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Equality of Higher-Order Diffusion Coefficients between Component and Composite Electron Swarms in Gas

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When an electron swarm in gas is composed of component electron swarms individually in drift equilibrium, the higher-order diffusion coefficients (HDCs) of the composite electron swarms are equal to those of the component electron swarms. We have derived this equality theoretically and have examined it by numerical simulation. The HDCs are the time derivatives of higher-order cumulants, which are quantities characterizing the shape of a spatial electron distribution. This fact seemingly indicates the dependence of the HDCs of the composite electron swarms on the arrangement of the component electron swarms. However, the equality holds irrespective of the relative positions and electron populations of the component electron swarms. We have given a consistent explanation to these facts. We have also discussed electron swarm development from dispersed or multiple electron sources that may appear in practical experiments.

KEYWORDS: electrical discharge, plasma, simulation, transport parameter, electron swarm, diffusion coefficient, drift equilibrium, cumulant, moment

1. Introduction

The electron diffusion coefficient in gas is an essential quantity for describing the electron transport in electrical discharges and plasmas. This quantity appears as the coefficient of the second-order gradient of the electron concentration in an electron continuity equation, with which numerical simulations are widely performed for various applications, such as processing plasmas and light sources.

A standard one-dimensional electron continuity equation is written as

$$\frac{\partial}{\partial t} f(x, t) = \left(R - W \frac{\partial}{\partial x} + D \frac{\partial^2}{\partial x^2} \right) f(x, t), \quad (1)$$

where $f(x, t)$ is the electron distribution at position x and time t , R is the effective ionization frequency, W is the drift velocity, and D is the diffusion coefficient. This equation is actually a truncated form of the following infinite series involving higher-order diffusion coefficients (HDCs):

$$\begin{aligned} \frac{\partial}{\partial t} f(x, t) = & \left(R - W \frac{\partial}{\partial x} + D_2 \frac{\partial^2}{\partial x^2} - D_3 \frac{\partial^3}{\partial x^3} \right. \\ & \left. + D_4 \frac{\partial^4}{\partial x^4} - \dots \right) f(x, t), \quad (2) \end{aligned}$$

where D_n is the n th-order diffusion coefficient ($D = D_2$).

A study on the HDCs of electron swarms is an approach to the formalization of the electron transport phenomenon in gas. Tagashira *et al.*¹⁾ presented a hierarchy of the HDCs, formulating D_n with the spatial moments of f . Kumar *et al.*²⁾ discussed the difference between physical models based on eqs. (1) and (2), and pointed out the relationship between D_3 and D_4 and the skewness and kurtosis of the shape of f . Blevin and Fletcher³⁾ obtained an expression of D_3 in an expansion of the electron energy distribution function in a hydrodynamic regime. Moreover, for ion diffusion in gas, Whealton and Mason⁴⁾ and Whealton⁵⁾ evaluated the influences of D_3 and D_4 , which appear as a deviation of the ion distribution from

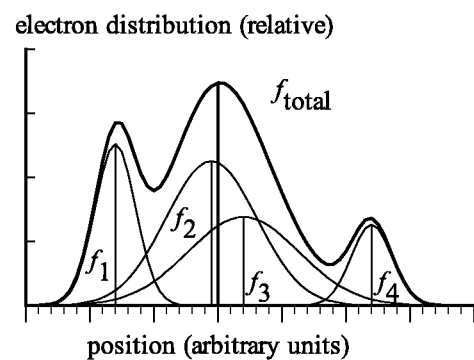


Fig. 1. Schematic of spatial electron distributions of component and composite electron swarms.

the Gaussian solution of the diffusion equation based on Fick's law.

$n!D_n$ is the time derivative of the n th-order cumulant κ_n of f .⁶⁾ The cumulant is a statistical quantity representing the deviation of distribution shape from a Gaussian distribution. This fact indicates a mutual dependence between D_n and the shape of f . Here, a question arises. Let us consider a composite electron swarm consisting of component electron swarms. In Fig. 1, f_k ($k = 1, 2, 3, \dots$) is the spatial electron distribution of the k th component electron swarm. The spatial arrangement of f_k is arbitrary and the shape of f_{total} ($= \sum_k f_k$) of the composite electron swarm can deviate from a Gaussian as much as one intends. The cumulant κ_n of f_{total} is dependent on the arrangement of f_k . When the component electron swarms are individually in drift equilibrium under a common uniform electric field, i.e., when the D_n value is common among all the component electron swarms (let the value be $D_{n,\text{eq}}$), is $D_n = (d\kappa_n/dt)/n!$ of the composite electron swarm always equal to $D_{n,\text{eq}}$?

Such a composite electron swarm may appear, for example, in a time-of-flight experiment using electrical shutters or UV irradiation for the release of initial electrons. When it has a temporal or spatial distribution, the

electron swarm can be regarded as a composite electron swarm developing from multiple electron sources. The typical gate periods 1–10 ns^{7–11}) and shutter grid spacing 1 mm¹²) can be comparable to the relaxation time and distance of electron swarms under certain conditions, although they are practically neglected by setting the observation time and distance relevantly long.

In general, the equality of transport parameters including the HDCs is expected from a thought that the electron transport parameters in drift equilibrium should be independent of initial conditions, such as the spatial arrangement of the electron sources. Therefore, the confirmation of the equality is meaningful not only for the formality of the electron swarm theory but also for the appropriate treatment of transport parameters in the practical simulations of plasma applications. In this paper, we analytically derive the equality of D_n between the component and composite electron swarms, and examine the result via numerical calculations of D_n . We give a consistent explanation to a fact that $D_n = (d\kappa_n/dt)/n!$ is independent of the arrangement of f_k , while κ_n is dependent on it.

2. Theory

We first describe the definitions of the quantities related to the HDCs and assumptions in the present analysis. Subsequently, we derive the equality of the HDCs by introducing the moment- and cumulant-generating functions to deal with all the HDCs collectively. Derivations for the zeroth-, first- and second-order parameters are presented in Appendix.

