PREDICTION OF FRACTIONAL BROWNIAN MOTION-TYPE PROCESSES

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Abstract. We introduce a class of continuous-time Gaussian processes with stationary increments via moving-average representation with good MA coefficient. The class includes fractional Brownian motion with Hurst index less than 1/2 as a typical example. It also includes processes which have different indices corresponding to the local and long-time properties, repsectively. We derive some basic properties of the processes, and, using the results, we establish a prediction formula for them. The prediction kernel in the formula is given explicitly in terms of MA and AR coefficients.

1. Introduction and main theorem

Fractional Brownian motion $(B_H(t) : t \in \mathbf{R})$ with Hurst index $H \in (0,1) \setminus \{1/2\}$ can be defined by the following "moving-average" representation: for $t \in \mathbf{R}$,

$$B_H(t) = \frac{1}{\Gamma(\frac{1}{2}+H)} \int_{-\infty}^{\infty} \left\{ ((t-s)_+)^{H-(1/2)} - ((-s)_+)^{H-(1/2)} \right\} dW(s), \tag{1.1}$$

where $(W(t) : t \in \mathbf{R})$ is a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ and $(x)_+ := \max(x, 0)$ for $x \in \mathbf{R}$. This process, abbreviated fBm, is a centered Gaussian process with stationary increments. It has been widely used to model various phenomena in hydrogy, network traffic, finance etc, which exhibit self-similarity. When $1/2 < H < 1$, fBm has also long-range dependence. We refer to Samorodnitsky and Taqqu [13] for its background.

In this paper, we consider a natural class of centered Gaussian processes with stationary increments, which includes fBm with $H \in (0, 1/2)$ as a typical example. The case $1/2 < H < 1$, which requires a different approach, is considered in a subsequent paper [4]. Thus we consider a Gaussian process $(X(t) : t \in \mathbf{R})$ with stationary increments that admits the following moving-average representation:

$$X(t) = \int_{-\infty}^{\infty} \{c(t-s) - c(-s)\}\, dW(s) \qquad (t \in \mathbf{R}), \tag{1.2}$$

where the MA (moving-average) coefficient $c(\cdot)$ is a function of the form

$$c(t) = \int_0^{\infty} e^{-ts} \nu(ds) \quad (t > 0), \qquad = 0 \quad (t \le 0) \tag{1.3}$$

with ν being a Borel measure on $(0, \infty)$ satisfying

$$\int_0^{\infty} \frac{1}{1+s} \nu(ds) < \infty \tag{1.4}$$

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and some extra conditions. In particular, in the main theorem (Theorem 1.1), we will assume

$$\lim_{t\to 0+} c(t) = \infty, \tag{1.5}$$

$$c(t) = O(t^q) \text{ as } t \to 0+ \text{ for some } q > -1/2, \tag{1.6}$$

$$c(t) \sim \frac{1}{\Gamma(\frac{1}{2}+H)} t^{-(\frac{1}{2}-H)} \ell(t) \qquad (t \to \infty), \tag{1.7}$$

where $\ell(\cdot)$ is a slowly varying function at infinity and H is a constant such that

$$0 < H < 1/2. \tag{1.8}$$

A typical example of such ν is

$$\nu(ds) = \frac{\cos(\pi H)}{\pi} s^{-(\frac{1}{2}+H)} ds \quad \text{on } (0, \infty) \tag{1.9}$$

with (1.8) (Example 2.5). For this ν, we have

$$c(t) = I_{(0,\infty)}(t) \frac{1}{\Gamma(\frac{1}{2}+H)} t^{-(\frac{1}{2}-H)} \qquad (t \in \mathbf{R}), \tag{1.10}$$

whence $(X(t))$ reduces to $(B_H(t))$. We can also choose ν so that the resulting process $(X(t))$ has two different indices H_0 and H corresponding to the local properties (such as path properties) and the long-time behavior, respectively (Example 2.6).

The central concern of this paper is the prediction of the process $(X(t))$. More specifically, our problem is to represent the conditional expectation

$$E\left[X(T)|\ \sigma(X(s) : -t_0 \le s \le t_1)\right],$$

by an integral consisting of the segment $(X(u) : -t_0 \le u \le t_1)$ of the process and some deterministic quantities, where t_0, t_1, and T are real constants such that

$$-\infty < -t_0 \le 0 \le t_1 < T < \infty, \qquad -t_0 < t_1. \tag{1.11}$$

The conditional expectation above stands for the predictor of the future value $X(T)$ based on the partial data $X(u)$ $(-t_0 \le u \le t_1)$. It should be noticed that computing such a finite-past predictor of a given process is generally a difficult problem (cf. Dym and McKean [6]). In fact, it has been computed explicitly only for very special processes. The result for fBm with $1/2 < H < 1$ by Gripenberg and Norros [10] and that for fBm with $0 < H < 1/2$ by Nuzman and Poor [11] (see also Anh and Inoue [2]) are among such results. The proofs in these references, however, cannot be applied to the process $(X(t))$ since they rely on special properties of fBm.

It turns out that, for the process $(X(t))$, the existence of a good AR (autoregressive) coefficient, in addition to the MA coefficient $c(\cdot)$, is a key to our solution to the problem above. Here we define the AR coefficient $a(\cdot)$ by

$$a(t) := -\frac{d\alpha}{dt}(t) \qquad (t > 0), \tag{1.12}$$

where the function $\alpha(\cdot)$ on $(0, \infty)$, in turn, is defined by

$$-iz \left(\int_0^\infty e^{izt} c(t) dt \right) \left(\int_0^\infty e^{izt} \alpha(t) dt \right) = 1 \qquad (\Im z > 0). \tag{1.13}$$

We will see that $a(\cdot)$ has a good integral representation similar to (1.3) (Corollary 3.3). In particular, $a(\cdot)$ is a positive decreasing function on $(0, \infty)$.

To state the main theorem, we introduce some functions which are given explicitly in terms of $c(\cdot)$ and $a(\cdot)$. We define $b(t,s)$ by

$$b(t,s) := \int_0^s c(u)a(t+s-u)du \qquad (t,s>0). \tag{1.14}$$

We will see that $b(\cdot,\cdot)$ is the kernel of the predictor based on the infinite past. We put

$$t_2 := t_0 + t_1, \quad t_3 := T - t_1. \tag{1.15}$$

For $t, s \in (0,\infty)$ and $n \in \mathbf{N}$, we define $b_n(t,s) = b_n(t,s;t_2)$ iteratively by

$$\begin{cases} b_1(t,s) := b(t,s), \\ b_n(t,s) := \int_0^\infty b(t,u) b_{n-1}(t_2+u,s)du \qquad (n=2,3,\dots). \end{cases} \tag{1.16}$$

We define $b_n(s) = b_n(s;t_3,t_2)$ by

$$b_n(s) := b_n(s,t_3) \qquad (s>0,\ n=1,2,\dots), \tag{1.17}$$

and a nonnegative function $h(s) = h(s;t_3,t_2)$ by

$$h(s) := \sum_{k=1}^\infty \{b_{2k-1}(t_2-s) + b_{2k}(s)\} \qquad (0<s<t_2). \tag{1.18}$$

For $s>0$, we define $D_n(s) = D_n(s;t_3,t_2)$ by

$$D_n(s) := \begin{cases} c(t_3-s) \qquad (n=0), \\ \int_0^\infty b_n(t_2+s+v)c(v)dv \qquad (n=1,2,\dots) \end{cases} \tag{1.19}$$

and $k(s) = k(s;t_3,t_2)$ by

$$k(s) = \sum_{n=0}^\infty D_n(s)^2 \qquad (s>0). \tag{1.20}$$

Here is the main theorem.

Theorem 1.1. *We assume* (1.2)–(1.8). *Then*

$$\int_0^{t_2} h(t)dt = 1, \tag{1.21}$$

$$E\left[X(T)|\ \sigma(X(s): -t_0 \le s \le t_1)\right] = \int_{-t_0}^{t_1} h(s+t_0)X(s)ds, \tag{1.22}$$

$$E\left|X_T - E\left[X(T)|\ \sigma(X(s): -t_0 \le s \le t_1)\right]\right|^2 = \int_0^\infty k(s)ds. \tag{1.23}$$

The prediction formula (1.22) and the corresponding result (1.23) for the mean-squared prediction error are explicit in the sense that both $h(\cdot)$ and $k(\cdot)$ are given explicitly by infinite series made up of the MA coefficient $c(\cdot)$ and the AR coefficient $a(\cdot)$. These explicit representations of $h(\cdot)$ and $k(\cdot)$ will be useful for, e.g., deriving precise asymptotics of $h(\cdot)$ and $k(\cdot)$ as the length of observation $t_2 = t_0 + t_1$ goes to infinity (cf. Inoue and Kasahara [9]). It is interesting that the equality (1.21) for the predictor kernel $h(\cdot)$, which has been known to hold for fBm with $0 < H < 1/2$ (see [11] and [2]), also holds for general $(X(t))$.

