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THE NINE MORSE GENERIC TETRAHEDRA*

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Abstract. There are two types of shapes for a generic triangle—acute and obtuse. These shapes are also distinguished by the (topological) Morse theory of the minimal distance function to the vertices. We can use the same method for a tetrahedron, and we show in this paper that there exist nine generic shapes. These can be described by a Morse poset or by a Gabriel graph. We also report on some computer experiments and compare our classification to another criterion used in computational geometry.

Key words. Morse theory, polyhedra, Voronoi diagram, tetrahedral shape

AMS subject classifications. Primary, 51M20; Secondary, 68U05

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1. Introduction.

Take N points $P_1, \ldots, P_N$ in $\mathbb{R}^n$ and consider the function $d : \mathbb{R}^n \to \mathbb{R}$ defined by

$$d(X) = \min_{j=1,\ldots,N} d(X, P_j).$$

We study the evolution of the sets $d_\epsilon = \{X \mid d(X) \le \epsilon\}$ as ϵ increases. In particular, we are interested in the Euler characteristic χ of d_ϵ. In case ϵ is very small, d_ϵ consists of N small solid spheres. Thus $\chi = N$. If ϵ is very big, then d_ϵ is contractible, and hence $\chi = 1$.

For a generic set of points, d is a topological Morse function. In that case, as ϵ grows, d passes through a number of nondegenerate critical values. When d passes a critical value of index i, an i-cell gets attached.

The number of critical points of index i is s_i. From Morse theory we know that

$$\sum (-1)^i s_i = 1.$$

As an example, take a triangle with an obtuse angle in the plane. This is a special case of the above problem with $n = 2$ and $N = 3$. Assume further that the two legs that encompass the obtuse angle have different lengths. In that case, $s_0 = 3$, $s_1 = 2$, and $s_2 = 0$. For an acute triangle where the edges have different lengths, we obtain $s_0 = 3$, $s_1 = 3$, and $s_2 = 1$. In this sense, there are two different generic triangles.

Returning to the n-dimensional case, each critical point of index i corresponds to a subset of size $i+1$ of $\{P_1, \ldots, P_N\}$, but not every subset of size $i+1$ corresponds to a critical point of index i. (With the obtuse triangle, the 2-face of the triangle, that is, the triangle itself, does not correspond to a local maximum of the function d.) Thus to the N-point set $P_1, \ldots, P_N$ we can associate a set of subsets that correspond to critical points of d. By extension we will call these subsets and the geometric simplices that they span critical as well. This set of subsets is partially ordered by inclusion, and thus it is a poset. We will call it the *Morse poset*.

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Question. In $\mathbb{R}^n$, for generic sets of N points, how many different Morse posets are there, up to combinatorial equivalence?

When $N = n + 1$ we thus ask how many "different" generic simplices there are. This is a natural first problem to consider.

In the plane the answer is two; see [Sie99, section 2]. For obtuse triangles, the Morse poset is

$$\{\{P_1\}, \{P_2\}, \{P_3\}, \{P_1, P_3\}, \{P_2, P_3\}\},$$

and for acute triangles, we get the Morse poset

$$\{\{P_1\}, \{P_2\}, \{P_3\}, \{P_1, P_2\}, \{P_1, P_3\}, \{P_2, P_3\}, \{P_1, P_2, P_3\}\}.$$

The main theorem of this article says that in $\mathbb{R}^3$, there are nine different generic tetrahedra (3-simplices).

In the first section, we recall the relevant Morse theory. Then we establish some notation and state what Voronoi diagrams and Gabriel graphs are. Next, we state and prove the main theorem. We also report on numerical experiments concerning volume data of the different compartments of the configuration space where the Morse poset is of a certain type. Finally, we compare our classification to the classification by shape types in [Ede01]. The Morse theoretic approach relates to alpha shapes as defined by Edelsbrunner [EM94].

2. Genericity conditions. We focus here on four points in $\mathbb{R}^3$, but most of the notation and definitions have straightforward extensions to the (n, N) general case.

We write P_{ij} for the middle of the interval P_iP_j, P_{ijk} for the center of the circumscribed circle of the triangle $P_iP_jP_k$, and P_{1234} for the center of the circumscribed sphere of the tetrahedron. For the tetrahedron itself, that is, the convex hull of $\{P_1, P_2, P_3, P_4\}$, we will use the notation $\mathbb{T}$.

