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Note on High Energy Deuteron-Stripping Process

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The in appearance over simplified calculation by R. SERBER of high energy deuteron-stripping cross section are examined. Also the nucleon density effects are considered.

§ 1. Introduction

New type of high energy nuclear reaction or "*stripping*" process is confirmed in 184- in. cyclotron by HELMHOLZ, MCMILLAN, and SEWELL[2], that is, when the deuteron grazes the edge of the nucleus, the proton, say, may strike it and be stripped off, while the neutron misses and continues to fly in a narrow cone with energy about half that of the deuteron. R. SERBER[1] asked for the stripping cross section in the following way, that is, at the instant of collision the proton, say, will be within a circle in the plane perpendicular to the deuteron motion and its radius equal to that of nucleus, while the neutron will be outside it. When calculates the cross section, he makes the assumption that the nuclear edge is considered to be straight, namely, $R \gg R_d$. But nuclear radius is not always so large as compared with that of deuteron. So we consider the effects of (1) the curvature of nuclear edge, (2) the nucleon distribution inside the nucleus only with SERBER approximation, (3) the both effects.

§ 2. The Effects of Curvature of Nuclear Edge

Using the notations in Fig. I that shows the projection of proton and neutron in a plane perpendicular to the deutron motion, the probability that the proton is in a domain $dxdl$ inside the nucleus and the neutron misses it is $2\theta\rho/2\pi\rho = \theta/\pi$. The total cross section for fixed ρ is

$$\sigma(\rho) = \iint \theta/\pi \, dx \, dl \,. \tag{1}$$

There is a geometrical relation for x,

$$(x-\rho\cos\theta)\left\{2R-(x-\rho\cos\theta)\right\} = (\rho\sin\theta)^2 \,,$$

or

$$x = \rho\cos\theta + R - \sqrt{R^2-\rho^2\sin^2\theta} \qquad \text{for} \quad x < R \,, \tag{2}$$

and

$$dl = (R-x)\, d\varphi \,.$$

Then

$$\sigma(\rho) = \frac{1}{\pi}\iint \theta\,(R-x)\, dx \, dl \,. \tag{3}$$

Hence,

$$\sigma(\rho) = \rho\sqrt{R^2-\rho^2} - \frac{\pi}{2}\rho^2 + R^2\sin^{-1}\frac{\rho}{R} \,. \tag{4}$$

Putting deuteron ground state waver function as $\Psi_d(r)$, then the total cross section in spherical polar coordinates is

$$\int_{r=0}^{\infty}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\left(\rho\sqrt{R^2-\rho^2} + R^2\sin^{-1}\frac{\rho}{R} - \frac{\pi}{2}\rho^2\right)|\Psi_d(r)|^2 \cdot r^2 dr \sin\theta \, d\theta \, d\phi \,. \tag{5}$$

After angular integrations are performed exactly ,

$$\sigma = \int |\Psi_d(r)|^2 \left\{2Rr\left[\frac{1}{3}(\boldsymbol{E}-\boldsymbol{D}) + (\boldsymbol{F}-\boldsymbol{D})\right] - \frac{2}{3}\frac{r^3}{R}\boldsymbol{D} - \frac{2}{3}\pi r^2\right\} d\vec{r} \,, \tag{6}$$

where $d\vec{r} = 4\pi r^2 dr$, $\boldsymbol{F}, \boldsymbol{E}$: complete elliptic integrals,

$$\boldsymbol{F} = \int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \,, \quad \boldsymbol{E} = \int_0^{\pi/2}\sqrt{1-k^2\sin^2\theta}\, d\theta \,, \quad \boldsymbol{D} = \frac{1}{k^2}(\boldsymbol{F}-\boldsymbol{E}) \,,$$

k = modulus of the elliptic integrals = r/R .

But, unfortunately these elliptic integrals are given only in numerical tables, hence pushing the calculation further is difficult.

So, we must take the approximation by power series of ρ, then

$$x = \rho\cos\theta + \frac{1}{2}\frac{\rho^2}{R}\sin^2\theta + \frac{1}{8}\frac{\rho^4}{R^3}\sin^4\theta + \cdots\cdots \,.$$

Also, here, we must consider two cases for the integration limits.

(A) No stripping occurs when the neutron and proton are in position like the Fig. II., or it is necessary that the proton is already at the depth of at least $R-\sqrt{R^2-\rho^2}$ inside the nucleus. Then the integration is performed between $\theta=\pi/2$ and 0. In the case of (B) x is allowed to be between 0 and ρ, that is, integration boundary is from $\theta=\pi/2+\varepsilon$ to 0, where $\sin\varepsilon=\rho/2R$, hence to $\varepsilon=\sin^{-1}(\rho/2R)$.

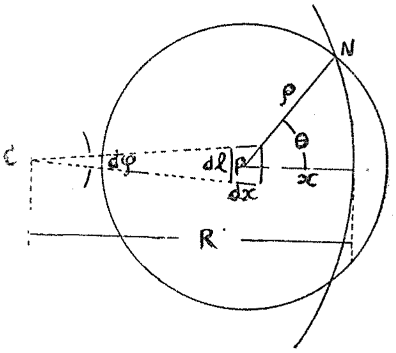

Fig. I.

