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# On the Cut-Off Methods in Meson Theory*

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In order to see its adequacy as a technique of cut-off in meson theory, FEYNMAN's convergent factor is introduced into finite integrals in various ways. Investigated are anomalous magnetic moment of a nucleon, decay of a neutral pion into two photons, and radiative corrections to the nucleon propagation function and the vertex operator. Results aer compared with GOTO's and it was found that similar values can be obtained for the magnitic moment and a pion decay if the manners of introduction of the convergent factor are appropriately chosen. The effect of this cut-off procedure is quite small for the radiative corrections to the nucleon propagation function and the vertex operator. Discussions are given of connections with the mixed field theory when they exist.

## § 1. Introduction

Since the recent discovery of many unstable heavy particles, it has become clearer that the interaction in which these particles take parts are not very weak[1]. As the result of which it has been recognized that, in contrast with quantum electrodynamics, the conventional meson theory does not constitute a closed system of the theory even in an approximate sense. Under such situation, the meaning of cut-off procedure in meson theory was discussed by FUKUDA,[2], CHEW[3], and others. According to them, it is felt that the cut-off procedure is not only a convenience to make divergent integrals finite, but has much more realistic physical meaning.

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* Part of the present work has been published in *Soryusiron-Kenkyu*** (Mimeographed circular in Japanese) 8 (1955), 258. Dr. S. GOTO's comments (private communication) on the article revived the author's interest which enabled him to complete the rest of the work. The author expresses his sicerest thanks for the comment.

** Numerical results on the anomalous magnetic moment of an electron are erroneously reported in this article. They should be corrected as Table II of the present paper.

Many results of the application of the cut-off procedure have been published on this standpoint. CHEW and his collaborators[3] investigated pion-nucleon scattering, anomalous magnetic moment of a nucleon (A.M.M.) etc., using gradient coupling pseudoscalar meson theory with a fixed nucleon to obtain improvemet over corresponding straightforward perturbation calculations. However, as admitted by CHEW[4] himself, $S$-wave scattering cannot be accounted for by his simple model and certain refinement is required on the point. Further, with regard to A.M.M., in which we are especially interested, some uncertainties about the treatment of renormalization effeect[5] and virtual nucleon current seem to be left unsettled. A.M.M. essentially depends on the mementum difference between initial and final nucleons, so it does not seem appropriate to make the nucleon completely fixed and certain refinement would also be necessary[6]. Finally, the model cannot be applied to processes in which nucleon pairs play essential role such as the decay of a neutral pion into photons.

The investigation recently published by GOTO[7] have covariant starting points and hence have no such defect. It should be noted, however, the calculation is much more complicated and any attempt to get out of perturbation method would be difficult to be sccessful. He first performed integration over the fourth component of the virtual four mementum appearing in calculations of various quantities and clarified correspondence between the covariant calculation and the old-fashioned perturbation one. Then he could cut off the integrals over the three components of virtual momentum. His results on A. M. M. and decay of a neutral pion show considerable improvement over the conventional calculations without cut off.

Although GOTO's cut-off procedure has a definite physical interpretation that the spacial components of a virtual quanta are cut off and the comparison with old-fashioned calculation is easily possible, the noncovariant charactor of the cut-off method makes the calculation involved (and perhaps some difficulties will be met in connection with renormalization technique in the calculation of some other quantities than those he has investigated).

The present paper, thus, intends to simplify the calculations using the covariant cut-off method. It is needless to say that the cut-off method thus introduced must really correspond to cut-off and must reproduce, *e.g.*, GOTO's results*. If such a method does exist, it is expected that the method may be applicable to many other processes and the calculations will be comparatively straightforward.

The most familiar covariant cut-off method will be the one introduced by FEYNMAN[8]. He introduced the covariant convergent factor $c(k^2)$ originally in order to make divergent integrals finite. However, if any physical meaning is to be attributed to the cut-off procedure as mentioned above, it will be of worth to investigate its effects when introduced into finite integrals. The effects of the introduction of FEYNMAN's convergent factor $c(k^2)$ into various finite integrals from this standpoint will be the object of the present investigations.

The factor $c(k^2)$ can be introduced in various ways. Originally, FEYNMAN[8] introduced the factor into propagation function of virtual quanta**. As will be mentioned in detail in what follows, this modification corresponds to a method of field mixture in which the third field with quanta of cut-off mass are coupled in an appropriate manner. The effect of this modification of the pion propagation function on A. M. M. problem is discussed in § 2. The corresponding modification of the propagation function of a fermion has no such a direct field theoretical interpretation and, as noted by FEYNMAN, leads to a certain difficulty related to the gauge invariancy of the theory. We, however, do not adhere to this point, since we are using the factor as a technique of reproducing the results of conventional cut-off procedure. Calculation of A. M. M. modified in this way is presented in § 3 and it will be seen that this procedure cannot correspond to physical cut-off. The third

* Of course, there is no reason why we should believe that GOTO's cut-off procedure, namely no cut-off for the fourth component of virtual momenta and straight cut-off for the spacial components, corresponds to physical reality. We believe, however, that any cut-off procedure must be such that the result of which shows a reasonable dependence on the cut-off parameter and improvement over the conventional ones without cut-off, since we have no other means to decide which procedure is correct other than to see to what degree it reproduces the experimental value at the present state of the meson theory.