The proof of Theorem 1.1 is based on a method which involves alternating projections associated with the infinite past and the infinite future, which we now explain. We write $M(X)$ for the real Hilbert space spanned by $\{X(t) : t \in \mathbf{R}\}$ in $L^2(\Omega, \mathcal{F}, P)$, and $\|\cdot\|$ for its norm. Let I be a closed interval of $\mathbf{R}$ such as $[-t_0, t_1]$, $(-\infty, t_1]$, and $[-t_0, \infty)$. Let $M_I(X)$ be the closed subspace of $M(X)$ spanned by $\{X(t) : t \in I\}$. We write P_I for the orthogonal projection operator from $M(X)$ onto $M_I(X)$, and $P_I^{\perp}$ for its orthogonal complement: $P_I^{\perp} Z = Z - P_I Z$ for $Z \in M(X)$. Notice that since $(X(t))$ is a Gaussian process, we have

$$P_{[-t_0,t_1]} Z = E\left[Z \mid \sigma(X(s) : -t_0 \le s \le t_1)\right] \qquad (Z \in M(X)).$$

In this method, the first priority is to show the following equality (Theorem 4.11):

$$M_{[-t_0,t_1]}(X) = M_{(-\infty,t_1]}(X) \cap M_{[-t_0,\infty)}(X). \tag{1.24}$$

By von Neumann's alternating projection theorem (cf. Pourahamadi [12, Section 9.6.3]), this is equivalent to

$$P_{[-t_0,t_1]} = \operatorname*{s-lim}_{n\to\infty} \left\{P_{[-t_0,\infty)} P_{(-\infty,t_1]}\right\}^n, \tag{1.25}$$

where s-lim denotes strong limits. The equality (1.25) enables us to represent quantities related to $P_{[-t_0,t_1]}$ in terms of the MA and AR coefficients since $P_{(-\infty,t_1]}$ and $P_{[-t_0,\infty)}$ themselves admit such representations. For example, we show that $P_{(-\infty,t]} X(T)$ is of the form $\int_{-\infty}^{t} b(t-s, T-t) X(s) dt$ with $b(\cdot,\cdot)$ in (1.14) (Theorem 3.7), an analogue of Wiener's prediction formula (see Wiener and Masani [14]).

Notice that the prediction formula (1.22) is given by an ordinary integral rather than a stochastic integral. The absolute convergence of this integral is not trivial, and a large part of this paper is devoted to showing it (see, e.g., Proposition 4.3).

We derive basic properties of $(X(t))$ in Section 2. In Section 3, we consider prediction of $(X(t))$ from an infinite part of the past. We prove Theorem 1.1 in Section 4, using the results in Sections 2 and 3. Finally, in Section 5, we prove an L^2-boundedness theorem which we need in the proof of Theorem 1.1.

2. Basic properties

The purpose of this section is to study basic properties of the process $(X(t))$ defined by (1.2). Throughout the section, we assume (1.3), (1.4) and

$$\int_0^1 c(t)^2 dt < \infty. \tag{2.1}$$

We put $f_t(s) := c(t-s) - c(-s)$ for $t, s \in \mathbf{R}$.

Lemma 2.1. *We have $\int_{-\infty}^{\infty} |f_t(s)|^2 ds < \infty$ for $t \in \mathbf{R}$.*

Proof. Since $\int_{-\infty}^{\infty} |f_t(s)|^2 ds = \int_{-\infty}^{\infty} |f_{-t}(s)|^2 ds$, we may assume $t > 0$. Then

$$\int_{-\infty}^{\infty} |f_t(s)|^2 ds = \int_0^{\infty} |c(t+s) - c(s)|^2 ds + \int_0^t c(s)^2 ds.$$

By (2.1), we have $\int_0^t c(s)^2 ds < \infty$. Since $c(s) - c(t+s) \le -tc'(s)$ for $s > 0$ and

$$\int_t^{\infty} c'(s)^2 ds \le -c'(t) \int_t^{\infty} (-c'(s)) ds = -c'(t) c(t) < \infty,$$

we have $\int_0^{\infty} |c(t+s) - c(s)|^2 ds < \infty$. $\square$

Let $(W(t) : t \in \mathbf{R})$ be a one-dimensional standard Brownian motion such that $W(0) = 0$. Lemma 2.1 allows us to have the next definition.

Definition 2.2. We define a centered Gaussian process $(X(t) : t \in \mathbf{R})$ with stationary increments by (1.2).

We define

$$F(z) := \int_0^\infty \frac{1}{s - iz} \nu(ds) \qquad (\Im z \geq 0,\ z \neq 0).$$

Lemma 2.3. *Let $t \in \mathbf{R}$. Then, in $L^2(\mathbf{R}, d\xi)$,*

$$(e^{-it\xi} - 1)F(\xi) = \lim_{N\to\infty} \int_{-N}^{N} e^{-is\xi} f_t(s) ds. \tag{2.2}$$

In particular, $(e^{-it\xi} - 1)F(\xi) \in L^2(\mathbf{R}, d\xi)$.

Proof. By Lemma 2.1, the limit on the right-hand side of (2.2) exists in $L^2(\mathbf{R}, d\xi)$. Hence we may prove (2.2), for $\xi \neq 0$, in the sense of pointwise convergence. For $N \geq |t|$ and $\xi \neq 0$,

$$\int_{-N}^{N} e^{-is\xi} f_t(s) ds = (e^{-it\xi} - 1)F(\xi) - e^{-it\xi}\mathrm{I}(\xi, N + t) + \mathrm{I}(\xi, N),$$

where $\mathrm{I}(\xi, u) := \int_0^\infty e^{-(s-i\xi)u}/(s - i\xi)\nu(ds)$ for $u > 0$. By the dominated convergence theorem, $\mathrm{I}(\xi, u) \to 0$ as $u \to \infty$. Thus we obtain (2.2) for $\xi \neq 0$. □

The Brownian motion $(W(t))$ has the following spectral representation as a process with stationary increments:

$$W(t) := \int_{-\infty}^{\infty} \frac{(1 - e^{-it\xi})}{i\xi} Z_0(d\xi) \qquad (t \in \mathbf{R}),$$

where Z_0 is a $\mathbf{C}$-valued Gaussian random measure such that

$$E[Z_0(A)\overline{Z_0(B)}] = \frac{1}{2\pi} \int_{A\cap B} d\xi.$$

By (1.2), Lemma 2.3 and the Parseval-type formula for integrals involving $Z_0(d\xi)$, we have the following spectral representation for $(X(t))$:

$$X(t) := \int_{-\infty}^{\infty} \frac{1 - e^{-it\xi}}{i\xi} (-i\xi)F(\xi)Z_0(d\xi) \qquad (t \in \mathbf{R}). \tag{2.3}$$

We recall some notation on random distributions (cf. [8, Section 2] and [3, Section 2]). We write $\mathcal{D}(\mathbf{R})$ for the space of all $\phi \in C^\infty(\mathbf{R})$ with compact support, endowed with the usual topology. We define the real Hilbert space M by

$$M := \{a \in L^2(\Omega, \mathcal{F}, P) : E[a] = 0\}, \quad (a, a) := E[Z_1 Z_2], \quad \|a\| := (a, a)^{1/2}.$$

A random distribution Y (with expectation zero) is a linear continuous map from $\mathcal{D}(\mathbf{R})$ to M. We write DY for its derivative. For a closed interval I of $\mathbf{R}$, we write $M_I(Y)$ for the closed linear hull of $\{X(\phi) : \phi \in \mathcal{D}(\mathbf{R}),\ \mathrm{supp}\ \phi \subset I\}$ in M.

Proposition 2.4. *The derivative DX of $(X(t))$ is a purely nondeterministic stationary random distribution, and the representation* (1.2) *is canonical in the sense that $M_{(-\infty,t]}(DX) = M_{(-\infty,t]}(DW)$ for every $t \in \mathbf{R}$.*

Proof. Since $\Re F(z) > 0$ for $z = x + iy$ with $y > 0$, $F(z)$, whence $-izF(z)$, is an outer function on $\Re z > 0$. Thus the proposition follows. □

Example 2.5. For $H \in (0, 1/2)$, let ν be as in (1.9). Then we have (1.10); and so (1.4)–(1.8) are satisfied. The resulting process $(X(t))$ is the fractional Brownian motion $(B_H(t))$ which has the representation (1.1).

