Charge simulation method for approximating the complex potential in a channel domain with multiple circular islands

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Figure 1: The Nile, an example of the river region with many sand islands.

1. Introduction

In dealing with environmental problems in rivers, it is important to describe how chemical and biological particles are advected by the river flows. However, the description of river flow itself is generally difficult, since the flow domain has a complex topography as we see in Figure 1. Moreover, the dispersion of such particles are in general non-uniform; Some pollutants spread over the whole river, while the others stay around stagnation points of the flow. Thus as the first step of the mathematical treatment toward the river environments, we need to develop a numerical method to generate flows in complex domains with which the particles float.

In the present article, we propose a numerical method to construct a uniform flow in a specific domain called “river region”, which is a channel region with many obstacles like sandbanks inside. The mathematical devices are the theory of perfect fluids in two-dimensional planar space and the elliptic functions.

2. Charge simulation method

Charge Simulation Method (CSM) is a well-known fast and accurate computational method to solve the Poisson equations[1]. For a given domain $\Omega \subset \mathbb{C}$, let us consider the Poisson equations for the function $g(z)$,

$$\Delta g(z) = 0 \quad \text{in } \Omega, \tag{1}$$

$$g(z) = b(z) \quad \text{on } \partial\Omega, \tag{2}$$

where $b(z)$ is a given function on the boundary $\partial\Omega$. CSM approximates the function $g(z)$ with a linear combination of fundamental solutions at N charge points $z = \zeta_1, \zeta_2, \dots, \zeta_N$ as follows.

$$G(z) = Q_0 + \sum_{i=1}^{N} Q_i \log|z - \zeta_i|, \tag{3}$$

in which $Q_1, \dots, Q_N$ are unknowns with the constraint $\sum_{i=1}^{N} Q_i = 0$. We determine Q_i numerically so that the equation (3) satisfies the boundary condition (2) at given collocation points $z_1, \dots, z_N$ along the boundary, i.e. $G(z_i) = b(z_i)$ for $i = 1, \dots, N$. This is equivalent to the following $(N+1)$-dimensional linear equation

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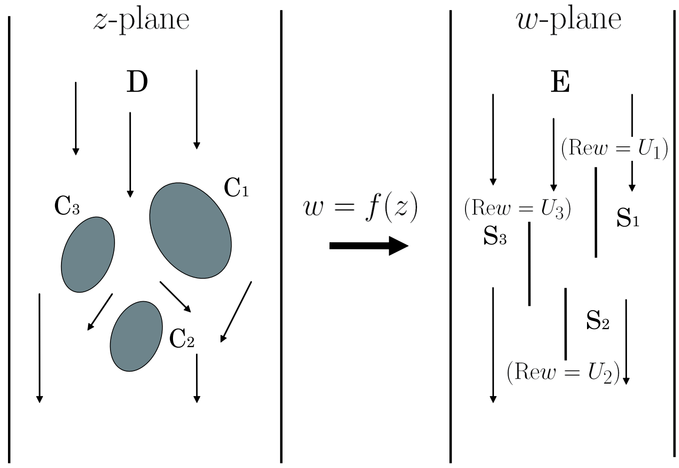

Figure 2: Conformal mapping from the complex z-plane to the complex w-plane

$$\begin{cases} Q_0 + \displaystyle\sum_{i=1}^{N} Q_i \log|z - z_i| = b(z_i), \qquad (i = 1, \cdots, N), \\ Q_1 + \cdots + Q_N = 0, \end{cases} \tag{4}$$

which is solved numerically by the LU-decomposition method. CMS has a nice disposition in terms of the error estimate between the exact solution and the approximate solution. Since the error estimate attains its maximum at the boundary of the domain due to the maximum principle, we can define the maximum error by $E = \max_{1 \le j \le N} |G(z_j) - g(z_j)|$ in which $g(z)$ and $G(z)$ represent the exact mapping and the approximate mapping respectively. When the charge points are properly set and the domain has sufficiently smooth boundaries, the maximum error decreases with $O(\tau^N)$ for some $0 < \tau < 1$, which depends on the shape of the domain. (See, e.g., Katsurada and Okamoto[3].)

3. Conformal mapping to the parallel slit domain

We propose how to construct the uniform flow in the river region by constructing a conformal mapping from the complex z-plane to the complex w-plane via CSM. see Figure 2. We consider the region D in the z-plane as a standard river region. Namely, the uniform flow is confined in a channel-like region with two long straight boundaries, in which there are cylindrical sandbanks $C_1, \ldots, C_d$. Next, we consider the region T in the w-plane as a channel region with parallel slits $S_1, \ldots, S_d$. The complex potential of the flow in D is mapped to a uniform flow in T by a conformal mapping $w = f(z) = z + H(z)$. (e.g., See Nehari[6].) CMS approximates the function $H(z)$.

