + \gamma_m$$

It can be easily seen by (3.16) and the assumption of ε_1 that

$f(U_m(y)) < 0$ for any $y \in (\Sigma_1(3\zeta_*/\eta_m)-\Sigma_1(1)) \cap \{ y \mid U_m(y) > a_1 \}$

$f(U_m(y)) > 0$ for any $y \in (\Sigma_1(3\zeta_*/\eta_m)-\Sigma_1(1)) \cap \{ y \mid U_m(y) < a_1 \}$

and that G_m is a harmonic function in $\Sigma_1(\zeta_*/\eta_m) - \Sigma_1(1)$ with the boundary condition $\dfrac{\partial G_m}{\partial y_1}(0,y') = 0 \quad (1 < |y'| < \dfrac{3\zeta_*}{\eta_m})$.

Then we can apply the standard argument similar to the Comparison Theorem to the function $U_m - a_1$ in the domain $\Sigma_1(3\zeta_*/\eta_m) - \Sigma_1(1)$ by using (3.14),(3.15) and the definition of γ_m and we obtain the following estimate (3.18) for sufficiently large m.

Recall $\lim_{m\to\infty} \zeta_m/\eta_m = 0$.

(3.18) $$| U_m(y) - a_1 | \leqq G_m(y) \quad \text{for } y \in \Sigma_1(3\zeta_*/\eta_m) - \Sigma_1(1) .$$

Applying the same argument as the the last part of the proof of Theorem 1 and moreover the diagonal argument to the family $\{ U_m \}_{m=1}^{\infty}$ in (3.14) with a-priori bound (3.17) by using $\lim_{m\to\infty} \eta_m = 0$ and $\lim_{m\to\infty} \zeta_m / \eta_m = 0$, we can choose a subsequence $\{ U_{m_j} \}_{j=1}^{\infty}$ such that there exists a smooth function U in

$C^\infty(\{(y_1,y') \in \mathbb{R}^n \mid y_1 \geqq 1 \} - \{p_1\})$

with the following conditions (3.19), (3.20), (3.21) and (3.22).

(3.19) $$M_* \leqq U(y) \leqq M^* \quad \text{in } \{ (y_1,y') \mid y_1 \geqq 1 \} - \{ p_1 \}$$

(3.20) $$\Delta_y U = 0 \quad \text{in } \{ (y_1,y') \in \mathbb{R}^n \mid y_1 > 1 \}$$

(3.21) $$\frac{\partial U}{\partial y_1}(1,y') = 0 \quad \text{for } y' \in \mathbb{R}^{n-1} \text{ such that } y' \neq 0$$

(3.22) $$\lim_{j\to\infty} U_{m_j} = U$$

in $C^\infty(\{ (y_1,y') \mid y_1 \geqq 1 , \eta \leqq | y - p_1 | \leqq \frac{1}{\eta} \})$

for any $\eta > 0$.

On the other hand, from the estimate (3.18), the convergence (3.22) and $\lim_{m\to\infty} \gamma_m = 0$, U satisfies the following estimates

$$(3.23) \qquad | U(y) - a_1 | \leqq \frac{\varepsilon_1}{| y - p_1 |^{n-2}}$$

$$\text{in } \{ (y_1,y') \in \mathbb{R}^n \mid y_1 \geqq 1 , | y - p_1 | \geqq 1 \}$$

$$(3.24) \qquad M_* \leqq U(y) \leqq M^* \text{ in } \{ (y_1,y') \in \mathbb{R}^n \mid y_1 > 1 \}$$

From (3.15), the convergence (3.22) and the compactness of the set $\{ (y_1,y') \in \mathbb{R}^n \mid y_1 \geqq 1 , | y - p_1 | = 1 \}$, it follows that

$$(3.25) \qquad \max_{y_1 \geqq 1, |y-p_1|=1} | U(y) - a_1 | = \varepsilon_1 .$$

Here we can define a function $\bar{U} \in C^\infty(\mathbb{R}^n - \{ p_1\})$ by using the Laplace equation (3.20) and the Neumann boundary condition (3.21) as follows

$$\bar{U}(y_1,y') = \begin{cases} U(y) & \text{for } y_1 \geqq 1 , y \neq p_1 \\ U(2-y_1,y') & \text{for } y_1 < 1 \end{cases}$$

By a simple calculation, we have,

$$\begin{cases} \Delta_y U = 0 & \text{in } \mathbb{R}^n - \{ p_1 \} \\ M_* \leqq U(y) \leqq M^* & \text{in } \mathbb{R}^n - \{ p_1 \} . \end{cases}$$

Therefore, applying the removable singularity theorem, we can extend $\bar{U}$ on $\mathbb{R}^n$ as a bounded harmonic function. We denote it also by $\bar{U}$.

Thus $\bar{U}$ must be a constant function by the Harnack Theorem. But it is impossible by (3.23) and (3.25). This is a contradiction and we complete the proof of Lemma 3.1.

By Lemma 3.1, we take a constant $\beta > 0$ such that

$$(3.26)\qquad \varliminf_{\zeta\to 0} \frac{\zeta}{\eta(\varepsilon_0,\zeta)} > \beta > 0\ ,\quad 0 < \beta < 1/2\ .$$

We change the variable x into y around p_1 by the following,

$$(3.27)\quad \begin{cases} x - p_1 = \zeta\cdot(y - p_1) \\ U_\zeta(y) = v_\zeta(\zeta(y-p_1)+ p_1) \end{cases}$$

By (3.27), the equation (3.1) is transformed into the following equation (3.28)-(3.29)

$$(3.28)\qquad \Delta_y U_\zeta + \zeta^2 f(U_\zeta) = 0 \quad \text{in } H_\zeta\ ,$$

$$(3.29)\qquad \frac{\partial U_\zeta}{\partial\nu} = 0 \quad \text{on } \partial H_\zeta \cap \partial H\ .$$

Here we have put,

$$\begin{aligned} H \equiv\ & \{ (y_1,y') \in \mathbb{R}^n \mid y_1 > 1 \} \\ \cup\ & \{ (y_1,y') \in \mathbb{R}^n \mid -1 < y_1 \leqq 1\ ,\ |y'| < \rho(y_1-1) \} \\ \cup\ & \{ (y_1,y') \in \mathbb{R}^n \mid y_1 \leqq -1\ ,\ |y'| < 1 \}\ , \end{aligned}$$

$$H_\zeta = H \cap \{ (y_1,y') \in \mathbb{R}^n \mid y_1 \leqq -1\ ,\ \text{or}\ | y - p_1| \leqq 3\zeta_*/\zeta \}$$

and ν denotes the unit outer normal vector on ∂H.

Here we define $\tau_\zeta = \max_{y_1 \geqq 1, |y-p_1|=3\zeta_*/\zeta} | U_\zeta(y) - a_1 |$

$$= \max_{x \in D_1, |x-p_1|=3\zeta_*} | v_\zeta(x) - a_1 |$$

It is easily seen by (3.7) that $\lim_{\zeta\to 0} \tau_\zeta = 0$.

From (3.26) and the definition of $\eta(\varepsilon_0,\zeta)$, we have,

(3.30) $\eta(\varepsilon_0,\zeta) < \zeta/\beta$ for sufficiently small $\zeta > 0$

and also we have,

(3.31) $$\max_{x \in \Sigma_1(\zeta/\beta)} | v_\zeta(x) - a_1 | = \max_{y_1 \geqq 1, |y-p_1| \leqq 1/\beta} |U_\zeta(y) - a_1|$$

$$\geqq \max_{x_1 \geqq 1, |x-p_1|=\eta(\varepsilon_0,\zeta)} | v_\zeta(x) - a_1 | = \varepsilon_0$$

(3.32) $$| U_\zeta(y) - a_1 | \leqq \varepsilon_0$$

in $\{ (y_1, y') \in \mathbb{R}^n \mid y_1 \geqq 1 , \frac{1}{\beta} \leqq | y - p_1 | \leqq \frac{3\zeta_*}{\zeta} \}$.

(3.33) $$M_* \leqq U_\zeta(y) \leqq M^* \quad \text{in } H_\zeta$$

(3.34) $$| U_\zeta(y) - a_1 | \leqq \frac{\varepsilon_0}{\beta^{n-2} | y - p_1 |^{n-2}} + \tau_\zeta$$

in $\Sigma_1(3\zeta_*/\zeta) - \Sigma_1(1/\beta)$

By the same argument in (3.28),(3.29),(3.33) and (3.34) as the proof of Lemma 3.1, we can choose a convergent subsequence $\{ U_{\zeta_m} \}_{m=1}^{\infty} \subset \{ U_\zeta \}_{0<\zeta<\zeta_*}$ such that $\lim_{m\to\infty} \zeta_m = 0$ and a function $U \in C^\infty(\overline{H})$ which satisfy the following equations

(3.35) $$\Delta_y U = 0 \quad \text{in } H$$

(3.36) $\frac{\partial U}{\partial \nu} = 0$ on ∂H

(3.37) $\lim_{m\to\infty} U_{\zeta_m} = U$ in $C^\infty(\overline{H}_\eta)$ for any $\eta > 0$

(3.38) $| U(y) - a_1 | \leqq \frac{\varepsilon_0}{\beta^{n-2} | y - p_1 |^{n-2}}$

in $\{ (y_1,y') \in \mathbb{R}^n \mid y_1 \geqq 1, | y - p_1 | \geqq 1/\beta \}$

(3.39) $M_* \leqq U(y) \leqq M^*$ in H

On the other hand, from (3.31),(3.37) and the compactness of the set $\{ (y_1,y') \in \mathbb{R}^n \mid y_1 \geqq 1, | y - p_1 | \leqq 1/\beta \}$, we obtain

(3.40) $\max_{y_1 \geqq 1, |y-p_1| \leqq \varepsilon_0} | U(y) - a_1 | \geqq 1/\beta$.

Thus (3.38) and (3.40) imply that U is a non-constant function in H . But this is impossible from (3.35),(3.36),(3.39) and the following Lemma 3.2.

Lemma 3.2. Let ψ be a bounded function which belongs to $C^\infty(\overline{H})$ and satisfies the following equations

(3.41) $\Delta_y \psi = 0$ in H

(3.42) $\frac{\partial \psi}{\partial \nu} = 0$ on ∂H

(3.43) $\lim_{y_1 \geqq 1, |y|\to\infty} | \psi(y) - a | = 0$

Then $\psi \equiv a$ in H

(Proof of Lemma 3.2) We assume the contrary. Without loss of generality, we may assume

$$(3.44) \qquad \sup_{y \in H} \psi(y) = M > a .$$

We choose a sequence of points $\{ r_m \}_{m=1}^{\infty} \subset H$ such that

$\lim_{m\to\infty} \psi(r_m) = a$. Using the Strong Maximum Principle, the Hopf Lemma (See [19]) and the equation (3.41)-(3.42), we can easily see that ψ cannot attain its maximum on $\overline{H}$, because ψ is a non-constant function. Moreover $\{ r_m \}_{m=1}^{\infty}$ does not have an accumulation point on $\overline{H}$ and so from (3.43), we obtain

$\lim_{m\to\infty} r_{m,1} = -\infty$.

We assume $r_{m,1} < 0$ for any m ;

here we denoted by $r_{m,i}$ the i-th component of the point r_m .
We define a family of functions $\{ \psi_m \}_{m=1}^{\infty}$ as follows,

$$\psi_m(y_1, y') = \psi(y_1 + r_{m,1} + 2, y') .$$

Each ψ_m satisfies the following equations,

$$(3.45) \qquad \Delta_y \psi_m = 0 \quad \text{in } H \cap \{ y_1 < 0 \}$$

$$(3.46) \qquad \frac{\partial \psi_m}{\partial \nu} = 0 \quad \text{on } \partial H \cap \{ y_1 < 0 \}$$

$$(3.47) \qquad \psi_m(y) \leqq M \quad \text{in } H$$

$$(3.48) \qquad \lim_{m\to\infty} \max_{H \cap \{ y_1 = -2 \}} \psi_m(y) = M$$

By the standard compactness argument cocerning the solutions of the elliptic boundary value problem and the Maximum Principle in (3.45)-(3.48), we deduce the following convergence,

$$(3.49) \qquad \lim_{m\to\infty} \psi_m = M \quad \text{in} \quad C^\infty(\overline{H \cap \{ -3 < y_1 < -1 \}})$$

On the other hand, integrating the equation (3.41) in y' on $\{ |y'| < 1 \}$ by using the Neumann boundary condition, we have,

$$\frac{d^2}{dy_1^2} \int_{|y'|<1} \psi(y_1,y')\, dy' = 0 \quad \text{for} \quad y_1 \leqq 0 .$$

But the boundedness of ψ implies the boundedness of

$$\int_{|y'|<1} \psi(y_1,y')\, dy' \quad \text{in} \quad -\infty < y_1 \leqq 0 .$$

Therefore $\int_{|y'|<1} \psi(y_1,y')\, dy'$ is independent of y_1 when y_1 is negative. We denote its value by K .