We define the positive constant $v(H)$ by

$$v(H) := \|B_h(1)\|^2 = \frac{\Gamma(2-2H)\cos(\pi H)}{\pi H(1-2H)} \qquad (0 < H < 1/2).$$

Example 2.6. Let $f(\cdot)$ be a nonnegative, locally integrable function on $(0, \infty)$. For $H_0, H \in (0, 1/2)$ and slowly varying functions $\ell_0(\cdot)$ and $\ell(\cdot)$ at infinity, we assume

$$f(s) \sim \frac{\cos(\pi H)}{\pi} s^{-(\frac{1}{2}+H)} \ell(1/s) \qquad (s \to 0+), \tag{2.4}$$

$$f(s) \sim \frac{\cos(\pi H_0)}{\pi} s^{-(\frac{1}{2}+H_0)} \ell_0(s) \qquad (s \to \infty). \tag{2.5}$$

We put $\nu(ds) = f(s)ds$ on $(0, \infty)$, so that $c(t) = \int_0^\infty e^{-st} f(s) ds$ for $t > 0$. By Abelian theorems for Laplace transforms (cf. Bingham et al. [5, Section 1.7]), (2.4) and (2.5) imply (1.7) and

$$c(t) \sim \frac{1}{\Gamma(\frac{1}{2}+H_0)} t^{-(\frac{1}{2}-H_0)} \ell_0(1/t) \qquad (t \to 0+), \tag{2.6}$$

respectively. In particular, (1.4)–(1.8) are satisfied. We put

$$\sigma(t) = E[|X(t+s) - X(s)|^2]^{1/2} \qquad (t \geq 0,\ s \in \mathbf{R}).$$

Then, by arguments similar to those of the proof of Lemma 2.7 below, (2.6) implies

$$\begin{aligned} \frac{\sigma(t)^2}{tc(t)^2} &= \int_0^1 \frac{c(tu)^2}{c(t)^2} du + \int_0^\infty \left| \frac{c(t(1+u))}{c(t)} - \frac{c(tu)}{c(t)} \right|^2 du \\ &\to \int_0^1 u^{H_0-(1/2)} du + \int_0^\infty \left\{ (1+u)^{H_0-(1/2)} - u^{H_0-(1/2)} \right\} du = v(H_0) \end{aligned}$$

as $t \to 0+$, or

$$\sigma(t) \sim t^{H_0} \ell_0(1/t) \sqrt{v(H_0)} \qquad (t \to 0+).$$

In particular,

$$H_0 = \sup\{\beta : \sigma(t) = o(t^\beta)\ (t \to 0+)\} = \inf\{\beta : t^\beta = o(\sigma(t))\ (t \to 0+)\}.$$

Thus H_0 is the index that describes the path properties of $(X(t))$ (cf. Adler [1, Section 8.4]). On the other hand, the index H describes the long-time behavior of $(X(t))$ (cf. Lemma 2.7 below).

We need the next lemma in Sections 3 and 4.

Lemma 2.7. *We assume* (1.7) *and* (1.8). *Then* $\|X(t)\| \sim t^H \ell(t) \sqrt{v(H)}$ *as* $t \to \infty$.

Proof. For $t \geq 0$, we have $\|X(t)\|^2 = \int_0^t c(u)^2 du + \int_0^\infty |c(t+u) - c(u)|^2 du$. It follows from (1.7) with (1.8) that $\int_0^t c(u)^2 du \sim tc(t)^2 \int_0^1 u^{2H-1} du$ as $t \to \infty$. We claim

$$\int_0^\infty |c(t+u) - c(u)|^2 du \sim tc(t)^2 w(H) \qquad (t \to \infty), \tag{2.7}$$

where $w(H) := \int_0^\infty \{(1+u)^{H-\frac{1}{2}} - u^{H-\frac{1}{2}}\}^2 du$. This claim implies

$$\|X(t)\| \sim t^{1/2} c(t) \left[\int_0^1 u^{2H-1} du + w_H \right]^{1/2} \sim t^H \ell(t) \sqrt{v(H)} \qquad (t \to \infty),$$

whence the lemma.

We complete the proof by proving (2.7). By the monotone density theorem (cf. [5, Theorem 1.7.5]), we have

$$\gamma(t) \sim t^{-(\frac{3}{2}-H)}\ell(t)\cdot\frac{\frac{1}{2}-H}{\Gamma(\frac{1}{2}+H)} \sim \left(\frac{1}{2}-H\right)t^{-1}c(t) \qquad (t\to\infty), \tag{2.8}$$

where $\gamma(t) := -dc(t)/dt$ for $t > 0$. We put $\tilde{\gamma}(t) := \gamma(1)$ for $0 < t < 1$, and $\tilde{\gamma}(t) := \gamma(t)$ for $1 \le t < \infty$. Then, by (2.1), we have, for $t > 1$,

$$\begin{aligned}
0 &\le \int_0^\infty |c(t+u)-c(u)|^2 du - \int_0^\infty \left|\int_u^{t+u}\tilde{\gamma}(v)dv\right|^2 du \\
&= \int_0^1 \left\{|c(u)-c(t+u)|^2 - |c(1)-c(t+u)+(1-u)\gamma(1)|^2\right\} du \\
&= \int_0^1 \{c(u)-c(1)-(1-u)\gamma(1)\}\{c(u)+c(1)-2c(t+u)+(1-u)\gamma(1)\}du \\
&\le \int_0^1 \{c(u)-c(1)-(1-u)\gamma(1)\}\{c(u)+c(1)+(1-u)\gamma(1)\}du < \infty.
\end{aligned}$$

Thus, instead of (2.7), we may prove

$$\int_0^\infty \left\{\int_u^{t+u}\tilde{\gamma}(v)dv\right\}^2 du \sim tc(t)^2 w_H \qquad (t\to\infty). \tag{2.9}$$

Choose $\epsilon > 0$ so that $\epsilon < \max(H, (1/2)-H)$. By applying [5, Theorem 1.5.6 (ii)] (Potter's theorem) to the slowly varying function $(1+t)^{(3/2)-H}\tilde{\gamma}(t)$, we easily find that there exists a positive constant A satisfying

$$\tilde{\gamma}(tv)/\tilde{\gamma}(t) \le A\max(v^{H-(3/2)+\epsilon}, v^{H-(3/2)-\epsilon}) \qquad (t\ge 1,\ v>0).$$

The dominated convergence theorem now yields, as $t\to\infty$,

$$\begin{aligned}
\int_0^\infty \left\{\int_u^{t+u}\tilde{\gamma}(v)dv\right\}^2 du &= t^3\tilde{\gamma}(t)^2\int_0^\infty \left\{\int_u^{u+1}\tilde{\gamma}(tv)/\tilde{\gamma}(t)dv\right\}^2 du \\
&\sim \left(\frac{1}{2}-H\right)^2 tc(t)^2\int_0^\infty \left\{\int_u^{u+1} v^{H-(3/2)}dv\right\}^2 du,
\end{aligned}$$

which implies (2.9). □

3. Prediction from an infinite segment of the past

In this section, we consider prediction of $(X(t))$ from an infinite segment of the past. We need such results in the proof of Theorem 1.1. Throughout the section, we assume (1.2)–(1.5), (1.7), (1.8), and (2.1). Notice that the processes $(X(t))$ in Examples 2.5 and 2.6 satisfy all these conditions.

Lemma 3.1. *Let $b(\cdot)$ and $f(\cdot,\cdot)$ be nonrandom real measurable functions such that*

$$\int_{-\infty}^{\infty} |b(\tau)| d\tau < \infty, \tag{3.1}$$

$$\int_{-\infty}^{\infty} |b(\tau)| \left(\int_{-\infty}^{\infty} f(u,\tau)^2 du \right) d\tau < \infty, \tag{3.2}$$

$$\int_{-\infty}^{\infty} |b(\tau)| \left(\int_{-\infty}^{\infty} f(u,\tau)^2 du \right)^{1/2} d\tau < \infty. \tag{3.3}$$

Then we have, almost surely,

$$\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} b(\tau) f(u,\tau) d\tau \right) dW(u) = \int_{-\infty}^{\infty} b(\tau) \left(\int_{-\infty}^{\infty} f(u,\tau) dW(u) \right) d\tau. \tag{3.4}$$

By (3.3), the integral on the right-hand side of (3.4) is well-defined as a Bochner integral taking values in $L^2(\Omega, \mathcal{F}, P)$. On the other hand, (3.1) and (3.2) imply

$$\left(\int_{-\infty}^{\infty} |b(\tau) f(u,\tau)| d\tau \right)^2 \le \left(\int_{-\infty}^{\infty} |b(\tau)| d\tau \right) \left(\int_{-\infty}^{\infty} |b(\tau)| f(u,\tau)^2 d\tau \right) < \infty.$$

Thus the integral on the left-hand side of (3.4) is also well-defined. We can prove this Fubini-type lemma in the same way as the proof of [7, Lemma 4.5]; so we omit the details.

