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Note on the Relativistic Wave Equation

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In order to investigate the possibility to construct some algebraically irreducible wave equation with mass levels, one of which has been obtained by BHABHA, studies are made for the wave function which is transformed according to the reducible representation of LORENTZ group $\mathfrak{R}(\frac{1}{2},\frac{1}{2})+\mathfrak{R}(\frac{1}{2},\frac{1}{2})$. It is shown that the irreducible equation can be obtained with positive definite norm, but the existence of two or more mass levels is incompatible with the irreducibility of the equation. Thus a important conclusion is obtained that it is impossible to construct any irreducible relativistic wave equation with mass levels, so long as the direct product of two wave functions which have the same transformation property uner LORENTZ transformation is adopted.

§ 1. Introduction

The general theories of relativistic wave equations have been developed by DIRAC[1], FIERZ[2] and PAULI[3),4]. As an essential feature of these theories, an assumption is made that an elementary particle must be described by an irreducible quantity in order that the theory may be invariant under the LORENTZ transformation. The irreducible quantity is the one which transforms according to the irreducible representation of the LORENTZ group in the LORENTZ transformation. Any reducible quantity, which is always written as a linear combination of some irreducible quantities, is considered to represent two or more kinds of elementary particles. In general case, however, the wave equation in these theories has rather complicated form so that the introduction of the interaction with other field is almost inhibited practically.

A theory based on a different standpoint has been proposed by BHABHA[5], who prescribed the theory in the form of wave equations. In this theory, the wave equations of free fields are considered to have the following form ;

$$(p_k\alpha^k + \chi m)\psi = 0 \ . \tag{1.1}$$

This theory has an advantage in the introduction of the interaction with other field. The guiding principle in this case is the irreducibility of the equation (1.1), that is the algebraic irreducibility of the matrix a^k. Developing this point of view, BHABHA has considered the cases of wave functions ψ which are transformed according to the reducible representations of the LORENTZ group,

$$\mathfrak{R}\left(\frac{3}{2}, \frac{1}{2}\right) + \mathfrak{R}\left(\frac{1}{2}, \frac{1}{2}\right)^{6)}, \tag{1.2}$$

$$\mathfrak{R}\left(\frac{3}{2}, \frac{1}{2}\right) + \mathfrak{R}\left(\frac{1}{2}, \frac{1}{2}\right) + \mathfrak{R}\left(\frac{1}{2}, \frac{1}{2}\right)^{7)}, \tag{1.3}$$

and obtained an interesting result in the latter case that an algebraically irreducible equation with two mass states and positive definite norm is possible[7]. In this formulation, however, the matrix forms of a^k are obtained in the exceedingly complicated manner so that the manipulation in the application is almost difficult. This difficulty was solved by K. K. GUPTA[8], who has succeeded to represent this BHABHA's equation in RARITA-SCHWINGER formalism using only DIRAC's γ matrices.

The characteristic feature of BHABHA's equation[7] is the introduction of several states of different spin values, which is evident from the transformation property of the wave function adopted.

In order to examine the generality of above mentioned feature of BHABHA's theory, the case is considered in this paper where the wave functions are transformed by the reducible representation of the LORENTZ group of the following type,

$$\mathfrak{R}\left(\frac{1}{2}, \frac{1}{2}\right) + \mathfrak{R}\left(\frac{1}{2}, \frac{1}{2}\right). \tag{1.4}$$

The formulation is done after the one of GUPTA[8] in RARITA-SCHWINGER-like form. In the next section the formulation and the results are discussed. In §3 the concluding remarks and the note concerning other arbitrary spin field are given.

§ 2. Formulations and Discussions

Consider the Lagrangian,

$$L = -\bar{\psi}_1(p_k\gamma^k + m)\psi_1 - a\bar{\psi}_2(p_k\gamma^k + \lambda m) - (\bar{\psi}_2 O\psi_1 + \text{conjugate}), \tag{2.1}$$

where ψ_1 and ψ_2 are 4-components DIRAC spinors, a, λ are the parameters without dimension, m the contant of dimension of mass. Operator O is defined by the relativistic invariance of the Lagrangian and the linearity of the field equation deduced from (2.1). From these requirements only two types of O are possible;

$$\text{(a)} \qquad O = \xi m \,, \tag{2.2a}$$

$$\text{(b)} \qquad O = \eta\, p_k \gamma^k \,, \tag{2.2b}$$

where ξ and η are some dimensionless parameters.