Therefore we have the following equality,

$$(3.50) \quad \int_{|y'|<1} \psi_m(-2,y')\, dy' = \int_{|y'|<1} \psi(r_{m,1},y')\, dy' \equiv K$$

We remark that the left hand side (3.50) tends to the value $M \int_{|y'|<1} 1\, dy'$ when m tends to ∞ . Then we obtain,

$$\int_{|y'|<1} \psi(r_{m,1},y')\, dy' = \int_{|y'|<1} M\, dy' \quad \text{for any} \quad m .$$

From (3.44), the above equality implies $\psi(r_{m,1}, y') = M$ for y' such that $|y'| < 1$. But this contradicts to the fact that ψ cannot attain its maximum on $\overline{H}$. This completes the proof of Lemma 3.2 and also the proof of $\lim_{\zeta \to 0} \sup_{x \in D_1} | v_\zeta(x) - a_1 | = 0$.

To prove $\lim_{\zeta \to 0} \sup_{x \in R_1(\zeta)} | v_\zeta(x) - a_1 | = 0$, we remember the transformation (3.27),(3.28),(3.29) and by means of a similar argument there and we get the compactness of the family $\{ U_\zeta \}_{0<\zeta<\zeta_*}$ in $C^\infty(H_\eta)$ for any $\eta > 0$. Let $\{ U_{\zeta_m} \}_{m=1}^{\infty}$ be any convergent subsequence of the above family such that $\lim_{m \to \infty} \zeta_m = 0$ and there exists $\overline{U} \in C^\infty(\overline{H})$ such that $\lim_{m \to \infty} U_{\zeta_m} = \overline{U}$ in $C^\infty(\overline{H}_\eta)$ for any $\eta > 0$. Then $\overline{U}$ is a harmonic function in H. (See (3.35) and (3.36).) But we have already proved $\lim_{\zeta \to 0} \sup_{x \in D_1} | v_\zeta(x) - a_1 | = 0$ which implies $\overline{U}(y_1, y') = a_1$ for any $y \in H \cap \{ y_1 > 1 \}$. Hence by the Unique Continuation Theorem, we get $\overline{U} \equiv a_1$ in H. Then we conclude $\lim_{\zeta \to 0} \sup_{y \in H_\eta} | U_\zeta(y) - a_1 | = 0$ for any $\eta > 0$. This implies $\lim_{\zeta \to 0} \sup_{x \in R_1(\zeta)} | v_\zeta(x) - a_1 | = 0$. Thus we complete the proof of Theorem 2.

(Proof of the Former Half of Theorem 3)

To analyze the asymptotic behavior of v_ζ in the thin part $Q(\zeta)$, we change the variable x into y as follows.

$$(3.51) \quad \begin{cases} y_1 = x_1 \\ \zeta\, y' = x' \\ U_\zeta(y) = v_\zeta(y_1, \zeta y') \end{cases}$$

We define $\iota(\zeta) = \sum_{i=1}^{2} \sup_{x \in R_i(\zeta)} | v_\zeta(x) - a_i |$ and so by Theorem 2, we have $\lim_{\zeta \to 0} \iota(\zeta) = 0$. We put $\omega = \max_{M_* \leq \xi \leq M^*} |f(\xi)|$.

By (3.51), the equation (3.1) is transformed into the following equation in the part corresponding to $Q(\zeta)$.

$$(3.52) \quad \left(\frac{\partial^2}{\partial y_1^2} + \frac{1}{\zeta^2} \sum_{j=2}^{n} \frac{\partial^2}{\partial y_j^2} \right) U_\zeta + f(U_\zeta) = 0 \quad \text{in } G(\zeta)$$

$$(3.53) \quad \frac{\partial U_\zeta}{\partial \nu} = 0 \qquad \text{on } \partial G \cap \{ -1 + \zeta < y_1 < 1 - \zeta \}$$

where we have put $G = \{ (y_1, y') \in \mathbb{R}^n \mid |y'| < 1 , -\infty < y_1 < \infty \}$ $G(\zeta) = G \cap \{ -1 + \zeta < y_1 < 1 - \zeta \}$ and denoted by ν the unit outer normal vector on ∂G .

We decompose U_ζ as $U_\zeta = U_{1,\zeta} + U_{2,\zeta}$ by the folloeing equations which determine $U_{1,\zeta}$ and $U_{2,\zeta}$ uniquely.

$$(3.54) \quad \left(\frac{\partial^2}{\partial y_1^2} + \frac{1}{\zeta^2} \sum_{j=2}^{n} \frac{\partial^2}{\partial y_j^2} \right) U_{1,\zeta}(y) = 0 \quad \text{in } G(\zeta)$$

$$(3.55) \quad \begin{cases} U_{1,\zeta}(y) = U_\zeta(y) & \text{on } G \cap \{ y_1 = 1 - \zeta \} \\ U_{1,\zeta}(y) = U_\zeta(y) & \text{on } G \cap \{ y_1 = -1 + \zeta \} \end{cases}$$

$$(3.56) \quad \frac{\partial U_{1,\zeta}}{\partial \nu}(y) = 0 \quad \text{on } \partial G \cap \{ -1 + \zeta < y_1 < 1 - \zeta \}$$

$$(3.57) \quad U_{2,\zeta} = U_\zeta - U_{1,\zeta}$$

By the above definition, $U_{2,\zeta}$ automaticaly satisfies the following equation

$$(3.58) \quad \left(\frac{\partial^2}{\partial {y_1}^2} + \frac{1}{\zeta^2} \sum_{j=2}^{n} \frac{\partial^2}{\partial {y_j}^2} \right) U_{2,\zeta} + f(U_\zeta) = 0 \quad \text{in } G(\zeta)$$

$$(3.59) \quad U_{2,\zeta}(1-\zeta, y') = U_{2,\zeta}(-1+\zeta, y') = 0 \qquad (|y'| < 1)$$

$$(3.60) \quad \frac{\partial U_{2,\zeta}}{\partial \nu} = 0 \quad \partial G \cap \{ -1 + \zeta \leqq y_1 \leqq 1 - \zeta \} .$$

Hereafter we denote the differential operator by P_ζ as follows

$$P_\zeta = \frac{\partial^2}{\partial {y_1}^2} + \frac{1}{\zeta^2} \sum_{j=2}^{n} \frac{\partial^2}{\partial {y_j}^2} .$$

We can deduce the following estimate by applying the comparison theorem in (3.54)-(3.55) by the aid of the definition of $\iota(\zeta)$.

Lemma 3.3. For any $\zeta \in (0, \zeta_*)$, we have,

$$(3.61) \quad \sup_{y \in G(\zeta)} \left| U_{1,\zeta}(y) - \frac{1-\zeta-y_1}{2-2\zeta} a_1 - \frac{1-\zeta+y_1}{2-2\zeta} a_2 \right| \leqq \iota(\zeta)$$

We define functions $\Phi_{\pm}$ in $G(\zeta)$ which estimate U_{ζ} roughly.

$$\Phi_{\pm,\zeta}(y_1,y') = \frac{y_1 + 1 - \zeta}{2 - 2\zeta} a_1 + \frac{1 - \zeta - y_1}{2 - 2\zeta} a_2$$

$$\pm \frac{\omega}{2}(y_1 + 1 - \zeta)(1 - \zeta - y_1) \pm \iota(\zeta)$$

Lemma 3.4. For any $\zeta \in (0,\zeta_*)$, we have the following estimate

$$\Phi_{-,\zeta}(y) \leqq U_{\zeta}(y) \leqq \Phi_{+,\zeta}(y) \quad \text{in } G(\zeta).$$

(Proof of Lemma 3.4) By an easy calculation, we have,

$$P_{\zeta} \Phi_{\pm} = \mp \omega \quad \text{in } G(\zeta)$$

$$\frac{\partial \Phi_{\pm}}{\partial \nu} = 0 \quad \text{on } \partial G \cap \{ -1 + \zeta < y_1 < 1 - \zeta \}$$ and by the definition of $\iota(\zeta)$, we also have

$$a_1 - \iota(\zeta) = \Phi_{-,\zeta}(1-\zeta,y') \leqq U_{\zeta}(1-\zeta,y') \leqq \Phi_{+,\zeta}(1-\zeta,y') = a_1 + \iota(\zeta)$$

$$a_2 - \iota(\zeta) = \Phi_{-,\zeta}(-1+\zeta,y') \leqq U_{\zeta}(-1+\zeta,y') \leqq \Phi_{+,\zeta}(-1+\zeta,y') = a_2 + \iota(\zeta)$$

Applying the comparison theorem, we have the consequence.

Lemma 3.5. There exists a positive constant c_1 such that

$$(3.63) \quad \int_{G(\zeta)} \left| \frac{\partial U_{2,\zeta}}{\partial y_1} \right|^2 dy + \frac{1}{\zeta^2} \sum_{j=2}^{n} \int_{G(\zeta)} \left| \frac{\partial U_{2,\zeta}}{\partial y'} \right|^2 dy$$

$$\leqq c_1 \quad \text{for any } \zeta \in (0,\zeta_*)$$

We can deduce this inequality by integrating the equation (3.58) after multiplying $U_{2,\zeta}$ and using the boundedness (3.3) and (3.61). We define a function which bounds $U_{2,\zeta}$ in $G(\zeta)$.

$$\Psi_\zeta(y_1, y') = \frac{\omega}{2}(1 - \zeta - y_1)(1 - \zeta + y_1)$$

Lemma 3.6. There exists a positive constant c_2 such that

(3.64) $$|U_{2,\zeta}(y)| \leqq \Psi_\zeta(y) \quad \text{in} \quad G(\zeta)$$

(3.65) $$\left|\frac{\partial U_{2,\zeta}}{\partial y_1}(1-\zeta, y')\right| \leqq c_2$$

$$(|y'| \leqq 1)$$

(3.66) $$\left|\frac{\partial U_{2,\zeta}}{\partial y_1}(-1+\zeta, y')\right| \leqq c_2$$

(Proof of Lemma 3.6)

Ψ_ζ satisfies the following equations,

(3.67) $$P_\zeta \Psi_\zeta + \omega = 0 \quad \text{in} \quad G(\zeta)$$

(3.68) $$\frac{\partial \Psi_\zeta}{\partial \nu} = 0 \quad \text{on} \quad \partial G \cap \{-1 + \zeta \leqq y_1 \leqq 1 - \zeta\}$$

(3.69) $$\Psi_\zeta(-1+\zeta, y') = \Psi_\zeta(1-\zeta, y') = 0 \qquad |y'| < 1 .$$

Then applying the comparison theorem to (3.58)-(3.60) and (3.67)-(3.69), we see that

(3.70) $$-\Psi_\zeta(y) \leqq U_{2,\zeta}(y) \leqq \Psi_\zeta(y) \quad \text{in} \quad G(\zeta) .$$

Then taking account of the boundary condition (3.59) and (3.69) we have,

$$\left|\frac{\partial U_{2,\zeta}}{\partial y_1}(1-\zeta, y')\right| \leqq \left|\frac{\partial \Psi_\zeta}{\partial y_1}(1-\zeta, y')\right| = \omega(1-\zeta) \leqq \omega .$$

By the same argument, we have $\left| \frac{\partial U_{2,\zeta}}{\partial y_1}(-1+\zeta, y') \right| \leqq \omega$.

Thus we conclude the result.

Lemma 3.7. For any $\delta \in (0,1)$, there exists a constant $c_{3,\delta} > 0$ such that

$$(3.71) \quad \left| \frac{\partial U_\zeta}{\partial y_1}(y) \right| \leqq c_{3,\delta} \quad \text{in } G(\delta) \quad (0 < \zeta \leqq \delta/2)$$

$$(3.72) \quad \left| \frac{\partial U_{1,\zeta}}{\partial y_1}(y) \right| \leqq c_{3,\delta} \quad \text{in } G(\delta) \quad (0 < \zeta \leqq \delta/2)$$

$$(3.73) \quad \left| \frac{\partial U_{2,\zeta}}{\partial y_1}(y) \right| \leqq c_{3,\delta} \quad \text{in } G(\delta) \quad (0 < \zeta \leqq \delta/2)$$

(Proof of Lemma 3.7) We will prove (3.71).

