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On Phase-Microscopic Images

(General Theory)

By

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The vector method, which has hitherto been used to estimate the degree of contrast of phase-microscopic image, can only be applied to sufficiently small objects. In Part I it is shown that, if certain idealization is allowed on the nature of phase plate, we can generalize the vector method so as to enable one to calculate the contrast of the image for objects which have arbitrary (finite) dimension. In Part II it is shown that by means of Fourier-transformation we can calculate easily the intensity distribution of the phase-microscopic image of transparent objects. This method leads to a vectorial calculation, whose relation to the ordinary vector method is explained. The interrelation between the contrast of image, width of the ring of the phase-plate, and the dimension of the object is discussed, with recourse to a few examples of calculation.

INTRODUCTION

In the prevailing literature on the phase-microscopy, so far as the author is aware, the mechanism of contrast-building in the phase-microscopic images is explained either by rather cumbersome calculation or by roughly approximated vector construction, and we have had to make use of these methods, especially on the latter, in deciding what sort of phase plate should be used to obtain the best image, i. e. the image with maximum contrast and clear-cutness. The former method,[1] being essentially based on Fourier-transformation, is entirely correct in itself, but on account of its two-dimensional character, the calculation becomes very much involved if we try to apply it to the actual cases. The detailed intensity distribution of the image can hardly be calculated, except rough order of contrast. By the latter, i. e., by KECK-BRICE and others' vector method or its mathematical equivalent,[2), 3)] which is very simple and can well be visualized, we can easily obtain the degree of contrast, but can never predict the details of the image. Moreover, this vector construction has in its original form only an unduely narrow range of validity, being strictly valid only in the case of a vanishingly small object. In Part I of this paper, it is shown that the vector method can be generalized, so that it may claim much broader validity than has so far been considered. There it permits us to evaluate the contrast of the image of any simple (uniform) "phase objects" (objects which are transparent and have only phase retardation relative to the surrounding medium), regardless of their dimensions, in so far as an "ideal" phase plate is introduced.

Even this adapted method (which we shall call generalized vector (GV) method) is not, however, adequate enough for our purpose. It is based on a too much idealized assumption that the phase retardation (or acceleration) and absorption take place on the phase plate only at the central infinitesimal portion of the Fraunhofer pattern of the object (ideal phase plate), which can never be realized in the actual experiment. On account of this limitation it can predict only the contrast, but not the *intensity distribution* caused by the finiteness of the portion where phase change takes place.

In order to be free from this serious limitation more general method which is based upon the principle of Fourier-transformation has been developed. It will be explained in Part II. It leads to a relatively simple vector-graphical calculation similar to GV method. It is shown that in the limiting case mentioned above this actually reduces to the GV method.

PART I

Generalization of the Vector Method.

Keck-Brice and others considered the mechanism of contrast-building in the phase microscopic images of transparent phase object as follows: We imagine the phase object that gives the plane light wave passing through the object plane a constant phase retardation ($2\pi\alpha$) within a finite region. Let $\overrightarrow{OS}$ in Fig. 1 represent the light wave corresponding to the uniformly bright background, and $\overrightarrow{OA}$ that referring to the object, which is retarded by $2\pi\alpha$ from $\overrightarrow{OS}$. $\overrightarrow{OA}$ can be split by the parallelogram of forces into two components, one of which is equal to $\overrightarrow{OS}$, and the other represented by $\overrightarrow{OD}$. In other words, the light wave, after passing through the object plane, can be regarded as consisting of two parts, one of which $\overrightarrow{OS}$ is the complete plane wave having the same amplitude and phase as the incident one, and the other ($\overrightarrow{OD}$) is the diffracted wave due to the presence of the phase object. We shall call these components S– and D– wave respectively. In case the object is sufficiently small, the D–wave will spread almost uniformly to infinity

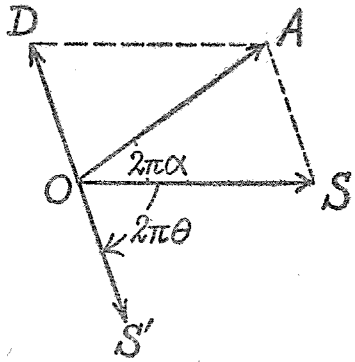

Fig. 1.

in the plane of Fraunhofer pattern. Therefore if the phase plate of the so-called A (+) type* is put just behind this plane, it exerts only an infinitesimal effect on the D–wave. On the other hand, the S–wave concentrates at one point (at the center) of the plane, and consequently suffers a phase retardation $2\pi\theta$ and amplitude reduction $(1-A)$ by the phase plate**. In other words, the vector $\overrightarrow{OS}$ suffers a contraction $(1-A)$ and rotation $2\pi\theta$. We can choose the values of A and θ so that $\overrightarrow{OS}$ becomes $\overrightarrow{OS'}$, which is just equal and opposite to $\overrightarrow{OD}$. The D–wave arrives at the image plane within a small definite region where it is distributed with uniform amplitude, while the S–wave spreads uniformly in the whole image plane. Thus we get the dark image on the bright background, since these two waves cancel each other in that definite region, and outside this region only the S–wave remains.

In the above explanation it was assumed that the object be sufciently small. If it is large, the D-wave will also more or less concentrate into the central region or at least spread only to *finite* extent in the plane of diffraction pattern. As the result some *finite* (not infinitesimal) portion of it suffers the same reduction and phase retardation as the S–wave, and the above argument no longer holds. It will be expected that in such a case the contrast of the image is to be diminished. In fact, however, it will be shown in Part II that in this case not only the diminution in contrast but also the

* By A (+) type is meant that phase retardation as well as absorption takes place at the central region of the Fraunhofer pattern.

** $A\,(\leqq 1)$ means here amplitude transmission.

modification of the intensity distribution takes place. Be that as it may, it is evident that here the afore-mentioned vector construction wholly breaks down, or at least retains only an approximate validity.

There is, however, another case in which the above argument holds good exactly. It is the case when the width of the "phase plate"* is not finite but vanishingly small (infinitesimal) (*ideal* phase plate), since then the portion of the D-wave which suffers amplitude reduction and phase retardation becomes again infinitesimal, even when the object is large, and yet the S-wave, which concentrates into the infinitesimal region, completely undergoes these effects.

Thus it becomes clear that the vector construction can claim its validity not only when the object is vanishingly small, but also when the "phase plate" has infinitesimal dimension. It seems to me more relevant to examine this latter case first and foremost, since then we might be able to treat the problem, without giving any restriction to the dimension of the object, or, in general, for any stepwise phase distribution in the object plane. This proves true, as will be seen in the following discussion.