We define

$$\mathcal{M} = \left\{ \mu : \begin{array}{l} \mu \text{ is a Borel measure on } (0,\infty) \text{ satisfying (1.4)} \\ \text{and } \mu(0,\infty) = \int_0^\infty s^{-1} \mu(ds) = \infty. \end{array} \right\}.$$

We consider the correspondence between $\mu \in \mathcal{M}$ and $\nu \in \mathcal{M}$ through the relation

$$-iz \left\{ \int_0^\infty \frac{1}{s - iz} \nu(ds) \right\} \left\{ \int_0^\infty \frac{1}{s - iz} \mu(ds) \right\} = 1 \qquad (\Im z > 0). \tag{3.5}$$

Theorem 3.2. *The relation* (3.5) *defines a one-to-one and onto map $\theta : \mathcal{M} \ni \mu \mapsto \theta(\mu) = \nu \in \mathcal{M}$.*

Proof. We write $F_\tau(z)$ for the Stieltjes transform $\int_0^\infty (s - iz)^{-1} \tau(ds)$. Let $\mu \in \mathcal{M}$. By [8, Theorem 3.1], there exist finite Borel measures ν_n $(n = 1, 2, \dots)$ on $(0,\infty)$ such that $-izF_{\nu_n}(z)\{(1/n) + F_\mu(z)\} = 1$ for $\Im z > 0$. Then, putting $z = i$, we find that $\sup_n \tilde{\nu}[0,\infty] < \infty$, where we define the measure $\tilde{\nu}_n$ on $[0,\infty]$ by $\tilde{\nu}_n(ds) := I_{(0,\infty)}(s)(1+s)^{-1}\nu_n(ds)$. By the Helly selection principle, there exists a subsequence n' such that $\tilde{\nu}_{n'}$ converges weakly to $\tilde{\nu}$, say, on $[0,\infty]$. It follows that

$$-iz \left[-\tilde{\nu}\{0\}/(iz) + \tilde{\nu}\{\infty\} + F_\nu(z) \right] F_\mu(z) = 1 \qquad (\Im z > 0),$$

where ν is the measure on $(0,\infty)$ defined by $\nu(ds) := I_{(0,\infty)}(s)(1+s)\tilde{\nu}(ds)$. Since $\lim_{y \to 0+} \int_0^\infty y/(s+y)\nu(ds) = 0$ and $\lim_{y \to 0+} \int_0^\infty 1/(s+y)\mu(ds) = \infty$, we see that $\tilde{\nu}\{0\} = 0$. In the same way, we have $\tilde{\nu}\{\infty\} = 0$ and $\int_0^\infty s^{-1}\nu(ds) = \nu(0,\infty) = \infty$. Thus there exists $\nu \in \mathcal{M}$ satisfying (3.5). Since F_ν determines ν, and (3.5) is symmetric in ν and μ, the theorem follows. □

It follows from (1.5) that $\nu(0,\infty) = c(0+) = \infty$, while (1.7) and (1.8) imply $\int_0^\infty s^{-1}\nu(ds) = \int_0^\infty c(t)dt = \infty$. Recall $\alpha(\cdot)$ and $a(\cdot)$ from (1.13) and (1.12), respectively. From Theorem 3.2, we immediately obtain the next corollary.

Corollary 3.3. *Define $\mu \in \mathcal{M}$ by $\mu = \theta(\nu)$. Then we have $\alpha(t) = \int_0^\infty e^{-ts}\mu(ds)$ and $a(t) = \int_0^\infty e^{-ts} s\mu(ds)$ for $t > 0$.*

Recall $b(\cdot,\cdot)$ from (1.14).

Lemma 3.4. *We have*

$$\int_0^t c(u)\alpha(t-u)du = 1 \qquad (t > 0), \tag{3.6}$$

$$\int_0^\infty b(t,s)dt = 1 \qquad (s > 0), \tag{3.7}$$

$$c(t+s) = \int_0^t c(t-u)b(u,s)du \qquad (t, s > 0). \tag{3.8}$$

Proof. For $\Im z > 0$, we have $-1/(iz) = \int_0^\infty e^{izs}ds$ and

$$\left(\int_0^\infty e^{izt}c(t)dt\right)\left(\int_0^\infty e^{izt}\alpha(t)dt\right) = \int_0^\infty e^{izt}\left(\int_0^t c(u)\alpha(t-u)du\right)dt.$$

Hence (3.6) follows from (1.13) and the uniqueness of the Laplace transform. Since $\int_0^\infty b(t,s)dt = \int_0^s c(u)\alpha(s-u)du = 1$ for $s > 0$, we obtain (3.7). From (3.6), we see that $\int_0^t \alpha(u)c(t+s-u)du = 1 - \int_0^s c(u)\alpha(t+s-u)du$ for $s, t > 0$, whence $\int_0^\infty e^{itz}\left(\int_0^t \alpha(u)c(t+s-u)du\right)dt$ is equal to

$$-\frac{1}{iz} - \int_0^\infty e^{itz}\left(\int_0^s c(u)\alpha(t+s-u)du\right)dt. \tag{3.9}$$

Since $\lim_{t\to\infty} e^{itz}\int_0^s c(u)\alpha(t+s-u)du = 0$, we see by integrating by parts that (3.9) is equal to $(-iz)^{-1}\int_0^\infty e^{itz}b(t,s)dt$. Therefore $\int_0^\infty e^{itz}c(t+s)dt$ is equal to

$$\begin{aligned}
&\left(\int_0^\infty e^{itz}c(t)dt\right)(-iz)\left(\int_0^\infty e^{itz}\alpha(t)dt\right)\left(\int_0^\infty e^{itz}c(t+s)dt\right)\\
&= \left(\int_0^\infty e^{itz}c(t)dt\right)(-iz)\left\{\int_0^\infty e^{itz}\left(\int_0^t \alpha(u)c(t+s-u)\right)dt\right\}\\
&= \int_0^\infty e^{itz}\left(\int_0^t c(t-u)b(u,s)du\right)dt.
\end{aligned}$$

This and the uniqueness of the Laplace transform imply (3.8). □

Lemma 3.5. *For $s > 0$, we have*

$$b(t,s) \sim \frac{H+\frac{1}{2}}{\Gamma(\frac{1}{2}-H)}\left(\int_0^s c(u)du\right)\frac{t^{-(H+\frac{3}{2})}}{\ell(t)} \qquad (t\to\infty).$$

Proof. By putting $z = iy$ in (1.13), we get

$$y\left(\int_0^\infty e^{-yt}c(t)dt\right)\left(\int_0^\infty e^{-yt}\alpha(t)dt\right) = 1 \qquad (y > 0). \tag{3.10}$$

Using Karamata's Tauberian theorem (cf. [5, Theorem 1.7.6]), we see from (3.10), (1.7) and (1.8) that

$$\alpha(t) \sim \frac{t^{-(\frac{1}{2}+H)}}{\ell(t)}\cdot\frac{1}{\Gamma(\frac{1}{2}-H)} \qquad (t\to\infty). \tag{3.11}$$

This and the monotone density theorem imply

$$a(t) \sim \frac{t^{-(H+\frac{3}{2})}}{\ell(t)} \cdot \frac{(H+\frac{1}{2})}{\Gamma(\frac{1}{2}-H)} \qquad (t \to \infty). \tag{3.12}$$

Since $a(t+s)\int_0^t c(u)du \le b(t,s) \le a(t)\int_0^t c(u)du$ and $a(t+s) \sim a(t)$ as $t \to \infty$, we obtain the lemma. □

Since the map $[0,\infty) \ni s \mapsto \|X(t)\|$ is continuous, the next proposition follows immediately from (3.7) and Lemmas 2.7 and 3.5.

Proposition 3.6. *For $s > 0$, we have $\int_0^\infty b(t,s)\|X(t)\|^2 dt < \infty$.*

Recall $P_{(-\infty,t]}$ from Section 1. Notice that $P_{(-\infty,t]}X(T)$ is equal to the conditional expectation $E\left[X(T)|\ \sigma(X(s) : -\infty < s \le t)\right]$. The Wiener-type prediction formula can now be given.

Theorem 3.7. *Let $0 \le t \le T$. Then*

$$P_{(-\infty,t]}X(T) = \int_{-\infty}^t b(t-s, T-t)X(s)ds,$$

the integral converging absolutely in $L^2(\Omega, \mathcal{F}, P)$.

Proof. By Proposition 2.4 and [3, Proposition 2.3 (2)], we have

$$P_{(-\infty,t]}X(T) - X(t) = \int_{-\infty}^t \{c(T-u) - c(t-u)\}\, dW(u).$$

For $\tau, u \in \mathbf{R}$, we put $f(u,\tau) = c(t-u-\tau) - c(t-u)$ and $b(\tau) = I_{(0,\infty)}(\tau) b(\tau, T-t)$. Then, by Lemma 2.7, Proposition 3.6 and the estimate

$$\int_{-\infty}^\infty f(u,\tau)^2 du = \|X(t-\tau) - X(t)\|^2 \le 4\left\{\|X(t-\tau)\|^2 + \|X(t)\|^2\right\},$$

(3.2) and (3.3) hold. By (3.7) and Lemma 3.1, we have

$$\begin{aligned}
&\int_0^\infty b(\tau, T-t)X(t-\tau)d\tau - X(t) = \int_0^\infty b(\tau, T-t)\{X(t-\tau) - X(t)\}\, d\tau \\
&= \int_0^\infty b(\tau, T-t)\left[\int_{-\infty}^\infty \{c(t-u-\tau) - c(t-u)\}\, dW(u)\right] d\tau \\
&= \int_{-\infty}^\infty \left[\int_0^\infty b(\tau, T-t)\{c(t-u-\tau) - c(t-u)\}\, d\tau\right] dW(u).
\end{aligned}$$

Now $c(t-u-\tau) - c(t-u) = 0$ for $u \ge t$ and $\tau > 0$. By Lemma 3.4, we have

$$c(T-u) - c(t-u) = \int_0^\infty b(\tau, T-t)\{c(t-u-\tau) - c(t-u)\}\, d\tau \qquad (u < t).$$

Hence $\int_0^\infty b(\tau, T-t)X(t-\tau)d\tau - X(t) = \int_{-\infty}^t \{c(T-u) - c(t-u)\}\, dW(u)$. Thus the theorem follows. □

Using the Hilbert space isomorphism $\theta : M(X) \to M(X)$ defined by $\theta(X(s)) = X(-s)$ for $s \in \mathbf{R}$, we easily obtain the next corollary to Theorem 3.7.

