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On Renormalization in the Field Theory with Non-Localized Interaction

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We have already found that the unobservable self-energy and vacuum polarization effects in quantum electrodynamics become finite, when the interaction terms in the usual field equations are replaced with the modified ones which are non-localized, averaged with Lorentz-invariant form functions. In this paper we investigate, in the second order perturbation calculation, if the finite self-energy and vacuum polarization effects are also made disappear with the use of the mass and charge renormalizations. For the purpose we begin with the expression of the second order radiative correction to the current operator which was worked out by one of us in the previous paper. In consequence it turns out that the renormalization procedure is always applicable to non-localized interactions with any arbitrary Lorentz-invariant form functions. In addition it is expected, through the general validity of this demonstration, that the possibility of renormalization still holds to all the higher orders in this non-local case as well.

§ 1. Introduction

In order to overcome the divergence difficulties in quantum electrodynamics, there have been proposed two alternative ways in the local field theory, one of which is the renormalization technique and the other is the regulator method. In the former technique unobservable infinities appearing in the theory are consistently subtracted out and the resulting formulation does not contain any more divergences, while the latter method aims at making infinite integrals appearing in the theory convergent by imposing some conditions on them. No satisfactory regulator method has been found and it seems rather hopeless to find a satisfactory way in this direction.

On the other hand, in view of the great successes thus far achieved by the renormalization procedure, more attention should be paid to it. It is the essence of this procedure to subtract the unobservable electromagnetic mass and vacuum polarization effects

by the renormalization of mass and charge. This possibility of renormalization is guaranteed in the current quantum electrodynamics. However, this is not always possible in meson theories. Therefore it is a very interesting problem to examine under what conditions renormalization technique is applicable.*

In this article we shall examine the renormalizability for a quantum electrodynamics with non-localized interaction, in which the field quantities appearing in the interaction terms are averaged over the whole four-dimensional space-time with some "smearing-out" Lorentz-invariant weight functions. In such a theory, there, of course, appear no divergence at all. However, it is desirable that the unobservable parts of electromagnetic mass and vacuum polarization should be subtracted away, if possible. One of the authors (S. M.) has given in the previous paper**[1] the expression for the second order electromagnetic mass and the second order radiative correction to the current operator. Using these results we shall show in the following that the second order mass and charge renormalizations are possible in general in the theory with such non-local interactions as considered in this paper. First we discuss mass renormalization in Section 2, and we have only to remark that the second order electromagnetic mass $\delta m^{(2)}$ do really become a pure number, which is of course finite in our treatment. Secondly, in Section 3, we show that the unobservable vacuum polarization current, which is also finite, is separable from the second order current operator and can, therefore, be renormalized by introducing new units of field strengths and charge.

§ 2. Mass Renormalization

In I the $\delta m\psi(x)$ has been, at the outset, separated from the mass term in the electron field equations with the view of canceling out the self-energy effect or performing mass renormalization (see (I, 5)). With these equations we have calculated the second order radiative correction to the current operator and its one-electron part, which is given by (I, 43.) In the first place we shall

* S. Sakata, H. Umezawa and S. Kamefuchi, Prog. Theor. Phys. 7 (1952), 377, investigated the applicability of the renormalization theory for the general types of local interactions.

** Hereafter we shall call this paper I.

observe the terms connected with the self-energy effect in (I, 43) :*

$$\frac{i}{2}\iint[\bar{\psi}^{(0)}(x),\ \gamma_\mu \bar{\bar{S}}(x-x')(\bar{\phi}(x')-\delta m^{(2)}\psi^{(0)}(x'))]_1\,dx' \\ +\frac{i}{2}\iint[(\bar{\psi}(x')-\delta m^{(2)}\bar{\psi}^{(0)}(x'))\bar{\bar{S}}(x'-x)\gamma_\mu,\ \bar{\psi}^{(0)}(x)]_1\,dx'\ , \tag{1}$$

where the $\bar{\phi}(x)$ is defined in (I, 41.) The possibility of mass renormalization is confirmed by seeing that the $\bar{\phi}(x)$ is merely a constant multiple of $\psi^{(0)}(x)$, i.e.,

$$\bar{\phi}(x) = \delta m^{(2)}\psi^{(2)}(x)\ , \tag{2}$$

and the constant $\delta m^{(2)}$ gives directly the second order electromagnetic mass.