** Several works on the singularity of propagation functions have been published recently. It seems, however, to the author that the conclusion is not decisive yet. See reference 9.

method of the introduction of $c(k^2)$ is to introduce it into the final integrations over four momenta. This might be expected to correspond most faithfully to the physical sense of cut-off. However, for A. M. M. at least, it is shown that this is not the case (§ 4).

Further, decay of a neutral pion (§ 5), and the radiative-corrected nucleon propagation function and the vertex operator (§ 6) are investigated. Finally, some tentative discussions are given about the most suitable method of introduction of the factor $c(k^2)$ in the final section, although the decisive conclusion is difficult to obtain.

## § 2. A. M. M. with Modified $\Delta_F$-Functions

In this section we shall see the effects of the convergent factor used to modify the pion propagation function. We first have the method of modification

$$\Delta_F(k) \equiv 1/(k^2+\mu^2) \rightarrow \Delta_F(k)\, c(k^2) \equiv \lambda^2/(k^2+\mu^2)(k^2+\lambda^2)\,, \qquad (2.1)$$

originally due to FEYNMAN[8].

We observe that the contribution of virtual pion current to A. M. M. contains two $\Delta_F$ functions, so the modification (2. 1) will be more effective to the meson current contribution than to the nucleon current one. On the other hand, the second order perturbation calculation without cut-off gives too large nucleon current contribution to A. M. M., wich is the main reason why it cannot reproduce experimental values. Therefore, the modification (2. 1) might show no improvement over the conventional results. Thus, we also consider the weaker modification*

$$\Delta_F(k)\, \Delta_F(k') \rightarrow (1/2)\, \Delta_F(k)\, \Delta_F(k')\, [c(k^2)+c(k'^2)]\,, \qquad (2.2)$$

which affects the meson current contribution only. In the following subsections we investigate the nucleon current contribution with (2. 1) and the meson current one with (2. 1) and (2. 2). All numerical results in this section are given in Table I and II.

(i) *nucleon current contribution*

The effective interaction Hamiltonian between the external field and the nucleon current is, to the order, $eg^2$,

* Similar modification has been used recently by S. COHEN [Phys. Rev., 98 (1955), 749] in the calculation of the life time of a neutral pion including PAULI interaction.

$$H_n(x)=(2\pi)^{-4}eg^2A_\mu(x)\bar{\psi}(x)\gamma_\nu(3-\tau_3)\int d^4k$$
$$\times\frac{k_\mu\,k_\nu}{[(p+k)^2+m^2][(p'+k)^2+m^2](k^2+\mu^2)}\,\psi(x)\,, \qquad (2.3)$$

where $p$ and $p'$ are the initial and final nucleon four momenta and $(i\gamma p+m)\,\psi(x)=\bar{\psi}(x)\,(i\gamma p'+m)=0$ .

Modifying (2.3) according to (2.1) we have, after lengthy but familiar calculations, (in nuclear magneton),

$$\delta\mathfrak{M}_n=-g^2/4\pi\cdot(3-\tau_3)/4\pi\cdot L_n\,, \qquad (2.4)$$

as the nucleon current contribution to A.M.M.. Here

$$L_n=\xi(\xi-\eta)^{-1}\int_0^1 dx\left[\frac{x^3}{x^2+\eta(1-x)}\right]-\left[\frac{x^3}{x^2+\xi(1-x)}\right]$$
$$\equiv\xi(\xi-\eta)^{-1}\left\{I_n(\eta)-I_n(\xi)\right\}\,, \qquad (2.5)$$

with $\xi=(\lambda/m)^2$ and $\eta=(\mu/m)^2$. It is quite clear that we have the familiar expression[10)]

$$\delta\mathfrak{M}_n=-g^2/4\pi\cdot(3-\pi)/4\pi\cdot I_n(\eta) \qquad (2.4)$$

for $\xi\to\infty$ .

The result (2.5) shows the close correspondence between the present cut-off procedure and the mixed field theory. We not the identity

$$\frac{1}{k^2+\mu^2}\cdot\frac{\lambda^2}{k^2+\lambda^2}=\xi/(\xi-\eta)\left[1/(k^2+\mu^2)-1/(k^2+\lambda^2)\right]. \qquad (2.6)$$

The fiirst term of the right hand side is, apart from factor $\xi/(\xi-\eta)$ the propagation function of pions, and the second term is the one of another quanta of mass $\lambda$. Thus (2.5) says that $\delta\mathfrak{M}_n$ consists of two contributions, one from quanta of mass $\mu$ and the other of $\lambda$, the relative sign being different. This interpretation can be used to simplify the calculations when $\Delta_F$ functions are modified according to (2.1) (see, for instance, §6).

(ii) *meson current contribution with* (2.1)

The effective interaction Hamiltonian beoween virtual meson current and the external field is

$$H_m(x) = -eg^2/(4\pi^4)\, A_\mu(x)\, \bar{\psi}(x)\, \gamma_\nu \tau_3 \psi(x) \int d^4k$$
$$\times \frac{(p-k)_\mu\, (p-k)_\nu}{(k^2+m^2)[(p-k)^2+\mu^2][(p'-k)^2+\mu^2]}\,, \tag{2.7}$$