Here let us analyse the afore-mentioned concepts of S– and D–wave from somewhat crucial point of view. For this purpose the exact form of Fraunhofer diffraction pattern of the phase object must be derived. Consider a "phase slit", which is characterized by the following formula:

$$\left.\begin{aligned} \psi(x) &= e^{2\pi i\alpha}\Delta'(x) + 1 - \Delta'(x)\ , \\ \Delta'(x) &= 1\ , \quad |x| < X/2\ , \\ &= 1/2\ , \quad |x| = X/2\ , \\ &= 0\ , \quad |x| > X/2\ . \end{aligned}\right\} \qquad (1)$$

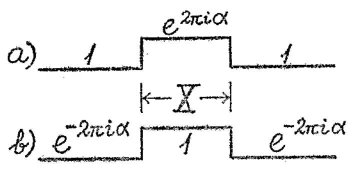

Fig. 2.

$\Delta'(x)$ is so-called Dirichlet's function. $\psi(x)$ is $e^{2\pi i\alpha}$ in the interval $|x| > X/2$, and unity everywhere outside this region, as schematically shown in Fig. 2(a). As is well-known, the Fraunhofer diffraction pattern of this object is represented by the Fourier transform

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* By the "phase plate" here we mean the region of the phase plate where the phase retardation $2\pi\theta$ and absorption $(1-A)$ take place. It is to be noticed that the same abbreviated expression will appear also in the following.

of $\psi(x)$, namely by

$$\Psi(\nu) = e^{2\pi i\alpha}\,\frac{\sin \pi\nu X}{\pi\nu} + \delta(\nu) - \frac{\sin \pi\nu X}{\pi\nu}$$
$$= \delta(\nu) + (e^{2\pi i\alpha} - 1)\,\xi(\nu)\,, \quad \xi(\nu) = \frac{\sin \pi\nu X}{\pi\nu}\,. \qquad (2)$$

It will be seen that the second term of this expression corresponds to the afore-mentioned D-wave including phase, and the first one to the S-wave. The second term represents exactly the same pattern $\xi(\nu)$ as that due to the slit of width X except for phase, and therefore the wave represented by this term evidently concentrates into the image at the image plane. The wave represented by the first term, on the other hand, spreads uniformly upon the whole image plane, after it has suffered amplitude and phase modulation by the phase plate. Thus in the image two waves interfere with each other to make it e. g. dark. This argument corresponds exactly to the one stated above.

It is clear from the above argument that in case the object has infinite extension, the vector construction as stated above will again break down even when the ideal phase plate is used, since then also the D-wave will include the δ-function, which suffers the same phase and amplitude modulation as the S-wave. For instance let us treat, by the vector method, the case in which the roles of object and background in the above example are interchanged. We can namely suppose the previous background as the "object" having phase retardation $-2\pi\alpha$, and the previous object as the "background" having no phase retardation (Fig. 2 (b)). The physical situations in these two cases are obviously identical, and therefore exactly the same result is to be obtained in the image plane. In this case the object wave $\overrightarrow{\mathrm{OA}}$ is retarded by $-2\pi\alpha$ from $\overrightarrow{\mathrm{OS}}$, so that the D-wave must be represented by $\overrightarrow{\mathrm{OD'}}$ in Fig. 3, instead of $\overrightarrow{\mathrm{OD}}$. But since A and θ, characterizing the phase plate, remain the same, the S-wave becomes $\overrightarrow{\mathrm{OS'}}$ as before. Thus in the "image" two waves *rein-*

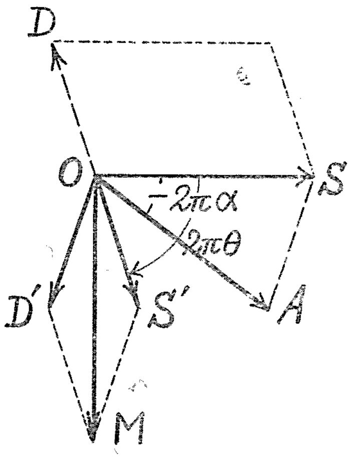

Fig. 3.

force each other, the resultant amplitude being $|OM|$, and in the background only the S–wave arrives, whose amplitude is $|OS'|$. Consequently the intensity distribution in the image plane is represented by the dashed curve in Fig. 4, in contrast to that in the previous case represented by the solid curve. This is evidently a contradictory result.

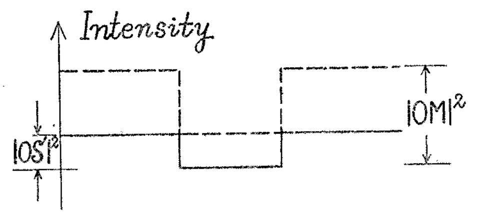

Fig. 4.

Mathematically speaking: The function which represents the diffraction pattern of our present object plane can be obtained simply by multiplying (2) by $e^{-2\pi i\alpha}$:

$$\Psi(\nu) = e^{-2\pi i\alpha}\,\delta(\nu) + (1 - e^{-2\pi i\alpha})\,\xi(\nu) \; . \tag{3}$$

This is easily transformed into

$$\left.\begin{aligned} &= \delta(\nu) + (e^{-2\pi i\alpha} - 1)\,\xi'(\nu) \; , \\ &\xi'(\nu) = \delta(\nu) - \xi(\nu) \; . \end{aligned}\right\} \tag{3'}$$

$\xi'(\nu)$ is the Fourier-transform of $1 - \Delta'(x)$, i. e., it represents the diffraction pattern of the opaque strip whose width is X. It is natural to regard this as the D–wave in the present case, since in the previous case D–wave has been represented by the diffraction pattern of the slit width X, which is the complementary figure of the opaque strip of width X. In fact (3′) shows this must be the case. The wave represented by the second term of (3′) wholly "concentrates" into the "image", and that represented by the first term spreads uniformly on the whole image plane, as in the previous case. Here, however, a part of the D–wave also undergoes phase and amplitude modulation, because of its inclusion of δ–function. This is the reason why the vector construction in Fig. 3 failed to be correct, as suggested at the beginning of the last paragraph.