2.1 Definitions of quantities and assumptions

We consider component and composite electron swarms in drift equilibrium in gas. The electrons drift under the uniform electric field $-E$ applied in the x direction, satisfying the balance between the energy gain from the electric field and the energy loss by collisions with gas molecules. The composite electron swarm is the sum of the component electron swarms. The quantities of the k th component electron swarm and the composite electron swarm are respectively indicated by subscripts “ k ” and “total” hereafter. The assumption that each component electron swarm is in drift equilibrium is interpreted as that the transport coefficients are constant and common among the component electron swarms. We denote such common values by symbols with subscript “eq”.

From f_k , we define the n th-order moments $m_{n,k}^L$ in laboratory systems and $m_{n,k}^C$ in centroid systems:

$$m_{n,k}^L(t) = \int_{-\infty}^{\infty} x^n f_k(x, t) dx, \quad (3)$$

$$m_{n,k}^C(t) = \int_{-\infty}^{\infty} [x - g_k(t)]^n f_k(x, t) dx, \quad (4)$$

$$g_k(t) = \frac{m_{1,k}^L(t)}{N_{e,k}(t)}, \quad (5)$$

where g_k is the centroid of f_k , and the zeroth-order moments $m_{0,k}^C$ and $m_{0,k}^L$ are both identical to the electron population $N_{e,k}$.

The effective ionization frequency R_k , centroid drift velocity W_k and n th-order longitudinal diffusion coefficients $D_{n,k}$ ($n \geq 2$) are defined as^{1,6})

$$R_k = \frac{\frac{d}{dt} N_{e,k}(t)}{N_{e,k}(t)}, \quad (6)$$

$$W_k = \frac{d}{dt} \frac{m_{1,k}^L(t)}{N_{e,k}(t)} = \frac{d}{dt} g_k(t), \quad (7)$$

$$D_{n,k} = \frac{1}{n!} \frac{d}{dt} \frac{m_{n,k}^C(t)}{N_{e,k}(t)} \quad (\text{for } n = 2, 3), \quad (8)$$

$$D_{n,k} = \frac{1}{n!} \frac{d}{dt} \frac{m_{n,k}^C(t)}{N_{e,k}(t)} - \sum_{i=2}^{n-2} \frac{D_{i,k}}{(n-i)!} \frac{m_{n-i,k}^C(t)}{N_{e,k}(t)} \quad (\text{for } n \geq 4). \quad (9)$$

From the assumption on equilibrium, they satisfy $R_k = R_{\text{eq}}$, $W_k = W_{\text{eq}}$ and $D_{n,k} = D_{n,\text{eq}}$ for all k 's, and $N_{e,k}$ grows exponentially as

$$N_{e,k}(t) = N_{e,k}(0) \exp(R_k t) = N_{e,k}(0) \exp(R_{\text{eq}} t). \quad (10)$$

The quantities of the composite electron swarm are given as

$$f_{\text{total}}(x, t) = \sum_k f_k(x, t), \quad (11)$$

$$\begin{aligned} m_{n,\text{total}}^L(t) &= \int_{-\infty}^{\infty} x^n f_{\text{total}}(x, t) dx \\ &= \sum_k m_{n,k}^L(t), \end{aligned} \quad (12)$$

$$\begin{aligned} m_{n,\text{total}}^C(t) &= \int_{-\infty}^{\infty} [x - g_{\text{total}}(t)]^n f_{\text{total}}(x, t) dx \\ &\neq \sum_k m_{n,k}^C(t), \end{aligned} \quad (13)$$

$$\begin{aligned} g_{\text{total}}(t) &= \frac{m_{1,\text{total}}^L(t)}{N_{e,\text{total}}(t)} \\ &= \frac{\sum_k N_{e,k}(t) g_k(t)}{\sum_k N_{e,k}(t)}. \end{aligned} \quad (14)$$

With these quantities, R_{total} , W_{total} and $D_{n,\text{total}}$ are defined in the same manner as eqs. (6)–(9) for R_k , W_k and $D_{n,k}$.

2.2 Relations between moments, cumulants and diffusion coefficients

The moment-generating functions M_k^L and M_k^C are defined as the series of the moments $m_{n,k}^L$ and $m_{n,k}^C$, respectively, with the parameter ϕ as

$$M_k^L(\phi, t) = \sum_{n=0}^{\infty} \frac{m_{n,k}^L(t)}{n!} \phi^n, \quad (15)$$

$$M_k^C(\phi, t) = \sum_{n=0}^{\infty} \frac{m_{n,k}^C(t)}{n!} \phi^n. \quad (16)$$

With the relation $\exp(x\phi) = 1 + x\phi + (x^2/2!) \phi^2 + (x^3/3!) \phi^3 + \dots$, M_k^L and M_k^C can also be written as

$$M_k^L(\phi, t) = \int_{-\infty}^{\infty} \exp(x\phi) f_k(x, t) dx, \quad (17)$$

$$\begin{aligned} M_k^C(\phi, t) &= \int_{-\infty}^{\infty} \exp\{[x - g_k(t)]\phi\} f_k(x, t) dx \\ &= \exp[-g_k(t)\phi] M_k^L(\phi, t). \end{aligned} \quad (18)$$

Moreover, for the composite electron swarm, M_{total}^L and M_{total}^C are defined in the same manner as eqs. (15)–(18), using $m_{n,\text{total}}^L$, $m_{n,\text{total}}^C$, f_{total} and g_{total} instead of $m_{n,k}^L$, $m_{n,k}^C$, f_k and g_k , respectively.

On the other hand, the cumulant generating function K and cumulant κ_n are formally defined using the generating function of normalized centered moments as

$$\begin{aligned} K_k(\phi, t) &= \sum_{n=1}^{\infty} \frac{\kappa_{n,k}(t)}{n!} \phi^n \\ &= \ln \frac{M_k^C(\phi, t)}{N_{e,k}(t)}, \end{aligned} \quad (19)$$

$$\begin{aligned} K_{\text{total}}(\phi, t) &= \sum_{n=1}^{\infty} \frac{\kappa_{n,\text{total}}(t)}{n!} \phi^n \\ &= \ln \frac{M_{\text{total}}^C(\phi, t)}{N_{e,\text{total}}(t)}, \end{aligned} \quad (20)$$

where $\kappa_{n,k}$ and $\kappa_{n,\text{total}}$ are n th-order cumulants.^{13–15} The cumulants can be represented with the normalized centered moments $\mu_n = m_n^C/N_e$ and related to statistical quantities:

$$\begin{aligned} \kappa_1 &= \mu_1 = 0 \quad (= \text{average}), \\ \kappa_2 &= \mu_2 \quad (= \text{variance} = \sigma^2), \\ \kappa_3 &= \mu_3 \quad (\text{cf. skewness} = \kappa_3/\sigma^3), \\ \kappa_4 &= \mu_4 - 3(\mu_2)^2 \quad (\text{cf. kurtosis} = \kappa_4/\sigma^4), \\ \kappa_5 &= \mu_5 - 10\mu_3\mu_2, \\ \kappa_6 &= \mu_6 - 15\mu_4\mu_2 - 10(\mu_3)^2 + 30(\mu_2)^3, \\ &\vdots, \end{aligned} \quad (21)$$

where σ is the standard deviation. The cumulant is a quantity characterizing the shape of a distribution. When a distribution is a Gaussian, $\kappa_n = 0$ for $n \geq 3$.^{13,14} Therefore, the higher-order cumulants are regarded as quantities representing the deviation of the distribution from a Gaussian.¹³ Note here that $\kappa_{n,\text{total}}$ may be non-zero even when $\kappa_{n,k} = 0$ for $n \geq 3$ for all k 's. Using $\kappa_{n,k}$ and $\kappa_{n,\text{total}}$, $D_{n,k}$ and $D_{n,\text{total}}$ ($n \geq 2$) are given as⁶)

$$D_{n,k} = \frac{1}{n!} \frac{d}{dt} \kappa_{n,k}(t), \quad (22)$$

$$D_{n,\text{total}} = \frac{1}{n!} \frac{d}{dt} \kappa_{n,\text{total}}(t). \quad (23)$$

Figure 2 shows the relations between the moments, cumulants and diffusion coefficients of the component and composite electron swarms. What to be derived is $D_{n,\text{total}} = D_{n,\text{eq}}$ for $n \geq 3$ when $D_{n,k} = D_{n,\text{eq}}$ for all k 's.

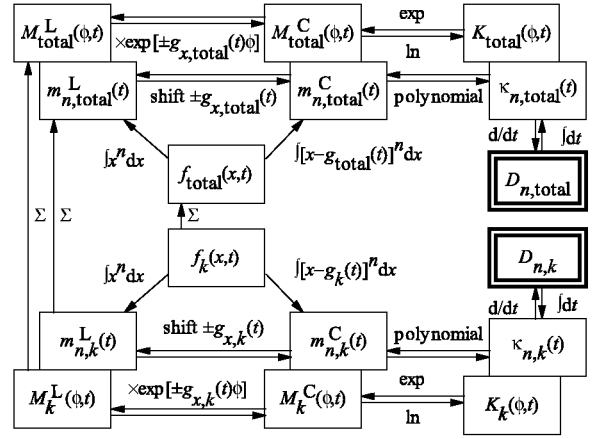


Fig. 2. Relations between quantities concerning the n th-order diffusion coefficients D_n : moments in laboratory and centroid systems, m_n^L and m_n^C ; cumulants, κ_n ; and their generating functions, M^L , M^C and K . Subscripts “ k ” and “total” indicate the k th component and composite electron swarms.

The relation between $D_{n,\text{total}}$ and $D_{n,k}$ is too indirect to realize their equality intuitively. We require several steps of substitutions and transformations of equations, as shown in the next subsection.

2.3 Derivation of higher-order diffusion coefficient equality

From eq. (22) and the assumption on equilibrium, we may let

$$\kappa_{n,k}(t) = n! D_{n,\text{eq}} t + \kappa_{n,k}(0), \quad (24)$$

where $\kappa_{n,k}(0)$ is a time-invariant constant. The objective is to show

$$\kappa_{n,\text{total}}(t) = n! D_{n,\text{eq}} t + \text{constant}. \quad (25)$$

We relate K_{total} and K_k through M_{total}^C , M_{total}^L , M_k^L and M_k^C in this sequence. Note that mediation by M_{total}^L and M_k^L is necessary because $M_{\text{total}}^L = \sum_k M_k^L$ but $M_{\text{total}}^C \neq \sum_k M_k^C$; $m_{n,k}^C$ is defined around g_k different for each k .

Equation (20) is rewritten as

$$\begin{aligned} M_{\text{total}}^C(\phi, t) &= N_{e,\text{total}}(0) \exp(R_{\text{eq}} t) \exp[K_{\text{total}}(\phi, t)]. \end{aligned} \quad (26)$$

Links from M_{total}^C to K_k are

$$M_{\text{total}}^C(\phi, t) = \exp[-g_{\text{total}}(t)\phi] M_{\text{total}}^L(\phi, t), \quad (27)$$

$$M_{\text{total}}^L(\phi, t) = \sum_k M_k^L(\phi, t), \quad (28)$$

$$M_k^L(\phi, t) = \exp[g_k(t)\phi] M_k^C(\phi, t), \quad (29)$$

$$M_k^C(\phi, t) = N_{e,k}(t) \exp[K_k(\phi, t)]. \quad (30)$$

Thus, we obtain

$$\begin{aligned} M_{\text{total}}^C(\phi, t) &= \sum_k \left(N_{e,k}(0) \exp(R_{\text{eq}} t) \right. \\ &\quad \left. \times \exp\{[g_k(t) - g_{\text{total}}(t)]\phi + K_k(\phi, t)\} \right). \end{aligned} \quad (31)$$

Here, the factor $g_k(t) - g_{\text{total}}(t) (\equiv d_k)$ in this equation is time-invariant because the relative velocity between the centroids is zero in equilibrium. Moreover, note that K_k has no first-order term because $\kappa_{1,k} = 0$. By equating the two expressions of M_{total}^C in eqs. (26) and (31), we obtain a description of K_{total} with $\kappa_{n,k}$:

$$\begin{aligned} \exp[K_{\text{total}}(\phi, t)] &= \sum_k \left\{ \frac{N_{e,k}(0)}{N_{e,\text{total}}(0)} \right. \\ &\quad \times \exp \left[d_k \phi + \frac{\kappa_{2,k}(t)}{2!} \phi^2 + \frac{\kappa_{3,k}(t)}{3!} \phi^3 + \dots \right] \left. \right\}. \end{aligned} \quad (32)$$