Case (a)

The case of (2.2a) is discussed in the first place. In this case the Lagrangian (2.1) is written in the following form,

$$L = -\Psi^+ D(p_k \Gamma^k + \Gamma m)\Psi \,, \tag{2.3a}$$

$$\Psi \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad D \equiv \begin{pmatrix} \gamma^0 & 0 \\ 0 & a\gamma^0 \end{pmatrix}, \quad \Gamma^k \equiv \begin{pmatrix} \gamma^k & 0 \\ 0 & \gamma^k \end{pmatrix}, \quad \Gamma \equiv \begin{pmatrix} 1 & \xi^* \\ \xi/a & \lambda \end{pmatrix}. \tag{2.4a}$$

The equation of motion is

$$(p_k \Gamma^k + \Gamma m)\Psi = 0 \,. \tag{2.5a}$$

Now it must be noticed that the theory following from (2.3) is invariant under the similarity transformation;

$$\Psi = S^{-1}\Psi' \,, \quad D = S^+ D' S \,, \quad \Gamma^k = S^{-1}\Gamma'^k S, \quad \Gamma = S^{-1}\Gamma' S \,, \tag{2.6a}$$

by means of which, therefore, the algebraic properties of the equation (2.5a) can be investigated.

I) *Reducibility.* Γ^k in Eq. (2.3a), regarding as the 2×2 matrices which are constructed from DIRAC's γ algebra, are merely the constant multiplicatives of unit matrix. If, therefore, the matrix Γ is diagonalizable regarding as 2×2 matrix of the similar nature, the equation (2.3a) could be separated into two DIRAC equations by some similarity transformation (2.6a), then the Eq. (2.3a) is reducible. On the other hand, it is well known that the DIRAC equation is the only one which is transformed according to the irredecible representation $\mathfrak{R}(\frac{1}{2}, \frac{1}{2})$ of the LORENTZ group. If, therefore, Eq. (2.3a) is reducible, then the equations after separation should be nothing else but two DIRAC equations, which should mean

that the Γ is deagonalizable by some transformation (2.6a). It follows, therefore, an important conclusion that the necessary and sufficient condition for Eq. (2.3a) to be reducible is the diagonalizability of the matrix Γ.

Furthermore, the condition of diagonalizability of Γ should refer to the eigenvalues. For if two eigenvalues of Γ are equal each other, the diagonalized form would be an unit matrix except for some multiplicative constant, which could not be realized by transformation (2.6a). Conversely, when two eigenvalues are not equal the similarity transformation which diagonalizes the matrix Γ can be obtained easily. Thus it has been shown that the necessary and sufficient condition for Γ to be diagonalizable is that the matrix Γ has two different eigenvalues. With the foregoing consideration, therefore, we can conclude that the necessary and suffecient condition for Eq. (2.3a) to be irreducible is that the two eigen values of Γ are equal each other.

Considering the secular equation of matrix Γ, this condition can be formulated as

$$(1-\lambda)^2+\frac{4}{a}|\xi|^2=0\ . \tag{2.7a}$$

The case in which this condition is satisfied is called *irreducible*, and the other case is *reducible*.

In the reducible case, Eq. (2.5a) separated by a similarity transformation (2.6a) into two DIRAC equations with different masses. The field described by Eq. (2.5a) is, therefore, merely a mixed field of two DIRAC fields. It must be noticed, in this case, that the mass values become complex number when

$$(1-\lambda)^2+\frac{4}{a}|\xi|^2<0\ ,$$

and therefore of no physical meaning. The irreducible case will be considered hereafter.

II) *Irreducible case.* Although Γ cannot be diagonalized by any similarity transformation, it may be able to find the transformation (2.6a) which pseudo-diagonalize the matrix Γ, *i.e.*,

$$\Gamma\rightarrow\Gamma'=\begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}, \tag{2.8a}$$

x is the eigenvalue of Γ and given by

$$x = \frac{1}{2}(1+\lambda)\,. \tag{2.9a}$$

Performing this transformation,

$$\Phi = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = S\Psi\,, \qquad \left[p_k\gamma^k\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + m\begin{pmatrix}x & 1\\ 0 & x\end{pmatrix}\right]\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = 0\,,$$

or

$$\begin{aligned}(p_k\gamma^k + xm)\,\phi_1 &= m\phi_2\,,\\ (p_k\gamma^k + xm)\,\phi_2 &= 0\,.\end{aligned} \tag{2.10a}$$

Here, the definition of mass value is important. Following BHABHA we define it by the relation: *mass value* $\equiv \sqrt{p_k p^k}$, in which p_k is the momentum eigenvalue corresponding to the non-vanishing solution of Eq. (2.10a). From this definition the mass value turns out to be given by

$$\textit{mass value} = xm$$

for all solutions of Eq. (2.10a).