For the purpose it is convenient to proceed with the SCHWINGER's calculation[2] in the local quantum electrodynamics. Now $\bar{\bar{S}}^{(1)}$ and $\bar{\bar{S}}$ are transformed by integration by parts into

$$\bar{\bar{S}}^{(1)}(x-x') = \left(\gamma\frac{\partial}{\partial x}-m\right)\bar{\bar{\Delta}}^{(1)}(x-x') \tag{3}$$

and

$$\bar{\bar{S}}(x-x') = \left(\gamma\frac{\partial}{\partial x}-m\right)\bar{\bar{\Delta}}(x-x')\ , \tag{4}$$

as is easily seen from their respective definitions. Using these expressions (3) and (4) and the identity

$$\gamma_\mu\left(\gamma\frac{\partial}{\partial x}-m\right)\gamma_\mu = -2\left(\gamma\frac{\partial}{\partial x}+2m\right)\ , \tag{5}$$

we can rewrite $\bar{\phi}(x)$ in a more tractable form

$$\bar{\phi}(x) = \int\Big\{\gamma_\lambda\Big[\bar{\bar{D}}(x-x')\frac{\partial}{\partial x_\lambda}\bar{\bar{\Delta}}^{(1)}(x-x')+\bar{\bar{D}}^{(1)}(x-x')\frac{\partial}{\partial x_\lambda}\bar{\bar{\Delta}}(x-x')\Big] \\ +2m\big[\bar{\bar{D}}(x-x')\bar{\bar{\Delta}}^{(1)}(x-x')+\bar{\bar{D}}^{(1)}(x-x')\bar{\bar{\Delta}}^{(1)}(x-x')\big]\Big\}\,\bar{\psi}^{(0)}(x')\,dx'\ . \tag{6}$$

It is worthy of note that all the associated D- and Δ-functions averaged once or twice over the whole space-time are functions of $\lambda = -(x-x')^2$ rather than $x_\mu - x'_\mu$, since they are of Lorentz-invariance like $D^{(1)}$- and $\bar{\Delta}$-functions themselves. Thus we can define a new quantity $\bar{P}(\lambda)$ as a function of λ by

* In the following we shall use the same notation as I and the natural units $\hbar=c=1$.

$$\frac{\partial \widetilde{P}(\lambda)}{\partial \lambda} = \widetilde{\widetilde{D}}(\lambda) \frac{\partial \widetilde{\widetilde{\Delta}}^{(1)}(\lambda)}{\partial \lambda} + \widetilde{\widetilde{D}}^{(1)}(\lambda) \frac{\partial \widetilde{\Delta}^{(1)}(\lambda)}{\partial \lambda} \ . \tag{7}$$

Since, as is easily seen,

$$\frac{\partial \widetilde{P}(\lambda)}{\partial x_\mu} = \widetilde{\widetilde{D}}(\lambda) \frac{\partial \widetilde{\widetilde{\Delta}}^{(1)}(\lambda)}{\partial x_\mu} + \widetilde{\widetilde{D}}^{(1)}(\lambda) \frac{\partial \widetilde{\Delta}(\lambda)}{\partial x_\mu} \ , \tag{8}$$