If, however, we perform the vector construction straightforwardly according to the original formula (3), instead of (3′), we naturally get the correct result, since it will differ from the previous one represented by Fig. 1 only by constant rotation $2\pi\alpha$ of all the vectors involved. Hence we are lead to the conclusion that we had better discard the concepts of S– and D–wave, which are likely to lead to incorrect results. Instead, we should resolve the function representing the diffraction pattern into as many separated terms as possible, and consider each term as a separate wave, and represent it by an appropriate vector. Thus the formula (2) is to be rewritten as

$$\Psi(\nu) = \delta(\nu) + e^{2\pi i\alpha}\xi(\nu) - \xi(\nu). \tag{2'}$$

The wave represented by the first term will spread uniformly everywhere in the image plane, after its phase (zero) and amplitude (unity) have been modified into $2\pi\theta$ and A respectively (from $\overrightarrow{OS}$ to $\overrightarrow{OS'}$ in Fig. 5). The one represented by the second term spreads uniformly into the region of slit image, its phase $2\pi\alpha$ and amplitude unity ($\overrightarrow{OA}$) being unchanged. The one represented by the third term spreads uniformly also in the region of slit image, with its phase π and amplitude unity ($\overrightarrow{OB}$). In the region of slit image these waves cancel each other, while outside the region there remains $\overrightarrow{OS'}$. To make the resultant of $\overrightarrow{OA}$ and $\overrightarrow{OB}$ and give it the physical meaning as "diffracted wave" is not, in general, a matter of course..

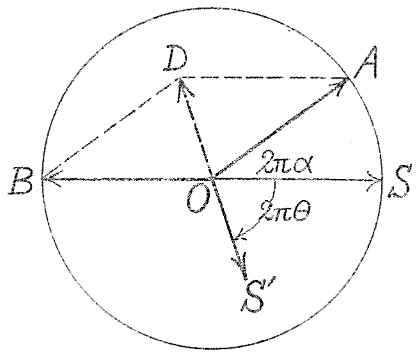

Fig. 5.

This method of obtaining the image shall be called GV (generalized vector) method.

Another illustration of the superiority of GV method is afforded by the following example: Consider the phase object which has phase retardation $2\pi\alpha$ in the right half relative to the left half of the whole plane. Let $D(x)$ be a function which is $-1/2$ on the left and $+1/2$ on the right side of the origin, then such an object can be represented by

$$\left.\begin{aligned} \psi(x) &= e^{2\pi i\alpha}D'(x) + 1 - D'(x)\ , \\ D'(x) &= D(x) + 1/2\ . \end{aligned}\right\} \tag{4}$$

Since the Fourier-transform of $D(x)$ (Heaviside's step function) is known to be $1/2\pi i\nu$*, the Fraunhofer pattern of this object is given by

* At least formally. The lack of mathematical rigor might be tolerated from the physical point of view.

$$\Psi(\nu) = \frac{e^{2\pi i a}}{2\pi i\nu} + \frac{1}{2}\,\delta(\nu)\,e^{2\pi i a} - \frac{1}{2\pi i\nu} + \frac{1}{2}\,\delta(\nu)\,. \quad (5)$$

Assume, for instance, that $2\pi a = \pi/3$, $2\pi\theta = -\pi/3$ and $A = 1$. Then the wave represented by the first term spreads uniformly into the right half of the image plane with amplitude 1/2 and phase $\pi/3$, and into the left half with amplitude 1/2 and phase $\pi+\pi/3$. (Since $1/2\pi i\nu$ is Fourier-transform of $D(x)$, it must reproduce $D(x)$ itself in the image plane*. Negative transmission $-1/2$ should be equivalent to the ordinary positive transmission 1/2 with reversed phase, i.e. π). This is therefore represented by vector 1 in the right, and by 1′ in the left half of the image plane (Fig. 6). The wave represented

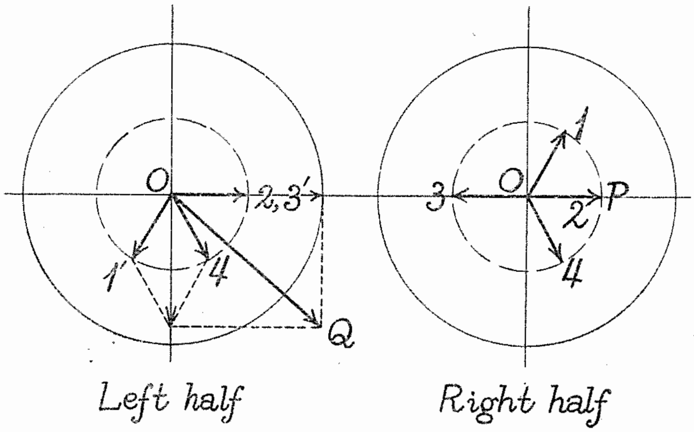

Fig. 6.

by the second term spreads uniformly over the whole image plane, with amplitude 1/2 and phase zero, since this wave suffers phase modulation $-\pi/3$ (absorption being zero). This is therefore represented by vector 2 both in the right and in the left side of the origin. In the same way the third term corresponds to vector 3 in the right and to 3′ in the left half, and the fourth term to 4 in both. The resultant of these four vectors is $\overrightarrow{OP}$ in the right and $\overrightarrow{OQ}$ in the left half of the image plane. The square of the magnitude of $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ is easily seen to be 1/4 and 7/4 respectively, i.e. in the left half of the image plane intensity is decreased by 3/4, and in the right half increased by 3/4, as compared with the normal illumi-

* The reversal of right and left in the image plane is left out of account.

nation (illumination which prevails when there is no object, in which case it must be unity everywhere). The energy conservation law is thus fulfilled (Fig. 7).* The same result will be obtained in Part II by Fourier-transformation method.

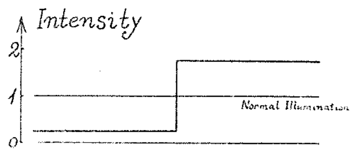

Fig. 7.

Thus GV method leads to a reasonable result also in this case, where evidently the original vector method cannot be applied, since the δ-functions necessarily appear in D-waves.**

PART II

Fourier-Transformation Method.

§ 1. General Principle.

As is well-known, the image of any object, obtained through a microscope, is, mathematically speaking, equivalent to the twice repeated Fourier-transform of the function which represents the amplitude transmission and phase retardation distribution of the object (Abbe's theory of microscopic vision)***. Twice repeated Fourier-transform of a function is nothing other than the original function itself, except that the coordinate is reversed. The whole image formation process may thus be regarded as consisting of two distinct steps: (1) formation of the Fraunhofer diffraction pattern of the original object, corresponding to the transformation of the function representing the characteristic amplitude transmission and phase retardation distribution of the original object into its Fourier-transform, and (2) composition of the elementary waves emitted

* In the previous case it seems at first sight as if the energy conservation failed to hold, since in Fig. 4 the lower curve shows an absolute decrease in intensity in the image. But this decrease is *infinitesimal* as compared with infinite total intensity outside the image (infinite aperture) Therefore there is *no essential decrease* of energy in comparison with the case of normal illumination. If the object plane were bounded (i. e., finite aperture), as is always the case in practice, such absolute decrease will not be able to occur. It will be shown in Part II this is actually the case.