Introducing (2.1) and performing familiar calculations, we obtain, as the meson current contribution to A.M.M.,

$$\delta\mathfrak{M}_m = g^2/(4\pi)\, \tau_3/\pi\, L_m^{(1)}\,, \tag{2.8}$$

where

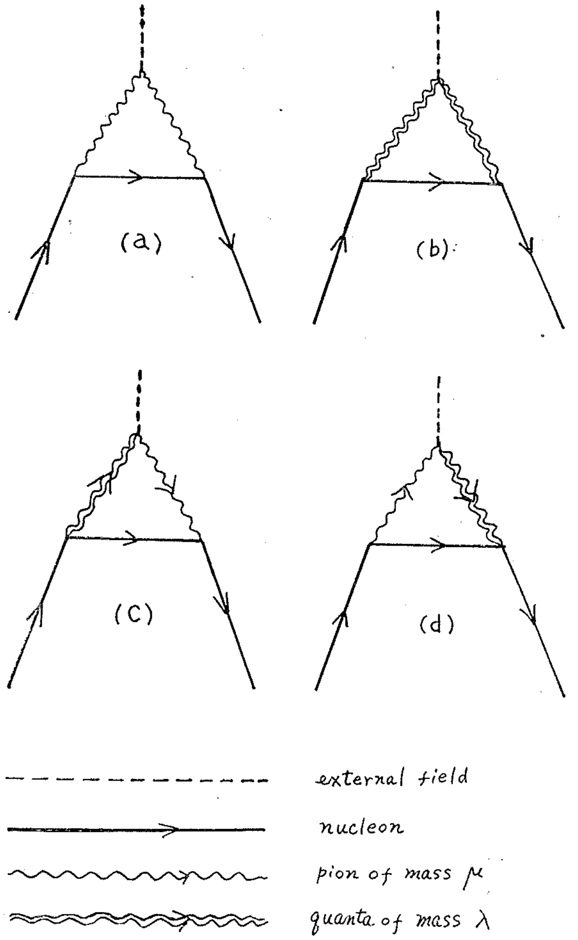


Fig. 1.

$$L_m^{(1)} = \xi^2/(\xi-\eta)^2 \left[I_m(\xi)+I_m(\eta)\right] - 2\xi^2/(\xi-\eta)^3 I_1 \,, \tag{2.9}$$

with

$$I_m(\eta) = \int_0^1 dx \frac{x^2(1-x)}{x^2+\eta(1-x)} \quad \text{and} \quad I_1 = \int_0^1 dx\, x^2 \ln \frac{x^2+\xi(1-x)}{x^2+\eta(1-x)} \,. \tag{2.10}$$

It is readily seen that for $\xi \to \infty$

$$\delta\mathfrak{M}_m = g^2/(4\pi)\,\tau_3/\pi\, I_m(\eta) \,, \tag{2.8$'$}$$

which is tne familiar result of conventional calculations[10].

We can find correspondence with field mixture as follows (see Fig. 1). The contribution $I_m(\eta)$ and $I_m(\xi)$ comes from (a) and (b), respectively. The difference in sign disappears because (b) contains two $\Delta_F$ functions. The contribution $-2I_1$, on the other hand, comes from (c) and (d). The negative sign before $I_1$ in (2.9) reflects the difference in sign between two terms in (2.6),

(iii) *meson current contributions with* (2.2)

In this case, the following replacement is made in (2.7):

$$\frac{1}{[(p-k)^2+\mu^2][(p'-k)^2+\mu^2]} \to \frac{\lambda^2}{2[(p-k)^2+\mu^2][(p'-k)^2+\mu^2]} \times \left\{\frac{1}{[(p-k)^2+\lambda^2]} + \frac{1}{[(p'-k)^2+\lambda^2]}\right\} . \tag{2.11}$$

It is easily seen that two terms on the right hand side of (2.11) give the same contributions. Final result is

$$\delta\mathfrak{M}_m = g^2/(4\pi)\tau_3/\pi\, L_m^{(2)} \,, \tag{2.12}$$

where

$$L_m^{(2)} = \xi/(\xi-\eta)\, I_m(\eta) + \xi/(\xi-\eta)^2 I_1 \,, \tag{2.13}$$

with $I_m(\eta)$ and $I_1$ defined in (2.10).

With regard to correspondence with mixed fiəld theory, we only note that $I_1$ comes from $1/2\{(c)+(d)\}$ of Fig. 1 and $I_m(\eta)$ from (a). That this interpretation is reasonable can be seen if (2.11) is rewritten in a similar way to (2.6).

(iv) *quantum electrodynamical analogue*

Goto[7] has shown that the high frequency part, which plays an essential role in pseudoscalar meson theory, is not so effective in quantum electrodynamics (Q.E.D.). In order that our covariant cut-off procedure can really have the meaning of cut-off, it is necessary that our procedure gives smaller effects in Q.E.D. than in meson theory. Therefore, we consider, in this subsection, the effect of modifications investigated in preceding subsections when applied to quantum electrodynamics. In this case we have no contribution analogous to the one of meson current.

