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Note on the Commutation Relations in Quantum Mechanics

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It is shown that if a physical system has a potential of a form $V=aq^n+b$, where n is an even integer the equation of motion does not necessarily imply ordinary commutation relation $i[p,q]=1$.

Consider a one-dimensional quantum-mechanical system possessing a hamiltonian $H=H(p,q)$ of the form*

$$H=\frac{1}{2}P^2+V(q). \tag{1}$$

The question as to whether the equation of motion

$$\dot{q}=P, \quad \dot{P}=-\partial V/\partial q\equiv -V_q, \quad (\cdot=d/dt) \tag{2}$$

or, equivalently,

$$i[H,q]=P, \quad i[H,P]=-V_q \tag{2'}$$

imply, as a consequence, the commutation relation

$$i[P,q]=1, \tag{3}$$

has been considered by WIGNER[1] and PUTNAM[2]. WIGNER has shown that, if $V=q^2$, so that Eq. (1) is the hamiltonian for the one-dimensional harmonic oscillator, and $V=0$, so that Eq. (1) is the hamiltonian for the free particle, then Eq. (2) fails to imply Eq. (3). And PUTNAM has pointed out that for an arbitrary potential $V=aq^n+b$, where n is an odd integer, Eq. (3) does follow from Eq. (2). In this note, we shall confirm in general way, but under some natural hypothesis, that for a potential $V=aq^n+b$, where n is an even integer, Eq. (3) does not necessarily follow from Eq. (2).**

* It is assumed that the particle has unit mass and that units are chosen so that PLANCK's constant is unity.

** Our treatment in this paper is quite analogous to the PUTNAM's one, and uses the same notations.

Suppose that there exist two sequence $a_1, a_2, \cdots$ and $b_1, b_2, \cdots$ of complex numbers and also two sequences $j_1, j_2, \cdots$ and $k_1, k_2, \cdots$ of non-negative integers with the property that

$$\sum_\alpha (a_\alpha q^{2m} + b_\alpha) V^{j_\alpha} V_q^{2k_\alpha} = 0, \quad m = 0, 1, 2, \cdots \tag{4}$$

with the understanding that $V^0 = V_q^0 = 1$, where V is a certain function of q. Our potential $V = aq^n + b$ satisfies a following relation, if $n = 2k$, $j = 2(k-1)$ and $m = k$,

$$q^{2m} V - b)^j - \text{const}\, V_q^{2k} = 0 . \tag{5}$$

Relation (5) is seen to be of the form (4), where $\sum_\alpha a_\alpha V_{j_\alpha} V_q^{2k_\alpha} = (V-b)^j$, thus our potential $V = aq^n + b$ satisfies Eq. (4). Eq. (2′) can be written, in virtue of Eq. (1), as

$$i[P^2, q] = 2P, \quad i[V, P] = -V_q . \tag{6}$$

The addition and subtraction of the quantity pqp to the left side of the first equation of Eq. (6) yields

$$PB + BP = 2P, \tag{7}$$

where

$$B \equiv i[P, q] . \tag{8}$$

From Eq. (7), we get

$$P^2B - BP^2 = 0 . \tag{9}$$

In Eq. (8) differentiation of B with respect to t (time), by the use of Eq. (2) and assuming that q commutes with V, gives

$$\dot{B} = i(\dot{P}q + P\dot{q} - \dot{q}P - q\dot{P}) = 0 . \tag{10}$$

It is assumed, as usual, that the following HEISENBERG equation holds for an arbitrary function $F = F(p, q)$

$$\dot{F} = i[H, F] . \tag{11}$$

Hence by Eqs. (10), (11), (1) and (9) it follows that

$$BV - VB = 0 . \tag{12}$$

Differentiation of Eq. (7) with respect to t and the use of Eq. (2) yield

$$BV_q + V_qB = 2V_q . \tag{13}$$

The similar procedure to the deduction of Eq. (9) gives

$$BV_q^2 - V_q^2 B = 0 \,. \tag{14}$$

It readily follows from Eqs. (12) and (14) that

$$BV^j - V^j B = 0 \quad \text{and} \quad BV_q^{2k} - V_q^{2k} B = 0 \tag{15}$$

hold for $j, k = 0, 1, 2, \cdots$. Multiplications of Eq. (4) by B from the left and from the right, and the use of Eq. (15), yield $Bq^{2m}C = q^{2m}BC$ where $C = \sum_\alpha a_\alpha V^{j_\alpha} V_q^{2k_\alpha}$.
The assumption that C^{-1} exists yields

$$Bq^{2m} = q^{2m}B \,. \tag{16}$$

Now Eq. (16) does not always imply the relation

$$Bq = qB \,. \tag{17}$$

If the relation (17) exists, as PUTNAM's conclusion, we have $B=1$ after some calculation.*** Our Eq. (16), however, does not necessarily imply Eq. (17), so that $B=1$ does not always be followed.

In conclusion, when the system has an even power potential, the equation of motion in quantum mechanics does not always imply the usual commutation relation.

Appendix

Instead of Eq. (4), we may take the form of

$$\sum_\alpha (a_\alpha q + b_\alpha) V_q^{j_\alpha} V_{q^2}^{2k_\alpha} = 0 \quad (V_{q^2} = \partial^2 V / \partial q^2) \,.$$

In this case, only when $BV_q^j = V_q^j B$, follows $B=1$. However, as seen from Eq. (15), this is the case only when j is an even integer. Therefore the same conclusive result will be followed.

References

1) E. P. WIGNER, *Phys. Rev.* 77 (1950), 711.
2) C. R. PUTNAM, *Phys. Rev.* 83 (1951), 1047.

*** Since $\dot{B} = 0$, differentiation of Eq. (17) with respect to t and the use of Eq. (2) imply $Bp = pB$. That $B=1$ now follows from Eq. (7) and the assumption that p^{-1} exists.