We transform the summation in eq. (32) into a single exponential function to show the dependence of $\kappa_{n,\text{total}}$ on t explicitly. Using eq. (24), the time-variant factor $\exp(D_{2,\text{eq}}t\phi^2 + D_{3,\text{eq}}t\phi^3 + \dots)$ commonly involved in the exponential functions in the right-hand side of eq. (32) is separated as

$$\begin{aligned} \exp[K_{\text{total}}(\phi, t)] &= \exp(D_{2,\text{eq}}t\phi^2 + D_{3,\text{eq}}t\phi^3 + \dots) \\ &\quad \times \sum_k \left\{ \frac{N_{e,k}(0)}{N_{e,\text{total}}(0)} \right. \\ &\quad \times \exp \left[d_k \phi + \frac{\kappa_{2,k}(0)}{2!} \phi^2 + \frac{\kappa_{3,k}(0)}{3!} \phi^3 + \dots \right] \left. \right\}. \end{aligned} \quad (33)$$

The time-invariant factor of eq. (33) is further rewritten by introducing A_k and B into the following forms:

$$\begin{aligned} \exp[K_{\text{total}}(\phi, t)] &= \exp(D_{2,\text{eq}}t\phi^2 + D_{3,\text{eq}}t\phi^3 + \dots) \\ &\quad \times \sum_k \frac{N_{e,k}(0)}{N_{e,\text{total}}(0)} \exp[A_k(\phi)] \quad (34) \\ &= \exp(D_{2,\text{eq}}t\phi^2 + D_{3,\text{eq}}t\phi^3 + \dots) B(\phi). \quad (35) \end{aligned}$$

A_k is a series of ϕ^n with time-invariant coefficients. Because an expansion of $\exp[A_k(\phi)]$ gives a series of $(A_k)^i$ ($i = 0, 1, 2, \dots$) as

$$\begin{aligned} \exp[A_k(\phi)] &= 1 + A_k(\phi) + \frac{1}{2!}[A_k(\phi)]^2 \\ &\quad + \frac{1}{3!}[A_k(\phi)]^3 + \dots, \quad (36) \end{aligned}$$

B is also a series of ϕ^n with time-invariant coefficients. Here, note that the terms of A_k are in the first order or higher with respect to ϕ . Therefore, the constant term free from ϕ in eq. (36) is unity, and the constant term in B is also unity since $\sum_k N_{e,k}(0)/N_{e,\text{total}}(0) = 1$.

At this time, using the formula $\log s = (s-1) - (s-1)^2/2! + (s-1)^3/3! - \dots$ for $0 < s < 2$, we can rewrite B as

$$\begin{aligned} B(\phi) &= \exp[\ln B(\phi)] \\ &= \exp \left\{ [B(\phi) - 1] - \frac{1}{2!}[B(\phi) - 1]^2 \right. \\ &\quad \left. + \frac{1}{3!}[B(\phi) - 1]^3 - \dots \right\}. \quad (37) \end{aligned}$$

Here, we may set the parameter ϕ to be as small as necessary to satisfy $0 < B(\phi) < 2$ by letting $B(\phi) \rightarrow 1$. The exponent in eq. (37) is again a series of ϕ^n , which does not include a constant term free from ϕ . Introducing the time-invariant coefficients C_n , we can rewrite $\exp[K_{\text{total}}(\phi, t)]$ as

$$\begin{aligned} \exp[K_{\text{total}}(\phi, t)] &= \exp(D_{2,\text{eq}}t\phi^2 + D_{3,\text{eq}}t\phi^3 + \dots) \\ &\quad \times \exp \left(C_1\phi + \frac{C_2}{2!}\phi^2 + \frac{C_3}{3!}\phi^3 + \dots \right). \quad (38) \end{aligned}$$

Therefore,

$$\begin{aligned} \exp[K_{\text{total}}(\phi, t)] &= \exp \left[\kappa_{1,\text{total}}(t)\phi + \frac{\kappa_{2,\text{total}}(t)}{2!}\phi^2 \right. \\ &\quad \left. + \frac{\kappa_{3,\text{total}}(t)}{3!}\phi^3 + \dots \right] \quad (39) \end{aligned}$$

$$\begin{aligned} &= \exp \left[C_1\phi + \left(D_{2,\text{eq}}t + \frac{C_2}{2!} \right) \phi^2 \right. \\ &\quad \left. + \left(D_{3,\text{eq}}t + \frac{C_3}{3!} \right) \phi^3 + \dots \right]. \quad (40) \end{aligned}$$

By comparing the coefficients of ϕ^n in eqs. (39) and (40), we obtain eq. (25).

We have shown that the HDCs of the composite electron swarm are equal to those of the component electron swarms. This result has been derived without conditions on the relative position and relative electron population between the component electron swarms. Moreover, the equality of the HDCs holds when the release of the initial electrons is asynchronous among the component electron swarms. For example, the time lag of the initial electron release would appear as the shortened flight distance of the component electron swarms. Therefore, the difference in the release timing is considered as the variation in the spatial arrangement of the component electron swarms.

3. Numerical Simulations

3.1 Simulation methods

To verify the theoretical result on the equality of the HDCs, we have performed D_n calculations up to $n = 6$ by a propagator method (PM) and Monte Carlo (MC) simulation. The PM^{6,16)} is a numerical technique for solving simultaneous moment equations derived from the Boltzmann equation. The D_n calculation by the PM was stable. On the other hand, the MC simulates the electron behavior using random numbers. D_n calculated by the MC was severely influenced by the statistical fluctuation, as discussed afterward. Equilibrium D_n values were recognizable up to $n = 4$ in the MC with electron samplings of 10^7 – 10^8 .

3.2 Simulation conditions

CF₄ has been chosen as a test gas. CF₄ has both ionization and attachment processes, and its vibrational excitation cross section lying in a low-energy region is expected to make the spatial electron distribution shape of

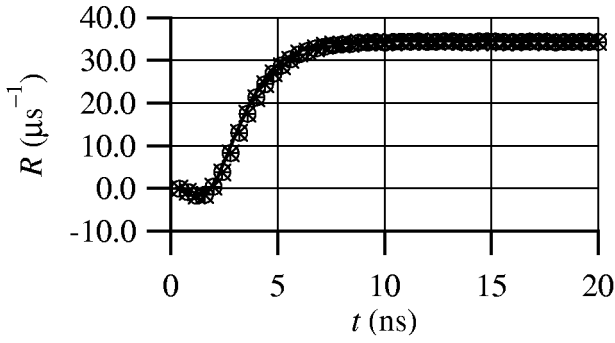


Fig. 3. Effective ionization frequencies of component and composite electron swarms: +, R_1 (MC); ×, R_2 (MC); o, R_{total} (MC); —, R_1 , R_2 and R_{total} (PM, overlapping).