For any $y_* \in [0, 1-\delta]$, we define a function W_1 which is defined on $G \cap \{ 2y_* - 1+\zeta \leqq y_1 \leqq y_* \}$ and satisfies the following equations

$$(3.74) \quad \frac{\partial W_1}{\partial y_1}(y_*, y') = \frac{\partial U_\zeta}{\partial y_1}(y_*, y') \quad \text{for } |y'| < 1$$

$$(3.75) \quad P_\zeta W_1 + \frac{1}{2} (f(U_\zeta(y)) - f(U_\zeta(2y_* - y_1, y'))) = 0$$

$$\text{in } G \cap \{ 2y_* - 1+\zeta < y_1 < y_* \}$$

$$(3.76) \quad \frac{\partial W_1}{\partial \nu} = 0 \quad \text{on } \partial G \cap \{ 2y_* - 1+\zeta \leqq y_1 \leqq y_* \}$$

$$(3.77) \quad W_1(y_*, y') = 0 \quad \text{for } |y'| < 1 .$$

We define a comparison function Θ_1 as follows

$$\Theta_1(y_1, y') = \frac{\omega}{2}(y_* - y_1)\cdot(y_1 - 2y_* + 1 - \zeta) + \frac{\overline{M}}{1 - y_* - \zeta}(y_* - y_1)$$

where we have put $\overline{M} = \max (|M_*|, |M^*|)$.

This satisfies the following equations

(3.78) $\quad P_\zeta \Theta_1 + \omega = 0 \quad$ in $G \cap \{ 2y_*-1+\zeta < y_1 < y_* \}$

(3.79) $\quad \dfrac{\partial \Theta_1}{\partial \nu} = 0 \quad$ on $\partial G \cap \{ 2y_*-1+\zeta \leqq y_1 \leqq y_* \}$

(3.80) $\quad \Theta_1(2y_*-1+\zeta, y') = \overline{M} \quad$ for $|y'| < 1$

(3.81) $\quad \Theta_1(y_*, y') = 0 \quad$ for $|y'| < 1$.

Applying the comparison theorem to (3.74)-(3.76) and (3.78) - (3.80) (Notice $P_\zeta(\Theta_1 - W_1)(y) \leqq 0$.), we obtain

$$- \Theta_1(y) \leqq W_1(y) \leqq \Theta_1(y) \quad \text{in } G \cap \{ 2y_*-1+\zeta \leqq y_1 \leqq y_* \}.$$

Taking notice of the boundary condition (3.77) and (3.81), we deduce from (3.82) by (3.74) that

(3.83) $$\left| \frac{\partial U_\zeta}{\partial y_1}(y_*, y') \right| = \left| \frac{\partial W_1}{\partial y_1}(y_*, y') \right| \leqq \left| \frac{\partial \Theta_1}{\partial y_1}(y_*, y') \right|$$

$$= \frac{\omega}{2} (1 - y_* - \zeta) + \frac{\overline{M}}{1 - y_* - \zeta} \leqq \frac{\omega}{2} + \frac{2\overline{M}}{\delta}$$

for any $\zeta \in (0, \delta/2]$.

The above estimate holds uniformly in $y_* \in [0, 1-\delta]$.

For the case that $y_* \in [-1+\delta, 0]$, the proof is the same as the above case. On the other hand, we can prove (3.72) and (3.73) by the completely same argument (reflection technique) as (3.71).

Lemma 3.8. For any $\delta \in (0,\zeta_*)$, there exists a positive constant $c_{4,\delta}$ such that

$$(3.84) \qquad \sum_{j=2}^{n} \left| \frac{\partial U_\zeta}{\partial y_j}(y) \right|^2 \leqq c_{4,\delta} \ \zeta^4$$

on $\partial G \cap \{ -1+\delta \leqq y_1 \leqq 1-\delta \}$ for any $\zeta \in (0,\delta/2]$.

(Proof of Lemma 3.8) For the sake of constructing a comparison function, we take a function $h \in C^\infty([0,\infty))$ which satisfies

(i) $h(0) = 0$, $h(1) = 1$

(ii) $\frac{d^k h}{d\xi^k}(0) = 0$ for any natural number k .

$\frac{dh}{d\xi}(\xi) > 0$ for any $\xi \in (0,1)$.

Take an arbitrary hyperplane π in $\mathbb{R}^n$ which contains the y_1-axis. By an appropriate orthogonal transformation of coordinate in $(y_2,\cdots,y_n)$, we can assume that π is expressed by the equation $y_2 = 0$ without loss of generality. Remark that the equation (3.52) is invariant under the above transformation.
Now we define a domain $G_+(\zeta)$ and a function $W_2(y)$ in $G_+(\zeta)$ as follows.

$$(3.85) \qquad G_+(\zeta) = G(\zeta) \cap \{ y_1 > 0 \}$$

$$(3.86) \qquad W_2(y) = \frac{1}{2}(U_\zeta(y_1,y_2,\cdots,y_n) - U_\zeta(y_1,-y_2,y_3,\cdots,y_n))$$

It is easily seen that W_2 satisfies the following equations

$$(3.87)\quad \frac{\partial W_2}{\partial y_2}(y) = \frac{\partial U_\zeta}{\partial y_2}(y) \quad \text{on} \quad \pi \cap \partial G_+(\zeta)$$

$$(3.88)\quad P_\zeta W_2 + \frac{1}{2}(f(U_\zeta) - f(U_\zeta(y_1,-y_2,y_3,\cdots,y_n)) = 0 \quad \text{in} \quad G_+(\zeta)$$

$$(3.89)\quad W_2(y) = 0 \quad \text{on} \quad \pi \cap \partial G_+(\zeta)$$

$$(3.90)\quad \frac{\partial W_2}{\partial \nu}(y) = 0 \quad \text{on} \quad \partial G_+(\zeta) \cap \partial G(\zeta)$$

We put a comparison function $\Theta_2(y)$ as follows

$$(3.91)\quad \Theta_2(y) = \begin{cases} e(\delta)\zeta^2 y_2(3-y_2) + \overline{M}\, h(\frac{y_1-1+\delta}{\delta-\zeta}) & (\ y_1 > 1-\delta\ ) \\ e(\delta)\ \zeta^2\ y_2(3-y_2) & (\ -1+\delta \leqq y_1 \leqq 1-\delta\ ) \\ e(\delta)\ \zeta^2\ y_2(3-y_2) + \overline{M}\, h(\frac{-y_1-1+\delta}{\delta-\zeta}) & (\ y_1 < -1+\delta\ ) \end{cases}$$

where $e(\delta) = 1 + \omega + \frac{2\overline{M}}{\delta^2} \sup_{\xi \in [0,1]} |h''(\xi)|$.

By a simple calculation, we obtain

$$(3.92)\quad P_\zeta\ \Theta_2(y) = \begin{cases} -2\ e(\delta) + \frac{\overline{M}}{(\delta-\zeta)^2}\, h''(\frac{y_1-1+\delta}{\delta-\zeta}) & (\ y_1 > 1-\delta\ ) \\ -2\ e(\delta) & (\ -1+\delta \leqq y_1 \leqq 1-\delta\ ) \\ -2\ e(\delta) + \frac{\overline{M}}{(\delta-\zeta)^2}\, h''(\frac{-y_1-1+\delta}{\delta-\zeta}) & (\ y_1 < -1+\delta\ ) \end{cases}$$

$$(3.93)\quad \frac{\partial \Theta_2}{\partial \nu}(y) > 0 \quad \text{on} \quad \partial G_+(\zeta) \cap \partial G$$

$$(3.94)\quad \Theta_2(y) \geqq \overline{M} \quad \text{on} \quad (\partial G_+(\zeta) \cap \{y_1=1-\zeta\}) \cup (\partial G_+(\zeta) \cap \{y_1=-1+\zeta\})$$

$$(3.95)\quad \Theta_2(y) = 0 \quad \text{on} \quad \partial G_+(\delta) \cap \pi$$

By using $0 < \zeta \leqq \delta/2$ and the definition of $e(\delta)$, we obtain from (3.88)-(3.90) and (3.92)-(3.94) that

$$(3.96)\quad P_\zeta(\Theta_2 - W_2)(y) < 0 \quad \text{in} \quad G_+(\zeta)$$

$$(3.97)\quad \frac{\partial}{\partial \nu}(\Theta_2 - W_2) > 0 \quad \text{on} \quad \partial G_+(\zeta) \cap \partial G$$

$$(3.98)\quad \Theta_2(y) - W_2(y) \geqq 0 \quad \text{on} \quad \partial G_+(\zeta) \cap \pi$$

$$(3.99)\quad \Theta_2(y) - W_2(y) \geqq 0 \quad \text{on} \quad (\partial G_+(\zeta)\cap\{y_1=1-\zeta\})\cup(\partial G_+(\zeta)\cap\{y_1=-1+\zeta\})$$

$$(3.100)\quad \Theta_2(y) - W_2(y) = 0 \quad \text{on} \quad \partial G_+(\delta) \cap \pi$$

Applying the Maximum Principle to (3.96)-(3.99), we obtain,

$$\Theta_2(y) - W_2(y) \geqq 0 \quad \text{in} \quad G_+(\zeta) .$$

By a similar argument with respect to $-\Theta_2$ and W_2 in $G_+(\zeta)$, we have $-\Theta_2(y) \leqq W_2(y)$ in $G_+(\zeta)$. Then we conclude that

$$(3.101)\quad | W_2(y) | \leqq \Theta_2(y) \quad \text{in} \quad G_+(\zeta) .$$

Thus by the inequality (3.101) with the boundary condition (3.100), we have $\left| \frac{\partial W_2}{\partial y_2}(y) \right| \leqq \frac{\partial \Theta_2}{\partial y_2}(y)$ on $\partial G_+(\delta) \cap \pi$.

Therefore we have from (3.87) that

$$(3.101)\quad \left| \frac{\partial U_\zeta}{\partial y_2}(y)\Big|_{\partial G_+(\delta)\cap\pi} \right| \leqq \left| \frac{\partial \Theta_2}{\partial y_2}(y)\Big|_{\partial G_+(\delta)\cap\pi} \right| = 3\ e(\delta)\ \zeta^2$$

Then we have $\left| \frac{\partial U_\zeta}{\partial y_2}(y)\Big|_{\partial G_+(\delta)\cap\pi\cap\partial G} \right| \leqq 3\ e(\delta)\ \zeta^2$.

On the other hand we have the Neumann boundary condition

$$\frac{\partial U_\zeta}{\partial \nu}(y) = 0 \quad \text{on} \quad \partial G_+ \cap \pi \cap \partial G \ .$$

Then by considering the arbitrariness of π (containing y_1-axis) and the uniformness of the above argument in taking the hyperplane π , we conclude that

$$\sum_{j=2}^{n} \left| \frac{\partial U_\zeta}{\partial y_j}(y)\ \Big|_{\partial G\cap\partial G_+(\delta)} \right| \leqq 3\ (n-1)\ e(\delta)\ \zeta^2 \quad (0 < \zeta \leqq \delta/2) \ .$$

We complete the proof of Lemma 3.8 by putting $c_{4,\delta} = (3(n-1)e(\delta))^2$.

By the aid of Lemma 3.3 - Lemma 3.8, we will obtain a convergent subsequence of $\{ U_{\zeta_m} \}_{m=1}^{\infty}$. From Lemma 3.4 and $\lim_{\zeta\to 0} \iota(\zeta) = 0$, it is easy to see that for $\varepsilon > 0$, there exists a constant $\overline{\zeta} = \overline{\zeta}(\varepsilon)$ and $\overline{\delta} = \overline{\delta}(\varepsilon)$ such that $\overline{\zeta}(\varepsilon)$ and $\overline{\delta}(\varepsilon)$ depend monotonously on ε and $\lim_{\varepsilon\to 0} \overline{\delta}(\varepsilon) = 0$, $\lim_{\varepsilon\to 0} \overline{\zeta}(\varepsilon) = 0$ and such that

$$(3.102)\quad \sup_{0<\zeta\leqq\overline{\zeta}} \left\{ \sup_{1-2\overline{\delta}\leqq y_1\leqq 1-\zeta} | U_\zeta(y)-a_1| + \sup_{-1+\zeta\leqq y_1\leqq -1+2\overline{\delta}} | U_\zeta(y)-a_2| \right\}$$

$$\leqq \varepsilon \ .$$

On the other hand we deal with the convergence on the domain $G(2\overline{\delta}(\varepsilon))$. From Lemma 3.5, $\{ U_{2,\zeta_m} \}_{m=1}^{\infty}$ is bounded in the Sobolev space $H^1(G(\overline{\delta}(\varepsilon)))$ and it is compact in $H^{1/2}(G(2\overline{\delta}(\varepsilon))$ by the Imbedding Theorem. Moreover $\{ U_{2,\zeta_m} |\partial G(2\overline{\delta})\cap\partial G \}_{m=1}^{\infty}$ is compact in $L^2(\partial G(2\overline{\delta})\cap\partial G)$ by the Trace Theorem. (Taylor [21] Chapter I)

Now take a sequence of positive values $\{ \varepsilon_k \}_{k=1}^{\infty}$ such that

$\varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_k > \varepsilon_{k+1} > \cdots > 0$ and $\lim_{k\to\infty} \varepsilon_k = 0$.