The Eq. (2.10a) has an analoguous form with that in GUPTA's formulation concerning the BHABHA's equation. This equation has two typical solutions. The one is such that $\phi_2=0$ and ϕ_1 satisfies $(p_k\gamma^k+xm)\phi_1=0$, and is nothing other than the ordinary DIRAC field with spin $\frac{1}{2}$ and mass xm. This may be called the solution (Ia). The other is the one in which $\phi_2\neq 0$, and ϕ_1 is determined by the DIRAC equation with ϕ_2 as its source. This also represents the state of spin $\frac{1}{2}$ and mass xm, which will be called as the solution (IIa). Thus our equation (2.10a) has two independent states with the same spin and mass value. Finally, the case in which Γ is singular must be noted. The singular property of Γ is represented as

$$a\lambda - |\xi|^2 = 0 \tag{2.11a}$$

From (2.7a) and (2.11a) it follows $x=0$. Thus the case of singular Γ correspond the particle of spin $\frac{1}{2}$ and vanishing mass.

III) *Norm.* It might be inquired that the transformations adopted so far are not the unitary transformations but the similarity ones, and that, therefore, it would change the norm of the field, as the result of which the system concerned might be changed in

the transformations. This difficulty, however, can be overcome by taking the norm of the field suitably.

From the Lagrangian (2.3a), the charge current 4-vector is obtained as

$$j^k = -\Psi^+ D\Gamma^k \Psi \,. \tag{2.12a}$$

The *charge density* is therefore

$$j^0 = -\Psi^+ D\Gamma^0 \Psi \,. \tag{2.13a}$$

This is the quantity which should be taken as the norm in our theory. It would be evident that the quantity (2.13a) is invariant under the transformation (2.6a).

It is interesting to examine whether this norm can be made positive definite or not. Concerning this question it is evident that the matrix $D\Gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$ cannot have absolutely definite sign, because it must be $a<0$ from irreducible condition (2.7a). As was mentioned by BHABHA, however, the *free charge density*, that is the expectation value of j^0 in the free particle state, may be shown to have definite sign. For performing the transformation by means of which Eq. (2.10a) is obtained from Eq. (2.5a), the free charge density becomes

$$\langle j^0 \rangle = a\,[(x-1)(\phi_1^+\phi_2 + \phi_2^+\phi_1) + (\xi + \xi^* + 1)\,\phi_2^+\phi_2] \,, \tag{2.14a}$$

where ϕ_1 and ϕ_2 are any solutions of Eq. (2.10a). For the solution (Ia) mentioned above it is evident that $\langle j^0 \rangle_{\mathrm{I}} = 0$. For the solution (IIa), after some manipulation with the equation (2.10a), (2.14a) becomes

$$\langle j^0 \rangle_{\mathrm{II}} = a\,[(\lambda^2 - 1)\phi_1^+\phi_1 + (\xi + \xi^* + 1)\,\phi_2^+\phi_2] \,. \tag{2.14a$'$}$$

Thus it is shown that $\langle j^0 \rangle$ has definite sign in both solutions by the suitable choice of the parameters ξ and λ.

It is interesting, finally, to note on the particle Hamiltonian to investigate the particle nature of our field. From our Lagrangian (2.3a) the canonical energy-momentum tensor gives as its 4–4 component the following Hamiltonian of the field,

$$\begin{aligned} H = T_{00} &= \Psi^+ D\,(p_i \Gamma^i + \Gamma m)\,\Psi \qquad (i \neq 0) \\ &= \Psi^+ D\Gamma^0 \cdot \Gamma^0 (p_i \Gamma^i + \Gamma m)\,\Psi \,. \end{aligned} \tag{2.15a}$$

Considering our choice of the norm the characteristic Hamiltonian for the particle picture of the field is shown to be $H=\Gamma^0(p_i\Gamma^i+\Gamma m)$. Furthermore it can be easily shown that the eigenvalue of this operator is given as a matter of course by $\sqrt{(xm)^2+p_ip^i}$.

Case (b)

Now the case of (2.2b) will be considered shortly. In this case the Lagrangian becomes

$$L=-\Psi^+\overline{D}\,(p_k\Gamma^k\overline{\Gamma}+m)\,\Psi\ , \tag{2.3b}$$

$$\Psi\equiv\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix},\quad \overline{D}\equiv\begin{pmatrix}\gamma^0 & 0\\ 0 & \lambda\gamma^0\end{pmatrix},\quad \Gamma^k\equiv\begin{pmatrix}\gamma^k & 0\\ 0 & \gamma^k\end{pmatrix},\quad \overline{\Gamma}\equiv\begin{pmatrix}1 & \eta^*\\ \eta/\lambda & a/\lambda\end{pmatrix}, \tag{2.4b}$$

where $\lambda\neq 0$ is assumed. In (2.3b) it should be noticed that the mass term ($\sim m$) is a unit matrix except for the multiplicative parameter m. This feature should be compared with the one in case (a), where the term $\sim p_k\gamma^k$ is proportional to the 2×2 unit matrix. The previous discussions concerning the matrix Γ in (2.4a), therefore, could be applied in this case to $\overline{\Gamma}$ in (2.4b).