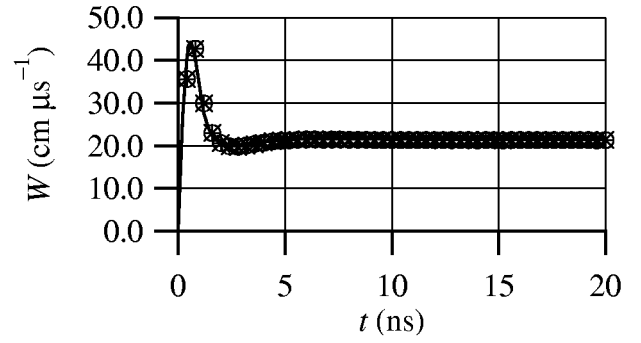


Fig. 4. Drift velocities of component and composite electron swarms: +, W_1 (MC); ×, W_2 (MC); o, W_{total} (MC); —, W_1 , W_2 and W_{total} (PM, overlapping).

the electron swarms complicated. The electron collision cross section set for CF_4 has been taken from ref. 17. The gas molecule number density N has been set at $3.54 \times 10^{16} \text{ cm}^{-3}$, assuming the gas pressure $p = 133 \text{ Pa}$ (1 Torr) at 0°C , and the applied electric field was $E = 100.0 \text{ V cm}^{-1}$. These values correspond to the reduced electric field E/N of 283 Td ($1 \text{ Td} = 10^{-17} \text{ V cm}^2$).

The initial electrons have been released from two point electron sources, P_1 at $(x, y, z) = (0, 0, 0)$ and P_2 at $(0.5 \text{ cm}, 1.0 \text{ cm}, 1.5 \text{ cm})$, at $t = 0$ with an uneven ratio of 60:40. This setting supposes a composite electron swarm consisting of two component electron swarms. The quantities concerning the x direction are the objects of the present interest. The energy distribution of the initial electrons was assumed to be a Maxwellian of the mean electron energy of 1.0 eV. The electron swarms were traced until $t = 40 \text{ ns}$, which is sufficiently long for the transport parameters to reach their equilibrium values.

4. Results and Discussion

4.1 Transport parameters

Figures 3–7 show the relaxation processes of R , W , D_2 , D_3 and D_4 of the component and composite electron swarms. It seems that their values have reached equilibrium by $t = 20 \text{ ns}$. The transport parameters of the composite electron swarm agree well with those of the component electron swarms. The statistical fluctuations in D_5 and D_6 sampled by the MC were too large to recognize their equilibrium values. The equality of D_5 and D_6 was verified by the result of the PM (Figs. 8 and 9).

Figures 10 and 11 show the temporal variations in κ_2 and κ_3 of the component and composite electron swarms. The difference between the initial $\kappa_{n,k}$ and $\kappa_{n,\text{total}}$ values represents the difference between the shapes of f_k and f_{total} . However, their time derivatives are common and constant in equilibrium. This result verifies eqs. (24) and (25).

The tendency observed in the MC result is that the statistical fluctuation is larger in the composite electron swarm than in the component electron swarms (see D_4 in Fig. 7), although the composite electron swarm contains more electrons than each component electron swarm. This is considered to be due to the fact that the distance

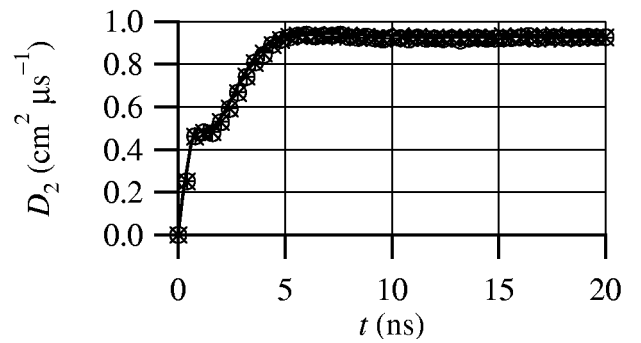


Fig. 5. Second-order diffusion coefficients of component and composite electron swarms: +, $D_{2,1}$ (MC); ×, $D_{2,2}$ (MC); o, $D_{2,\text{total}}$ (MC); —, $D_{2,1}$, $D_{2,2}$ and $D_{2,\text{total}}$ (PM, overlapping).

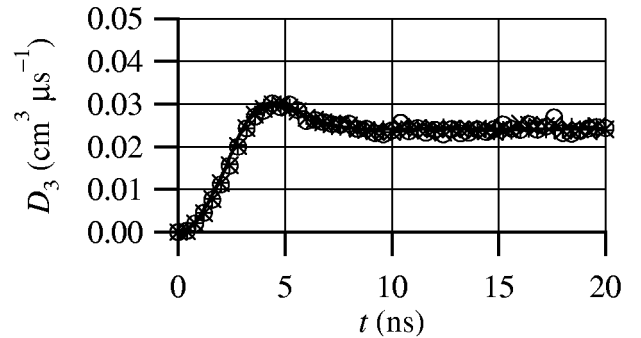


Fig. 6. Third-order diffusion coefficients of component and composite electron swarms: +, $D_{3,1}$ (MC); ×, $D_{3,2}$ (MC); o, $D_{3,\text{total}}$ (MC); —, $D_{3,1}$, $D_{3,2}$ and $D_{3,\text{total}}$ (PM, overlapping).

factor $[x - g_{\text{total}}(t)]^n$ in eq. (13) for the composite electron swarm is larger than $[x - g_k(t)]^n$ in eq. (4) for each component electron swarm. D_n becomes more sensitive to this distance factor as n increases. The fluctuations in $D_{4,\text{total}}$, $D_{4,1}$ and $D_{4,2}$ increase with t probably for the same reason. The distance from the centroid also increases for electron diffusion. Although the exponential increase in N_e due to ionization often stabilizes the MC sampling result, the instability due to the statistical fluctuation appears strongly in the calculation of the HDCs.