By the above compactness argument for $\varepsilon = \varepsilon_1$, we have a subsequence $\{ \zeta_m^{(1)} \}_{m=1}^{\infty} \subset (0, \overline{\zeta}(\varepsilon_1))$ such that $\{ U_{2,\zeta_m^{(1)}} \}_{m=1}^{\infty}$ is convergent in $H^{1/2}(G(2\overline{\delta}(\varepsilon_1)))$ and also in $L^2(\partial G(2\overline{\delta}(\varepsilon_1))\cap\partial G)$ and its limit function is independent of y' by Lemma 3.5 . Therefore by Lemma 3.3, $\{ U_{\zeta_m^{(1)}} \}_{m=1}^{\infty}$ is convergent in $H^{1/2}(G(2\overline{\delta}(\varepsilon_1)))$ and also in $L^2(\partial G(2\overline{\delta}(\varepsilon_1))\cap\partial G)$

(Recall $U_{\zeta_m^{(1)}} = U_{1,\zeta_m^{(1)}} + U_{2,\zeta_m^{(1)}}$) .

Then there exists a function $U^{(1)} \in H^{1/2}(G(2\overline{\delta}(\varepsilon_1)))$ which is independent of y' , such that

$$\lim_{m\to\infty} U_{\zeta_m^{(1)}} = U^{(1)} \quad \text{in} \quad H^{1/2}(G(2\overline{\delta}(\varepsilon_1))) .$$

Again applying the same argument to the sequence $\{ U_{\zeta_m^{(1)}} \}_{m=1}^{\infty}$ for $\varepsilon = \varepsilon_2$, we get a subsequence $\{ \zeta_m^{(2)} \}_{m=1}^{\infty} \subset \{ \zeta_m^{(1)} \}_{m=1}^{\infty} \cap (0, \overline{\zeta}(\varepsilon_2))$ and a function $U^{(2)} \in H^{1/2}(G(2\overline{\delta}(\varepsilon_2)))$ which is independent of y', such that

$$\lim_{m\to\infty} U_{\zeta_m^{(2)}} = U^{(2)} \quad \text{in} \quad H^{1/2}(G(2\overline{\delta}(\varepsilon_2)))$$

$$U^{(2)}{}_{|G(2\overline{\delta}(\varepsilon_2))} = U^{(1)}$$

Repeating this process inductively , we obtain a sequence of subsequences of $\{ \zeta_m \}_{m=1}^{\infty}$ such that

$$\{ \zeta_m \}_{m=1}^{\infty} \supset \{ \zeta_m^{(1)} \}_{m=1}^{\infty} \supset \{ \zeta_m^{(2)} \}_{m=1}^{\infty} \supset \cdots \supset \{ \zeta_m^{(q)} \}_{m=1}^{\infty} \supset \cdots$$

and a function V which is independent of y' such that

$\lim_{m\to\infty} U_{\zeta_m^{(q)}} = V$ in $H^{1/2}(G(2\overline{\delta}(\varepsilon_q)))$ for any natural number q .

Determine the subsequence $\{ \varkappa_m \}_{m=1}^{\infty} \subset \{ \zeta_m \}_{m=1}^{\infty}$ by $\varkappa_m = \zeta_m^{(m)}$ $(m \geqq 1)$. From the way of the construction $\{ \zeta_m^{(q)} \}_{m=1}^{\infty}$, we have

$$\text{(3.105)} \quad \lim_{m\to\infty} U_{\varkappa_m} = V \quad \text{in} \quad H^{1/2}(G(2\overline{\delta}(\varepsilon_q))) \qquad (q \geqq 1)$$

$$\text{(3.106)} \quad \lim_{m\to\infty} U_{\varkappa_m} = V \quad \text{in} \quad L^2(\partial G(2\overline{\delta}(\varepsilon_q))\cap\partial G) \qquad (q \geqq 1)$$

$$\text{(3.107)} \quad \sup_{1-2\overline{\delta}(\varepsilon_q)\leqq y_1\leqq 1-\varkappa_m} | U_{\varkappa_m}(y)-a_1| + \sup_{-1+\varkappa_m \leqq y_1\leqq -1+2\overline{\delta}(\varepsilon_q)} | U_{\varkappa_m}(y)-a_2|$$

$$\leqq \varepsilon_q \qquad (m \geqq q \geqq 1) .$$

From now on, we will investigate the uniform convergence of $\{ U_{\varkappa_m} \}_{m=1}^{\infty}$. By Lemma 3.7 and Lemma 3.8 and the Ascoli-Arzera Theorem, $\{ U_{\varkappa_m} |\partial G(2\overline{\delta}(\varepsilon_q))\cap\partial G \}_{m=1}^{\infty}$ is compact in $C^0(\partial G(2\overline{\delta}(\varepsilon_q))\cap\partial G)$ for any natural number q. On the other hand we already have (3.106). Then we conclude that V is continuous in the interval $(-1,1)$ and

$$(3.108)\quad \lim_{m\to\infty} \sup_{y\in \partial G(2\overline{\delta}(\varepsilon_q))\cap\partial G} | U_{\varkappa_m}(y_1,y') - V(y_1) | = 0$$

for any integer $q \geqq 1$.

Then, let m tend to ∞ in (3.107) and we have

$$(3.109)\quad \sup_{1-2\overline{\delta}(\varepsilon_q)\leqq y_1<1} | V(y_1)-a_1| + \sup_{-1<y_1\leqq -1+2\overline{\delta}(\varepsilon_q)} | V(y_1)-a_2| \leqq \varepsilon_q .$$

This concludes that V is continuous on $[-1,1]$ and

$$(3.110)\quad V(1) = a_1 , \quad V(-1) = a_2 .$$

Therefore from (3.107),(108),(3.109) and (3.110), we have,

$$\overline{\lim_{m\to\infty}} \sup_{y\in\partial G\cap\partial G(\varkappa_m)} | U_{\varkappa_m}(y_1,y') - V(y_1) |$$

$$\leqq \overline{\lim_{m\to\infty}} \sup_{y\in\partial G(2\overline{\delta}(\varepsilon_q))\cap\partial G} | U_{\varkappa_m}(y) - V(y_1) |$$

$$+ \overline{\lim_{m\to\infty}} \sup_{1-2\overline{\delta}(\varepsilon_q)\leqq y_1\leqq 1-\varkappa_m, |y'|=1} | (U_{\varkappa_m}(y)-a_1) + (a_1-V(y_1)) |$$

$$+ \overline{\lim_{m\to\infty}} \sup_{-1+\varkappa_m \leqq y_1 \leqq -1+2\bar{\delta}(\varepsilon_q),\, |y'|=1} |\, (U_{\varkappa_m}(y)-a_2) + (a_2-V(y_1))\, |$$

$$\leqq \varepsilon_q + \sup_{1-2\bar{\delta}(\varepsilon_q) \leqq y_1 \leqq 1} |V(y_1)-a_1| + \sup_{-1\leqq y_1 \leqq -1+2\bar{\delta}(\varepsilon_q)} |V(y_1)-a_2|$$

for any $q \geqq 1$. Then $\lim_{q\to\infty} \varepsilon_q = 0$ and $\lim_{q\to\infty} \bar{\delta}(\varepsilon_q) = 0$ imply

$$\lim_{m\to\infty} \sup_{y \in \partial G(\varkappa_m)\cap\partial G} |\, U_{\varkappa_m}(y_1,y') - V(y_1)\, | = 0 .$$

Again by (3.107) and (3.109) we conclude that

$$\lim_{m\to\infty} \sup_{y \in \partial G(\varkappa_m)} |\, U_{\varkappa_m}(y_1,y') - V(y_1)\, | = 0 .$$

From the equation (3.52) and (3.53), we have

$$\left(\frac{\partial^2}{\partial y_1^{\,2}} + \frac{1}{\varkappa_m^{\,2}} \sum_{j=2}^{n} \frac{\partial^2}{\partial y_j^{\,2}} \right) U_{\varkappa_m} + f(U_{\varkappa_m}) = 0 \quad \text{in} \quad G(\varkappa_m)$$

$$\frac{\partial U_{\varkappa_m}}{\partial \nu}(y) = 0 \qquad \text{on} \quad \partial G(\varkappa_m)\cap\partial G$$

Take any $\phi \in C_0^\infty((-1,1))$ and integrate the above equation in $G(\varkappa_m)$ after multiplying $\phi(y_1,y') = \phi(y_1)$. Then we have for sufficiently large m so that $\operatorname{supp}\phi \subset (-1+\varkappa_m, 1-\varkappa_m)$,

$$\int_{G(\varkappa_m)} U_{\varkappa_m}(y)\, P_{\varkappa_m} \phi \, dy + \int_{G(\varkappa_m)} \phi \, f(U_{\varkappa_m})\, dy = 0 .$$

(Remark that $P_{\varkappa_m} \phi(y) = \frac{\partial^2 \phi}{\partial y_1{}^2}(y_1)$)

Let m tend to ∞ and we get by (3.105) that

$$\int_{|y'| \leqq 1} dy' \int_{-1}^{1} \left(V(y_1) \frac{\partial^2}{\partial y_1{}^2}\phi(y_1) + \phi(y_1)\, f(V(y_1)) \right) dy_1 = 0$$

By the arbitrariness of ϕ , we have

$$\frac{d^2}{dy_1{}^2} V(y_1) + f(V(y_1)) = 0 \quad \text{in} \quad (-1,1) \quad .$$

Lemma 3.9.

$$\lim_{m\to\infty} \sup_{y \in G(\varkappa_m)} | U_{\varkappa_m}(y_1,y') - V(y_1) | = 0$$

(Proof of Lemma 3.9) We define a comparison function $\Theta_{\pm,m}$ by

$$\Theta_{\pm,m}(y) = V(y_1) \pm \frac{\omega}{n-1} (1 - |y'|^2) \varkappa_m{}^2$$
$$\pm \sup_{y \in \partial G(\varkappa_m)} | U_{\varkappa_m}(y_1,y') - V(y_1) |$$

$\Theta_{\pm,m}$ satisfy the following equations by (3.112)

$$P_{\varkappa_m} (\Theta_{\pm,m} - U_{\varkappa_m}) = - f(V) \mp 2\,\omega + f(U_{\varkappa_m}) \lesseqgtr 0 \quad \text{in} \quad G(\varkappa_m)$$

$$\Theta_{\pm,m}(y) - U_{\varkappa_m}(y) \gtreqless 0 \quad \text{on} \quad \partial G(\varkappa_m)$$

Then applying the Maximum Principle, we have

$$\Theta_{\pm,m}(y) - U_{\varkappa_m}(y) \gtreqless 0 \quad \text{in} \quad G(\varkappa_m)$$

$$\text{or} \quad \Theta_{-,m}(y) \leqq U_{\varkappa_m}(y) \leqq \Theta_{+,m}(y) \quad \text{in} \quad G(\varkappa_m) \; .$$

By the definition of $\Theta_{\pm,m}$, we conclude that

$$\lim_{m\to\infty} \sup_{y \in G(\varkappa_m)} | U_{\varkappa_m}(y) - V(y_1) | = 0$$

and complete the proof of Lemma 3.9 .

Expressing the equality in Lemma 3.9 in the original variable x , we complete the proof of the former assertion of Theorem 3.

(Proof of the Latter Half of Theorem 3)

(1) The case $\lambda_V < 0$.

We will prove that the first eigenvalue $\mu_1(\varkappa_m)$ in (3.2) for $v_{\varkappa_m}$ is bounded from below by a negative constant for sufficiently large m .

It is well-known that

$$(3.113) \quad \mu_1(\varkappa_m) = \inf_{\psi \in H^1(\Omega(\varkappa_m))} \frac{\int_{\Omega(\varkappa_m)} (|\nabla\psi|^2 - f'(v_{\varkappa_m})\psi^2) \, dx}{\int_{\Omega(\varkappa_m)} |\psi|^2 \, dx}$$

Here we define a function

$$\psi_m(x_1,x') = \begin{cases} 0 & x \in D_1 \cup D_2 \cup R_1(\varkappa_m) \cup R_2(\varkappa_m) \\ \Phi_V(x_1) - \Phi_V(1-2\varkappa_m) & x \in \Gamma(\varkappa_m) \end{cases}$$

Remark that $\Phi_V(z) = \Phi_V(-z)$ on $(-1,1)$ and $\psi_m \in H^1(\Omega(\varkappa_m))$.