We impose the following condition.

Genericity Condition 2.1. *We require the set of points P_1, P_2, P_3, P_4 to be in general position, so that the convex hull of P_1, P_2, P_3, P_4 is 3-dimensional. Moreover, the points P_{ijk} do not lie on one of the edges of the triangle $P_iP_jP_k$, and also P_{1234} does not lie on one of the planes of the triangles $P_iP_jP_k$.*

Genericity Condition 2.1 means that the function d is not too badly behaved. To express more carefully what that means we recall the definition of a topological Morse function; see [Mor59]. Let $P \in \mathbb{R}^n$, and let f be a continuous real-valued function on $\mathbb{R}^n$.

Definition 2.2. *The function f is* topologically regular *at $P \in \mathbb{R}^n$ if there is some neighborhood U of P and a homeomorphism $\phi: U \to U$ such that the composition $f \circ \phi$ is a nonconstant affine function on U. The function f has a critical point at $P \in \mathbb{R}^n$ if f is not topologically regular at P. In that case, P is called a* nondegenerate critical point of index i *if there is a neighborhood U of P and a homeomorphism $\phi: U \to U$ such that*

$$f \circ \phi = f(P) - \sum_{j=1}^{i} x_j^2 + \sum_{j=i+1}^{n} x_j^2.$$

A topological Morse function is a continuous function that has only regular and nondegenerate critical points.

For topological Morse functions the two crucial statements that hold in the differentiable case—the regular interval theorem and the attachment of cells (see Chapter 5

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in [Mil63])—are true as well, as Morse proves in [Mor73]. So, if we show that d is topologically regular, we can apply those theorems, just as was done in [Sie99].

We will need the following notation.

Let

$$\mathrm{Terr(P_i) = \{X \in \mathbb{R}^3 | d(X, P_i) \leq d(X, P_k) \text{ for all } k\}}$$

and $V_i = \mathrm{Terr(P_i)}$, $V_{ij} = V_i \cap V_j$ (i different from j), $V_{ijk} = V_i \cap V_j \cap V_k$ (i, j, k *different from one another*), and $V_{1234} = V_1 \cap V_2 \cap V_3 \cap V_4$.

PROPOSITION 2.3. *The function d is a topological Morse function if Genericity Condition* 2.1 *is fulfilled. In that case, d is topologically regular in all points of* $\mathbb{R}^3$, *except in P_1, P_2, P_3, P_4, where d has a minimum, and (perhaps) in the points P_{ij}, P_{ijk}, and P_{1234}. Moreover, d has*

- *a minimum exactly in the points P_1, P_2, P_3, P_4,*
- *a* 1*-saddle (saddle point of index* 1*) in P_{ij} iff $P_{ij} = V_{ij} \cap P_iP_j$,*
- *a* 2*-saddle (saddle point of index* 2*) in P_{ijk} iff $P_{ijk} = V_{ijk} \cap P_iP_jP_k$, and*
- *a maximum in P_{1234} iff $P_{1234} = V_{1234} \cap \mathbb{T}$ (equivalently, $P_{1234} \in \mathbb{T}$).*

Proof. If $x = P_i$, then because the points lie in general position all P_j with $i \neq j$ lie at some positive distance from x, so d has a minimum there. If $x \in \mathrm{Terr(P_i)}$ but $x \neq P_i$, then x is obviously topologically regular.

The function d restricted to the interior V_{ij} has a minimum if P_{ij} lies in that interior. This can happen only when $P_{ij} \neq P_{ijk}$, which is ensured by Genericity Condition 2.1.

In the directions, orthogonal to V_{ij}, d decreases, so we see that d has a critical point of index 1 at P_{ij}.

The function d restricted to the interior of V_{ijk} has a minimum at P_{ijk} if P_{ijk} lies in that interior. This can be the case only when $P_{ijk} \neq P_{1234}$. In the directions orthogonal to V_{ijk}, d decreases, so d has a critical point of index 2 at P_{ijk}.

Finally, if $P_{1234} \in \mathbb{T}$, d obviously has a local maximum there. ∎

Remark 2.4. This statement is well known. One can find it in several places; see, for instance, [Ede04].