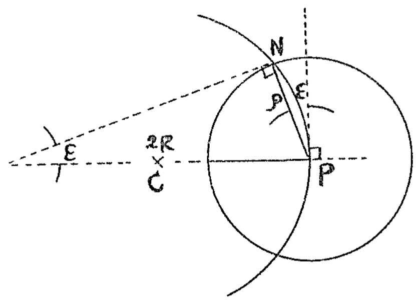

Fig. II.

After θ integrations, total cross section for fixed ρ is, for case (A)

$$\sigma(\rho)=2R\rho-\frac{\pi}{2}\rho^2-\frac{1}{3}\frac{\rho^3}{R}-\frac{1}{20}\frac{\rho^5}{R^3}-\cdots\cdots, \tag{7}$$

and for (B)

$$\sigma(\rho)=2R\rho-\frac{1}{12}\frac{\rho^3}{R}-\frac{7}{40}\frac{\rho^5}{R^3}-\cdots\cdots. \tag{8}$$

Next, we perform the integration of the type (5), in which we take cylindrical coordinate, i.e., $r^2=\rho^2+z^2$, then we meet the typical integral as

$$4\pi\int_0^\infty|\Psi_d(r)|^2\cdot rdr\int_0^r\frac{\rho^n}{\sqrt{r^2-\rho^2}}\rho\, d\rho\ , \tag{9}$$

where ρ integral becomes to

$$r^{n+1}\frac{\sqrt{\pi}}{2}\cdot\frac{n}{n+1}\cdot\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}\ , \tag{10}$$

and if

$$\Psi_d(r)=\sqrt{\frac{\alpha}{2\pi}}\cdot\frac{e^{-\alpha r}}{r},\quad \alpha=\frac{\sqrt{M\varepsilon_d}}{\hbar},\quad \bar{r}_n\equiv\int_0^\infty r^n|\Psi_d(r)|^2 d\vec{r}=\Gamma(n+1)R_d^n, \tag{10'}$$

then, (9) becomes to

$$\frac{\sqrt{\pi}}{2}\cdot\frac{n}{n+1}\cdot\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}\cdot\Gamma(n+1)R_d^n. \tag{11}$$

Thus the total stripping cross sections are, for case (A)

$$\sigma=\frac{\pi}{2}RR_d\left\{1-\frac{2}{3}\left(\frac{R_d}{R}\right)-\frac{3}{2}\left(\frac{R_d}{R}\right)^2-15\left(\frac{R_d}{R}\right)^4-\cdots\right\}, \tag{12 A}$$

and for case (B),

$$\sigma=\frac{\pi}{2}RR_d\left\{1-\frac{3}{8}\left(\frac{R_d}{R}\right)^2-\frac{7\cdot5\cdot3}{2}\left(\frac{R_d}{R}\right)^4-\cdots\right\}. \tag{12 B}$$

Compared with the Serber's result $\sigma=\frac{\pi}{2}RR_d$, the higher terms in brackets are the corrections from the effects of nuclear curvature. Relative corrections to the third order in % are given in Table I. They are negative.

Table I.

	Be	Al	Cu	Mo	Sn	Ta	Pb	U
R_d/R	0.68	0.47	0.35	0.31	0.28	0.25	0.24	0.23
(A)	−114%	−64	−41	−34	−31	−26	−25	−24%
(B)	− 17%	− 8	− 4	−3.5	− 3	−2.5	− 2	− 2%

In experiment, the ratio of stripping cross sections is 4/5 between Be and U, where Serber's approximation gives 1/3 for the same pair. Thus these negative corrections, especially in case (A), make the situation worse under the condition that the experimental ratio is reliable. So we can say that Serber's approximation corresponds to our case (B), though in outlook to the case (A), and aims at approaching the experimentals.

§ 3. Effect of the Density Distribution of Nucleons

We considered in the preceding section the effect of the curvature of nuclear edge on the stripping cross section. Next, we estimate the influences of the density distribution of nucleons on the penetration of proton into the nucleus.

(i) Case of Linearly Falling

At first, we consider simply the effect of proton density distribution is a linear function, i.e.

$$f(x)=1-\frac{x}{a}=\begin{cases}1 & \text{for} \quad x=0\\ 0 & \text{for} \quad x\geqq a\,,\end{cases} \tag{13}$$

where a is a certain distance into the nucleos. Then, the total cross section for fixed ρ is

$$\sigma(\rho)=\iint\frac{\theta}{\pi}f(x)dx\,dl=\iint\frac{\theta}{\pi}\,dx\,dl-\iint\frac{\theta}{\pi}\,\frac{x}{a}\,dx\,dl\,. \tag{14}$$

The second term is after integrating over dx, dl, which contributes to the total cross section as $\rho^2\to 2/3\cdot R_d^2$. Thus, the total cross section is

$$\sigma=\frac{\pi}{2}RR_d\left\{1-\frac{1}{3}\left(\frac{R_d}{a}\right)\right\}\,. \tag{15}$$

Now, $a\approx R_d$, then the total cross section decreases by the factor 2/3.