To estimate $\mu_1(\varkappa_m)$ from above by using (3.113), we calculate

$$(3.114) \quad \int_{\Omega(\varkappa_m)} (|\nabla \psi_m|^2 - f'(v_{\varkappa_m}) \psi_m{}^2) \, dx$$

$$= \int_{|x'| \leqq \varkappa_m} dx' \int_{-1+2\varkappa_m}^{1-2\varkappa_m} (| \frac{\partial \Phi_V}{\partial x_1} |^2 - f'(v_{\varkappa_m}) |\Phi_V(x_1)-\Phi_V(1-2\varkappa_m)|^2) \, dx_1$$

$$= - \int\int dx' dx_1 \{ \frac{d^2\Phi_V}{dx_1{}^2} + f'(v_{\varkappa_m})(\Phi_V(x_1)-\Phi_V(1-2\varkappa_m))\} (\Phi_V(x_1)-\Phi_V(1-2\varkappa_m))$$

$$= \int_{|x'| \leqq \varkappa_m} \left\{ \int_{-1+2\varkappa_m}^{1-2\varkappa_m} \{ \lambda_V + f'(V(x_1)) - f'(v_{\varkappa_m}(x_1,x')) \} \Phi_V(x_1)^2 dx_1 \right.$$

$$+ \int_{-1+2\varkappa_m}^{1-2\varkappa_m} \{-\lambda_V\Phi_V(x_1)+(2f'(v_{\varkappa_m})-f'(V))\Phi_V(x_1) - f'(v_{\varkappa_m})\Phi_V(1-2\varkappa_m)\} \times$$

$$\left. \Phi_V(1-2\varkappa_m)dx_1 \right\} dx'$$

Using the former assertion of Theorem 3 which we have already proved we have $| f'(V) - f'(v_{\varkappa_m}) | \leqq - \lambda_V/4$ in $\Gamma(\varkappa_m)$ for sufficiently large m . On the other hand, $\lim_{m\to\infty} \Phi_V(1-2\varkappa_m) = 0$ holds from the boundary condition $\Phi_V(1) = 0$ and then we have the inequality,

$$| \int_{-1+2\varkappa_m}^{1-2\varkappa_m} \{ -\lambda_V\Phi_V(x_1) + (2f'(v_{\varkappa_m})-f'(V))\Phi_V(1-2\varkappa_m)- f'(v_{\varkappa_m})\Phi_V(1-2\varkappa_m)\}$$

$$\times \Phi_V(1-2\varkappa_m) dx_1| \leqq - \frac{\lambda_V}{4} \int_{-1+2\varkappa_m}^{1-2\varkappa_m} \Phi_V(x_1)^2dx_1 \qquad \text{for large m.}$$

Then we have

$$\int_{\Omega(\varkappa_m)} (|\nabla\psi_m|^2 - f'(v_{\varkappa_m}) \psi_m{}^2) dx \leqq \frac{\lambda_V}{2} \int\int_{\Gamma(\varkappa_m)} \Phi_V(x_1)^2 dx_1 dx'$$

for large m .

On the other hand one can easily check that

$$\int_{\Omega(\varkappa_m)} \psi_m(x)^2dx \leqq 2 \int\int_{\Gamma(\varkappa_m)} \Phi_V(x_1)^2 dx_1 dx' \quad \text{for large } m .$$

Then we conclude that $\mu_1(\varkappa_m) \leqq \lambda_V/4$ for sufficiently large m . This concludes the result the case (1).

(2) The case $\lambda_V > 0$.

From now on we will prove that $\mu_1(\varkappa_m)$ is bounded from below by a positive constant for sufficiently large m. To prove by the contradiction we assume that there exists a subsequence $\{ m(j) \}_{j=1}^{\infty}$ such that

$$(*) \qquad \lim_{j\to\infty} m(j) = \infty , \quad \lim_{j\to\infty} \mu_1(\varkappa_{m(j)}) \leqq 0 .$$

Let ψ_j be the corresponding eigenfunction of (3.2) to the eigenvalue $\mu_1(\varkappa_{m(j)})$ such that

$$(3.115) \qquad \| \psi_j \|_{L^2(\Omega(\varkappa_{m(j)}))} = 1 \quad (j \geqq 1).$$

Lemma 3.10. Under the condition (*),

$$\lim_{j\to\infty} \psi_j = 0 \text{ in } C^{\infty}((\overline{D_1-\Sigma_1(\eta)})\cup(\overline{D_2-\Sigma_2(\eta)})) \quad \text{for any } \eta > 0.$$

(Proof of Lemma 3.10) Applying the bootstrap argument by the a-priori estimate in S.Agmon,A.Douglas and L.Nirenberg [1], we see

$$(3.116) \qquad \{ \psi_j \}_{j=1}^{\infty} \text{ is compact in } C^{\infty}((\overline{D_1-\Sigma_1(\eta)})\cup(\overline{D_2-\Sigma_2(\eta)})) \text{ for any } \eta > 0 .$$

On the other hand, we take two functions $\phi_1 , \phi_2 \in C^{\infty}(\mathbb{R}^n)$ such that

$$\phi_1(x) = 1 \text{ in } D_1 , \quad \phi_1(x) = 0 \text{ in } D_2 , \quad \phi_2(x) = 0 \text{ in } D_1 ,$$
$$\phi_2(x) = 1 \text{ in } D_2 , \quad \operatorname{supp} \phi_1 \cap \operatorname{supp} \phi_2 = \emptyset .$$

We put , for $i = 1,2$ and $j = 1,2,3,\cdots$,

$$\theta_j^{(i)} \equiv \| (\Delta + f'(v_{\varkappa_{m(j)}}))\phi_i - f'(a_i)\phi_i \|_{L^2(\Omega(\varkappa_{m(j)}))} / \| \phi_i \|_{L^2(\Omega(\varkappa_{m(j)}))}$$

and we can easily check that $\lim_{j\to\infty} \theta_j^{(i)} = 0 \quad (i = 1, 2)$ by Theorem 2 and a simple calculation.

Therefore the eigenvalue problem (3.2) for $\zeta = \varkappa_{m(j)}$ has eigenvalues $\mu^{(1)}(j)$ and $\mu^{(2)}(j)$ for large j such that

$$\mu^{(i)}(j) \in [-f'(a_i) - \theta_j^{(i)^{1/2}}, -f'(a_i) + \theta_j^{(i)^{1/2}}] \equiv I_j^{(i)}$$

$$\| P_{I_j^{(i)}} \phi_i - \phi_i \|_{L^2(\Omega(\varkappa_{m(j)}))} / \| \phi_i \|_{L^2(\Omega(\varkappa_{m(j)}))} \leqq \theta_j^{(i)^{1/2}}$$

for $i = 1, 2$ and large $j \geqq 1$, where $P_{I_j^{(i)}}$ is the eigenprojection (associated with the self-adjoint operator $-\Delta - f'(v_{\varkappa_{m(j)}})$), onto the subspace of $L^2(\Omega(\varkappa_{m(j)}))$ corresponding to the the interval $I_j^{(i)}$.

We have $\mu_1(\varkappa_{m(j)}) \notin \bigcup_{i=1}^{2} [-f'(a_i)-\theta_j^{(i)}, -f'(a_i)+\theta_j^{(i)}]$ for large j by (*) and then $(\psi_j, P_{I_j^{(i)}} \phi_i)_{L^2(\Omega(\varkappa_{m(j)}))} = 0$ for large j and $i = 1,2$. Therefore we have for $i = 1,2$,

$$|(\psi_j, \phi_i)_{L^2(\Omega(\varkappa_{m(j)}))}| / \|\phi_i\|_{L^2(\Omega(\varkappa_{m(j)}))} \leqq \theta_j^{(i)^{1/2}}$$

for large j and so we can easily deduce $\lim_{j\to\infty} \int_{D_i} \psi_j \, dx = 0$ ($i = 1,2$). Remark that $\psi_j(x) > 0$ in $\overline{\Omega(\varkappa_{m(j)})}$ and we have that $\lim_{j\to\infty} \psi_j(x) = 0$ for a.e. $x \in D_1 \cup D_2$.

By the compactness (3.116), we conclude the result of Lemma 3.10.

By using Lemma 3.10, we can choose a monotone sequence of positive values $\{ \ell_j \}_{j=1}^{\infty}$ such that

$$(3.117) \begin{cases} \lim_{j\to\infty} \ell_j = 0 , \quad \ell_j > \varkappa_{m(j)} \\ \lim_{j\to\infty} K(j) = 0 \\ \text{where } K(j) = \sup_{x \in (D_1-\Sigma_1(2\ell_j))\cup(D_2-\Sigma_2(2\ell_j))} |\psi_j(x)| > 0 . \end{cases}$$

Here we define two sets,

$$S_j = (Q(\varkappa_{m(j)}) \cup \Sigma_1(2\ell_j) \cup \Sigma_2(2\ell_j)) \cap \{ \ |x_1| < 1 + ((2\ell_j)^2 - (\varkappa_{m(j)})^2)^{1/2} \ \}$$

$$T_j = \{ (x_1, x') \in \mathbb{R}^n \mid \ |x'| < \varkappa_{m(j)}, \ |x_1| \leqq 1 + ((2\ell_j)^2 - (\varkappa_{m(j)})^2)^{1/2} \}$$

Now we decompose eigenfunction ψ_j uniquely as follows

$\psi_j(x) = \psi_j^{(1)} + \psi_j^{(2)}$ in S_j , by the following equations,

$$(3.118) \begin{cases} \Delta \psi_j^{(1)} = 0 \quad \text{in} \quad S_j \\ \psi_j^{(1)}(x) = \psi_j(x) \quad \text{on} \quad \partial S_j - \partial\Omega(\varkappa_{m(j)}) \\ \dfrac{\partial \psi_j^{(1)}}{\partial \nu}(x) = 0 \quad \text{on} \quad \partial S_j \cap \partial\Omega(\varkappa_{m(j)}) \end{cases}$$

$$(3.119) \qquad \psi_j^{(2)}(x) = \psi_j(x) - \psi_j^{(1)}(x) \quad \text{in} \quad S_j$$

Apply the maximum principle to (3.118), we obtain the inequality,

$$(3.120) \quad 0 < \psi_j^{(1)}(x) \leqq K_j \quad \text{in} \quad S_j .$$

Now we calculate as follows.

$$(3.121) \quad \mu_1(\varkappa_{m(j)}) = \int_{\Omega(\varkappa_{m(j)})} (\ |\nabla\psi_j|^2 - f'(v_{\varkappa_{m(j)}}) \ \psi_j^2 \ ) \, dx$$

$$= \int_{\Omega(\varkappa_{m(j)})-S_j} (\ |\nabla\psi_j|^2 - f'(v_{\varkappa_{m(j)}}) \ \psi_j^2 \ ) \, dx$$

$$+ \int_{S_j} |\nabla\psi_j^{(1)}|^2 \, dx$$

$$+ \int_{S_j} (|\nabla\psi_j^{(2)}|^2 - f'(v_{\varkappa_{m(j)}}) \ |\psi_j^{(2)}|^2 \ ) \, dx$$

$$- \int_{S_j} f'(v_{\varkappa_{m(j)}}) \, (2 \psi_j^{(2)} - \psi_j^{(1)}) \, \psi_j^{(1)} \, dx$$

$$\equiv B_1(j) + B_2(j) + B_3(j) + B_4(j)$$

We have used $\int_{S_j} \nabla\psi_j^{(1)} \nabla\psi_j^{(2)} \, dx = 0$ in the above.