the first term of (6) is simplified

$$\begin{aligned} &\int \gamma_\lambda \Big[\widetilde{\widetilde{D}}(x-x') \frac{\partial}{\partial x_\lambda} \widetilde{\widetilde{\Delta}}^{(1)}(x-x') + \widetilde{\widetilde{D}}^{(1)}(x-x') \frac{\partial}{\partial x_\lambda} \widetilde{\Delta}(x-x') \Big] \bar{\psi}^{(0)}(x') dx' \\ &= -m \int \widetilde{P}(\lambda) \bar{\psi}^{(0)}(x') dx' \ , \end{aligned} \tag{9}$$

where we have employed the fact that $\bar{\psi}^{(0)}(x)$ as well as $\psi^{(0)}(x)$ satisfies the free Dirac equation. Hence,

$$\bar{\phi}(x) = m \int \widetilde{Q}(\lambda) \bar{\psi}^{(0)}(x') dx' \ , \tag{10}$$

where

$$\widetilde{Q}(\lambda) = 2[\widetilde{\widetilde{D}}(x-x') \widetilde{\widetilde{\Delta}}^{(1)}(x-x') + \widetilde{\widetilde{D}}^{(1)}(x-x') \widetilde{\Delta}^{(1)}(x-x')] - \widetilde{P}(\lambda) \ . \tag{11}$$

It is not only impossible but also unnecessary to evaluate the explicit forms of $\widetilde{P}(\lambda)$ and $\widetilde{Q}(\lambda)$, because we have given no unique functional forms of the smearing-out functions and because what is required is to demonstrate that the $\delta m^{(2)}$ is actually a mere constant rather than to evaluate its complete expression. Now we shall give the Fourier transform of $\widetilde{Q}(\lambda)$ as follows

$$\widetilde{Q}(\lambda) = \frac{1}{(2\pi)^4} \int e^{ik(x-x')} R(k^2) \, dk \ , \tag{12}$$

where it should be noted that the R is, in fact, a function of k^2 only. Since the zeroth order spinor field $\psi^{(0)}(x)$ satisfies the free Dirac equation, this, of course, satisfies the Klein-Gordon differential equation

$$(\Box - m^2) \psi^{(0)}(x) = 0 \ . \tag{13}$$

This implies that the Fourier component $\psi^{(0)}(k)$ is proportional to $\delta(k^2+m^2)$, i.e.,

$$(k_\mu^2 + m^2) \psi^{(0)}(k) = 0 \ . \tag{14}$$

Thus, for any arbitrary function R of k^2, the remarkable relation

$$R(k^2) \psi^{(0)}(k) = R(-m^2) \psi^{(0)}(k) \tag{15}$$

holds. Hence, (10) leads to

$$\tilde{\phi}(x) = mR(-m^2)\, g(-m^2)\, \psi^{(0)}(x)\,. \tag{16}$$

Accordingly, we have obtained from (2) and (16) as a finite second order electromagnetic mass

$$\delta m^{(2)} = mR(-m^2)\, g(-m^2)\,. \tag{17}$$

This is surely a pure constant and renormalization is possible to the second order in the case where the field quantities appearing in the interaction terms are averaged with any arbitrary Lorentz-invariant scalar function.

§ 3. Charge Renormalization

After disposing of the self-energy effect by mass renormalization, as was shown in Section 2, the one-particle part of the second order current operator leads, from (I, 43) and (I, 44), to

$$\begin{aligned}\langle j_\mu^{(2)}(x)\rangle_{1.0} = &-\frac{i}{4}\iint[\tilde{\psi}^{(0)}(x'), \widetilde{K}_\mu(x'-x, x-x'')\,\tilde{\psi}^{(0)}(x'')]_1\, dx'dx'' \\ &+\frac{i}{4}\iint[\tilde{\psi}^{(0)}(x''), \gamma_\lambda\tilde{\psi}^{(0)}(x'')]_1\, Tr\,\{\tilde{\tilde{S}}^{(1)}(x'-x)\gamma_\mu\tilde{\tilde{S}}(x-x')\gamma_\lambda \\ &+\tilde{\tilde{S}}(x'-x)\gamma_\mu\tilde{\tilde{S}}^{(1)}(x-x')\gamma_\lambda\}\,\tilde{\tilde{D}}(x'-x'')dx'dx''\,,\end{aligned} \tag{18}$$