Corollary 3.8. *Let $0 \le t \le T$. Then*

$$P_{[-t,\infty)}X(-T) = \int_{-t}^{\infty} b(t+s, T-t)X(s)ds,$$

the integral converging absolutely in $L^2(\Omega, \mathcal{F}, P)$.

Example 3.9. As in Example 2.5, we consider $(B_H(t))$ with $0 < H < 1/2$ as $(X(t))$. Then $\int_0^\infty e^{izt}c(t)dt = (-iz)^{-H-(1/2)}$ for $\Im z > 0$. It follows from (1.13) that $\int_0^\infty e^{izt}\alpha(t)dt = (-iz)^{H-(1/2)}$ or $\alpha(t) = \Gamma(\frac{1}{2} - H)^{-1}t^{-(H+\frac{1}{2})}$ for $t > 0$. Hence

$$a(t) = \frac{H + \frac{1}{2}}{\Gamma(\frac{1}{2} - H)} t^{-(H+\frac{3}{2})} \qquad (t > 0).$$

By change of variable $u = sv$, $\int_0^s (s-u)^{H-\frac{1}{2}}(t+u)^{-H-\frac{3}{2}}du$ is equal to

$$s^{H+\frac{1}{2}}t^{-H-\frac{3}{2}} \int_0^1 (1-v)^{H-\frac{1}{2}}\{1 + (s/t)v\}^{-H-\frac{3}{2}}dv = \frac{1}{(H+\frac{1}{2})}\left(\frac{s}{t}\right)^{H+\frac{1}{2}} \frac{1}{t+s},$$

where we have used the equality

$$\int_0^1 (1-v)^{p-1}(1+xv)^{-p-1}dv = \frac{1}{p(x+1)} \qquad (p > 0,\ x > -1), \tag{3.13}$$

which, in turn, is obtained by change of variable $(x+1)u = t/(1-t)$. Thus

$$b(t,s) = \frac{\cos(\pi H)}{\pi}\left(\frac{s}{t}\right)^{H+\frac{1}{2}} \frac{1}{t+s} \qquad (t > 0,\ s > 0).$$

From Theorem 3.7, we see that, for $0 \le t < T$,

$$E\left[B_H(T)|\ \sigma(B_H(s) : -\infty < s \le t)\right] = \frac{\cos(\pi H)}{\pi}\int_{-\infty}^{t}\left(\frac{T-t}{t-s}\right)^{H+\frac{1}{2}} \frac{B_H(s)}{T-s}ds.$$

This prediction formula for fractional Brownian motion with $0 < H < 1/2$ was obtained by Yaglom [15, (3.41)] by a different method.

4. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. Throughout the section, we assume (1.2)–(1.8). Notice that (1.6) implies (2.1). Let t_0, t_1, and T be as in (1.11).

Recall t_2, t_3, and $b_n(t,s) = b_n(t,s;t_2)$ from (1.15) and (1.16). We define

$$\beta_n(u) := \int_{t_2}^{\infty} b_n(s,u)ds \qquad (u > 0,\ n = 1, 2, \dots).$$

By (3.7), we have, for $n = 2, 3, \dots$ and $u > 0$,

$$\int_0^\infty \left(\int_{t_2}^\infty b(s,v)ds\right) b_{n-1}(t_2+v,u)dv \le \int_0^\infty b_{n-1}(t_2+v,u)dv$$

or $\beta_n(u) \le \beta_{n-1}(u)$. Hence the sequence $(\beta_n(u))_{n=1}^\infty$ is decreasing for every $u > 0$. In particular, $\lim_{n\to\infty}\beta_n(u)$ exists and is finite. Recall $h(s) = h(s;t_3,t_2)$ from (1.18).

Lemma 4.1. *We have*

$$\int_0^{t_2} h(s)ds = 1 - \lim_{n\to\infty}\beta_n(t_3). \tag{4.1}$$

In particular, $\int_0^{t_2} h(s)ds \le 1$.

Proof. We claim that, for $u > 0$,

$$
(4.2) \qquad \begin{cases} \displaystyle\int_0^{t_2} b_1(s,u)ds = 1 - \int_{t_2}^{\infty} b_1(s,u)ds, \\ \displaystyle\int_0^{t_2} b_n(s,u)ds = \int_{t_2}^{\infty} b_{n-1}(s,u)ds - \int_{t_2}^{\infty} b_n(s,u)du \quad (n = 2, 3, \dots). \end{cases}
$$

In fact, the first assertion follows immediately from (3.7), while we have

$$
\int_0^{\infty} b_n(s,u)ds = \int_0^{\infty} \left(\int_0^{\infty} b(s,v)ds \right) b_{n-1}(t_2 + v, u)dv = \int_{t_2}^{\infty} b_{n-1}(s,u)ds,
$$

whence the assertions for $n = 2, 3, \dots$ follow. From (4.2), it follows that

$$
\int_0^{t_2} h(s)ds = \lim_{n\to\infty} \sum_{k=1}^{n} \int_0^{t_2} b_k(s, t_3)ds = 1 - \lim_{n\to\infty} \beta_n(t_3).
$$

Thus we obtain the lemma. □

For simplicity, we put

$$
\beta_n := 0, \qquad \beta_n := \beta_n(t_3) \quad (n = 1, 2, \dots).
$$

Recall $b_n(s) = b_n(s; t_3, t_2)$ from (1.17). We define the random variables G_n by

$$
G_n := \begin{cases} \displaystyle\int_{-t_0}^{t_1} b_n(t_1 - s)X(s)ds - \beta_{n-1}X(t_1) + \beta_n X(-t_0) \qquad (n = 1, 3, \dots), \\ \displaystyle\int_{-t_0}^{t_1} b_n(t_0 + s)X(s)ds - \beta_{n-1}X(-t_0) + \beta_n X(t_1) \qquad (n = 2, 4, \dots). \end{cases}
$$

We also define the random variables ϵ_n $(n = 0, 1, \dots)$ by $\epsilon_0 := X(T)$ and

$$
\epsilon_n := \begin{cases} \displaystyle\int_{-\infty}^{-t_0} b_n(t_1 - s)X(s)ds - \beta_n X(-t_0) \qquad (n = 1, 3, \dots), \\ \displaystyle\int_{t_1}^{\infty} b_n(t_0 + s)X(s)ds - \beta_n X(t_1) \qquad (n = 2, 4, \dots). \end{cases}
$$

Recall $P_{(-\infty,t_1]}$ and $P_{[-t_0,\infty)}$ from Section 1. We set

$$
P_n := \begin{cases} P_{(-\infty,t_1]} & (n = 1, 3, 5, \dots), \\ P_{[-t_0,\infty)} & (n = 2, 4, 6, \dots). \end{cases}
$$

.

Proposition 4.2. *Let $n \in \mathbf{N}$. Then*

$$
(4.3) \qquad P_n P_{n-1} \cdots P_1 X(T) = \epsilon_n + \sum_{k=1}^{n} G_k.
$$

Proof. We use induction. By Theorem 3.7, (4.3) holds for $n = 1$. Suppose that (4.3) holds for $n = m \in \mathbf{N}$. Recall $M_I(X)$ from Section 1. Since

$$
(4.4) \qquad M_{[-t_0,t_1]}(X) \subset M_{(-\infty,t_1]}(X) \cap M_{[-t_0,\infty)}(X)
$$

and $G_k \in M_{[-t_0,t_1]}(X)$ for $k = 1, 2, \dots,$ we have $P_{m+1}G_k = G_k$ for $k = 1, 2, \dots.$ Thus

$$
P_{m+1}P_m \cdots P_1 X(T) = P_{m+1}\left(\epsilon_m + \sum\nolimits_{k=1}^{m} G_k\right) = P_{m+1}\epsilon_m + \sum\nolimits_{k=1}^{m} G_k.
$$

If m is odd, then, by Corollary 3.8, we have

$$\begin{aligned} P_{m+1}\epsilon_m &= \int_{-\infty}^{-t_0} b_m(t_1-u)P_{[-t_0,\infty)}X(u)du - \beta_m X(-t_0) \\ &= \int_{-\infty}^{-t_0} b_m(t_1-u)\left(\int_{-t_0}^{\infty} b(t_0+s,-u-t_0)X(s)ds\right)du - \beta_m X(-t_0) \\ &= \int_{-t_0}^{\infty} b_{m+1}(t_0+s)X(s)ds - \beta_m X(-t_0) = G_{m+1}+\epsilon_{m+1}, \end{aligned}$$

whence (4.3) with $n = m+1$. If m is even, then, using Theorem 3.7, we have similarly $P_{m+1}\epsilon_m = G_{m+1}+\epsilon_{m+1}$. Thus again we have (4.3) with $n = m+1$. By induction, the proposition follows. □

Proposition 4.3. *We have* $\int_0^\infty b(s,u)\|X(s)\|^2 ds \sim \|X(u)\|^2$ *as* $u \to \infty$.