By Theorem 2, $- f'(v_{\varkappa_{m(j)}}) \geqq \beta_* / 2$ in $\Omega(\varkappa_{m(j)}) - S_j$ for j ,

where $\beta_* = \min(-f'(a_1), -f'(a_2))$. Then we have ,

(3.122) $B_1(j) \geqq \min(1, \beta_*/2) \, (\|\psi_j\|_{L^2(\Omega(\varkappa_{m(j)})-S_j)})^2$ for large j

By (3.117) and the boundedness of $\|\psi_j^{(2)}\|_{L^2(S_j)}$ ($j = 1,2,3,\cdots$),

(3.123) $\lim_{j\to\infty} B_4(j) = 0$

Hereafter we estimate $B_3(j)$ from below.

$$B_3(j) = \int_{T_j} (|\nabla\psi_j^{(2)}|^2 - f'(v_{\varkappa_{m(j)}}) \, |\psi_j^{(2)}|^2) \, dx$$

$$+ \int_{S_j - T_j} (|\nabla\psi_j^{(2)}|^2 - f'(v_{\varkappa_{m(j)}}) \, |\psi_j^{(2)}|^2) \, dx$$

Again from Theorem 2, the second term of $B_3(j)$

$$\geqq \min(1, \beta_*/2) \, (\|\psi_j^{(2)}\|_{H^1(S_j - T_j)})^2 .$$

To estimate the first term of $B_3(j)$, we change the variable x into y in T_j as follows.

$$\begin{cases} x_1 = (1+\sigma_j)\, y_1 \\ x' = y' \\ \Xi_j(y_1,y') = \psi_j^{(2)}(\sigma_j y_1, y') \quad \text{for} \quad |y_1| \leqq 1\,,\ |y'| < \varkappa_{m(j)} \end{cases}$$

where $\sigma_j = 1 + (\ (2\ell_j)^2 - (\varkappa_{m(j)})^2\ )^{1/2}$.

Remark that $\lim_{j\to\infty} \sigma_j = 1$, $\Xi_j(y) = 0$ for $y_1 = \pm 1$, $|y'| < \varkappa_{m(j)}$.

Then we estimate as follows,

$$\int_{T_j} (\ |\nabla\psi_j^{(2)}|^2 - f'(v_{\varkappa_{m(j)}})\ |\psi_j^{(2)}|^2\ )\, dx$$

$$\geqq \int_{|y'| \leqq \varkappa_{m(j)}} dy' \int_{-1}^{1} dy_1 \ \{\ \frac{1}{\sigma_j^{\,2}}\ |\ \frac{\partial \Xi_j}{\partial y_1}(y_1,y')|^2$$

$$- f'(v_{\varkappa_{m(j)}}(\sigma_j y_1, y'))\ \ \Xi_j^{\,2}\ \}\ \sigma_j$$

$$= \frac{1}{\sigma_j} \int_{|y'|\leqq \varkappa_{m(j)}} dy' \int_{-1}^{1} (\,|\ \frac{\partial \Xi_j(y_1,y')}{\partial y_1}\,|^2 - f'(V(y_1))\,|\Xi_j(y_1,y')|^2)\,dy_1$$

$$+ \int_{|y'|\leqq \varkappa_{m(j)}} dy' \int_{-1}^{1} (\frac{1}{\sigma_j^{\,2}}\ f'(V(y_1)) - f'(v_{\varkappa_{m(j)}}(\sigma_j y_1, y')))\ \Xi_j^{\,2}\ \sigma_j\ dy_1$$

$$\geqq \frac{1}{\sigma_j} \int_{|y'|\leqq \varkappa_{m(j)}} dy'\ \ \lambda_V \int_{-1}^{1} \Xi_j(y)^2\ dy_1 \ - \int_{|y'|\leqq \varkappa_{m(j)}} dy' \int_{-1}^{1} \Xi_j(y)^2 dy_1$$

$$\times \sup_{|y'|\leqq \varkappa_{m(j)},\,|y_1|<1} |\ \frac{1}{\sigma_j^{\,2}}\ f'(V(y_1)) - f'(v_{\varkappa_{m(j)}}(\sigma_j y_1, y'))|$$

$$= \frac{\lambda_V}{\sigma_j^{\,2}}\ (\ \|\psi_j^{(2)}\|_{L^2(T_j)})^2 - (\ \|\psi_j^{(2)}\|_{L^2(T_j)})^2\ \frac{1}{\sigma_j} \cdot \sup\ |\ \cdot\cdot\ |$$

By the first half of Theorem 3 and $\lim_{j\to\infty} \sigma_j = 0$, the second term of the above line is minor to the first for large j. Then we have the following inequality (3.124) for large j.

$$(3.124) \quad B_3(j) \geqq \frac{\lambda_V}{2} \left(\|\psi_j^{(2)}\|_{L^2(T_j)} \right)^2 .$$

Therefore from the inequalities (3.121),(3.122) and (3.124), we have

$$\mu_1(\varkappa_{m(j)}) - B_4(j) \geqq \min(1, \beta_*/2) \left(\|\psi_j\|_{L^2(\Omega(\varkappa_{m(j)})-S_j)} \right)^2$$

$$+ \int_{S_j} |\nabla\psi_j^{(1)}|^2 dx + \min(1,\beta_*/2) \left(\|\psi_j^{(2)}\|_{H^1(S_j-T_j)} \right)^2$$

$$+ \frac{\lambda_V}{2} \left(\|\psi_j^{(2)}\|_{L^2(T_j)} \right)^2 .$$ Let j tend to ∞ , we have ,

$\lim_{j\to\infty} \|\psi_j\|_{L^2(\Omega(\varkappa_{m(j)}))} = 0$ by using $\lim_{j\to\infty} \mu_1(\varkappa_{m(j)}) \leqq 0$ and (3.123) . But this contradicts to the fact $\|\psi_j\|_{L^2(\varkappa_{m(j)})} = 1$ for $j \geqq 1$ (See (3.115)). Then we have completed the proof of $\underline{\lim}_{m\to\infty} \mu_1(\varkappa_m) > 0$ and we conclude the result of the case $\lambda_V > 0$.

Therefore we have completed the proof of Theorem 3.

§ 4 Construction of Unstable Solution.

In this section, we will consider the equation (3.1) on the domain $\Omega(\zeta)$ established in Section 3 where we choose f in (3.1) as the one we will establish below. We will construct a family of solutions $\{ v_\zeta \}_{\zeta>0}$ in (III-4) where v_ζ is an unstable solution of (3.1) under the condition $a_1 = a_2 = b_1$ for small $\zeta > 0$.

We determine the nonlinear term f in the following form.

(4.1) $$f(\xi) = \vartheta\, g(\xi) \qquad (\vartheta > 0)$$

where $g \in C^\infty(\mathbb{R})$ satisfies the following conditions (IV-1)-(IV-2) and the parameter ϑ will be chosen later.

(IV-1) There exist three points $b_1 < b_2 < b_3$ such that

$$g(b_i) = 0 \ (1 \leqq i \leqq 3) \ , \ g'(b_1) < 0 \ , \ g'(b_3) < 0$$

$$g(\xi) > 0 \ \text{ in } \ (-\infty,b_1) \cup (b_2,b_3)$$

$$g(\xi) < 0 \ \text{ in } \ (b_1,b_2) \cup (b_3,\infty).$$

(IV-2) $$\int_{b_1}^{b_3} g(\xi)\, d\xi > 0$$

From (IV-1)-(IV-2), there exists a unique $d \in (b_2,b_3)$ such that $\int_{b_1}^{d} g(\xi)\, d\xi = 0$.

Above all things we seek for the solutions of the following two point boundary value problem of the ordinary differential equation (4.2) up to their linearized stability where the nonlinear term f is that in (4.1).

(4.2) $$\begin{cases} \dfrac{d^2V}{dz^2} + f(V) = 0 & \text{in } -1 < z < 1 \\ V(1) = b_1 \ , \ V(-1) = b_1 & \end{cases}$$

Proposition 2. There exists a positive values ϑ_0 such that for any $\vartheta \geqq \vartheta_0$, (4.2) has exactly three solutions

$$V^{(0)}(z) \ (\equiv b_1) < V^{(1)}(z) < V^{(2)}(z) \qquad (-1 < z < 1)$$

with the following stability properties,

$$\lambda_{V^{(0)}} > 0 \ , \quad \lambda_{V^{(1)}} < 0 \ , \quad \lambda_{V^{(2)}} > 0 \ .$$

(See Definition 3 in Section 3 as for $\lambda_{V^{(0)}}$, $\lambda_{V^{(1)}}$, $\lambda_{V^{(2)}}$.)

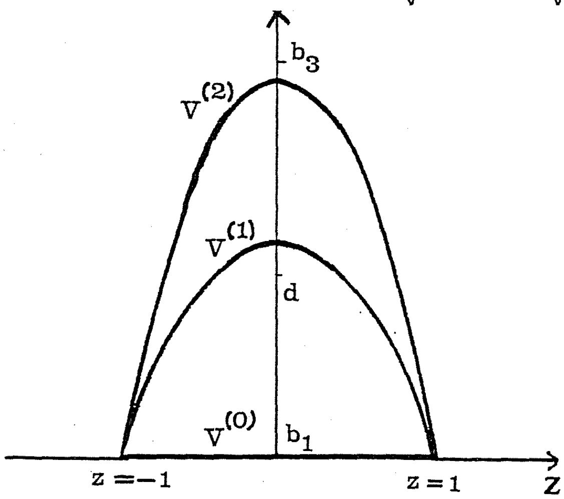


Figure 5

(Proof of Proposition 2) To construct nontrivial solutions, we must search for the value $\xi \in (d,b_3)$ which satisfies the following equation.

$$(4.3) \quad \int_{b_1}^{\xi} \left(2\int_{\sigma}^{\xi} f(\rho)d\rho \right)^{-1/2} d\sigma = 1 \quad (d < \xi < b_3) \text{ (See Maginu [14].)}$$

To examine the left hand side as a function of ξ, we define $s(\xi)$ which is defined in (d,b_3) as follows.

$$s(\xi) = \int_{b_1}^{\xi} \left(2 \int_{\sigma}^{\xi} g(\rho) \ d\rho \right)^{-1/2} d\sigma$$

$s(\xi)$ is well-defined by (IV-1) and (IV-2) and moreover we have the following properties concerning $s(\xi)$.

Lemma 4.1. $s(\xi)$ is a positively valued differentiable function on (d,b_3) with the following asymptotic conditions,

$$\begin{cases} \lim_{\xi \uparrow b_3} \dfrac{s(\xi)}{\left(-1/g'(b_3)\right)^{1/2} \log \dfrac{1}{b_3 - \xi}} = 1 \\ \lim_{\xi \downarrow d} \dfrac{s(\xi)}{\left(-1/4g'(b_1)\right)^{1/2} \log \dfrac{1}{\xi - d}} = 1 \\ \lim_{\xi \uparrow b_3} \dfrac{d}{d\xi} s(\xi) = +\infty , \quad \lim_{\xi \downarrow d} \dfrac{d}{d\xi} s(\xi) = -\infty . \end{cases}$$

(Proof of Lemma 4.1) First we deal with the case that ξ is near b_3 i.e. $d' < (d' + b_3)/2 \leqq \xi < b_3$ where d' is a point in (b_2,b_3) which will be determined later.

$$(4.4)\quad s(\xi) = \int_{b_1}^{d'} \left(2 \int_{\sigma}^{\xi} g(\rho)\, d\rho \right)^{-1/2} d\sigma + \int_{d'}^{\xi} \left(2 \int_{\sigma}^{\xi} g(\rho)\, d\rho \right)^{-1/2} d\sigma$$

It is easily seen that the first term belongs to $C^{\infty}([(d'+b_3)/2,b_3])$ then the second term is essential to the asymptotic behavior of $s(\xi)$ when $\xi \uparrow b_3$. Expand $g(\rho)$ around $\rho = b_3$ as follows,

$$g(\rho) = g'(b_3)(\rho - b_3) + r_1(\rho)(\rho - b_3)^2 \equiv g_1(\rho) + g_2(\rho) .$$

By the simple calculation, we have,

$$(4.5)\quad \int_{d'}^{\xi} \left(2\int_{\sigma}^{\xi} g_1(\rho)\, d\rho \right)^{-1/2} d\sigma = \int_{d'}^{\xi} \left(2\int_{\sigma}^{\xi} g'(b_3)(\rho - b_3)\, d\rho \right)^{-1/2} d\sigma$$

$$= \frac{1}{\left(-g'(b_3)\right)^{1/2}} \log \frac{b_3 - d' + \left((d'-b_3)^2 - (\xi-b_3)^2 \right)^{1/2}}{b_3 - \xi}$$

$$(4.6)\quad \left| \frac{\int_\sigma^\xi g_2(\rho)\,d\rho}{\int_\sigma^\xi g_1(\rho)\,d\rho} \right| = \left| \frac{\int_\sigma^\xi r_1(\rho)(\rho-b_3)^2\,d\rho}{\frac{1}{2} g'(b_3)\,((\sigma-b_3)^2 - (\xi-b_3)^2)} \right|$$

$$\leqq \frac{2 r_*}{-3g'(b_3)} (2b_3 - \sigma - \xi) \leqq \frac{4 r_*}{-3g'(b_3)} (b_3 - d')$$

where $r_* = \max_{b_2 \leqq \rho \leqq b_3} | r_1(\rho)|$, $d' < (d' + b_3)/2 \leqq \xi < b_3$.