The effective interaction Hamiltonian for this case is

$$H_{qed}(x) = (2\pi)^{-4}(-4e^3)\, A_\mu(x)\,\bar{\psi}(x) \int d^4k \times \frac{(2p_\mu + k_\mu)(\gamma\cdot k) - imk_\mu}{[(p+k)^2+\kappa^2]\,[(p'+k^2)+\kappa^2]k^2}\,\psi(x)\,, \tag{2.14}$$

where $\kappa$ is electron mass. We make the replacement

$$D_F(k) = 1/k^2 \rightarrow D_F(k)\,c(k^2) \equiv \lambda^2/[k^2(k^2+\lambda^2)]\,, \tag{2.15}$$

corresponding to (2.1). A.M.M. is given by

$$\delta\mathfrak{M} = e^2/(4\pi)\zeta/\pi \int_0^1 dx \frac{(1-x)^2}{x^2+(1-x)\zeta}\,, \tag{2.16}$$

in Bohr magnton. Here $\zeta = (\lambda/\kappa)^2$ and if $\zeta\rightarrow\infty$ we have the well-known result[11]

$$\delta\mathfrak{M} = e^2/(4\pi)\cdot 1/(2\pi)\,. \tag{2.16'}$$

(v) *numerical results and discussions*

All numerical results obtained in the preceding four subsections are tabulated in Tables I and II. Table I contains values of $\delta\mathfrak{M}_n$ and $\delta\mathfrak{M}_m$ obtained by modifying $\Delta_F$ functions according to (2.1) and (2.2). In Table II are given $\delta\mathfrak{M}$ of an electron calculated according to the replacement (2.15).

From Table I we see that the introduction of $c(k^2)$ corresponds to cut-off in some way. In fact, $\delta\mathfrak{M}_n$ and $\delta\mathfrak{M}_m$ are monotonically increasing functions of the cut-off parameter $\lambda$ and their values are qualitatively plausible. Quantitatively, we note the following features. The introduction of $c(k^2)$ according to (2.1) is, as expected, too effective to $\delta\mathfrak{M}_m$, and reduction rates of $\delta\mathfrak{M}_n$ and $\delta\mathfrak{M}_m$ (2.1) are almost proportional. (For $\xi=1$, for instance, $\delta\mathfrak{M}_n$ and $\delta\mathfrak{M}_m$ (2.1) are

TABLE I.

$\delta\mathfrak{M}_n$ and $\delta\mathfrak{M}_m$ with the cut-off (2.1) and (2.2). $\delta\mathfrak{M}_n$ is measured in unit of $(-g^2/4\pi)\,(3-\tau_3)/4\pi$ while $\delta\mathfrak{M}_m$ in $(g^2/4\pi)\,(\tau_3/\pi)$. $G$ means GOTO's result[7] with cut-off at $m$.

| | $\delta\mathfrak{M}_n$ | $\delta\mathfrak{M}_m$ (2.1) | $\delta\mathfrak{M}_m$ (2.2) |
|---|---|---|---|
| $\lambda = 0.5\,m$ | 0.10 | 0.07 | 0.13 |
| $1.0\,m$ | 0.19 | 0.14 | 0.19 |
| $1.5\,m$ | 0.24 | 0.18 | 0.21 |
| $2.0\,m$ | 0.29 | 0.22 | 0.24 |
| $\infty$ | 0.47 | 0.35 | 0.35 |
| $G$ | 0.19 | 0.22 | |

TABLE II.

A. M. M. of an electron as calculated according to (2.15). Unit used is $(e^2/4\pi)\,(1/2\pi)$, so that $\delta\mathfrak{M}=1$ if no cut-off is introduced. $G$ is GOTO's[7] result with the cut-off at electron mass $\kappa$.

| Cut off | $\lambda = 0.5\,\kappa$ | $\kappa$ | $1.5\,\kappa$ | $2\,\kappa$ | $G$ |
|---|---|---|---|---|---|
| $\delta\mathfrak{M}$ | 0.62 | 0.81 | 0.86 | 0.91 | 1.25 |

respectively 39.5% and 38.9% of the values without cut-off). So (2.1) does not show any improvement over the conventional calculation without cut-off and we can conclude that (2.1) cannot be used as the cut-off technique at least for A. M. M. problem.

The method (2.2), on the other hand, gives results similar to those of GOTO[7]. Numerically we have*

$$\begin{aligned}\varDelta\mu_p &= 0.47\\ \varDelta\mu_n &= -1.87\end{aligned} \quad \text{or} \quad |\varDelta\mu_p/\varDelta\mu_n| = 0.27\,, \tag{2.17}$$

for $\xi = 1$ and $g^2/4\pi = 15$. These values are close to those of GOTO,

$$\begin{aligned}\varDelta\mu_p &= 0.61\\ \varDelta\mu_n &= -1.90\end{aligned} \quad \text{or} \quad |\varDelta\mu_p/\varDelta\mu_n| = 0.32. \tag{2.18}$$

Slight difference between (2.17) and (2.18) is, as is seen from Table

* The experimental values are $\varDelta\mu_p = 1.79$ and $\varDelta\mu_n = -1.94$, or $|\varDelta\mu_p/\varDelta\mu_n| = 0.92$, while the theoretical ones without cut-off are (for $g^2/4\pi=15$), $\varDelta\mu_p=0.52$ and $\varDelta\mu_n=-3.9$, or $|\varDelta\mu_p/\varDelta\mu_n|=0.13$.

I, due to the difference in $\delta\mathfrak{M}_m$.

If the values in Table I are compared with those in Table II, it is quite clear that the high frequency part is not effective in Q.E.D., which is the same conclusion as GOTO's[7].

## § 3. A.M.M. with Modified $S_F$ Functions

In this section we shall see the effects of the convergent factor used to modify the nucleon propagation functions. Corresponding to the modification (2.1) we first consider

$$S_F(k) \rightarrow S_F(k)\, c(k^2)\ . \tag{3.1}$$

Since the nucleon current contribution contains two $S_F$ functions, we can alternatively modify as

$$S_F(k)\, S_F(k') \rightarrow 1/2\, S_F(k)\, S_F(k')\, [c(k^2)+c(k'^2)]\ , \tag{3.2}$$

corresponding to (2.2). We now investigate the meson current contribution with (3.1) and nucleon current one with both (3.1) and (3.2). All numerical results are found in Table III.