where $\widetilde{K}_\mu$ was given by (I, 42). The first term of (18) corresponds to the radiative modification of the electromagnetic properties of electron and the second term represents the current induced in the vacuum. The latter is, of course, finite in the present treatment, though it produced an infinite charge renormalization in the usual theory. However, the part proportional to the zeroth order current in the vacuum induced one is to be renormalized, since it is unobservable. In order to study the renormalizability we shall transform the expression of the vacuum polarization current. To begin with, we evaluate the trace factor in the integrand of (18), taking note of (3) and (4),

$$\begin{aligned}Tr\{\cdots\} = &-8\left\{\frac{\partial\tilde{\tilde{\Delta}}^{(1)}(x-x')}{\partial x_\mu}\,\frac{\partial\tilde{\Delta}(x-x')}{\partial x_\lambda}+\frac{\partial\tilde{\tilde{\Delta}}^{(1)}(x-x')}{\partial x_\lambda}\,\frac{\partial\tilde{\Delta}(x-x')}{\partial x_\mu}\right\} \\ &+8\delta_{\lambda\mu}\left\{\frac{\partial\tilde{\tilde{\Delta}}^{(1)}(x-x')}{\partial x_\nu}\,\frac{\partial\tilde{\Delta}(x-x')}{\partial x_\nu}+m^2\tilde{\tilde{\Delta}}^{(1)}(x-x')\,\tilde{\Delta}(x-x')\right\}\,.\end{aligned} \tag{19}$$

From the fact that $\tilde{\tilde{\Delta}}^{(1)}$ and $\tilde{\Delta}$ are functions of $\lambda = -(x-x')^2$, we can

define $G(\lambda)$, a function of λ, according to

$$2\frac{\partial\bar{\bar{\Delta}}^{(1)}(x-x')}{\partial\lambda}\frac{\partial\bar{\bar{\Delta}}(x-x')}{\partial\lambda}=\frac{\partial^2 G(\lambda)}{\partial\lambda^2}\ . \tag{20}$$

Note first that

$$\begin{aligned}&\frac{\partial\bar{\bar{\Delta}}^{(1)}(x-x')}{\partial x_\mu}\frac{\partial\bar{\bar{\Delta}}(x-x')}{\partial x_\lambda}+\frac{\partial\bar{\bar{\Delta}}^{(1)}(x-x')}{\partial x_\lambda}\frac{\partial\bar{\bar{\Delta}}(x-x')}{\partial x_\mu}\\&=\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\lambda}G(\lambda)+2\delta_{\lambda\mu}\frac{\partial G(\lambda)}{\partial\lambda}\ ,\end{aligned} \tag{21}$$

so that (19) reduces to

$$-8\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\lambda}G(\lambda)+8\delta_{\lambda\mu}H(\lambda)\ , \tag{22}$$

where

$$H(\lambda)=-2\frac{\partial G(\lambda)}{\partial\lambda}-2\lambda\frac{\partial^2 G(\lambda)}{\partial\lambda^2}+m^2\bar{\bar{\Delta}}^{(1)}(\lambda)\bar{\bar{\Delta}}(\lambda)\ . \tag{23}$$