Remark 2.5. In the 2-dimensional case the distance function d is always a topological Morse function, whether the points are in general position or not. For a proof, see [Sie99]. It would be interesting to know whether the distance function would always be a topological Morse function in higher dimensions also, irrespective of general position considerations.

The conditions of Proposition 2.3 tell us exactly the positions of the critical points. As explained in the introduction, a critical point determines a subset of the points P_1, P_2, P_3, P_4, which we call *critical subset*. The *Morse poset* is the set of critical subsets, partially ordered by inclusion. Two sets of points in $\mathbb{R}^3$ are called *combinatorially equivalent* if there exists a bijection between points that sends the critical subsets onto each other. We want to give a classification with respect to this equivalence relation.

Throughout this paper we deal with four points, generic in the sense of Genericity Condition 2.1 *and contained in* $\mathbb{R}^3$. However, it seems relevant to point out that our problem can be formulated in a more general setting. In the next few definitions we will deal with a point set $\{P_1, \dots, P_N\}$ in $\mathbb{R}^n$ without making any genericity assumptions.

DEFINITIONS 2.6. *A subset $\{P_i\}_{i \in I}$ belongs to the* Delaunay triangulation *if there are points $X \in \mathbb{R}^n$ with $d(X, P_i) = d(X, P_j)$ for all pairs $\{i, j\} \subset I$ and $d(X, P_i) < d(X, P_j)$ for all pairs $\{i, j\}$ with $i \in I$ and $j \notin I$. The* Voronoi cell *of such a subset $\{P_i\}_{i \in I}$ consists of the points $X \in \mathbb{R}^n$ that make the subset part of the Delaunay*

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triangulation. The Delaunay cell *of such a subset* $\{P_i\}_{i\in I}$ *is simply the convex hull of* $\{P_i\}_{i\in I}$.

In particular, all subsets of cardinality 1 *are part of the Delaunay triangulation. For each* $1 \leq i \leq N$ *we denote the closure of the Voronoi cell of* P_i *by* $\mathrm{Terr(P_i)}$, *because such a Voronoi cell is best thought of as a territory with capital* P_i.

The union of all $< n$ *dimensional Voronoi cells is usually called the* Voronoi diagram. *In fact, the closure of any* $(n-1)$*-dimensional Voronoi cell is always the intersection of exactly two sets* $\mathrm{Terr(P_i)}$ *and* $\mathrm{Terr(P_j)}$. *Thus, dual to the Voronoi diagram we have the* 1*-skeleton*

$$\{P_iP_j \quad | \quad \mathrm{dim(Terr(P_i)\cap Terr(P_j)) = n-1}\}.$$

The Gabriel graph *(see* [BKOS97], *where it is defined in* $\mathbb{R}^2$*) is a subset of this skeleton: it consists of those edges* P_iP_j *such that the circumball with poles* P_i *and* P_j *does not contain any of the other points in the point set* $\{P_1,\ldots,P_N\}$. *The* Gabriel complex *(see* [AGJ00]*) is the subcomplex of the Delaunay triangulation, generated by those* $(n-1)$*-dimensional Delaunay cells whose smallest circumball does not contain any of the other points.*

Remark 2.7. General position is defined in this more general case as follows. We require that for each subset $\{P_i\}_{i\in I}$ in the Delaunay triangulation the possibly empty intersection of the Delaunay cell and the Voronoi cell is contained in both the relative interior of the Delaunay and the relative interior of the Voronoi cell. For more details, see [Ede01] and [EM94].

QUESTION 2.8. *What is the relation between the Morse poset and the Gabriel graph and complex? The edge between* P_i *and* P_j *is part of the Gabriel graph iff the point* P_{ij} *is a saddle point (index* 1*) of* d. *These are exactly the critical subsets of size* 2 *of the Morse poset. In the* 3*-dimensional case a face between* P_i, P_j, *and* P_k *is part of the Gabriel complex iff the point* P_{ijk} *is a saddle point (index* 2*) of* d. *These are precisely the critical subsets of size* 3 *of the Morse poset. But the Gabriel complex also contains by definition the boundaries of those* 2*-simplices, while for the Morse poset this is not automatically the case. So the Morse poset is not a simplicial complex and can differ from the Gabriel complex. See examples of tetrahedrons of type* $(4,5,3,1)$ *in the next section.*

The Morse poset contains the information from both the Gabriel graph and the Gabriel complex. Two Morse posets can be combinatorially equivalent only if the underlying Gabriel graph and Gabriel complex are the same.