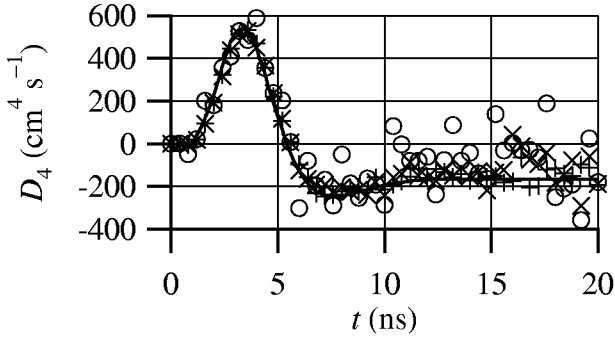


Fig. 7. Fourth-order diffusion coefficients of component and composite electron swarms: +, $D_{4,1}$ (MC); \times , $D_{4,2}$ (MC); \circ , $D_{4,\text{total}}$ (MC); —, $D_{4,1}$, $D_{4,2}$ and $D_{4,\text{total}}$ (PM, overlapping).

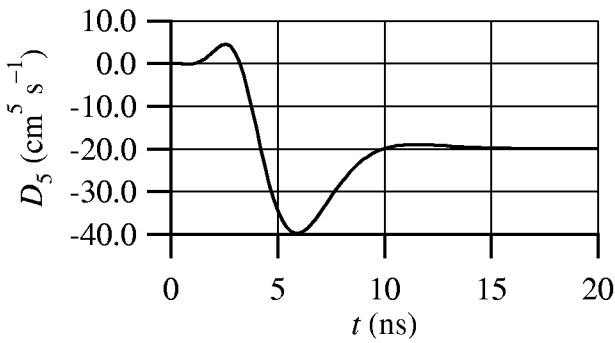


Fig. 8. Fifth-order diffusion coefficients of component and composite electron swarms calculated by PM: $D_{5,1}$, $D_{5,2}$ and $D_{5,\text{total}}$ (overlapping).

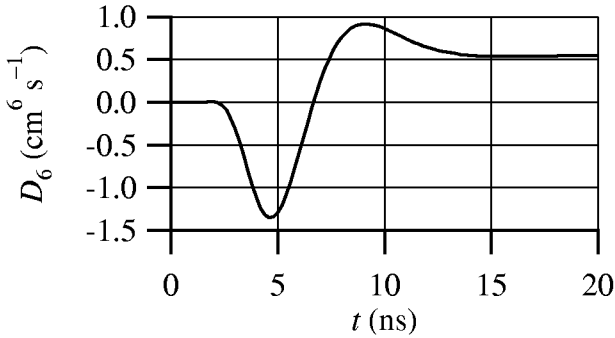


Fig. 9. Sixth-order diffusion coefficients of component and composite electron swarms calculated by PM: $D_{6,1}$, $D_{6,2}$ and $D_{6,\text{total}}$ (overlapping).

4.2 Spatial electron distribution

Figure 12 shows the development of f_{total} composed by f_1 and f_2 . The shape of f_{total} is still far from a Gaussian at $t = 20$ ns, while the transport parameters of the component and composite electron swarms have reached their equilibrium values by this moment. This supports one of the theoretical results that the equality of the transport parameters is independent of the spatial arrangement of the component electron swarms.

It is generally thought that the shape of the spatial

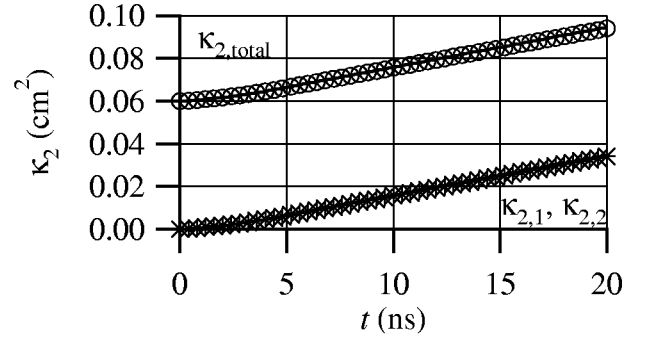


Fig. 10. Second-order cumulants of component and composite electron swarms: +, $\kappa_{2,1}$ (MC); \times , $\kappa_{2,2}$ (MC); \circ , $\kappa_{2,\text{total}}$ (MC); —, $\kappa_{2,1}$, $\kappa_{2,2}$ and $\kappa_{2,\text{total}}$ (PM).

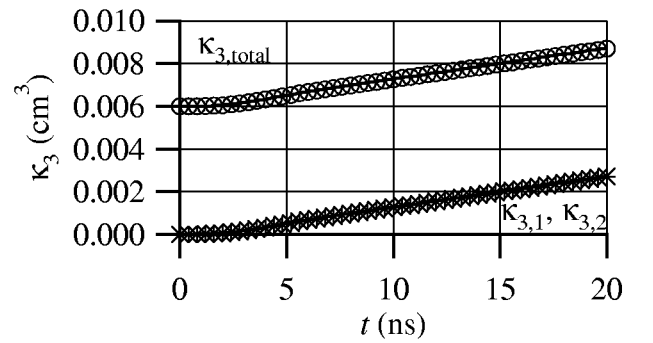


Fig. 11. Third-order cumulants of component and composite electron swarms: +, $\kappa_{3,1}$ (MC); \times , $\kappa_{3,2}$ (MC); \circ , $\kappa_{3,\text{total}}$ (MC); —, $\kappa_{3,1}$, $\kappa_{3,2}$ and $\kappa_{3,\text{total}}$ (PM).

electron distribution of an electron swarm tends to be a Gaussian. Sugawara *et al.*⁶⁾ theoretically showed this tendency for an isolated electron swarm developing from a point electron source. This tendency is also observed in f_{total} of the composite electron swarm in Fig. 12. As the electrons diffuse, the distance between P_1 and P_2 would eventually become negligible relative to a spatial scale (e.g., the standard deviation $\sigma = \sqrt{\kappa_{2,\text{total}}}$) of f_{total} . This is a basis with which we can approximate the electron source as a point in swarm experiments. Here, this tendency does not conflict with the infinite increase of $|\kappa_{n,\text{total}}|$ with t represented by eq. (25), although $\kappa_{n,\text{total}}$ is a quantity characterizing the deviation of f_{total} from a Gaussian. Sugawara *et al.*⁶⁾ showed that while the absolute amount of the deviation increases, its dimensionless value normalized by σ tends to be zero.