The upper curve in Fig. 4 is not reasonable also from this point of view, since it does not fulfill the conservation law (infinite absolute increase in total energy).

** D-waves may as well be imagined as in (3′), but also here it bears only a *formal* meaning.

*** This is of course only approximately true, but for simplicity we assume its strict validity throughout this paper, which would not essentially affect the correctness of several conclusions therein obtained.

from this Fraunhofer pattern into the final image, corresponding to the Fourier transformation just reverse to the above. The image reproduces, therefore, just the same intensity distribution as the original object, except that the image is reversed.

Phase-microscopy enables one to give remarkable contrast to the images of "phase object". As has been already illustrated in Part I, its principle is essentially to give, by means of phase plate, phase retardation and amplitude regulation to the elementary waves which originate from the neighborhood of the center of the Fraunhofer pattern, i. e., the zeroth order spectrum of the original object, relative to the ones originating from its wings, or the higher order spectrum.

Mathematically, this procedure can be stated as follows. If $\psi(x)$ represents the amplitude transmission and phase retardation characteristics of the original object*, the amplitude and phase distribution of its Fraunhofer pattern is given by

$$\Psi(\nu) = \int_{-\infty}^{\infty} \psi(x)\, e^{-2\pi i \nu x}\, dx \ . \tag{6}$$

Here ν is to be regarded as the order of the spectrum of the object. If there were no phase plate, this Fraunhofer pattern is again transformed by Fourier integral into the image of the object $F_0(x')$:

$$\left.\begin{aligned} F_0(x') &= \int_{-\infty}^{\infty} \Psi(\nu)\, e^{-2\pi i \nu x'}\, d\nu \\ &= \int_{-\infty}^{\infty} d\nu \int_{-\infty}^{\infty} \psi(x)\, e^{-2\pi i (x+x')\nu}\, dx \\ &= \psi(-x') \ . \end{aligned}\right\} \tag{7}$$

But if the phase plate is inserted, the situation becomes different. Let $\varphi(\nu)$ represent the amplitude transmission and phase retardation characteristics of the phase plate. The secondary light waves emitted from the Fraunhofer pattern are modified to $\varphi(\nu)\Psi(\nu)$, and consequently the final image becomes

$$\left.\begin{aligned} F(x') &= \int_{-\infty}^{\infty} \varphi(\nu)\Psi(\nu)\, e^{-2\pi i \nu x'}\, d\nu \\ &= \int_{-\infty}^{\infty} \Phi(-x'-t)\, \psi(t)\, dt \ , \end{aligned}\right\} \tag{8}$$

* We restrict our formulation and calculation to the one-dimensional cases only. This restriction does not imply, we suppose, any loss of generality of the argument.

by the well-known theorem that the Fourier-transform of the product of two functions is equal to the "Faltung" of their respective Fourier-transforms, $\Phi(x)$ being the Fourier-transform of $\varphi(\nu)$. The reversal of the image is of course included in $F(x')$ given by (8), but in order to avoid the confusion arising from this, we shall use the function $F_r(x')$ given by the expression

$$F_r(x') = \int_{-\infty}^{\infty} \Phi(x'-t)\,\psi(t)\,dt\ , \tag{8'}$$

which represents the image free from reversal. Moreover, we shall not in the following discussion distinguish the coordinate of image plane (x') from that of object plane (x).

Evidently $F_r(x)$ differs in general from $\psi(x)$. The origin of contrast-building in the phase-microscopic images lies in this fact.

In Part I, we treated, and could only treat, the extreme case in which the phase and amplitude regulation by phase plate is confined to the central infinitesimal portion of the diffraction pattern, i. e. the case in which $\varphi(\nu)$ has the form :

$$\left.\begin{aligned} \varphi(\nu) = \lim_{N\to 0} & \left\{ A e^{2\pi i\theta} \Delta(\nu) + 1 - \Delta(\nu) \right\}\ , \\ \Delta(\nu) &= 1\ , \quad |\nu| < N/2\ , \\ &= 1/2\ , \quad |\nu| = N/2\ , \\ &= 0\ , \quad |\nu| > N/2\ . \end{aligned}\right\} \tag{9}$$

But now we are able to calculate the image for any form of $\varphi(\nu)$, at least in principle.

§ 2. Energy Conservation.

Before proceeding to the actual calculation, it is convenient to demonstrate the energy conservation during the image-formation process. In ordinary (not phase-microscopic) image-formation, energy conservation during the first step is guaranteed by the so-called Parseval formula

$$\int_{-\infty}^{\infty} |\psi(x)|^2\,dx = \int_{-\infty}^{\infty} |\Psi(\nu)|^2\,d\nu\ , \tag{10}$$

and that during the second step is out of question. In phase-microscopic image-formation, the energy conservation during the first step is also guaranteed by (10), but for the second step it needs a proof.

If the phase plate has phase retardation characteristics only, having transmission coefficient unity everywhere, the energy ought to be conserved in this step. Actually, in this case $\varphi(\nu)$ should have the form $e^{2\pi i f(\nu)}$, so that, using (8) and (10),

$$
\left.\begin{aligned}
\int_{-\infty}^{\infty} |F(x)|^2\,dx &= \int_{-\infty}^{\infty} F(x)\bar{F}(x)\,dx \\
&= \int_{-\infty}^{\infty} \bar{F}(x)\,dx \int_{-\infty}^{\infty} e^{2\pi i f(\nu)}\,\Psi(\nu)\,e^{-2\pi i\nu x}\,d\nu \\
&= \int_{-\infty}^{\infty} \Psi(\nu)\,e^{2\pi i f(\nu)}\,d\nu \int_{-\infty}^{\infty} \bar{F}(x)\,e^{-2\pi i\nu x}\,dx \\
&= \int_{-\infty}^{\infty} \Psi(\nu)\,e^{2\pi i f(\nu)}\,\bar{\Psi}(\nu)\,e^{-2\pi i f(\nu)}\,d\nu \\
&= \int_{-\infty}^{\infty} |\Psi(\nu)|^2\,d\nu \\
&\left(= \int_{-\infty}^{\infty} |\psi(x)|^2\,dx\right) .
\end{aligned}\right\} \quad (11)
$$

§ 3. Calculation of the Image of a Phase Slit.