Proof. We put $a^*(x) := a(1)$ for $0 < x \le 1$ and $a^*(x) := a(x)$ for $1 < x < \infty$. Notice that $a(x) \ge a^*(x)$ for $x > 0$. By (3.7), we have, for $u > 0$,

$$\begin{aligned} &\int_0^\infty \left(\int_0^u |a(s+u-\tau) - a^*(s+u-\tau)|c(\tau)d\tau\right)\|X(s)\|^2 ds \\ &= \int_0^1 \left(\int_0^u |a(s+u-\tau) - a^*(s+u-\tau)|c(\tau)d\tau\right)\|X(s)\|^2 ds \\ &\le 2\left(\max_{0\le s\le 1}\|X(s)\|^2\right)\int_0^1\left(\int_0^u a(s+u-\tau)c(\tau)d\tau\right)ds \\ &\le 2\left(\max_{0\le s\le 1}\|X(s)\|^2\right)\int_0^1 b(s,u)ds \le 2\left(\max_{0\le s\le 1}\|X(s)\|^2\right). \end{aligned}$$

By this and Lemma 2.7, we may assume that $a(\cdot)$ is a positive constant on $(0,1]$.

We put $c^*(x) := c(1)$ for $0 < x \le 1$ and $c^*(x) := c(x)$ for $1 < x < \infty$. In the same way as above, we have, for $u > 0$,

$$\int_0^1 \left(\int_0^u |c(\tau) - c^*(\tau)|a(s+u-\tau)d\tau\right)\|X(s)\|^2 ds \le 2\left(\max_{0\le s\le 1}\|X(s)\|^2\right).$$

On the other hand, by Lemma 2.7 and (3.12), we have

$$\int_1^\infty a(s)\|X(s)\|^2 ds < \infty, \tag{4.5}$$

whence, for $u > 1$,

$$\begin{aligned} &\int_1^\infty \left(\int_0^u |c(\tau) - c^*(\tau)|a(s+u-\tau)d\tau\right)\|X(s)\|^2 ds \\ &= \int_1^\infty \left(\int_0^1 |c(\tau) - c^*(\tau)|a(s+u-\tau)d\tau\right)\|X(s)\|^2 ds \\ &\le \left(\int_0^1 |c(\tau) - c^*(\tau)|d\tau\right)\left(\int_1^\infty a(s)\|X(s)\|^2 ds\right) < \infty. \end{aligned}$$

Thus we may also assume that $c(\cdot)$ is a positive constant on $(0,1]$.

By the substitutions $s = us'$ and $\tau = u\tau'$, we see that $\int_0^\infty b(s,u)\|X(s)\|^2 ds$ is equal to

$$u^2 c(u)a(u)\|X(u)\|^2 \int_0^\infty \left(\int_0^1 \frac{c(u\tau)}{c(u)} \cdot \frac{a(u(s+1-\tau))}{a(u)}d\tau\right)\frac{\|X(su)\|^2}{\|X(u)\|^2}ds.$$

Choose $\delta > 0$ so that $H + 3\delta < 1/2$. By applying [5, Theorem 1.5.6 (ii)] to the slowly varying functions $a(t)(1+t)^{H+\frac{3}{2}}$, $c(t)(1+t)^{\frac{1}{2}-H}$ and $\|X(t)\|^2(1+t)^{-2H}$, we see that there exists a positive constant M such that, for $u > 1$,

$$\begin{aligned} c(u\tau)/c(u) &\le M\tau^{H-\frac{1}{2}-\delta} \qquad (0 < \tau < 1), \\ a(u(s+1-\tau))/a(u) &\le Mf_1(s,\tau) \qquad (s > 0,\ 0 < \tau < 1), \\ \|X(su)\|^2/\|X(u)\|^2 &\le Mf_2(s) \qquad (s > 0), \end{aligned}$$

where we define $f_1(s,\tau) := \max\{(s+1-\tau)^{-H-\frac{3}{2}+\delta}, (s+1-\tau)^{-H-\frac{3}{2}-\delta}\}$ and $f_2(s) := (1+s)^{2H}\max(s^{\delta}, s^{-\delta})$. By (3.13), we have

$$\begin{aligned} &\int_1^\infty \left(\int_0^1 \tau^{H-\frac{1}{2}-\delta} f_1(s,\tau)d\tau\right) f_2(s)ds \\ &= \int_1^\infty \left(\int_0^1 \tau^{H-\frac{1}{2}-\delta}(s+1-\tau)^{-H-\frac{3}{2}+\delta}\right)(1+s)^{2H}s^{\delta}ds \\ &= \left(H+\frac{1}{2}-\delta\right)^{-1}\int_1^\infty \frac{s^{-H-\frac{1}{2}+\delta}}{(1+s)}(1+s)^{2H}s^{\delta}ds < \infty. \end{aligned}$$

In the same way,

$$\begin{aligned} &\int_0^1 \left(\int_0^s \tau^{H-\frac{1}{2}-\delta} f_1(s,\tau)d\tau\right) f_2(s)ds \\ &= \int_0^1 \left(\int_0^s \tau^{H-\frac{1}{2}-\delta}(s+1-\tau)^{-H-\frac{3}{2}+\delta}\right)(1+s)^{2H}s^{-\delta}ds \\ &\le \left(H+\frac{1}{2}-\delta\right)^{-1}\int_0^1 \frac{s^{-H-\frac{1}{2}+\delta}}{(1+s)}(1+s)^{2H}s^{-\delta}ds < \infty. \end{aligned}$$

Moreover,

$$\begin{aligned} &\int_0^1 \left(\int_s^1 \tau^{H-\frac{1}{2}-\delta} f_1(s,\tau)d\tau\right) f_2(s)ds \\ &= \int_0^1 \left(\int_s^1 \tau^{H-\frac{1}{2}-\delta}(s+1-\tau)^{-H-\frac{3}{2}-\delta}\right)(1+s)^{2H}s^{-\delta}ds \\ &\le \left(\int_0^1 \tau^{H-\frac{1}{2}-\delta}(1-\tau)^{-\frac{1}{2}(H+\frac{3}{2}+\delta)}d\tau\right)\left(\int_0^1 s^{-\frac{1}{2}(H+\frac{3}{2}+\delta)}(1+s)^{2H}s^{-\delta}ds\right) < \infty. \end{aligned}$$

By the dominated convergence theorem, we obtain

$$\begin{aligned} &\lim_{u\to\infty}\int_0^\infty \left(\int_0^1 \frac{c(u\tau)}{c(u)}\cdot\frac{a(u(s+1-\tau))}{a(u)}d\tau\right)\frac{\|X(su)\|^2}{\|X(u)\|^2}ds \\ &= \int_0^\infty \left(\int_0^1 \tau^{H-\frac{1}{2}}(s+1-\tau)^{-H-\frac{3}{2}}d\tau\right)s^{2H}ds \\ &= \frac{1}{(H+\frac{1}{2})}\int_0^\infty \frac{1}{(1+s)s^{H+\frac{1}{2}}}s^{2H}ds = \frac{\pi}{(H+\frac{1}{2})\sin(\pi(H+\frac{1}{2}))}. \end{aligned}$$

Thus the proposition follows. □

Proposition 4.4. *For $n = 1, 2, \ldots,$ we have*

$$\int_0^\infty b_n(s,u)\|X(s)\|^2 ds < \infty \qquad (u > 0). \tag{4.6}$$

Proof. We use induction. The assertion (4.6) with $n = 1$ follows from Proposition 3.6. Suppose (4.6) holds for $n = k \geq 1$. Then

$$\int_0^\infty b_{k+1}(s,u)\|X(s)\|^2 ds = \int_0^\infty \left(\int_0^\infty b(s,\tau)\|X(s)\|^2 ds\right) b_k(t_2+\tau,u)d\tau.$$

Now $b(s,\tau) = \int_0^\tau c(v)a(s+\tau-v)dv \leq a(s)\int_0^\tau c(v)dv$ for $s,\tau > 0$, whence we get

$$\int_1^\infty b(s,\tau)\|X(s)\|^2 ds \leq \left(\int_1^\infty a(s)\|X(s)\|^2 ds\right)\int_0^\tau c(v)dv.$$

By (3.7), $\int_0^1 b(s,\tau)\|X(s)\|^2 ds \leq \max_{0\leq s\leq 1}\|X(s)\|^2$. Thus, by (4.5), the function $\tau \mapsto \int_0^\infty b(s,\tau)\|X(s)\|^2 ds$ is locally bounded on $[0,\infty)$. From this, Lemma 2.7 and Proposition 4.3, we obtain (4.6) with $n = k+1$. □

Proposition 4.5. *We have, for* $n = 1, 3, 5, \ldots,$

$$\int_{-\infty}^{-t_0} b_n(t_1-s)X(s)ds = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{-t_0} b_n(t_1-s)\left[-c(u-s)+c(u)\right]ds\right)dW^*(u),$$

and, for $n = 2, 4, 6, \ldots,$

$$\int_{t_1}^{\infty} b_n(t_0+s)X(s)ds = \int_{-\infty}^{\infty}\left(\int_{t_1}^{\infty} b_n(t_0+s)\left[c(s-u)-c(-u)\right]ds\right)dW(u).$$

Proof. Suppose that $n = 1, 3, 5, \ldots$. We put $f(u,s) := -c(u-s)+c(u)$ and $b(s) := I_{(-\infty,-t_0)}(s)b_n(t_1-s)$ for $s,u \in \mathbf{R}$. Then since $\int_{-\infty}^{\infty} f(u,s)^2 ds = \|X(-s)\|^2$, we see from Lemma 2.7 and Proposition 4.4 that (3.1)–(3.3) hold. Hence Lemma 3.1 implies the first assertion. The proof of the second assertion is similar. □

Recall $D_n(s) = D_n(s; t_2, t_3)$ from (1.19).