Hence, the second term of (18), denoted by "vac. pol.", is rewritten as

$$\begin{aligned}vac.\,pol.=4\iint\Big[&-\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\lambda}G(\lambda)+\delta_{\lambda\mu}H(\lambda)\Big]\\&\times\bar{\bar{D}}(x'-x'')\langle j_\lambda^{(0)}(x'')\rangle_{1,0}dx'dx''\ ,\end{aligned} \tag{24}$$

where, owing to the continuity equation for the zeroth order current, the first term in the brackets vanishes by integration by parts. If we add the one-particle part of the zeroth order current to (24), we obtain that of the total current including radiative correction up to the second order (note that $\langle j_\mu^{(1)}(x)\rangle_{1,0}=0$, as is readily seen from (I, 22))

$$\langle j_\mu(x)\rangle_{1,0}=e\langle j_\mu^{(0)}(x)\rangle_{1,0}+e^3\int\bar{\bar{H}}(x-x')\langle j_\mu^{(0)}(x')\rangle_{1,0}dx'\ , \tag{25}$$

where we put

$$\bar{\bar{H}}(x-x')=\bar{\bar{H}}(\lambda)=4\int H(x-x'')\,\bar{\bar{D}}(x''-x')\,dx''\ . \tag{26}$$

It is helpful in the following discussion to expand $\bar{\bar{H}}(x-x')$, i.e.,

$$\begin{aligned}\bar{\bar{H}}(x-x')&=\frac{1}{(2\pi)^4}\int e^{ik(x-x')}\bar{\bar{H}}(k^2)\,dk\\&=\frac{1}{(2\pi)^4}\int e^{ik(x-x')}\{C_0+C_1(-k^2)+C_2(-k^2)^2+\cdots\}\,dk\\&=C_0\delta(x-x')+C_1\Box\delta(x-x')+C_2\Box^2\delta(x-x')+\cdots\ .\end{aligned} \tag{27}$$

Then we get

$$\langle j_\mu(x)\rangle_{1,0} = e\langle j_\mu^{(0)}(x)\rangle_{1,0} + e^3C_0\langle j_\mu^{(0)}(x)\rangle_{1,0} + e^3C_1\Box\langle j_\mu^{(0)}(x)\rangle_{1,0} + e^3C_2\Box^2\langle j_\mu^{(0)}(x)\rangle_{1,0} + \cdots , \tag{28}$$

where the second term gives nothing but the unobservable part of vacuum polarization.

As the final step, it is necessary to show that the term may be dropped out by charge renormalization. If we introduce correlatively new units of field strengths and charge according to

$$e_0F_{\mu\nu 0} = eF_{\mu\nu} \tag{29}$$

and

$$e_0^2(1+C_0e_0^2) = e^2 , \tag{30}$$

a subscript zero indicating the quantity thus far used, then, in view of the field equation (I, 6),

$$\begin{aligned} e_0\frac{\partial}{\partial x_\nu}F_{\mu\nu 0} &= e\frac{\partial}{\partial x_\nu}F_{\mu\nu} \\ &\sim e_0^2(1+C_0e_0^2)\langle j_\mu^{(0)}(x)\rangle_{1,0} + e_0^4C_1\Box\langle j_\mu^{(0)}(x)\rangle_{1,0} + \cdots \\ &= e^2\langle j_\mu^{(0)}(x)\rangle_{1,0} + e^4C_1\Box\langle j_\mu^{(0)}(x)\rangle_{1,0} + \cdots , \end{aligned} \tag{31}$$

where we put $e_0^4 = e^4$ in the second term since our calculation may be restricted to the second order inclusive. Hence

$$\frac{\partial}{\partial x_\nu}F_{\mu\nu} \sim e\langle j_\mu^{(0)}(x)\rangle_{1,0} + e^3C_1\Box\langle j_\mu^{(0)}(x)\rangle_{1,0} + \cdots . \tag{32}$$

Thus we have no more unobservable part to the approximation considered here.

We have so far carried out only the second order perturbation calculation, and it is still open whether the renormaliziaton procedure may be applicable to all the higher order. It seems, however, to us that our non-local structure of the interaction between the electron and electromagnetic fields does not mar the renormalizability.

References

1) S. Minami, *J. Fac. Sci. Hokkaido Univ.*, Ser. II, 4 (1952), 153.

2) J. Schwinger, *Phys. Rev.* 75 (1949), 651.