Consider the "phase slit", which is represented by formula (1) of Part I:

$$
\left.\begin{aligned}
\psi(x) &= e^{2\pi i\alpha}\,\Delta'(x) + 1 - \Delta'(x)\,, \\
\Delta'(x) &= 1\,, \qquad |x| < X/2\,, \\
&= 1/2\,, \quad\; |x| = X/2\,, \\
&= 0\,, \qquad |x| > X/2\,,
\end{aligned}\right\} \quad (1)
$$

and the phase plate characterized by

$$
\left.\begin{aligned}
\varphi(\nu) &= Ae^{2\pi i\theta}\,\Delta(\nu) + B - B\Delta(\nu)\,, \\
\Delta(\nu) &= 1\,, \qquad |\nu| < N/2\,, \\
&= 1/2\,, \quad\; |\nu| = N/2\,, \\
&= 0\,, \qquad |\nu| > N/2\,,
\end{aligned}\right\} \quad (2)
$$

which gives phase and amplitude modulation in the central *finite* portion (width N) of diffraction pattern. B indicates the uniform absorption, if any, outside this portion.* The Fourier-transforms of

* When θ is negative, it means that phase retardation is given outside the central region of the pattern. B ($\leqq 1$) stands for the amplitude transmisssion, so that $(1 - B)$ gives the amplitude absorption. B is less than unity, while A = 1, in the so-called "B" type phase plates, and conversely, A < 1 (B = 1) in the "A" type phase plates.

the functions $\psi(x)$ and $\varphi(\nu)$ are

$$\Psi(\nu) = e^{2\pi i\alpha} \frac{\sin \pi\nu X}{\pi\nu} + \delta(\nu) - \frac{\sin \pi\nu X}{\pi\nu} , \tag{13}$$

and

$$\Phi(x) = Ae^{2\pi i\theta} \frac{\sin \pi x N}{\pi x} + B\delta(x) - B \frac{\sin \pi x N}{\pi x} , \tag{14}$$

respectively. We have, therefore, for the amplitude distribution of the image the following integral as before:

$$\begin{aligned} F_r(x) &= \int_{-\infty}^{\infty} \Phi(x-t)\,\psi(t)\,dt \\ &= \int_{-\infty}^{\infty} \left[Ae^{2\pi i\theta} \frac{\sin \pi(x-t)N}{\pi(x-t)} + B\delta(x-t) - B \frac{\sin \pi(x-t)N}{\pi(x-t)} \right] \\ &\qquad \times \left[e^{2\pi i\alpha} \Delta'(t) + 1 - \Delta'(t) \right] dt . \end{aligned} \tag{15}$$

Performing the integration, we get

$$F_r(x) = (Ae^{2\pi i\theta} - B)(e^{2\pi i\alpha} - 1)\,\sigma(x) + (Ae^{2\pi i\theta} - B) + B\psi(x) , \tag{16}$$

where

$$\sigma(x) = \int_{x-X/2}^{x+X/2} \frac{\sin \pi t N}{\pi t}\,dt = \frac{1}{\pi}\left[\mathrm{Si}\left\{\pi N\left(x+\frac{X}{2}\right)\right\} - \mathrm{Si}\left\{\pi N\left(x-\frac{X}{2}\right)\right\} \right]. \tag{17}$$

The value of $\sigma(x)$ may be calculated from that of the "integral sine" $\mathrm{Si}(x)$, which is found in tables of functions. After this has been done for several values of x, final intensity distribution, $|F_r(x)|^2$, can be obtained by the graphical (vectorial) method, as explained below.

Let C, in Fig. 8, be the unit circle on the complex plane. First we consider the affair occurring in the region $|x| < X/2$. In this region the last term of (16) is represented by the vector $\overrightarrow{\mathrm{OE}}$, which has the magnitude B and makes the angle $2\pi\alpha$ with the positive real axis. The first of the second term $Ae^{2\pi i\theta}$ is represented by the vector $\overrightarrow{\mathrm{OF}}$, with magnitude A and argument $2\pi\theta$. The whole second term is the difference between this vector and the vector $\overrightarrow{\mathrm{OQ}}$ lying on the positive real axis, i.e. the vector $\overrightarrow{\mathrm{QF}}$ or $\overrightarrow{\mathrm{OF'}}$. In the same way the first factor of the first term is also represented by $\overrightarrow{\mathrm{OF'}}$ and

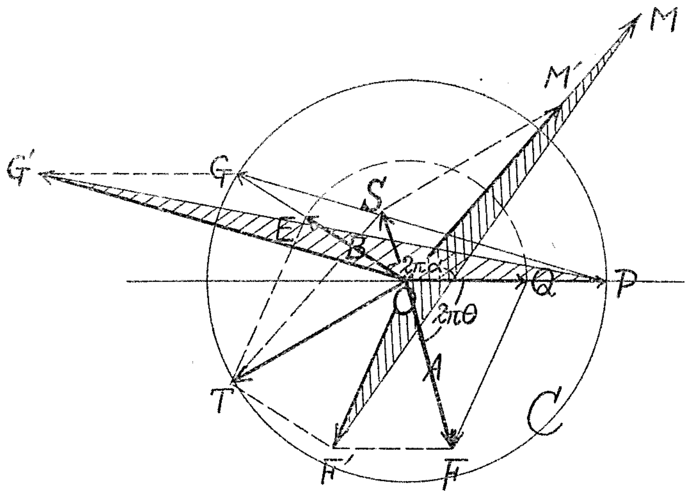

Fig. 8.

the second by $\overrightarrow{OG'}$. The product $(Ae^{2\pi i\theta}-B)(e^{2\pi i\alpha}-1)$, i. e. $\overrightarrow{OF'}\cdot\overrightarrow{OG'}$ is then obtained according to the ordinary rule, that is, if we construct $\triangle$OFM which is similar to $\triangle$OPG, then the vector $\overrightarrow{OM}$ must represent this product. Since $\sigma(x)$ is real, the whole first term of (16) will be represented by $\overrightarrow{OM'}$, which is the contraction or magnification of $\overrightarrow{OM}$, according as $\sigma(x)$ is smaller or larger than unity. After this has been done, we have only to add three vectors $\overrightarrow{OE}$, $\overrightarrow{OF'}$ and $\overrightarrow{OM'}$, as indicated in the figure, resulting in the vector $\overrightarrow{OS}$. Square of the length of this vector represents the intensity of light reaching to the point indicated by a particular value of x in the image plane. Repeating the same procedure (making $\overrightarrow{OM'}$ to get $\overrightarrow{OT}$) for each value of x, we obtain the intensity distribution of the image in the interval $|x|<X/2$.