Proposition 4.6. *We have*

$$P_{n+1}^{\perp}\epsilon_n = \begin{cases} \displaystyle\int_{t_1}^{\infty} D_n(s-t_1)dW(s) & (n = 0, 2, 4, \ldots), \\ \displaystyle -\int_{-\infty}^{-t_0} D_n(-t_0-s)dW^*(s) & (n = 1, 3, 5, \ldots). \end{cases} \tag{4.7}$$

Proof. It is easy to see that (4.7) holds for $n = 0$. Suppose that $n = 1, 3, \ldots$. Then, by Proposition 4.5 and [3, Proposition 2.3 (7)], $P_{n+1}^{\perp}\epsilon_n$ is equal to

$$-\int_{-\infty}^{-t_0}\left(\int_{-\infty}^{u} b_n(t_1-s)c(u-s)ds\right)dW^*(u) = -\int_{-\infty}^{-t_0} D_n(-t_0-s)dW^*(s),$$

whence (4.7) for $n = 1, 3, \ldots$. The proof of (4.7) for $n = 2, 4, \ldots$ is similar. □

Proposition 4.7. *For* $s > 0$ *and* $n = 0, 1, \ldots,$ *we have*

$$b_{n+1}(s) = \int_0^\infty a(s+u)D_n(u)du. \tag{4.8}$$

Proof. We easily find that (4.8) holds for $n = 0$ and $s > 0$. We assume that $n \geq 1$. Then, by the Fubini–Tonelli theorem, we see that, for $s > 0$, $b_{n+1}(s)$ is equal to

$$\begin{aligned}\int_0^\infty b(s,u)b_n(t_2+u)du &= \int_0^\infty\left(\int_0^u c(u-v)a(s+v)dv\right)b_n(t_2+u)du \\ &= \int_0^\infty a(s+v)\left(\int_0^\infty b_n(t_2+u+v)c(u)du\right)dv = \int_0^\infty a(s+v)D_n(v)dv.\end{aligned}$$

Thus (4.8) holds. $\square$

We define

$$K(x,y) := \int_0^\infty \{c(x) - c(x+s)\} a(t_2 + s + y) ds \qquad (x, y > 0).$$

Proposition 4.8. *We have*

$$P_{n+1}\epsilon_n = \begin{cases} \int_{-t_0}^\infty \left(\int_0^\infty K(t_0 + s, u) D_{n-1}(u) du \right) dW^*(s) & (n = 1, 3, 5, \dots), \\ -\int_{-\infty}^{t_1} \left(\int_0^\infty K(t_1 - s, u) D_{n-1}(u) du \right) dW(s) & (n = 2, 4, 6, \dots). \end{cases}$$

Proof. We assume $n = 1, 3, \dots$. Then, by [3, Proposition 2.3 (7)], we have

$$\begin{aligned} P_{n+1}\epsilon_n &= P_{[-t_0,\infty)} \int_{-\infty}^\infty \left\{ \int_{-\infty}^{-t_0} b_n(t_1 - s) \left[c(u + t_0) - c(u - s) \right] ds \right\} dW^*(u) \\ &= \int_{-t_0}^\infty \left\{ \int_{-\infty}^{-t_0} b_n(t_1 - s) \left[c(u + t_0) - c(u - s) \right] ds \right\} dW^*(u). \end{aligned}$$

By Proposition 4.7, for $u > -t_0$, $\int_{-\infty}^{-t_0} b_n(t_1 - s) \left[c(u + t_0) - c(u - s) \right] ds$ is equal to

$$\begin{aligned} &\int_0^\infty ds \left[c(u + t_0) - c(u + t_0 + s) \right] \int_0^\infty a(t_2 + s + v) D_{n-1}(v) dv \\ &= \int_0^\infty K(t_0 + u, v) D_{n-1}(v) dv. \end{aligned}$$

This proves the case $n = 1, 3, \dots$. The proof of the case $n = 2, 4, \dots$ is similar. $\square$

Proposition 4.9. *We have* $\lim_{n\to\infty} \int_0^\infty D_n(s)^2 ds = 0$.

Proof. We write Q for the orthogonal projection operator from $M(X)$ onto the closed subspace $M_{(-\infty,t_1]}(X) \cap M_{[-t_0,\infty)}(X)$. Then, by von Neumann's alternating projection theorem (cf. Pourahamadi [12, Section 9.6.3]), we have

$$Q = \operatorname*{s-lim}_{n\to\infty} P_n P_{n-1} \cdots P_1. \tag{4.9}$$

From this and Propositions 4.6 and 4.2, we have

$$\begin{aligned} \int_0^\infty D_{2n}(s)^2 ds &= \left\| P_{2n+1}^\perp \epsilon_{2n} \right\|^2 = \left\| P_{(-\infty,t_1]}^\perp P_{2n} P_{2n-1} \cdots P_1 X(T) \right\|^2 \\ &\to \left\| P_{(-\infty,t_1]}^\perp Q X(T) \right\|^2 = 0 \qquad (n \to \infty). \end{aligned}$$

Similarly, we have $\lim_{n\to\infty} \int_0^\infty D_{2n+1}(s)^2 ds = 0$. Thus the proposition follows. $\square$

Here is a key lemma.

Lemma 4.10. *We have* $\lim_{n\to 0} \|\epsilon_n\| = 0$.

Proof. By integration by parts, we have $K(x,y) = \int_0^\infty \gamma(x+s)\alpha(t_2+s+y)ds$ for $x, y > 0$, where $\gamma(t) := -dc(t)/dt$ for $t > 0$ as in the proof of Lemma 2.7. Since $\gamma(t) = \int_0^\infty e^{-ts} s\nu(ds)$ for $t > 0$, $\gamma(\cdot)$ is a positive and decreasing function on $(0,\infty)$. By (2.8), (3.11) and (1.6), the conditions of Theorem 5.1 in the next section are satisfied with $p = (1/2) - H$. Hence the integral operator $Kf(x) =$

$\int_0^\infty K(x,y)f(y)dy$ is a bounded operator on $L^2((0,\infty),dx)$. By Propositions 4.6, 4.8 and 4.9, we have

$$\begin{aligned}\|\epsilon_n\|^2 &= \int_0^\infty D_n(s)^2 ds + \int_0^\infty \left(\int_0^\infty K(s,u)D_{n-1}(u)du\right)^2 ds \\ &\le \int_0^\infty D_n(s)^2 ds + \|K\|^2 \int_0^\infty D_{n-1}(s)^2 ds \to 0 \qquad (n\to\infty).\end{aligned}$$

Thus the lemma follows. □

Recall from Section 1 that $P_{[-t_0,t_1]}$ is the orthogonal projection operator from $M(X)$ onto $M_{[-t_0,t_1]}(X)$.

Theorem 4.11. *We have* (1.24) *and the following equalities:*

$$P_{[-t_0,t_1]} = \operatorname*{s\text{-}lim}_{n\to\infty} P_n P_{n-1}\cdots P_1, \tag{4.10}$$

$$\|P^{\perp}_{[-t_0,t_1]} Z\|^2 = \left\|P_1^{\perp} Z\right\|^2 + \sum_{n=1}^{\infty} \left\|(P_{n+1})^{\perp} P_n \cdots P_1 Z\right\|^2 \qquad (Z\in M(X)). \tag{4.11}$$

Proof. Let Q be as in the proof of Proposition 4.9. For $t\in(t_1,\infty)$, we claim $P_{[-t_0,t_1]}X(t) = QX(t)$. In fact, (4.4) implies $P_{[-t_0,t_1]} = P_{[-t_0,t_1]}P_nP_{n-1}\cdots P_1$, whence $P_nP_{n-1}\cdots P_1 - P_{[-t_0,t_1]} = P^{\perp}_{[-t_0,t_1]}P_nP_{n-1}\cdots P_1$. On the other hand, we have $P^{\perp}_{[-t_0,t_1]}G_k = 0$ for $k\in\mathbf{N}$. Hence, it follows from Proposition 4.2 and Lemma 4.10 that

$$\begin{aligned}\left\|P_nP_{n-1}\cdots P_1X(t) - P_{[-t_0,t_1]}X(t)\right\| &= \left\|P^{\perp}_{[-t_0,t_1]}P_nP_{n-1}\cdots P_1X(t)\right\| \\ &= \left\|P^{\perp}_{[-t_0,t_1]}\left(\epsilon_n + \sum\nolimits_1^n G_k\right)\right\| = \left\|P^{\perp}_{[-t_0,t_1]}\epsilon_n\right\| \le \|\epsilon_n\| \to 0 \qquad (n\to\infty).\end{aligned}$$