3. The main classification. From Proposition 2.3 we know the maximal number of critical points of each type. Moreover, the Euler characteristic is +1: $s_0 - s_1 + s_2 - s_3 = 1$. This gives a priori the following 9 possibilities:

s_0	s_1	s_2	s_3
4	6	4	1
4	5	3	1
4	4	2	1
4	3	1	1
4	2	0	1
4	6	3	0
4	5	2	0
4	4	1	0
4	3	0	0

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But not all possibilities will occur. Since we start with four vertices and the result should be a connected space, we need at least three saddle points of index 1. This rules out the possibility $(4, 2, 0, 1)$.

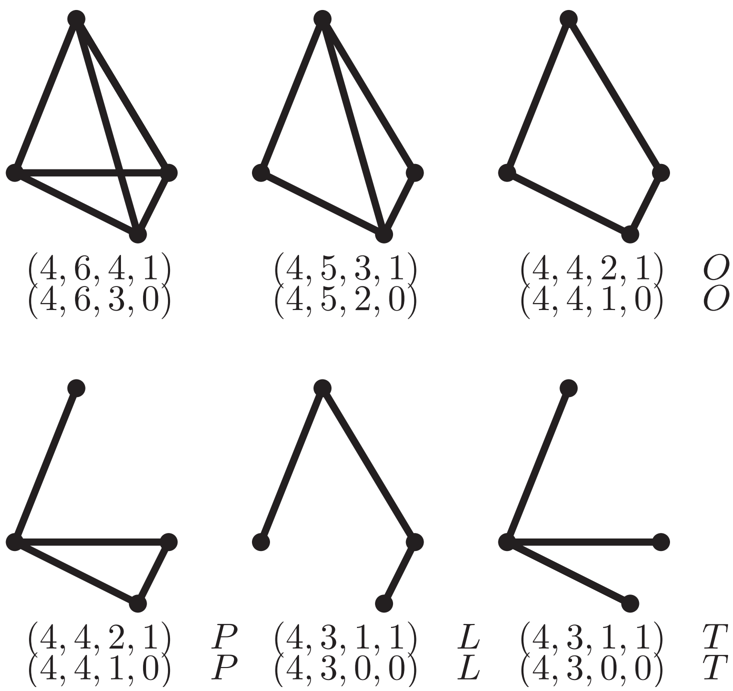

Fig. 1. *List of possible Gabriel graphs. The vertices are the minima of the distance function. Each vertex of the tetrahedron $\mathbb{T}$ is a minimum of d. There are no other minima. The edges of the graph are the index 1 critical points of the distance function. Not each midpoint of an edge of $\mathbb{T}$ is an index 1 critical point of the distance function d. The graph $(4410)\,O$ can be laid out so as to form the letter "O." The names of the other graphs are chosen similarly.*

We list in Figure 1 the (a priori) possible Gabriel graphs for the above cases; they are the connected graphs with four vertices. Just as the cases $(4, 2, 0, 1)$ cannot occur, we will prove in this section by deriving a contradiction that the cases $(4, 4, 2, 1)\,P$, $(4, 3, 1, 1)\,T$, and $(4, 3, 1, 1)\,L$ do not occur. We will prove our main theorem.

Theorem 3.1. *Up to combinatorial equivalence of their Morse posets there are nine generic tetrahedra. They are uniquely described by the nine Gabriel graphs* $(4, 6, 4, 1)$, $(4, 6, 3, 0)$, $(4, 5, 3, 1)$, $(4, 5, 2, 0)$, $(4, 4, 2, 1)\,O$, $(4, 4, 1, 0)\,O$, $(4, 4, 1, 0)\,P$, $(4, 3, 0, 0)\,L$, *and* $(4, 3, 0, 0)\,T$, *drawn in Figure* 1.

Proof. We have to exclude $(4, 3, 1, 1)\,T$, $(4, 3, 1, 1)\,L$, and $(4, 4, 2, 1)\,P$ from the list. In these cases M is critical. We first pay attention to the following.

3.1. Saddle points of index 1. We consider the plane E through $P_1P_2P_3$ (and set it as the "ground plane" in the picture). The half space containing P_4 is called "above," and the other is called "below."