(i) *meson current contribution*

The integrand of (2.7) is multiplied by the factor $\lambda^2(k^2+\lambda^2)^{-1}$. The contribution to the A.M.M. is given as

$$\delta\mathfrak{M}_m = (g^2/4\pi)\,(\tau_3/\pi)\, J_m\ , \tag{3.3}$$

where

$$J_m = \xi(\xi-1)^{-1}[I_m(\eta)-I_2]\ , \tag{3.4}$$

with $I_m(\eta)$ defined by (2.10) and

$$I_2 = \int_0^1 dx \frac{x^2(1-x)}{x^2+(\xi-\eta-1)\,x+\eta}\ . \tag{3.5}$$

For $\xi \rightarrow \infty$, $I_2$ vanishes and $\delta\mathfrak{M}_m$ reduces to (2.8′). On the other hand, we have, for $\xi=1\,(\lambda=m)$*

$$J_m(\xi=1) = \int_0^1 dx \frac{x^3(1-x)}{[x^2+\eta\,(1-x)]^2}\ . \tag{3.6}$$

* If we put $\eta=0$ in (3.6) $J_m$ diverges. (The same is true about (3.10)). This infrared type divergence has been frequently met in the fourth order calculation of an electron A.M.M.[12].

(ii) *nucleon current contribution with* (3.1)

The integrand in (2.3) gets the extra-factor $\lambda^4[(p+k)^2+\lambda^2]^{-1}\times[(p'+k)^2+\lambda^2]^{-1}$. Calculations are similar to those in §2 (ii) and the result is

$$\delta\mathfrak{M}_n = -(g^2/4\pi)(3-\tau_3)/4\pi\cdot J_n^{(1)}\,, \tag{3.7}$$

with

$$J_n^{(1)} = \xi^2(\xi-1)^{-2}[I_n(\eta)+I_3-2\,I_4/(\xi-1)]\,, \tag{3.8}$$

where

$$I_3 = \int_0^1 dx\,\frac{1}{x^2+(\xi-\eta-1)\,x+\eta}$$

and (3.9)

$$I_4 = \int_0^1 dx\;ln\,\frac{x^2+(\xi-\eta-1)\,x+\eta}{x^2+\eta\,(1-x)}\,.$$

It is quite obvious that $\delta\mathfrak{M}_n$ reduces to (2.4′) for $\xi\to\infty$. For $\xi=1$, $J_m^{(1)}$ becomes

$$J_m^{(1)}(\xi=1) = 1/3\cdot\int_0^1 dx\,\frac{x^5}{[x^2+\eta\,(1-x)]^3}\,. \tag{3.10}$$

(iii) *nucleon current contribution with* (3.2)

In this case the integrand of (2.3) is multiplied by factor $\lambda^2/2\cdot\left\{[(p+k)^2+\lambda^2]^{-1}+[(p'+k)^2+\lambda^2]^{-1}\right\}$. Arising two terms are readily seen to give the same contributions to $\delta\mathfrak{M}_n$. We only write down the result.

$$\delta\mathfrak{M}_n = -(g^2/4\pi)(3-\tau_3)/4\pi\cdot J_n^{(2)}\,, \tag{3.11}$$

where

$$J_n^{(2)} = \xi\,(\xi-1)^{-1}\{I_n(\eta)-I_4/(\xi-1)\}\,. \tag{3.12}$$

For $\xi=1$, we have

$$J_n^{(2)} = 1/2\cdot\int_0^1 dx\,\frac{x^4}{[x^2+\eta\,(1-x)]^2}\,. \tag{3.13}$$

Although we have no relation as (2.6) in this case, we note remarkable similarities between (3.4) and (2.5), (3.8) and (2.9), (3.12) and (2.13).

(iv) *Q.E.D. analogue*

We have two possible substitutions (3.1) and (3.2) in this case. For the sake of simplicity, we have investigated the case $\lambda=\kappa$. In this particular case calculations are straightforward and we find that $\delta\mathfrak{M}$ contains the divergent factor

$$\int_0^1 \frac{1+x}{x^2}\,dx$$

for the substitution (3.1), and

$$\int_0^1 \frac{1+x}{x}\,dx$$

for (3.2). In either case there would appear no divergence if a photon were of nonvanishing mass. We have noted above that expressions (3.6) and (3.10) for $\xi=1$ have infrared divergence if we put $\eta=0$ (vanishing meson mass). This exactly corresponds to the situation we have just met.

This kind of divergence was of no harm in the fourth order calculation of A.M.M. in Q.E.D.. KARPLUS and KROLL[12] could avoid the difficulties by letting photon have an infinitesimal virtual mass $\varepsilon$ and it was shown that the contributions containing $\varepsilon$ cancell each other when added altogether and the final results are independent of $\varepsilon$. In our case, however, we have no term to be added and hence nothing to do with the difficulty.

Therefore, we may conclude that modification of $S_F$ functions cannot correspond to the physical cut-off procedure in Q.E.D.. It will be seen in the next subsection that this is true for the meson theory, too, although we have no divergence difficulty in this case.