In the region $|x|>X/2$, the last term of (16) is not $\overrightarrow{OE}$ but $\overrightarrow{OQ}$. Except for this replacement the whole procedure remains the same. The sum of the last two term of (16) becomes now $\overrightarrow{OF}$ instead of $\overrightarrow{OT}$ ($\because Ae^{2\pi i\theta}-B+B=Ae^{2\pi i\theta}$), and adding $\overrightarrow{OM'}$ to $\overrightarrow{OF}$ makes the final vector $\overrightarrow{OS}$.

The intensity distribution curves for several couples of values of N and X, with arbitrarily chosen values of θ and α ($2\pi\theta=77°$, $2\pi\alpha=158°$), are shown in Fig. 9 and 10 (A and B were put equal to unity). It can be seen from Fig. 9 that when X is given, the image becomes more and more deformed and obscure as N becomes larger. When $N=0$, the image is the exact reproduction of the object as it ought to be, and that with the highest contrast. The curve for $N=2$ shows deformation to such an extent that these appear excessive intensity maxima both inside and outside the limb. But the limb itself retains more or less remarkable sharpness. When N becomes infinitely large the image naturally disappears. (Mathematically speaking: As $N \to 0$, the curve $\sin \pi t N / \pi t$ becomes flatter and flatter, until in the end it becomes completely flattened, when $\sigma(x)$ also becomes zero regardless of the value of x, so that $\overrightarrow{\mathrm{OT}}$ and $\overrightarrow{\mathrm{OF}}$ represent the respective *constant* amplitude inside and outside the limb. When $N \to \infty$ the function $\sin \pi t N / \pi t$ reduces to δ-function and $\sigma(x)$ to $\Delta'(x)$, so that $F_r(x) = e^{2\pi i \theta}\varphi(x)$.)

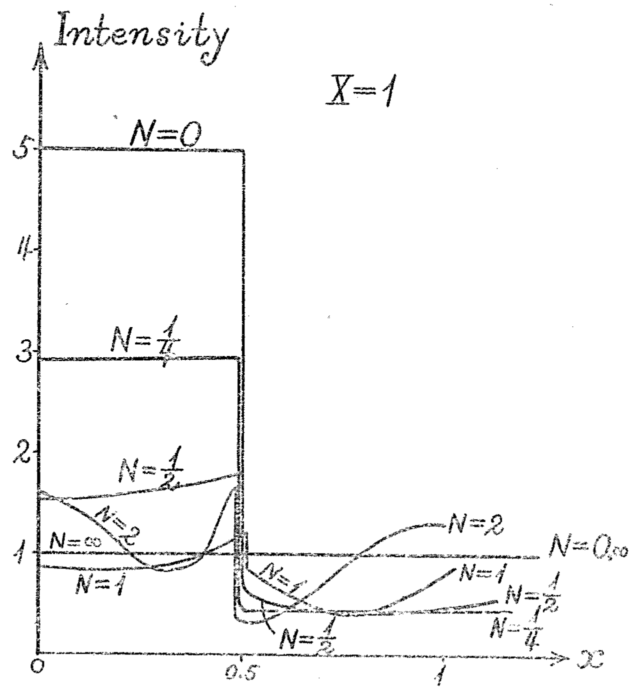

Fig. 9.

Similar affairs occur also when N is fixed at a finite value and X varied. As Fig. 10 shows, the image becomes more indistinct and deformed as X increases. In the limiting case $X \to \infty$, the image

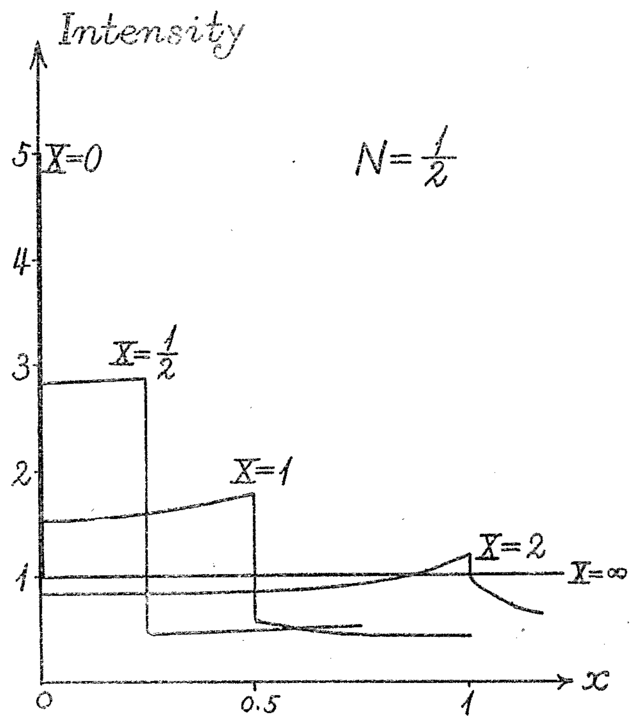

Fig. 10.

disappears completely, as is naturally expected ($\sigma(x)=1$ and $F_r(x)=Ae^{2\pi i(\theta+\alpha)}$). In the other extreme $X\rightarrow 0$, the image becomes of infinitesimal width with complete sharpness ($\sigma(x)$ is always zero; $\psi(x)=1$ except at $x=0$, where $\psi(0)=e^{2\pi i\alpha}$; $F_r(x)=Ae^{2\pi i\theta}$, $F_r(0)=Ae^{2\pi i\theta}+B(e^{2\pi i\alpha}-1)$).

The foregoing discussion illustrates clearly the effect of finite width of phase plate (finite N), which is embodied in the function $\sigma(x)$. Only in the limit $N\rightarrow 0$, the image is exactly the same as that obtained by GV method. The plate with finite N gives phase and amplitude modulation to a certain finite portion of each wave which does not include the δ-function. This brings about the deformation of images. The larger N becomes, or the larger X becomes, the more appreciable portion of non-δ-wave suffers the modulation (in the latter case because of the concentration of these waves in the central region of diffraction pattern). In particular, it is to be noted that the curve for $N=1/4$, $X=1$ and that for $N=1$, $X=1$ are completely similar to the curve for $N=1/2$, $X=1/2$ and that for $N = 1/2$, $X = 2$, respectively. It means that for a definite value of $N \cdot X$ the form of the images should be identical except for their proper dimensions depending on X. This is to be expected since the smaller X is, the larger N should be to modulate the non-δ-waves in the same proportion. The value of $N \cdot X$ might be regarded as the measure of deformation of the image, in the sense that the smaller the value of $N \cdot X$ is, the better image we have.