We are going to consider the condition that the point $Y = P_{i4}$ is critical. We first make no assumptions about the triangle $P_1P_2P_3$ except that it does not have a right angle. We consider the point $X = P_4$ as a variable and denote its projection on the E-plane by X'. Let $Y = P_{14}$ and its projection be Y'. From the condition

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that $Y = P_{14}$ is critical we get

$$d(P_1, Y) = d(P_4, Y) < \min\{d(P_2, Y), d(P_3, Y)\}.$$

The inequality is strict because of Genericity Condition 2.1.

For the projection, this means that in terms of Voronoi diagrams in E,

$$Y' \in \text{Interior}(\text{Terr}(\text{P}_1)).$$

In terms of the projection X', this means

$$X' \in \text{Interior}(2\text{Terr}(\text{P}_1)),$$

where by $2\text{Terr}(\text{P}_1)$ we mean scalar multiplication of $\text{Terr}(\text{P}_1)$ by a factor 2 from the point P_1.

We can do the same for P_{24} and P_{34}. We get three subsets $2\text{Terr}(\text{P}_1)$, $2\text{Terr}(\text{P}_2)$, and $2\text{Terr}(\text{P}_3)$, which cover the plane E. They divide the plane into regions, where one, two, or three of the points P_{14}, P_{24}, or P_{34} are critical.

Inside the plane E, the border of $\text{Terr}(\text{P}_\text{i})$ consists of two half lines that meet in P_{123}. The scalar multiplication by 2 maps P_{123} to P_i^*, the antipodal point of P_i on the circle through P_1, P_2, and P_3.

Next, look at the case of an acute triangle drawn in Figure 2.

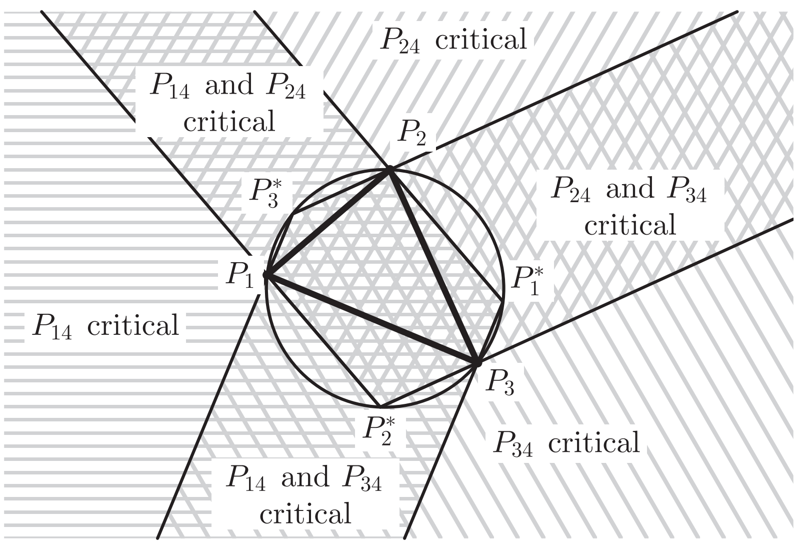

FIG. 2. *The plane E when the triangle is acute.*

The regions where only one P_{i4} is critical are all outside the disc D, which is bounded by the circumscribed circle of triangle $P_1P_2P_3$.

The picture for the obtuse case is in Figure 3.

Let P_2 be the vertex with the obtuse angle. We see that the region where only P_{24} is critical is outside D but that the regions where P_{14} or P_{34} is the only critical point can have some intersections with D.

3.2. The center M of the circumscribed sphere. We assume again that $P_1P_2P_3$ lies in the plane E. We know that $M = P_{1234}$ belongs to the axis of the triangle $P_1P_2P_3$, so its projection is $M' = P_{123}$. Fix M for the moment, and consider