The condition for irreducibility thus obtained in the analoguous manner as in (a) is

$$\left(1-\frac{a}{\lambda}\right)^2+\frac{4}{\lambda}\,|\eta|^2=0\ . \tag{2.7b}$$

Again as the result of the similarity transformation we obtain the following field equation corresponding to (2.10a),

$$\left(p_k\gamma^k+\frac{1}{y}m\right)\phi_1=-p_k\gamma^k\phi_2\ ,$$
$$\left(p_k\gamma^k+\frac{1}{y}m\right)\phi_2=0\ , \tag{2.10b}$$

or

$$\left(p_k\gamma^k+\frac{1}{y}m\right)\phi_1=\frac{1}{y}m\phi_2\ ,$$
$$\left(p_k\gamma^k+\frac{1}{y}m\right)\phi_2=0\ , \tag{2.10b$'$}$$

where y is the eigenvalue of $\overline{\Gamma}$ and

$$y = \frac{1}{2}\left(1+\frac{a}{\lambda}\right). \tag{2.9b}$$

Thus we obtain two typical solutions as in the case (a): the solution (Ib), in which $\phi_2=0$, $\left(p_k\gamma^k+\frac{1}{y}m\right)\phi_1=0$, that is ordinary DIRAC field, and the solution (IIb), in which $(p_k\gamma^k+m/y)\phi_2=0$, $(p_k\gamma^k+m/y)\phi_1=(m/y)\phi_2$. Both of these solutions correspond the states of spin $\frac{1}{2}$ and mass m/y.

The norm of the field should be defined by

$$j^0 = -\Psi^+\overline{D}\Gamma^0\overline{\Gamma}\Psi. \tag{2.13b}$$

Again the matrix $\overline{D}\Gamma^0\overline{\Gamma}=\begin{pmatrix}1 & \eta^* \\ \eta & a\end{pmatrix}$ cannot have absolutely definite sign. The free charge density is, however, obtained as

$$\langle j^0\rangle = \lambda\left[4y(1-y)\phi_1^+\phi_2+\phi_2^+\phi_1)+\{1-y(\eta^*+\eta)\}\phi_2^+\phi_2\right], \tag{2.14b}$$

after the similarity transformation which gives the Eq. (2.10b). For the solution (Ib) this gives $\langle j^0\rangle_{\mathrm{I}}=0$. For the solution (IIb), after some manipulation this becomes

$$\langle j^0\rangle_{\mathrm{II}} = \lambda\left[4y(1-y)\phi_1^+\phi_1+\{1-y(\eta+\eta^*)\}\phi_2^+\phi_2\right]. \tag{2.14b'}$$

Thus it is shown that $\langle j^0\rangle$ has definite sign in both solutions by the suitable choice of parameters, which is the same conclusion as in the case (a).

§ 3. Conclusions

From the considerations in previous sections it can be concluded that, according to BHABHA's point of view, it is possible to construct the algebraically irreducible wave equation with positive definite norm using the wave function which transforms according to the reducible representation of the LORENTZ group $\mathfrak{R}(\frac{1}{2},\frac{1}{2})+\mathfrak{R}(\frac{1}{2},\frac{1}{2})$. In this case, however, the mass levels which was obtained by BHABHA when the reducible representation of the LORENTZ group $\mathfrak{R}(\frac{3}{2},\frac{1}{2})+\mathfrak{R}(\frac{1}{2},\frac{1}{2})+\mathfrak{R}(\frac{1}{2},\frac{1}{2})$ was adopted[7], cannot appear in any way. The reason is shown to be *the incompatibility of the algebraic irreducibility of the equation and the existence of the mass levels.* Thus it has been proved that in BHABHA's equation[7] it was essential to

mix the wave functions of several different spin values.

Finally it should be noted that in the foregoing parts of this paper the discussions have been performed quite independent from the special nature of DIRAC's γ algebra. If, for example, therefore, we replace the γ's in this paper by the DUFFIN-KEMMER matrices β, the conclusion should not be altered at all. Thus we obtain the important conclusion that it is not possible to construct any algebraically irreducible wave equation with two or more mass levels, so long as we adopt the direct product of two wave functions which have the same transformation property under the LORENTZ transformations.

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