This result is also a natural consequence in harmony with the requirement that for contrast-building in phase-microscopy the modification of *zero, and only zero* order spectrum is essential. Modification of higher order spectrum plays merely destructive role, as it reduces the contrast and sharpness of the image.

It must however be emphasized here that this statement is correct only when the aperture of the lens is assumed to be infinitely large. In the case of finite aperture, the situation becomes different, as will be seen in § 4. There, N should necessarily be finite.

§ 4. Comparison with GV Method.

It has already been pointed out in the last section that the GV method developed in Part I actually corresponds to the limiting case $N \to 0$ of Fourier-transformation method. This correspondence will become much clearer if we treat the same case as considered in Part I by the latter method. Namely, let α, θ, A and B take the value as indicated in Fig.5 ($B = 1$), and consider the limiting case $N \to 0$. Then (16) becomes simply

$$F_r(x) = Ae^{2\pi i\theta} - 1 + \psi(x)\,. \tag{18}$$

In the region $|x| < X/2$, $\psi(x) = e^{2\pi i\alpha}$ and $F_r(x)$ is represented by the resultant of three vectors $\overrightarrow{OA}$, $\overrightarrow{OS'}$ and $\overrightarrow{OB}$, which is equal to zero. Outside this region $\psi(x) = 1$ and $F_r(x)$ is represented by the vector $\overrightarrow{OS'}$ (Fig. 11). The situation is just the same as that in Fig. 5.

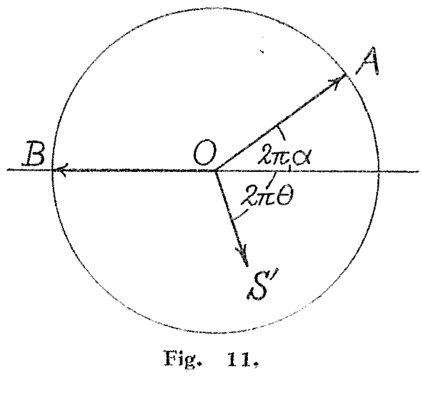

Fig. 11.

In Fig. 12 dark contrast images of the object given by (1) for the values $A = B = 1$, $2\pi\theta = -\pi/3$, $2\pi\alpha = \pi/3$, $N = 0, 1, 2$ are given. The deformation of images and inapplicability of the GV method for finite values of N may clearly be seen also from this figure.

Next we consider the case in which the object is represented by (4). Phase plate is given by (12) as before. Then

$$
\begin{aligned}
F_r(x) &= \int_{-\infty}^{\infty} \Phi(x-t)\,\psi(t)\,dt \\
&= \int_{-\infty}^{\infty} \left[Ae^{2\pi i\theta}\,\frac{\sin\pi(x-t)N}{x-t} + B(x-t) - B\frac{\sin\pi(x-t)N}{\pi(x-t)} \right] \\
&\qquad \times \left[e^{2\pi i\alpha} D'(t) + 1 - D'(t) \right] dt\,.
\end{aligned}
$$

This reduces to

$$
\left.
\begin{aligned}
F_r(x) &= B\psi(x) + Ae^{2\pi i\theta} - B + (Ae^{2\pi i\theta} - B)(e^{2\pi i\alpha} - 1)\left\{\frac{1}{2} + \eta(x)\right\}\,, \\
\eta(x) &= \int_0^x \frac{\sin\pi tN}{\pi t}\,dt = \frac{1}{\pi}\,\mathrm{Si}\,(\pi Nx)\,.
\end{aligned}
\right\} \quad (19)
$$

This formula provides a vectorial method of calculation similar to that in the last section. Here we have only to consider the limiting

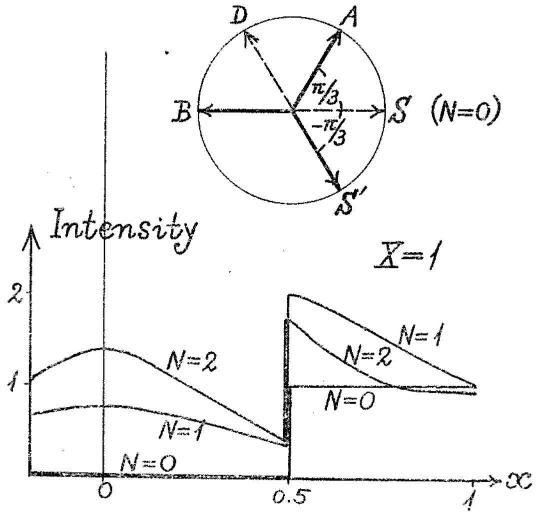

Fig. 12.

case $N \to 0$. In this case $\eta(x)$ becomes infinitesimal everywhere. If we put $A = B = 1$, $2\pi\alpha = \pi/3$, $2\pi\theta = -\pi/3$ as before, (19) becomes

$$F_r(x) = \psi(x) + e^{-\pi i/3} - 1/2\,. \tag{20}$$

In the right half of the image plane, each term of (20) is represented by the vectors 1, 2, 3 in Fig. 13 (a), respectively, the resultant being $\overrightarrow{OP}$. In the left half of the plane the first vector becomes

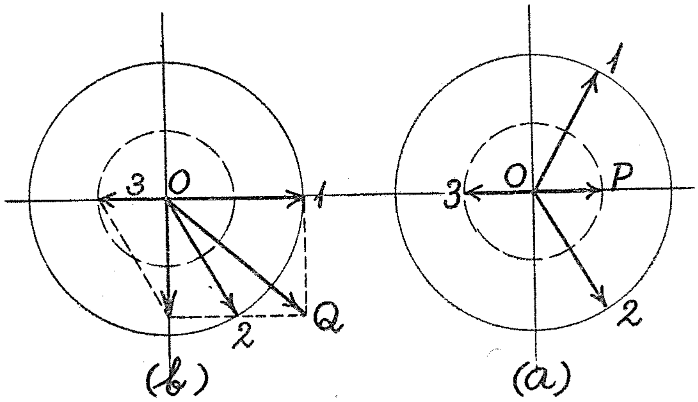

Fig. 13.

the real unit vector as shown in Fig. 13 (b), so that we get the resultant $\overrightarrow{OQ}$. The result is again the same as before (Fig. 6).

Thus we have demonstrated that Fourier-transformation method reduces to GV method in the limiting case $N\to 0$ (or $X\to 0$ as well). The former can, therefore, be regarded as the generalization of the latter, having in principle no limit of validity.

Thus far, it is true, we have confined ourselves to treating the one-dimensional case. But it might well be inferred that general behavior would be nearly the same also in the two-dimensional cases.

§ 3. Effect of Finite Aperture.