This and (4.9) imply the claim above. The rest of the proof is the same as that of [3, Theorem 4.6], and so we omit it. □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. By (4.10), Proposition 4.2 and Lemma 4.10, we have

$$P_{[-t_0,t_1]}X(T) = \lim_{n\to\infty}\sum_{k=1}^{n} G_k \qquad \text{in } M(X). \tag{4.12}$$

It holds that

$$\sum_{k=1}^{n} G_k = \begin{cases} \int_{-t_0}^{t_1} h_n(t_0+s)X(s)ds + \beta_n X(-t_0) & (n=1,3,5,\dots), \\ \int_{-t_0}^{t_1} h_n(t_0+s)X(s)ds + \beta_n X(t_1) & (n=2,4,6,\dots), \end{cases}$$

where, for $0<s<t_2$, we define

$$h_n(s) = \begin{cases} b_1(t_2-s) + b_2(s) + \cdots b_n(t_2-s) & (n=1,3,5,\dots), \\ b_1(t_2-s) + b_2(s) + \cdots b_n(s) & (n=2,4,6,\dots). \end{cases}$$

Notice that $h_n(s) \uparrow h(s)$ as $n \to \infty$. From (4.12) and Lemma 4.1, we see that $P_{[-t_0,t_1]}X(T)$ has the following two kinds of representations:

$$\int_{-t_0}^{t_1} h(t_0+s)X(s)ds + \beta_\infty X(-t_0) = \int_{-t_0}^{t_1} h(t_0+s)X(s)ds + \beta_\infty X(t_1),$$

where $\beta_\infty := \lim_{n\to\infty} \beta_n$. However β_∞ must be zero since $X(-t_0)$ and $X(t_1)$ are linearly independent. Thus (1.22) follows. The assertion (1.21) follows from (4.1). By (4.11) and Propositions 4.2 and 4.6, $\|P_{[-t_0,t_1]}^{\perp}X(T)\|^2$ is equal to

$$\left\|P_1^{\perp}X(T)\right\|^2 + \sum_{n=1}^{\infty} \left\|(P_{n+1})^{\perp}P_n\cdots P_1 X(T)\right\|^2 = \sum_{n=0}^{\infty}\int_0^\infty D_n(s)^2 ds.$$

Thus we obtain (1.23). □

5. L^2-BOUNDEDNESS THEOREM

In this section, we prove the L^2-boundedness theorem that we need in the proof of Lemma 4.10.

Theorem 5.1. *Let $p \in (0,1/2)$ and let $\ell(\cdot)$ be a slowly varying function at infinity. Let $C(\cdot)$ and $A(\cdot)$ be nonnegative and decreasing functions on $(0,\infty)$. We assume*

$$A(t) \sim \frac{t^{-(1-p)}}{\ell(t)} \cdot \frac{1}{\Gamma(p)} \qquad (t\to\infty), \tag{5.1}$$

$$C(t) \sim t^{-(1+p)}\ell(t) \cdot \frac{p}{\Gamma(1-p)} \qquad (t\to\infty). \tag{5.2}$$

We also assume $A(0+) < \infty$ and

$$\textstyle\int_t^\infty C(s)ds = O(t^q) \quad (t\to 0+) \quad \text{for some } q > -1/2.$$

Then

$$\sup_{0<x<\infty} \int_0^\infty K(x,y)\,(x/y)^{1/2}\,dy < \infty, \tag{5.3}$$

$$\sup_{0<y<\infty} \int_0^\infty K(x,y)\,(y/x)^{1/2}\,dx < \infty, \tag{5.4}$$

where $K(x,y) := \int_0^\infty C(x+u)A(u+y)du$ for $x,y>0$. In particular, the integral operator K defined by $(Kf)(x) := \int_0^\infty K(x,y)f(y)dy$ for $x>0$ is a bounded operator on $L^2((0,\infty),dy)$.

Proof. Step 1. Since $K(x,y) \le A(y)\int_x^\infty C(s)ds$, we have

$$\int_0^\infty K(x,y)(x/y)^{1/2}dy \le x^{1/2}\left(\int_x^\infty C(s)ds\right)\left(\int_0^\infty \frac{A(y)}{y^{1/2}}dy\right) \to 0 \quad (x\to 0+),$$

whence

$$\lim_{x\to 0+}\int_0^\infty K(x,y)(x/y)^{1/2}dy = 0. \tag{5.5}$$

We have

$$\sup_{0<x<\infty}\int_0^\infty K(x,y)x^{-1/2}dx \le \int_0^\infty K(x,0)x^{-1/2}dx$$
$$\le A(0+)\int_0^1\left(\int_x^\infty C(s)ds\right)x^{-1/2}dx + \int_1^\infty\left(\int_0^\infty C(x+u)A(u)du\right)x^{-1/2}dx.$$

By (5.1) and (5.2), $\int_0^\infty C(x+u)A(u)du \sim \pi^{-1}\sin(\pi p)x^{-1}$ as $x \to \infty$ (cf. [8, Proposition 4.3]), whence

$$\sup_{0<y<\infty} \int_0^\infty K(x,y)x^{-1/2}dx < \infty. \tag{5.6}$$

Step 2. We claim

$$\lim_{x\to\infty} \int_0^\infty K(x,y)\,(x/y)^{1/2}\,dy = \frac{\sin^2(p\pi)}{\cos(p\pi)}. \tag{5.7}$$

The assertion (5.3) follows from this and (5.5). We have

$$\begin{aligned}
&\int_0^\infty K(x,y)\,(x/y)^{1/2}\,dy \\
&= x^2C(x)A(x)\int_0^\infty \left(\int_0^\infty \frac{C(x(1+u))}{C(x)}\cdot\frac{A(x(u+y))}{A(x)}du\right)y^{-1/2}dy.
\end{aligned}$$

By standard arguments that involve Theorem 1.5.6 (ii) of [5] and the dominated convergence theorem (cf. the proofs of Lemma 2.7 and Proposition 4.3), we get

$$\begin{aligned}
&\lim_{x\to\infty}\int_0^\infty \left(\int_0^\infty \frac{C(x(1+u))}{C(x)}\cdot\frac{A(x(u+y))}{A(x)}du\right)y^{-1/2}dy \\
&= \int_0^\infty \left(\int_0^\infty \frac{1}{(1+u)^{1+p}(u+y)^{1-p}}du\right)y^{-1/2}dy \\
&= \left(\int_0^\infty \frac{1}{(1+y)^{1-p}y^{1/2}}dy\right)\left(\int_0^\infty \frac{1}{(1+u)^{1+p}u^{(1/2)-p}}du\right) = \frac{\pi}{p}\tan(p\pi).
\end{aligned}$$

Since (5.1) and (5.2) imply

$$\lim_{x\to\infty} x^2C(x)A(x) = \frac{p\sin(p\pi)}{\pi}, \tag{5.8}$$

we obtain (5.7).

Step 3. We claim

$$\lim_{y\to\infty} \int_0^\infty K(x,y)\,(y/x)^{1/2}\,dx = \frac{\sin^2(p\pi)}{\cos(p\pi)}. \tag{5.9}$$

The assertion (5.4) follows from this and (5.6). We put $C^*(x) := C(1)$ for $0 < x \le 1$, and $C^*(x) := C(x)$ for $1 < x < \infty$. Then

$$\begin{aligned}
&\int_0^\infty \left(\int_0^\infty |C(x+u) - C^*(x+u)|A(u+y)du\right)(y/x)^{1/2}\,dx \\
&\le y^{1/2}A(y)\int_0^1\left(\int_0^1 [C(x+u)+C^*(x+u)]\,du\right)x^{-1/2}dx \\
&\le y^{1/2}A(y)\left\{\int_0^1\left(\int_x^\infty C(u)du\right)x^{-1/2}dx + C(1)\int_0^1 x^{-1/2}dx\right\} \to 0 \quad (y\to\infty).
\end{aligned}$$

Thus, to prove (5.9), we may assume that $C(\cdot)$ is a positive constant on $(0,1]$. As in Step 2, $\int_0^\infty K(x,y)\,(y/x)^{1/2}\,dx$ is equal to

$$y^2C(y)A(y)\int_0^\infty\left(\int_0^\infty \frac{C(y(x+u))}{C(y)}\cdot\frac{A(y(u+1))}{A(y)}du\right)x^{-1/2}dx.$$

By [5, Theorem 1.5.6 (ii)] and the dominated convergence theorem, we have

$$\lim_{y\to\infty}\int_0^\infty\left(\int_0^\infty \frac{C(y(x+u))}{C(y)}\cdot\frac{A(y(u+1))}{A(y)}du\right)x^{-1/2}dx$$
$$=\int_0^\infty\left(\int_0^\infty \frac{1}{(1+u)^{1-p}(x+u)^{1+p}}du\right)x^{-1/2}dx=\frac{\pi}{p}\tan(p\pi).$$

Thus, from (5.8), we obtain (5.9) as desired. □

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