4.3 Higher-order transverse diffusion coefficients

The derivation of the equality of D_n presented in §2 can also be applied to the transverse diffusion coefficient D_{Tn} ($n \geq 2$). D_{Tn} quantifies the electron diffusion in the y direction lateral to the applied electric field and is defined in the same manner as that for D_n ; the definition of D_n becomes that of D_{Tn} by replacing x with y . Figure 13 shows the schematic of $f_{\text{total}}(y)$ under the present simulation conditions.

Date *et al.*¹⁸⁾ derived the characteristic feature of D_{Tn}

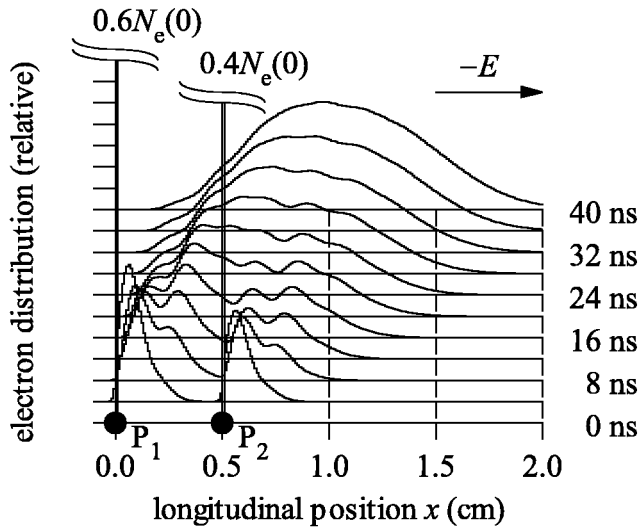


Fig. 12. Spatiotemporal development of electron swarm in CF_4 calculated by MC at $E/N = 283$ Td. Sixty percent of the initial electrons has been released from electron source P_1 at $x = 0$ cm, and forty percent from electron source P_2 at $x = 0.5$ cm.

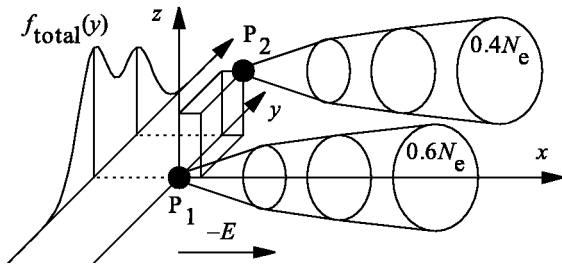


Fig. 13. Schematic of asymmetrical transverse electron distribution.

by Fourier-transformed Boltzmann equation analysis:

$$D_{Tn,\text{total}} = 0 \quad (\text{for } n = 3, 5, 7, \dots). \quad (41)$$

This feature can also be derived from the equality of D_{Tn} between the component and composite electron swarms. Equation (41) is evident when $f_{\text{total}}(y)$ itself is symmetrical in $y = g_{T,\text{total}}$, because D_{Tn} 's ($n = 3, 5, 7, \dots$) are given by integrations for the odd functions of y .^{6,19} Moreover, $f_{\text{total}}(y)$ may be asymmetrical as long as its components $f_k(y)$ are all individually symmetrical, because $D_{Tn,\text{total}} = 0$ for odd n when $D_{Tn,k} = 0$ for all k 's (the equality of D_{Tn}). Figure 14 shows $D_{T3,k}$ and $D_{T3,\text{total}}$ calculated by the MC for the same component and composite electron swarms as have been examined. While each component electron swarm is symmetrical, the composite electron swarm is asymmetrical. Comparing with D_3 plotted together, we can observe that $D_{T3,k} = D_{T3,\text{total}} = 0$.

Furthermore, it is considered that the symmetry of $f_k(y)$ is always established in equilibrium even though the initial electrons are given with asymmetrical spatial and velocity distributions. Let us assume that the k th initial electron generates the k th component electron swarm and that electron scattering by gas molecules

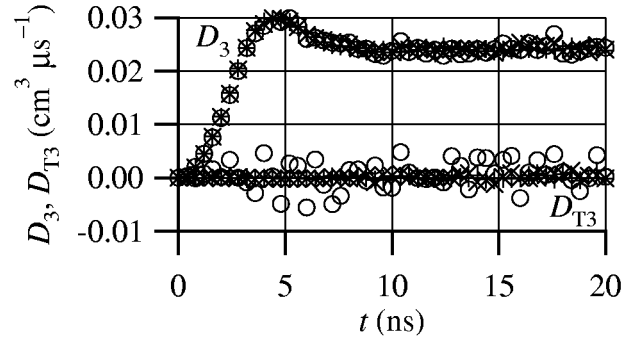


Fig. 14. Third-order longitudinal and transverse diffusion coefficients of component and composite electron swarms calculated by MC: +, $D_{3,1}$ and $D_{T3,1}$; ×, $D_{3,2}$ and $D_{T3,2}$; and ○, $D_{3,\text{total}}$ and $D_{T3,\text{total}}$.

is isotropic. The expectation of $f_k(y)$ becomes symmetrical after the first collision of the k th initial electron, because the asymmetry in the velocity vector of the initial electron vanishes from the expectation of $f_k(y)$ at the scattering. Consequently, eq. (41) always holds even in asymmetrical electron swarms. Actually, eq. (41) was derived in ref.18 with no assumption on the shape of $f_{\text{total}}(y)$.

5. Conclusions

A theorem on the higher-order diffusion coefficients (HDCs) has been derived; the n th-order diffusion coefficient D_n ($n \geq 2$) of a composite electron swarm consisting of component electron swarms are equal to that of the component electron swarms when they are individually in drift equilibrium. This theorem represents the equality of the HDCs between an electron swarm developing from multiple electron sources and another originating from a single electron source. Although the HDCs are derived from the higher-order cumulants characterizing the shape of the spatial electron distribution of the composite electron swarm, the equality of the HDCs holds irrespective of the spatial arrangement and electron population of the component electron swarms.