(v) *numerical results and discussions*

Numerical results obtained in the foregoing subsections are given in Table III below. We readily see the inadequacy of the procedure considered in this section as a cut-ff technique. In fact the dependence of $\delta\mathfrak{M}_n$ and $\delta\mathfrak{M}_m$ on the cut-off parameter $\lambda$ shows, if we remember GOTO's analysis[7], that (3.1) and (3.2) cannot correspond to cut-off physically. Further, as mentioned in §3 (iv), the same method gives unreasonable divergent results when applied to Q.E.D.. Therefore, we may conclude that the modification of $S_F$ functions in any way cannot be used as a cut-off technique at least for A.M.M..

TABLE III.

$\delta\mathfrak{M}_n$ and $\delta\mathfrak{M}_m$ calculated with the cut-off procedure (3.1) and (3.2). Units used are the same as those in Table I.

| $\lambda$ | $\delta\mathfrak{M}_n$ (3.1) | $\delta\mathfrak{M}_n$ (3.2) | $\delta\mathfrak{M}_m$ |
|---|---|---|---|
| $m$ | 0.45 | 0.37 | 0.82 |
| $2m$ | 0.36 | 0.39 | 0.40 |
| $\infty$ | 0.47 | 0.47 | 0.35 |

## § 4. A.M.M. Cut Off in the Final Integrations

In this section we investigate the third method to introduce the factor $c(k^2)$ into calculations, that is to introduce it into the final integration over the virtual four momenta. Intuitively this procedure appears to correspond to the physical sense of cut-off most faithfully. Further we have no trouble on the consistency of the renormalization because in this procedure we deal with the final expressions after required subtractions*. Unfortunately, however, we shall see that this procedure cannot be used as a cut-off method in the following at least for A.M.M..

If we start from the effective Hamiltonian (2. 3) and (2. 6) and perform conventional calculations without any modification, we obtain the following expressions:

$$\delta\mathfrak{M}_n = i(g^2/4\pi)(3-\tau_3)/2\pi^3\cdot m^2\int_0^1 dx\, x^2\int d^4k\, \frac{1}{[k^2+\Lambda]^2}\,, \quad (4.1)$$

$$\delta\mathfrak{M}_m = -i(g^2/4\pi)(2\tau_3/\pi^3)\, m^2\int_0^1 dx\, x^2(1-x)\int d^4k\, \frac{1}{[k^2+\Lambda]^3}\,, \quad (4.2)$$

where

$$\Lambda = \mu^2(1-x) + m^2x^2\,.$$

* This is not the case for procedures considered in §2 and §3, where $c(k^2)$'s are introduced into calculations before renormalization. In the conventional calculations of A.M.M. we drop infinite terms of the form $L\bar{\psi}\gamma_\mu\psi A_\mu$ ($L=\infty$) since these can be incoorporated into the electric charge renormalization. More rigorously, these terms cancell each other due to the gauge invariancy of the theory and we need no charge renormalization as a whole. This situation is clearly unchanged by the procedure of §3. It must be noted, however, that the procedures considered in §2 and §3 do change this character of the theory, although we do not adhere to this point.

We easily get (2.4)′ and (2.8)′ by performing the integrations over $k_\mu$.

Now we multiply the integrands in (4.1) and (4.2) by the factor $c(k^2)$ to obtain

$$\delta\mathfrak{M}_n = -(g^2/4\pi)(3-\tau_3)/4\pi\cdot\int_0^1 dx\, x^3 I(\xi,\eta)\,,$$

$$\delta\mathfrak{M}_m = (g^2/4\pi)(\tau_3/\pi)\int_0^1 dx\, x^2(1-x)\, I(\xi,\eta)\,, \tag{4.3}$$

where

$$I(\xi,\eta) = \xi\left[\eta(1-x)+x^2-\xi\right]^{-3}\left\{x^2+\eta(1-x) + 2\xi\, ln\frac{\xi}{x^2+\eta(1-x)} - \frac{\xi^2}{x^2+\eta(1-x)}\right\}.$$

$\delta\mathfrak{M}_n$ and $\delta\mathfrak{M}_m$ given here are easily seen to reduce to (2.4)′ and (2.8)′ for $\xi\to\infty$.

Numerical values are given in Table IV. We see that the procedure considered in this section can correspond to cut-off in some sense. Quantitatively, however, we note that, for $\lambda=m$, $\delta\mathfrak{M}_n$ and $\delta\mathfrak{M}_m$ are respectively 49% and 26% of the values obtained without any modification, that is $\delta\mathfrak{M}_m$ is much more damped than $\delta\mathfrak{M}_n$. Thus the procedure considered in this section does not better the situation about the second order A.M.M.. It has become clear by Goto's investigation that the high frequency part is not so effective in $\delta\mathfrak{M}_m$ as in $\delta\mathfrak{M}_n$. Comparing this fact with Table IV, we see that the procedure cuts the contributions off in different ways for $\delta\mathfrak{M}_n$ and $\delta\mathfrak{M}_m$. It is too effective for meson current contribution.

(4.1) and (4.2) were derived after many transformations, so that the four momentum $k_\mu$ appearing in these equations is far from

Table IV.

$\delta\mathfrak{M}_n$ and $\delta\mathfrak{M}_m$ cut-off in the final integration.
Units are the same as used in Table I.

| $\lambda$ | $m$ | $2m$ | $\infty$ |
|---|---|---|---|
| $\delta\mathfrak{M}_n$ | 0.23 | 0.34 | 0.47 |
| $\delta\mathfrak{M}_m$ | 0.09 | 0.25 | 0.35 |

the quantity to be interpreted as four momentum of virtual quantum. Therefore, it might be not surprising that we were led to the conclusion stated above.