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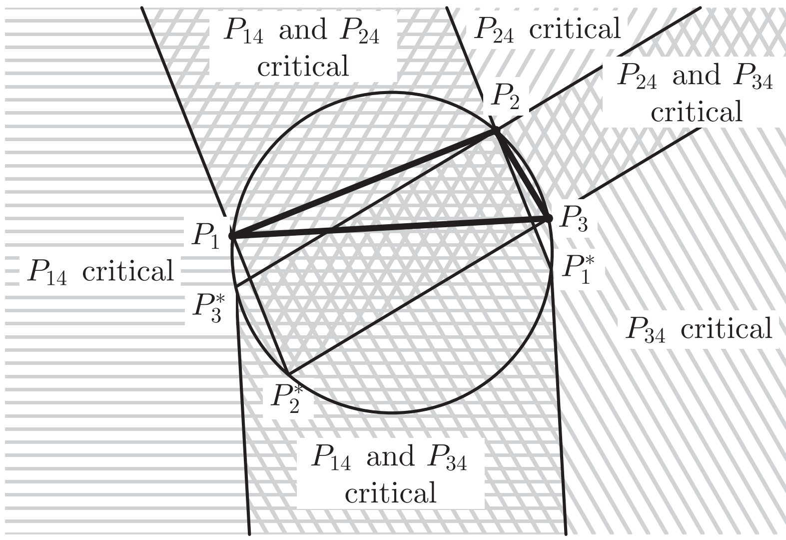

FIG. 3. *The plane E when the triangle is obtuse. Indicated are the territories* $2\mathrm{Terr(P_i)}$*, $i = 1, \ldots, 3$, and their intersections.*

$X = P_4$ as a variable. Possible positions of $X = P_4$ are on the sphere with center M and radius $r = d(P_1, M) = d(P_2, M) = d(P_3, M)$. Denote by $P_1^* P_2^* P_3^*$ the mirror image of $P_1P_2P_3$ after central reflection through P_{123}.

In the cases we wish to exclude, M is critical. In other words, the distance function has a local maximum in M, and M lies inside the tetrahedron. The point X must lie in the double cone with apex M and base the triangle $P_1P_2P_3$, but also on the sphere (M, r). Consider the plane E'', which is the image of E by central reflection through M. E'' is parallel to E. The cone intersects the plane in a triangle, whose orthogonal projection onto the plane E is the triangle $P_1^* P_2^* P_3^*$. X belongs to a part of the cap of the sphere, which lies "above" E'' (seen from M). It follows that X projects to a point X' which lies inside the disc D. This is true for any position of M. For the acute case this is all we need to conclude that it is not possible that M and only one P_{i4} are critical. In the obtuse case we have to use additional arguments to get to this same conclusion. Let P_2 be the obtuse angle. Recall that M is critical iff it belongs to the interior of the tetrahedron $P_1P_2P_3X$. Equivalently, X' lies in the interior of the reflected triangle, that is, in $P_1^* P_2^* P_3^*$. This triangle is contained in the sector $P_1^* M' P_3^*$, which implies that M is critical only if X' lies in the interior of this sector (the one which does not contain P_2). We combine the conditions for M to be critical and one single P_{i4} is critical. From the geometric observations above it follows that the combination of M being critical and a single P_{i4} being critical cannot occur together.

Next choose the appropriate labels for the vertices of the tetrahedron: In cases $(4, 3, 1, 1)\,T$ and $(4, 3, 1, 1)\,L$ the ground plane should contain two adjacent critical edges, and in case $(4, 4, 2, 1)\,P$ it should contain the triangle with the three critical edges. This observation rules out the graphs $(4, 3, 1, 1)\,T$, $(4, 3, 1, 1)\,L$, and $(4, 4, 2, 1)\,P$.

3.3. Positions of index 2 saddles. We are left with nine possibilities of Gabriel graphs. The Gabriel graphs do not give a complete picture of the combinatorics of the Morse points. The Gabriel graph, together with the information on whether M is a critical point or not, tells us only the number of saddles of index 2, but not the position.

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Lemma 3.2. *Generic tetrahedra with isomorphic Gabriel graphs have combinatorially equivalent Morse posets.*

Proof. We need to prove that in all cases there are unique positions for the 2-saddles.

$(4, 6, 4, 1)$: Unique positions (no choices).

$(4, 5, 3, 1)$: There are two triangles, where all three midpoints are critical. Both triangles must be acute. Take one of them in the "ground plane" and assume it to be $P_1P_2P_3$. Since P_{1234} is critical, P_{1234} lies on the same side as P_4. It follows that V_{123} must intersect the ground plane in the point P_{123}, so that point is critical. The same reasoning applies to the second triangle; this fixes the position of two 2-saddles. The two positions left for the third saddle are combinatorially equivalent.