By the power series expansion, we have,

$$\left(1 + Y \right)^{-1/2} = \sum_{j=0}^{\infty} c_j Y^j \quad \text{for } |Y| < 1 \quad (\text{ radius of convergence })$$

where $c_j = (-\frac{1}{2})(-\frac{1}{2}-1)(-\frac{1}{2}-2)\cdots(-\frac{1}{2}-(j-1)) / j!$.

Then by using the above expansion with the estimate (4.6), we have the following expansion .

$$(4.7)\quad \left(2 \int_\sigma^\xi g(\rho)\,d\rho \right)^{-1/2} = \left(2 \int_\sigma^\xi g_1(\rho)\,d\rho \right)^{-1/2}$$

$$\times \sum_{j=0}^{\infty} c_j \left(\frac{\int_\sigma^\xi g_2(\rho)\,d\rho}{\int_\sigma^\xi g_1(\rho)\,d\rho} \right)^j \quad \text{for } d' < (d'+b_3)/2 \leqq \xi < b_3 .$$

For any $\varepsilon > 0$, take d' near to b_3 and fix it, so that we have by the estimate (4.6), the following estimate (4.8)

$$(4.8)\quad \left| 1 - \sum_{j=0}^{\infty} c_j \left(\frac{\int_\sigma^\xi g_2(\rho)\,d\rho}{\int_\sigma^\xi g_1(\rho)\,d\rho} \right)^j \right| \leqq \varepsilon$$

for $\xi \in [(d'+b_3)/2 ,b_3)$ and $\sigma \in [d',\xi]$.

Integrating (4.7) with σ from d' to ξ, we have

$$(4.9)\quad 1-\varepsilon \leqq \frac{\int_{d'}^{\xi}\left(2\int_{\sigma}^{\xi} g(\rho)\,d\rho\right)^{-1/2} d\sigma}{\int_{d'}^{\xi}\left(2\int_{\sigma}^{\xi} g_1(\rho)\,d\rho\right)^{-1/2} d\sigma} \leqq 1+\varepsilon$$

for $\xi \in [(d'+b_3)/2, b_3)$.

Using (4.4),(4.5) and (4.9), we have the following estimate (4.10)

$$(4.10)\quad 1-\varepsilon \leqq \varliminf_{\xi\uparrow b_3} \frac{s(\xi)}{(-g'(b_3))^{-1/2}\log\frac{1}{b_3-\xi}}$$

$$\leqq \varlimsup_{\xi\uparrow b_3} \frac{s(\xi)}{(-g'(b_3))^{-1/2}\log\frac{1}{b_3-\xi}} \leqq 1+\varepsilon$$

for any $\varepsilon > 0$.

Then we have $\lim_{\xi\uparrow b_3} \dfrac{s(\xi)}{(-g'(b_3))^{-1/2}\log\frac{1}{b_3-\xi}} = 1$.

Hereafter we take d' near b_3 and fix it so that $B(\xi,\sigma) \geqq 1/2$ for $\xi \in [(d'+b_3)/2, b_3)$ and $\sigma \in [d',\xi]$.

We put
$$B(\xi,\sigma) = \left(1 + \frac{\int_{\sigma}^{\xi} g_2(\rho)\,d\rho}{\int_{\sigma}^{\xi} g_1(\rho)\,d\rho}\right)^{-1/2}$$

$$F(\xi,\sigma) = \left(2\int_{\sigma}^{\xi} g_1(\rho)\,d\rho\right)^{1/2}$$

and then, the second term of (4.4) $= \int_{d'}^{\xi} \frac{1}{F(\xi,\sigma)} B(\xi,\sigma)\,d\sigma$ $= \int_0^{\xi-d'} \frac{1}{F(\xi,\xi-\eta)} B(\xi,\xi-\eta)\,d\eta \equiv I(\xi)$. Then we have the follwing.

$$(4.11)\quad \frac{d}{d\xi} I(\xi) = \frac{1}{F(\xi,d')} B(\xi,d') + \int_0^{\xi-d'} \frac{\partial}{\partial\xi}\frac{1}{F(\xi,\xi-\eta)}\cdot B(\xi,\xi-\eta)\,d\eta$$

$$+ \int_0^{\xi-d'} \frac{1}{F(\xi,\xi-\eta)} \frac{\partial}{\partial\xi} B(\xi,\xi-\eta)\,d\eta$$

$$\geqq \frac{1}{2} \frac{1}{F(\xi,d')} + \frac{1}{2} \int_0^{\xi-d'} \frac{\partial}{\partial\xi} \frac{1}{F(\xi,\xi-\eta)} \, d\eta$$

$$+ \int_0^{\xi-d'} \frac{1}{F(\xi,\xi-\eta)} \frac{\partial}{\partial\xi} B(\xi,\xi-\eta) \, d\eta$$

Here we have used that $\frac{\partial}{\partial\xi} \frac{1}{F(\xi,\xi-\eta)} \geqq 0$.

On the other hand, one can easily check that

$\frac{d}{d\xi} \left(\frac{\int_{\xi-\eta}^{\xi} g_2(\rho) \, d\rho}{\int_{\xi-\eta}^{\xi} g_1(\rho) \, d\rho} \right)$ is bounded for $(d'+b_3)/2 \leqq \xi < b_3$ and

$0 < \eta \leqq \xi - d'$ and also is $\frac{\partial}{\partial\xi} B(\xi,\xi-\eta)$, then we have some constant M such that $\left| \frac{\partial}{\partial\xi} B(\xi,\xi-\eta) \right| \leqq M$

for $\xi \in [(d'+b_3)/2, b_3)$ and $0 < \eta \leqq \xi - d'$. Therefore we have

$$(4.12) \quad \frac{\partial}{\partial\xi} I(\xi) \geqq \frac{1}{2} \frac{\partial}{\partial\xi} \int_0^{\xi-d'} \frac{1}{F(\xi,\xi-\eta)} \, d\eta - M \int_0^{\xi-d'} \frac{1}{F(\xi,\xi-\eta)} \, d\eta$$

$$= \frac{1}{2} \frac{\partial}{\partial\xi} \int_{d'}^{\xi} \frac{1}{F(\xi,\sigma)} \, d\sigma - M \int_{d'}^{\xi} \frac{1}{F(\xi,\sigma)} \, d\sigma$$

$$= \frac{1}{2} (-g'(b_3))^{-1/2} \left(\frac{1}{b_3 - \xi} + \frac{(b_3-\xi)\cdot((d'-b_3)^2-(b_3-\xi)^2)^{-1/2}}{b_3-d'+ ((d'-b_3)^2-(b_3-\xi)^2)^{1/2}} \right)$$

$$- \frac{M}{2} (-g'(b_3))^{-1/2} \log \left(\frac{b_3 - d' + ((d'-b_3)^2-(b_3-\xi)^2)^{1/2}}{b_3 - \xi} \right)$$

Then we conclude by (4.4) that

$$\lim_{\xi \uparrow b_3} \frac{d}{d\xi} s(\xi) = \infty \quad .$$

In the case that ξ approaches d, we can deal as the same procedure as above by the following decomposition.

$$s(\xi) = \int_{b_1}^{e_1} \left(2 \int_{\sigma}^{b_1} g(\rho)\, d\rho + 2 \int_{d}^{\xi} g(\rho) d\rho \right)^{-1/2} d\sigma$$

$$+ \int_{e_1}^{d} \left(2 \int_{\sigma}^{b_1} g(\rho) d\rho + 2 \int_{d}^{\xi} g(\rho) d\rho \right)^{-1/2} d\sigma + \int_{d}^{\xi} \left(2 \int_{\sigma}^{\xi} g(\rho) d\rho \right)^{-1/2} d\sigma$$

Then we omit the proof of this case and we complete the proof of Lemma 4.1.

By Lemma 4.1, the equation (4.3) which is rewritten as follows

$$s(\xi) = \vartheta^{1/2}$$

has exactly two solutions $\xi_1 < \xi_2$ in the interval (d, b_3) by taking the parameter $\vartheta > 0$ adequately large and at the same time

(4.13) $\quad s'(\xi_1) < 0$ and $s'(\xi_2) > 0$ hold.

We fix this ϑ and also $f(\xi) = \vartheta\, g(\xi)$. Therefore corresponding to ξ_1 and ξ_2 , we obtain two solutions $V^{(1)}$ and $V^{(2)}$ of (4.2) for $f(\xi) = \vartheta\, g(\xi)$ determined above and one can easily check that

$b_1 < V^{(1)}(z) < V^{(2)}(z) < b_3$ in $-1 < z < 1$.

By the aid of the almost same method as in K.Maginu [14], we can investigate the signature of the linearized first eigenvalues $\lambda_{V^{(0)}}$ $\lambda_{V^{(1)}}$ and $\lambda_{V^{(2)}}$ (See (3.5) for definition) by (4.13) and we conclude that $\lambda_{V^{(0)}} > 0$, $\lambda_{V^{(1)}} < 0$ and $\lambda_{V^{(2)}} > 0$ where $V^{(0)}(z) \equiv b_1$. We complete the proof of Proposition 2.

Hereafter first we will construct a family of solutions of (3.1) $\{ v_\zeta^{(2)} \}_{0<\zeta<\zeta_*}$ such that $v_\zeta^{(2)}$ behaves like $V^{(2)}$ in $Q(\zeta)$ and takes values near b_1 in $D_1 \cup D_2$ and moreover $\mu_1(v_\zeta^{(2)}) > 0$ holds for small $\zeta > 0$. Here we denoted by $\mu_1(v_\zeta^{(2)})$ the first eigenvalue of the eigenvalue problem (3.5) for the family $\{ v_\zeta^{(2)} \}_{0<\zeta<\zeta_*}$.

We set the function $\Psi_*(x_1) = \Phi_{V^{(2)}}(x_1) + \rho_*$ where $\rho_* > 0$ is a small constant such that $\lambda_{V^{(2)}} \Phi_{V^{(2)}}(x_1) - \rho_* f'(V^{(2)}(x_1)) > 0$ for any $x_1 \in [-1,1]$. (Recall that $V^{(2)}(-1) = V^{(2)}(1) = b_1$ and $f'(b_1) < 0$.)

Now we define a function $W_\zeta(x)$ which is defined in $\Omega(\zeta)$ as follows

$$W_\zeta(x_1,x') \equiv$$

$$\begin{cases} b_1 + \dfrac{\zeta}{2}\left(\dfrac{dV^{(2)}}{dx_1}(1-2\zeta) - \delta_*(\zeta)\,\dfrac{d\Psi_*}{dx_1}(1-2\zeta)\right) \\ \qquad \text{for } x \in D_1 \cup D_2 \cup (R_1(\zeta)\cap\{x_1 \geqq 1-\zeta\}) \cup (R_2(\zeta)\cap\{x_1 \leqq -1+\zeta\}) \\ b_1 - \dfrac{1}{2\zeta}\left(\dfrac{dV^{(2)}}{dx_1}(1-2\zeta) - \delta_*(\zeta)\,\dfrac{d\Psi_*}{dx_1}(1-2\zeta)\right)\cdot(x_1-1+2\zeta)\cdot(x_1-1) \\ \qquad \text{for } x \in R_1(\zeta) \cap \{ 1-2\zeta \leqq x_1 < 1-\zeta \} \\ V^{(2)}(x_1) - \delta_*(\zeta)\,\Psi_*(x_1) \qquad \text{for } x \in \Gamma(\zeta) \\ W_\zeta(-x_1,x') \qquad \text{for } x \in R_2(\zeta) \cap \{ -1+\zeta < x_1 \leqq -1+2\zeta \} \end{cases}$$

where we have put $\delta_*(\zeta) = (V^{(2)}(1-2\zeta) - b_1)/ \Psi_*(1-2\zeta)$.

It is easily seen that $\delta_*(\zeta) > 0$ and $\lim_{\zeta\to 0} \delta_*(\zeta) = 0$.