## § 5. Decays of a Neutral Pion into Two Photons

The lifetime of a neutral pion is one of the most remarkable examples in which conventional meson theory fails to show agreement with experimental values even in order of magnitude. In fact the conventional calculation gives

$$\tau = \alpha^2(g^2/4\pi)(m^2\mu^3/4\pi^6)\,|\,I\,|^2\,, \tag{5.1}$$

where $\alpha = (e^2/4\pi)$ and

$$I = \int d^4k\,\frac{1}{[(p-k)^2+m^2][(q-k)^2+m^2](k^2+m^2)} \tag{5.2}$$

in which $q_\mu$ is the four momentum of the neutral pion and $p_\mu$ is that of emitted photon. If the integral (5.2) is performed neglecting the contributions of the order $(\mu/m)^2$ or higher, we obtain

$$I \approx \int d^4k\,\frac{1}{(k^2+m^2)^3} = i\pi^2/(2m^2)\,. \tag{5.3}$$

Substituting (5.3) into (5.1), the life time becomes $\tau = 4.2\times10^{-17}$ *sec.* for $(g^2/4\pi)=15$, which is too short compared with the experimental value[13] $\tau\sim10^{-15}$ *sec.*.

Now, we want to modify the integral (5.3) by $c(k^2)$ as

$$I = \lambda^2\int d^4k\,\frac{1}{(k^2+m^2)(k^2+\lambda^2)}\,. \tag{5.4}$$

The integration is performed quite easily and the result is

$$I = i\pi^2/(2m^2)\,\xi\int_0^1 dx\,x^2\Big/\big[\xi+(1-\xi)x\big]^2 \equiv (i\pi^2/2m^2)A(\xi)\,. \tag{5.5}$$

The life time calculated by substituting this value into (5.1) is tabluated in Table V for several values of $\xi$. We see that the values are very satisfactory. Especially the lifetime for $\lambda = m$ is very close to that of Goto's ($3.4\times10^{-16}$ *sec.*), which shows that the procedure adopted in this section can really correspond to cut-off for the process considered.

The modification (5.5) is the same as the one used in §4 or

(3.2). Both of these did not give reasonable results and were concluded not to correspond to cut-off in the problem of A.M.M. Thus we find that how to introduce $c(k^2)$ must depend on the process considered in order that it can have the meaning of a cut-off.

TABLE V.

Values of $A(\xi)$ in (5.5) and the corresponding lifetime of a neutral pion for $g^2/4\pi=15$.

| $\lambda/m$ | 0.5 | 1 | 1.5 | 2 | $\infty$ |
|---|---|---|---|---|---|
| $A(\xi)$ | 1/6.9 | 1/3 | 1/2.1 | 1/1.7 | 1 |
| $\tau$ (*sec.*) | $2.0\times10^{-15}$ | $3.7\times10^{-16}$ | $1.8\times10^{-16}$ | $1.3\times10^{-16}$ | $4.2\times10^{-17}$ |

## §6. Radiative Corrections to $S_F$ and $\gamma_5$

According to BRUECKNER *et al.*[14], the lowest order radiative correction to the nucleon propagation function is, by scattering process at low energies, of the order $(\mu/m)(g^2/4\pi)$ for $P_{1/2}$ state and $g^2/4\pi$ for $S_{1/2}$ state*. Recently, WYLD[15] has applied similar argument to show that quite the same situations obtains for the lowest order radiative correction to the vertex operator $\gamma_5$. These favorable features of the theory arise from the renormalization procedure and it is desirable that the cut-off method which we have used does not alter the features. In this section we shall investigate how the cut-off method used in §2 changes the situation.

Without cut-off, the lowest order radiative correction to $S_F(p)$ is[14]

$$S_F^{(2)}(p) = -3g^2/(16\pi^2)\, S_F(p) f(p,\mu)\,, \tag{6.1}$$

where

$$f(p,\mu) = \int_0^1 dx \left[ -\frac{(1-x)\, i\gamma p - m}{i\gamma p + m}\, ln\, \frac{\phi(p^2,\mu^2)}{\phi(-m^2,\mu^2)} + \frac{2m^2x^2(1-x)}{\phi(-m^2,\mu^2)} \right],$$

$$\phi(p^2,\mu^2) = (1-x)\mu^2 + xm^2 + x(1-x)p^2\,.$$

Noting the identity (2.6), we see that the same quantity becomes

$$S_F^{(2)}(p) = -3g^2/(16\pi^2)\, \xi/(\xi-\eta)\, S_F(p)\left[f(p,\mu) - f(p,\lambda)\right], \tag{6.2}$$

* We do not speak of the damping of pair formation effect, since it has become clear that it is quite inadequate to discuss the effect by taking into account only this simple correction. See reference 15 and 16.

if we use the cut-off procedure of §2.