(ii) Case of Exponentially Falling

Next, we consider the case of

$$f(x)=e^{-\frac{x}{a}}\,, \tag{16}$$

$$\sigma(\rho)=\iint\frac{\theta}{\pi}\,e^{-\frac{x}{a}}\,dx\,dl\,. \tag{17}$$

After integrating over dx, dl,

$$\sigma(\rho)=2R\rho\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n\,!}\cdot\left(\frac{\rho}{a}\right)^{n-1}\cdot\frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}\,. \tag{18}$$

ρ^n gives the contribution to the total cross section as (11), whence the n-th term in series is

$$\frac{\pi}{2}\cdot\frac{1}{n+1}\cdot\left(\frac{R_d}{a}\right)^{n-1}\cdot R_d\,.$$

Hence

$$\sigma = \pi RR_d \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n+1} \cdot \left(\frac{R_d}{a}\right)^{n-1}. \tag{19}$$

If we take $a = R_d$, then

$$\sigma = \pi/2 \cdot RR_d \cdot (1 - ln2) = \frac{\pi}{2} RR_d \cdot 0.614 \cdots . \tag{20}$$

This is about the same result as the preceding case (i).

§ 4. Effects of both Curvature and Density at Nuclear Edge

In the higher corrections we consider the effects of both curvature and density distribution at nuclear edge.

(i) Case of Linearly Falling

Here the cross section for fixed ρ is

$$\sigma(\rho) = \iint \frac{\theta}{\pi} \left(1 - \frac{x}{a}\right)(R - x)\, dx\, d\varphi . \tag{21}$$

And the total cross section

$$\sigma = \frac{\pi}{2} RR_d \left\{1 - \frac{2}{3}\frac{R_d}{R} - \frac{1}{3}\frac{R_d}{a} - \frac{3}{2}\left(\frac{R_d}{R}\right)^2 + 2\frac{R_d{}^2}{aR}\right\} . \tag{22}$$

If we put $a = R_d$, then

$$= \frac{\pi}{2} RR_d \left\{\frac{2}{3} + \frac{3}{4}\frac{R_d}{R} - \frac{3}{2}\left(\frac{R_d}{R}\right)^2\right\} . \tag{23}$$

And the relative corrections in % (negative) to SERBER cross sections are in TABLE II.

TABLE II.

Be	Al	Cu	Mo	Sn	Ta	Pb	U
−12%	−4.4	− 5	−6.3	−7.7	− 9	−10	−11%

These may change the cross section to favour the experimentally nearly independent yield ratios of target elements.

(ii) Case of Exponentially Falling

Here

$$\sigma(\rho)=\iint\frac{\theta}{\pi}\,(R-x)\,e^{-\frac{x}{a}}\,dx\,d\varphi\,. \tag{24}$$

$$\begin{aligned}&=2R\rho\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n\,!}\cdot\left(\frac{\rho}{a}\right)^{n-1}\frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}\\&+2\rho\,a\sum_{n=1}^{\infty}\frac{(-1)^{n}}{n\,!}\cdot\left(\frac{\rho}{a}\right)^{n}\frac{\sqrt{\pi}}{2}\cdot\frac{n}{n+1}\cdot\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}\\&+2\rho\,a\sum_{n=1}^{\infty}\frac{(-1)^{n}}{n\,!}\cdot\left(\frac{\rho}{a}\right)^{n-1}\frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}\,.\end{aligned} \tag{25}$$

And the total cross section is

$$\begin{aligned}\sigma&=\pi RR_d\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n+1}\cdot\left(\frac{R_d}{a}\right)^{n-1}+\pi a\,R_d\sum_{n=0}^{\infty}(-1)^n\left(\frac{R_d}{a}\right)^{n}\\&+\pi a\,R_d\,2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n+1}\cdot\left(\frac{R_d}{a}\right)^{n-1}\,.\end{aligned} \tag{26}$$

For $a=R_d$, this becomes to

$$\sigma=\frac{\pi}{2}\,RR_d\left\{1-0.228\,\frac{R_d}{R}\right\}\,. \tag{27}$$

And the corrections to SERBER cross sections are in TABLE III. Negative large corrections in lighter elements make the situation worse.

TABLE III.

Be	Al	Cu	Mo	Sn	Ta	Pb	U
−15%	−11	− 8	−6.5	−6.5	−5.7	−5.4	−5.2

§ 5. Conclusion

We may say from the preceding research that SERBER's in outlook over simplified cross section is pretty well as far as the yield ratios given by HELMHOLZ et al. is reliable, and this cross section may be a good approximation of our case which consider the higher corrections of the effect of either nuclear curvature or necessary modifications of integration limit, and the density distribution of linearly changing type.

References

1) R. SERBER, *Phys. Rev.* **72** (1947), 1007.
2) A. C. HELMHOLZ, E. M. MCMILLAN and D. C. SEWELL, *Phys. Rev.* **72** (1947), 1003.
3) Cf. S. FLÜGGE, *Z. Physik* **96** (1935), 459.