Lemma 4.2. $W_\zeta \in C^1(\overline{\Omega(\zeta)})$ and we have, for small $\zeta > 0$,

$$\Delta W_\zeta + f(W_\zeta) > 0$$

in $\Omega(\zeta) - R_1(\zeta)\cap\{x_1=1-2\zeta \text{ or } 1-\zeta\} - R_2(\zeta)\cap\{x_1= -1+2\zeta \text{ or } -1+\zeta\}$

$$\frac{\partial W_\zeta}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega(\zeta) .$$

(Proof of Lemma 4.2) One can check $W_\zeta \in C^1(\overline{\Omega(\zeta)})$ by a simple calculation. In $D_1 \cup D_2 \cup (R_1(\zeta)\cap\{x_1 > 1-\zeta\}) \cup (R_2(\zeta)\cap\{x_1 < -1+\zeta\})$ $\Delta W_\zeta = 0$ and $W_\zeta(x) \leqq b_1$ for small $\zeta > 0$ from $\frac{dV^{(2)}}{dx_1}(1) < 0$ and $\lim_{\zeta\to 0} \delta_*(\zeta) = 0$. Then by (IV-1), we obtain the inequality.

In $R_1(\zeta)\cup\{1-2\zeta < x_1< 1-\zeta\}$, $\Delta W_\zeta = \frac{-1}{2\zeta}(\frac{dV^{(2)}}{dx_1}(1-2\zeta) - \delta_*(\zeta)\frac{d\Psi_*}{dx_1}(1-2\zeta))$ > 0 and $W_\zeta(x) \leqq b_1$ for small $\zeta > 0$. Therefore we have the inequality by the same way above also in $R_2(\zeta)\cup \{-1+\zeta<x_1< -1+2\zeta\}$.

In $\Gamma(\zeta)$, we calculate as follows,

$$\Delta W_\zeta + f(W_\zeta) = \frac{\partial^2}{\partial {x_1}^2}(V^{(2)}(x_1) - \delta_*(\zeta)\, \Psi_*(x_1))$$

$$+ f(V^{(2)}) - \delta_*(\zeta)\Psi_*\, f'(V^{(2)}) + \delta_*(\zeta)^2\, {\Psi_*}^2\, \Xi_\zeta$$

(where $\Xi_\zeta(x_1) = \int_0^1 (1-\tau)\, f''(V^{(2)}(x_1)-\tau\delta_*(\zeta)\Psi_*(x_1))\, d\tau$)

$$= \frac{d^2V^{(2)}}{d{x_1}^2} + f(V^{(2)}) - \delta_*(\zeta)(\frac{d^2\Psi_*}{d{x_1}^2} + f'(V^{(2)})) + \delta_*(\zeta)^2\, {\Psi_*}^2\, \Xi_\zeta$$

$$= \delta_*(\zeta) \left\{(\lambda_{V^{(2)}}\Phi_{V^{(2)}}(x_1) - \rho_*\, f'(V^{(2)}(x_1))) + \delta_*(\zeta)\, \Xi_\zeta(x_1)\right\} > 0$$

holds in $\Gamma(\zeta)$ for sufficiently small $\zeta > 0$ by $\lim_{\zeta\to 0} \delta_*(\zeta) = 0$. Therefore we have completed the proof of Lemma 4.2.

By Lemma 4.2, W_ζ is a " weak lower solution " in the sense of D.H.Sattinger [20] for small $\zeta > 0$ and we have the following comparison property by the argument used in Section 2 and the comparison theorem.
The set $E_1(\zeta) = \{ \psi \in C^1(\overline{\Omega(\zeta)}) \cap C^2(\Omega(\zeta)) \mid \psi(x) \geqq W_\zeta(x) \text{ in } \Omega(\zeta) \}$ is a positively invariant set under the flow defined by the evolution equation (1.2) for $\Omega(\zeta)$ and f in this section and also is the set $E_*(\zeta) = E(\delta,\zeta) \cap E_1(\zeta)$ ($\delta \in [\delta(\zeta), 2\delta(\zeta)]$) for sufficiently small $\zeta > 0$, where $E(\delta,\zeta)$ and $\delta(\zeta)$ are the ones constructed in Section 2. Therefore applying again Theorem 4.2 in [15], we have at least one stable equilibrium solution in $E_*(\zeta)$ for small $\zeta > 0$. Moreover we have the following.

Lemma 4.3. For small $\zeta > 0$, there exists exactly one solution $v_\zeta^{(2)}$ of (3.1) in $E_*(\zeta)$ and the linealized first eigenvalue $\mu_1(v_\zeta^{(2)})$ is bounded from below by a positive constant.

Here we put $v_\zeta^{(0)}(x) \equiv b_1$ in $\Omega(\zeta)$ which is a stable solution.

One can easily check that $v_\zeta^{(0)}(x) < v_\zeta^{(2)}(x)$ in $\Omega(\zeta)$ by the Strong Maximum Principle and by Theorem 4.4 in Matano [15], we obtain another solution $v_\zeta^{(1)}$ between the above two solutions.

($v_\zeta^{(0)}(x) < v_\zeta^{(1)}(x) < v_\zeta^{(2)}(x)$ for $x \in \Omega(\zeta)$)

$\varliminf_{\zeta\to 0} \mu_1(v_\zeta^{(2)}) > 0$ implies that $v_\zeta^{(2)}$ is locally unique for small

$\zeta > 0$ and then $v_\zeta^{(1)}$ must be asymptotically near to $V^{(1)}$ on $Q(\zeta)$ by Theorem 3 and Proposition 2. Therefore we have obtained the following result by Theorem 3.

Theorem 4. There exist three distinct solutions (for small ζ) $v_\zeta^{(0)} < v_\zeta^{(1)} < v_\zeta^{(2)}$ of (3.1) where $f = \vartheta g$ ($\vartheta \geqq \vartheta_0$) and the solutions satisfy the following asymptotic conditions.

$$\lim_{\zeta\to 0} \sup_{x \in D_1 \cup D_2} | v_\zeta^{(i)}(x) - b_1 | = 0 \quad (i = 0, 1, 2)$$

$$\lim_{\zeta\to 0} \sup_{x \in Q(\zeta)} | v_\zeta^{(i)}(x_1,x') - V^{(i)}(x_1) | = 0 \quad (i = 1, 2)$$

$$\underline{\lim}_{\zeta\to 0} \mu_1(v_\zeta^{(0)}) > 0 \quad , \quad \overline{\lim}_{\zeta\to 0} \mu_1(v_\zeta^{(1)}) < 0 \quad , \quad \underline{\lim}_{\zeta\to 0} \mu_1(v_\zeta^{(2)}) > 0$$

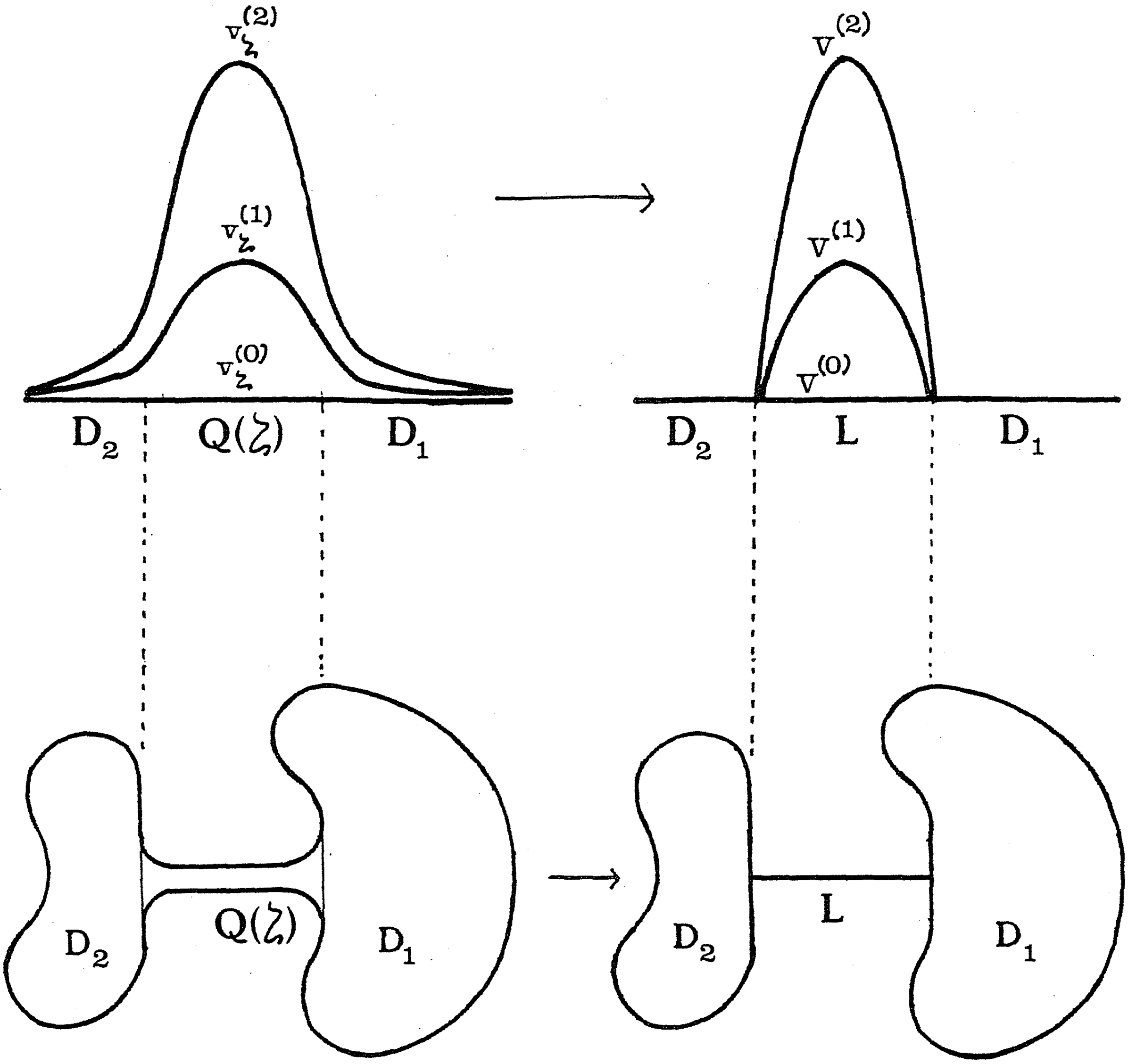
where we denoted by $\mu_1(v_\zeta^{(1)})$ the first eigenvalue of the eigenvalue problem (3.5) for the family $\{ v_\zeta^{(1)}\}_{0<\zeta<\zeta_*}$.

§5 Concluding Remarks.

In Section 4, by choosing an appropriate nonlinear term f in the situation of Section 3 plus a condition $a_1 = a_2 = b_1$, we have constructed three distinct solutions $v_\zeta^{(0)} < v_\zeta^{(1)} < v_\zeta^{(2)}$ of (3.1) for small $\zeta > 0$ such that $v_\zeta^{(i)}$ takes values near b_1 in $D_1 \cup D_2$ and behaves like $V^{(i)}$ in $Q(\zeta)$ for small $\zeta > 0$ ($i = 0,1,2$) and $\mu_1(v_\zeta^{(0)}) \geqq c$, $\mu_1(v_\zeta^{(1)}) \leqq -c$ and $\mu_1(v_\zeta^{(2)}) \geqq c$ hold for small $\zeta > 0$ and a positive constant c which is independent of ζ . All $v_\zeta^{(i)}$ ($i = 0,1,2$) take almost same values near b_1 in $D_1 \cup D_2$ ($f'(b_1) < 0$) but $v_\zeta^{(1)}$ is unstable while $v_\zeta^{(0)}$ and $v_\zeta^{(2)}$ are stable for small $\zeta > 0$. This phenomenon is owing to the fact that the asymptotic behavior of $v_\zeta^{(1)}$ on $Q(\zeta)$ corresponds to $V^{(1)}$ which is an unstable solution of the ordinary differential equation (3.4) on L while those of $v_\zeta^{(0)}$ and $v_\zeta^{(2)}$ correspond to the stable ones $V^{(0)}$ and $V^{(2)}$. See Figure 6.

From these results, we see that the dependence of the stability of the solution, upon the moving part $Q(\zeta)$, does not vanish when $\zeta \to 0$. Moreover the behavior of the solution on $Q(\zeta)$ plays an important role to determine the stability and on the other hand, it is described as the solution of the ordinary differential equation (3.4) on L in this case.

Therefore we conclude that it natural to regard $\Omega(\zeta)$ as a perturbation from $\Omega_* = D_1 \cup D_2 \cup L$ (See Figure 4 in Section 1) if we cosider the behavior of the structure of the solutions of (3.1).



$v_\zeta^{(2)}$
$v_\zeta^{(1)}$
$v_\zeta^{(0)}$
D_2
$Q(\zeta)$
D_1
$V^{(2)}$
$V^{(1)}$
$V^{(0)}$
D_2
L
D_1
D_2
$Q(\zeta)$
D_1
D_2
L
D_1

Figure 6

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