The HDCs of a composite electron swarm consisting of two component electron swarms have been calculated by a propagator method and a Monte Carlo simulation. It has been numerically demonstrated that D_n 's up to $n = 6$ are equal between the component and composite electron swarms.

Furthermore, the present theorem is applicable not only to the longitudinal diffusion coefficients D_n but also to the transverse diffusion coefficients D_{Tn} . The fact that $D_{Tn} = 0$ for $n = 3, 5, 7, \dots$ has been explained from the equality of the HDCs.

Acknowledgement

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Appendix: Equality of Lower-Order Transport Parameters

The equalities of the zeroth-order ($R_{\text{total}} = R_{\text{eq}}$) and the first-order ($W_{\text{total}} = W_{\text{eq}}$) parameters are almost evident. The former is based on the exponential growth of $N_{e,\text{total}}$ derived from eq. (10):

$$\begin{aligned} N_{e,\text{total}}(t) &= \sum_k N_{e,k}(0) \exp(R_{\text{eq}}t) \\ &= N_{e,\text{total}}(0) \exp(R_{\text{eq}}t), \end{aligned} \quad (\text{A.1})$$

$$R_{\text{total}} = \frac{\frac{d}{dt} N_{e,\text{total}}(t)}{N_{e,\text{total}}(t)} = R_{\text{eq}}. \quad (\text{A.2})$$

The latter is concluded from $W_k = W_{\text{eq}}$ for all k 's; i.e., the relative velocity between g_k 's of any two component electron swarms is zero. With eqs. (14) and (A.1), W_{total} is equated to W_{eq} as

$$\begin{aligned} g_{\text{total}}(t) &= \frac{\sum_k N_{e,k}(0) \exp(R_{\text{eq}}t) g_k(t)}{N_{e,\text{total}}(0) \exp(R_{\text{eq}}t)} \\ &= \frac{\sum_k N_{e,k}(0) g_k(t)}{N_{e,\text{total}}(0)}, \end{aligned} \quad (\text{A.3})$$

$$\begin{aligned} W_{\text{total}} &= \frac{\sum_k N_{e,k}(0) \frac{d}{dt} g_k(t)}{N_{e,\text{total}}(0)} \\ &= \frac{\sum_k N_{e,k}(0) W_{\text{eq}}}{N_{e,\text{total}}(0)} = W_{\text{eq}}. \end{aligned} \quad (\text{A.4})$$

The equality of the second-order parameter ($D_{2,\text{total}} = D_{2,\text{eq}}$) is not immediately evident, because $m_{2,\text{total}}^C$ for deriving $D_{2,\text{total}}$ involves the distance from the centroid as a nonlinear weight:

$$\begin{aligned} D_{2,k} &= \frac{1}{2} \frac{d}{dt} \frac{m_{2,k}^C(t)}{N_{e,k}(t)} \\ &= \frac{1}{2} \frac{d}{dt} \frac{\int_{-\infty}^{\infty} [x - g_k(t)]^2 f_k(x, t) dx}{N_{e,k}(t)} \\ &= D_{2,\text{eq}}, \end{aligned} \quad (\text{A.5})$$

$$\begin{aligned} D_{2,\text{total}} &= \frac{1}{2} \frac{d}{dt} \frac{m_{2,\text{total}}^C(t)}{N_{e,\text{total}}(t)} \\ &= \frac{1}{2} \frac{d}{dt} \frac{\int_{-\infty}^{\infty} [x - g_{\text{total}}(t)]^2 f_{\text{total}}(x, t) dx}{N_{e,\text{total}}(t)}. \end{aligned} \quad (\text{A.6})$$

The shape of f_{total} is dependent on the arrangement of f_k , and f_k may be placed anywhere relative to the other component electron swarms. However, $D_{2,\text{total}} = D_{2,\text{eq}}$ is derived independently of this arrangement.

Expanding the factor $[x - g_k(t)]^2$ in eq. (A.5) and integrating its result, we obtain

$$\begin{aligned} D_{2,k} &= \frac{1}{2} \frac{d}{dt} \left\{ \frac{m_{2,k}^L(t)}{N_{e,k}(t)} - [g_k(t)]^2 \right\} \\ &= \frac{1}{2} \frac{\frac{d}{dt} m_{2,k}^L(t) - R_{\text{eq}} m_{2,k}^L(t)}{N_{e,k}(t)} - W_{\text{eq}} g_k(t). \end{aligned} \quad (\text{A.7})$$

In the same manner, from eq. (A.6), $D_{2,\text{total}}$ is rewritten as

$$\begin{aligned} D_{2,\text{total}} &= \frac{1}{2} \frac{\frac{d}{dt} m_{2,\text{total}}^L(t) - R_{\text{eq}} m_{2,\text{total}}^L(t)}{N_{e,\text{total}}(t)} \\ &\quad - W_{\text{eq}} g_{\text{total}}(t). \end{aligned} \quad (\text{A.8})$$

By decomposing $m_{2,\text{total}}^L$ and g_{total} into $m_{2,k}^L$ and g_k , this equation becomes

$$\begin{aligned} D_{2,\text{total}} &= \frac{1}{2} \frac{\frac{d}{dt} \sum_k m_{2,k}^L(t) - R_{\text{eq}} \sum_k m_{2,k}^L(t)}{N_{e,\text{total}}(t)} \\ &\quad - W_{\text{eq}} \frac{\sum_k N_{e,k}(t) g_k(t)}{N_{e,\text{total}}(t)} \\ &= \sum_k \left\{ \frac{N_{e,k}(t)}{N_{e,\text{total}}(t)} \right. \\ &\quad \times \left. \left[\frac{1}{2} \frac{\frac{d}{dt} m_{2,k}^L(t) - R_{\text{eq}} m_{2,k}^L(t)}{N_{e,k}(t)} - W_{\text{eq}} g_k(t) \right] \right\}. \end{aligned} \quad (\text{A.9})$$

Because the terms in brackets in eq. (A.9) are identical to the description of $D_{2,k}$ in eq. (A.7), we obtain

$$D_{2,\text{total}} = \frac{\sum_k N_{e,k}(t) D_{2,\text{eq}}}{N_{e,\text{total}}(t)} = D_{2,\text{eq}}. \quad (\text{A.10})$$

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