$(4, 4, 1, 0)\,O$: All positions equivalent.

$(4, 4, 2, 1)$: Suppose P_{12} is not critical. Now P_3 or P_4 must be contained in the ball $B = B(P_{12}, r_{12})$, where $r_{12} = d(P_1, P_{12})$. If P_{123} is critical, then triangle $P_1P_2P_3$ must be acute; this means that P_3 is outside the ball B. If P_{124} is critical, then triangle $P_1P_2P_4$ must be acute; this means that P_4 is outside the ball B. It follows that the situation where two 2-saddles are separated by a noncritical edge cannot occur. This fixes the places of the 2-saddles up to permutation.

$(4, 6, 3, 0)$: All positions equivalent.

$(4, 5, 2, 0)$: Choose a triangle, say, $P_1P_2P_3$, where all three edges are critical. If P_{123} is not critical, then it follows that P_{1234} lies below the ground plane E. We look again at the projection P_4'. First P_4' must lie inside D. But since two of the P_{i4} (say, P_{14} and P_{24}) must be critical, we know that P_4' also must lie in the region described in section 3.1. The intersection is a subset of D, which is contained in the region D^* bounded by the arc $P_1P_3^*P_2$ and by the edges $P_2P_3^*$ and $P_3^*P_1$. There are still other conditions to meet:

- P_4 must lie outside the ball $B(P_{12}, r_{12})$, since P_{12} is critical.
- P_4 must lie inside the ball $B(P_{123}, r_{123})$, where $r_{123} = d(P_1, P_{123})$.

This is not simultaneously possible if P_4' lies in D^*. It follows that there is only one possibility: the 2-saddles are the centers of two triangles with all edges critical.

$(4, 4, 1, 0)\,P$: Let $P_1P_2P_3$ be the triangle with all three midpoints of the edges critical. This triangle is acute; take its plane as ground plane. If P_{1234} is above the ground plane, then P_{123} is critical. Assume now that P_{1234} is below the ground plane. It follows that P_4' lies inside the disc D. But the fact that only P_{14} is critical means that P_4' is outside. This is a contradiction, so only P_{123} can be critical.

$(4, 3, 0, 0)\,T$: All places equivalent.

$(4, 3, 0, 0)\,L$: All places equivalent. ∎

3.4. Existence and statistics. We carried out some statistical experiments to see how the different types of tetrahedra are distributed among different 4-tuples of points. That all the cases do occur follows from the computer experiments described below. As a consequence all nine generic tetrahedrons are realized and the proof of Theorem 3.1 is complete. ∎

Four points in $\mathbb{R}^3$ do not form a bounded space, and thus there lives no uniform probability distribution on it. As the Morse poset is invariant under translations, scaling, and rotations, without loss of generality we can assume that all four points lie on S^2. By translation and scaling, this can always be achieved. We consider the configuration space of different points in S^2. The nongeneric tetrahedra form a hypersurface (of measure zero) in this space—the discriminant hypersurface.

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The uniform distribution on $[0,1]^2$ leads (as Archimedes already used) to a uniform distribution on the 2-sphere by the map $\gamma : [0,1]^2 \to S^2 \subset \mathbb{R}^3$ given by

$$\gamma(a_1, a_2) = (\sin(\arccos(2a_1 - 1)) \sin(2\pi a_2), \sin(\arccos(2a_1 - 1)) \cos(2\pi a_2), 2a_1 - 1).$$

We take two random numbers in $[0,1]$ and map them to S^2 using γ.

For our experiment we used the Gnu Scientific Library; see [Gal06]. This library has an implementation of the apparently very reliable MT19937 random number generator. Two random numbers in $[0,1]$ were mapped to S^2 using γ. We took samples of 10^8 tetrahedra. Here is one:

1	$(4,3,0,0)\,L$	17,807,919
2	$(4,3,0,0)\,T$	898,689
3	$(4,4,1,0)\,O$	26,224,574
4	$(4,4,1,0)\,P$	16,421,773
5	$(4,5,2,0)$	24,350,101
6	$(4,6,3,0)$	1,797,721
7	$(4,4,2,1)$	3,266,345
8	$(4,5,3,1)$	2,697,783
9	$(4,6,4,1)$	6,535,095

Other samples gave approximately the same numbers, with a maximum difference of 3000.