(6.2) vanishes if $p_\mu$ is the momentum of a free nucleon. Therefore, it is clear that (6.2) is of the order $(\mu/m)(g^2/4\pi)$ for $P_{1/2}$ state in the low energy pion-nucleon scattering. The correction amounts

$$-3g^2/(16\pi^2)2\xi/(\xi-\eta)\,[I_m(\eta)-I_m(\xi)]\equiv-(g^2/4\pi)D(\xi)\,, \qquad (6.3)$$

for $S_{1/2}$ state, where the integral $I_m$ is defined by (2.10). As is seen from Table IV, the correction cutoff at nucleon mass is about 80% of the value with no cut-off. We can say that the radiative corrections to the nucleon propagation function is not so sensitiv to the cut-off as A.M.M.. This shows that the renormalization procedure, which gave the expression (6.1), has removed most of the contributions from high frequency part.

Let us now consider the vertex operator. The lowest order radiative correction to $\gamma_5$ is, without cut-off,

$$\begin{aligned}Y_5(p,p',\mu)=g^2/(16\pi^2)\gamma_5\int_0^1 dx\int_0^1 xdy\Big\{&\{\mu^2(1-x)+m^2(1+x)-(i\gamma p)(i\gamma p')(1-x)\\ &+m(-i\gamma p+i\gamma p')\}/\Lambda(x,y,\mu)-\mu^2(1-x)/[m^2x^2+\mu^2(1-x)]\\ &-2\,ln\big[\Lambda(x,y,\mu)/[m^2x^2+\mu^2(1-x)]\big]\Big\}\,, \qquad (6.4)^{15)}\end{aligned}$$

where

$$\begin{aligned}\Lambda(x,y,\mu)=&\ \mu^2(1-x)+m^2x+p^2xy(1-x)\\ &+p'^2x(1-x)(1-y)+(p-p')^2x^2y(1-y)\,.\end{aligned}$$

Noting the identity (2.6), we find that the same quantity becomes

$$\xi\,(\xi-\eta)^{-1}\,[Y_5(p,p',\mu)-Y_5(p,p',\lambda)]\,, \qquad (6.5)$$

if we introduce the cut-off procedure in §2.

Following the argument of Wyld[15], we see that both $Y_5(\mu)$ and $Y_5(\lambda)$ are of the order $(\mu/m)(g^2/4\pi)$ for $P_{1/2}$ state at low energy. For $S_{1/2}$ state, on the other hand, (6.5) becomes

$$(g^2/16\pi^2)\,2\xi^2(\xi-\eta)^{-1}\gamma_5\int_0^1 dx\frac{(1-x)}{\xi\,(1-x)+x^2}\equiv(g^2/4\pi)\gamma_5 G\,(\xi)\,. \quad (6.6)$$

The effect of cut-off is not large as can be seen from Table VI.

Thus we may conclude that the favorable character of the theory mentioned at the begining of this section is not altered by the cut-off method used here.

TABLE VI.

Radiative corrections to $S_F$ and $\gamma_5$. Tabulated values are those of $D(\xi)$ and $G(\xi)$ defined in (6.3) and (6.6), respectively.

| $\lambda/m$ | 1 | 2 | $\infty$ |
|---|---|---|---|
| $D(\xi)$ | 0.18 | 0.21 | 0.23 |
| $G(\xi)$ | 0.10 | 0.12 | 0.16 |

## §7. Discussions and Conclusions

Nobody knows what kind of cut-off corresponds to physical reality, if the procedure is necessary not only to avoid the infinities but also for the internal consistency of the theory. At present, we have no means to answer the problem other than to see to what degree the method reproduces the experimental values. We have been investigating the cut-off method of introducing FEYNMAN's convergent factor $c(k^2)$ from this standpoint. Although this method has an advantage over GOTO's in that it is covariant, it is not clear if the method can have the practical meaning of cut-off just for that reason.

Conclusions we have obtained are the following: (1) Regarding A.M.M., the moification of the pion propagation function by $c(k^2)$ gives reasonable and favorable result. Here, one $c(k^2)$ must be introduced for one virtual pion (not one $\Delta_F$). No other method gives reasonable result. (2) We have seen, applying the same method to Q.E.D., that the high frequency part is not so effective in Q.E.D. as in pseudoscalar meson theory. (3) As for the decay of a netural pion, most satisfactory result has been obtained by introducing one $c(k^2)$ into the final integration. We may conclude that the method should depend on the process we consider, if we compare this result with that in the prolem of A.M.M.. (4) The feature, that the lowest order radiative corrections to $S_F$ and $\gamma_5$ are both of the order $(\mu/m)$ for $P_{1/2}$ state and 1 for $S_{1/2}$ state in the pion-nucleon csattering at low energies, is not altered by the cut-off method of modifying $\Delta_F$ function.

Since the examples we have considered are much limited, it is very difficult to obtain a general conclusion on the method. As a plausible conjecture, however, we may say that the introduction of one $c(k^2)$ for one virtual pion (not $\Delta_F$) gives a good result if the process contains any virtual pion. If, on the other hand, the process contains no virtual pion at all, the introduction of one $c(k^2)$ into the final integration over a momentum will do. (This is equivalent to modify $S_F$ by $c(k^2)$ if the contribution on the order $(\mu/m)^2$ or higher is neglected). It is desirable in order to establish these statements to investigate further application of the method.

The profound difficulty about the renormalization theory recently discussed by LEE[17] made some physisists believe that the cut-off is necessary besides renormalization in order to secure the internal consistency of the theory*. If this view point is right, more thorough investigation of the cut-off method will be needed. In fact, we postponed in this paper the consideration on the consistency problem of the theory arising from the introduction of $c(k^2)$. This, with others, must be investigated in order that any cut-off method can be established.

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