So far we have treated only the cases in which both the object plane and the plane upon which the Fraunhofer pattern is formed extend to infinity on both sides. But in practical cases this is obviously impossible. Thus it becomes necessary to consider the effect of finite field of view, in other words, finite aperture. In this section a brief account of this will be given.

Taking into account the finiteness of aperture is equivalent to put amplitude transmission equal to zero (to cut off) outside a certain interval both in $\psi(x)$ and $\varphi(\nu)$. But for simplicity we treat them separately. First we cut off the function $\psi(x)$ outside the interval whose width is $Y(>X)$ (finite object phane). Then the object plane will be characterized by

$$\left.\begin{aligned} \psi'(x) &= e^{2\pi i\alpha}\Delta'(x) - \Delta'(x) + \Delta''(x)\,, \\ \Delta''(x) &= 1\,, \qquad |x| < Y/2\,, \\ &= 1/2\,, \quad |x| = Y/2\,, \\ &= 0\,, \qquad |x| > Y/2\,, \end{aligned}\right\} \tag{21}$$

as can easily be checked. The image becomes

$$\begin{aligned} F_r(x) = \int_{-\infty}^{\infty} &\left[Ae^{2\pi i\theta}\,\frac{\sin\pi(x-t)N}{\pi(x-t)} + B\delta(x-t) - B\,\frac{\sin\pi(x-t)N}{\pi(x-t)}\right] \\ &\times\left[e^{2\pi i\alpha}\Delta'(t) - \Delta'(t) + \Delta''(t)\right] dt \end{aligned}$$

$$\left.\begin{aligned} &= (Ae^{2\pi i\theta} - B)(e^{2\pi i\alpha} - 1)\,\sigma(x) + B\psi'(x) + (Ae^{2\pi i\theta} - B)\,\sigma'(x)\,, \\ &\sigma'(x) = \int_{x-Y/2}^{x+Y/2} \frac{\sin\pi tN}{\pi t}\,dt\,. \end{aligned}\right\} \tag{22}$$

The difference between (22) and (16) is substancially the replacement of the coefficient of the term $(Ae^{2\pi i\theta} - B)$, i. e. unity, by $\sigma'(x)$. This does not alter the image substancially as long as N remains appropriately finite, and Y is sufficiently large, since then $\sigma'(x)$ retains approximately constant value close to unity, and (22) becomes substancially equal to (16). Here the product $N\cdot Y$ is the measure of the secondary deformation of the image due to the finiteness of aperture, which manifests itself along with the primary one effected by the finiteness of $N\cdot X$. The effect of $N\cdot Y$ is, contrary to that of $N\cdot X$, such that the larger the product is, the smaller the secondary deformation becomes. If, however, we make N too large in order to make $N\cdot Y$ large, $N\cdot X$ becomes also too large and the primary deformation will come out drastic.

In the extreme case $N\to 0$, $\sigma'(x)$ as well as $\sigma(x)$ is everywhere infinitesimal, and $F(x)$ becomes simply

$$F_r(x) = B\psi(x)\,, \tag{23}$$

which evidently shows no contrast! At first glance this behavior might appear curious. But it turns out quite natural and reasonable, if we take into account the fact that as long as Y remains finite at all, the diffraction pattern contains no δ-function, and the phase plate with infinitesimal N cannot exert any effect on the pattern. Actually, if this disappearance of the contrast did not occur, the energy would cease to be conserved, while in the case of finite object plane it must necessarily be conserved strictly in accordance with (11). (See the footnote at the end of Part I).

Thus we have reached to the conclusion that in the cases of finite object plane, finiteness of N is even necessary, the value of which must be chosen so as to get the best image, that is, so as to obtain as much as possible both the clear-cutness and the contrast of the image.

Next we cut off the function $\varphi(\nu)$ outside the interval of width $N'\,(> N)$, leaving the object plane infinite. Then $\varphi(\nu)$ becomes

$$\left.\begin{aligned} \varphi'(\nu) &= (e^{2\pi i\theta}-1)\,\varDelta(\nu)+\varDelta'(\nu)\,, \\ \varDelta'(\nu) &= 1\,, \quad |\nu| < N'/2\,, \\ &= 1/2\,, \quad |\nu| = N'/2\,, \\ &= 0\,, \quad |\nu| > N'/2\,. \end{aligned}\right\} \tag{23}$$

Hence

$$\left.\begin{aligned} F_r(x) &= \int_{-\infty}^{\infty}\left[Ae^{2\pi i\theta}-B)\,\frac{\sin\pi\,(x-t)\,N}{\pi\,(x-t)}+B\,\frac{\sin\pi\,(x-t)\,N'}{\pi\,(x-t)}\right] \\ &\qquad \times\left[(e^{2\pi i\alpha}-1)\,\varDelta'(t)+1\right]dt \\ &= (Ae^{2\pi i\theta}-B)(e^{2\pi i\alpha}-1)\,\sigma(x)+Ae^{2\pi i\theta}+B(e^{2\pi i\alpha}-1)\sigma''(x)\,, \\ \sigma''(x) &= \int_{x-X/2}^{x+X/2}\frac{\sin\pi tN'}{\pi t}\,dt\,. \end{aligned}\right\} \tag{24}$$

This formula is obtained if we replace $\varDelta'(x)$ by $\sigma''(x)$ in (16). When N' is sufficiently large, $\sigma''(x)$ is almost equivalent to $\varDelta'(x)$, except that the discontinuities in the first derivative disappear and consequently the sharp edges are more or less rounded. The effect of finite phase plate (finiteness of the plane of diffraction pattern) is thus seen to somewhat diminish the clear-cutness of the image. But it is expected that in usually encountered conditions this effect can practically be ignored.

SUMMARY

The so-called GV method, which may be regarded as the generalization of the original vector method, and the Fourier-trans-

formation method,which not only claims the most general validity but in typical cases leads to a simple vector construction, have been established. It was illustrated that the Fourier-transformation method include the GV method as valid in certain limiting cases. The intensity distribution of images in typical cases was calculated and illustrated in the figures, and the interrelation between the width of phase plate, width of the aperture, and form and contrast of the image was argued in detail.

It is not intended to claim any special originality in the conceptual content of this paper, but, I hope, these methods would cast some light upon practical phase-microscopy.

I am very much indebted to Prof. T. Hori for his giving me the first suggestion on this work and his detailed discussion on the nature of the vector method and on several other questions. It is to be added here that the main features of the results obtained in this paper have been actually observed by him by means of a phase-microscope prepared in his own laboratory.

Finally, it is my pleasure to express my hearty thanks to Prof. K. Imahori for the kind advices and valuable comments he has given in the present work.

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