For three random points on the circle, it is a simple exercise to see that chances are 3 out of 4 that one gets an obtuse triangle. These results indicate that for four random points on the unit sphere the chances are 7 out of 8 that the center of the unit sphere is a local maximum for the distance function. Indeed, the last three entries of the above table add up to $12,499,223$. Doubtlessly, a more general statement can be proven here.

4. Notes and remarks.

4.1. Discrete Morse functions. A natural question is the existence of a discrete Morse function (in the sense of Forman) [For02] on the tetrahedron, which realizes the same effect as the minimal distance function. The question is discussed extensively in [vMS05]. It turns out that the corresponding discrete function contains more information than the Morse poset. It also includes, e.g., information about the critical values. Taking into account the critical values, one can make a much finer classification, which distinguishes, e.g., in the $(4,6,4,1)$ case, between an almost regular tetrahedron and one with a small acute base triangle and a top vertex far away. Another related concept is the alpha shape as discussed in, e.g., [EM94], which has applications in molecular shape analysis.

4.2. Higher dimensional results. We have not been able to prove a classification theorem in $\mathbb{R}^n$. However, upon request the authors will send interested readers a computer program that calculates the list in higher dimensions by just trying a lot of random point sets.

Except for the results in [Sie99] on four points in the plane we have no results on the number of Morse posets for N points when $N > n + 1$.

4.3. Edelsbrunner ratio.

DEFINITION 4.1. *The Edelsbrunner ratio ρ is the radius R of the circumsphere of $\mathbb{T}$ divided by the minimal edge length* $\min_{i \neq j} d(P_i, P_j)$.

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The ratio is used by Edelsbrunner (see [Ede01, section 6.2]) to classify tetrahedra into "shape types." This article has the same objective, so it is worthwhile to compare his criterion to ours.

For each of the nine generic types of tetrahedra the ratio ρ is bounded from below by the values in the table below. The infimum corresponds to the quadruple of points in the third column. These quadruples are not generic tetrahedra, except for the case $(4,6,4,1)$, which corresponds to the global minimum of ρ.

Type	Ratio ρ	Infimum
$(4,3,0,0)\,L$	$\frac{1}{2}\sqrt{3}$	(0,0,0),(1,0,0),(1,1,0),(1,1,1)
$(4,3,0,0)\,T$	$\frac{1}{2}\sqrt{3}$	(1,0,0), (0,1,0), (0,0,1), (0,0,0)
$(4,4,1,0)\,P$	$\sqrt{\frac{7}{12}}$	$(\frac{1}{2}\sqrt{3},-\frac{1}{2},\ 0),(-\frac{1}{2}\sqrt{3},-\frac{1}{2}),\ (0,1,0),(0,1,\sqrt{3})$
$(4,4,1,0)\,O$	$\frac{1}{2}\sqrt{2}$	(0,0,0),(1,0,0),(1,1,0),(0,1,0)
$(4,5,2,0)$	$\frac{1}{2}\sqrt{2}$	(0,0,0),(1,0,0),(1,1,0),(0,1,0)
$(4,6,3,0)$	$\frac{1}{2}\sqrt{2}$	$(\cos\alpha_j,\sin\alpha_j,0)\ j=1,\cdots,3,\ (0,0,1)$
$(4,4,2,1)$	$\frac{1}{2}\sqrt{2}$	$(1,0,0),\ (0,1,0),\ (0,\text{-}1,0),\ (\cos\alpha,0,\sin\alpha)$
$(4,5,3,1)$	$\frac{1}{2}\sqrt{2}$	(1,0,0), (0,1,0), (0,-1,0), (0,0,1)
$(4,6,4,1)$	$\frac{1}{4}\sqrt{6}$	$(0,1,0),(\frac{1}{2}\sqrt{3},-\frac{1}{2},0),(-\frac{1}{2}\sqrt{3},-\frac{1}{2},0),(0,0,\sqrt{2})$

For the case $(4,6,3,0)$ the triangle $P_1P_2P_3$ is acute. For the case $(4,4,2,1)$ the angle α should satisfy $-\frac{\pi}{2} \le \alpha \le 0$.

Hence, small values of ρ can be attained by all the types listed in Theorem 3.1. One can see from this that the two classifications are incomparable and conclude that they define different features.

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