For the sake of simplicity, we assume that the left and right doundaries of the region D correspond to the imaginary axis and Re$z = \alpha$ in the z-plane respectively, and that the flow is periodic in the imaginary direction with period 2π. Then, because of the principle of reflection, the flow in D must be symmetric with respect to both imaginary axis and the other right boundary as we see in Figure 3. Therefore, let D' denote the reflecting image of D. The union $D \cup D'$ is the basic computational region that covers the whole z-plane double periodically. As the fundamental solution to describe this flow in CSM, we adopt the elliptic functions. First, Weierstrass ζ-function is defined by

$$\zeta(z) = \frac{1}{z} + \sum_{\omega \in \Omega'} \left(\frac{1}{\omega} + \frac{1}{z - \omega} + \frac{z}{\omega^2} \right), \tag{5}$$

in which $\omega_1 = \alpha,\ \omega_2 = 2\pi i$ and $\Omega' = \{n\omega_1 + m\omega_2 \mid n, m \in \mathbb{Z}\} \setminus \{0\},$ Second, the elliptic theta function of type 1 is given by

$$\vartheta_1(z) = 2 \sum_{n=1}^{\infty} (-1)^n h^{\frac{(2n-1)^2}{4}} \sin{(2n-1)\pi z}, \tag{6}$$

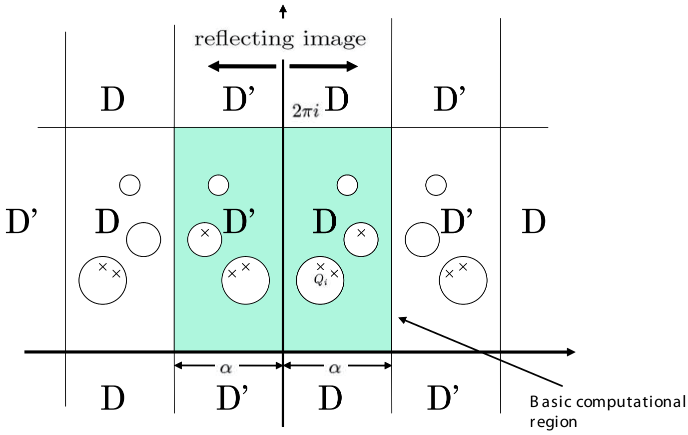

Figure 3: Doubly periodic computational region.

in which $h = e^{\frac{\omega_2}{\omega_1} i\pi}$. These two elliptic functions are connected through the following relation,

$$\zeta(u) = \frac{2\eta}{\omega_1} u + \frac{d}{du} \log \vartheta_1 \left(\frac{u}{\omega_1} \right), \qquad \eta = \zeta \left(\frac{\omega_1}{2} \right). \tag{7}$$

Now, the complex potential for the flow is approximated by the linear combination of the elliptic theta functions:

$$\begin{aligned} H(z) &= Q_0 + \sum_{l=1}^{n} \sum_{i=1}^{N_l} Q_{li} \sum_{\omega} (\log |z - \zeta_{li} - \omega| - \log |z + \overline{\zeta_{li}} - \omega|) \\ &= Q_0 + \sum_{l=1}^{n} \sum_{i=1}^{N_l} Q_{li} \left\{ -\frac{z}{\omega_1} + \log \left(\frac{\vartheta_1 \{(z - \zeta_{li})/\omega_1\}}{\vartheta_1 \{(z + \overline{\zeta_{li}})/\omega_1\}} \right) \right\}, \end{aligned} \tag{8}$$

in which n is the number of the islands, N_l is that of the charge points in C_l and the collocation points on C_l and ζ_{li} is the position of the i-th charge point inside C_l. When the channel domain has the infinite length in the imaginary direction, namely $\omega_2 = \infty$, the approximating function (8) is equivalent to

$$H(z) = Q_0 + \sum_{l=1}^{n} \sum_{i=1}^{N_l} Q_{li} \log \left(\frac{\sin \{(z - \zeta_{li})\pi/\omega_1\}}{\sin \{(z + \overline{\zeta_{li}})\pi/\omega_1\}} \right). \tag{9}$$

(e.g., see Hurwitz and Courant [5].) Note that the actual computation of the ϑ_1 function can be carried out by truncating the inifinite product representation of ϑ_1.

4. Numerical method of infinite channel

Here we consider the river region with $\omega_2 = \infty$, the infinite channel. The equation (9) is not suitable for actual numerical computations because of the logarithmic singularity in the fundamental solution. To avoid the appearance of the branch singularities, we substract

$$0 = \sum_{l=1}^{n} \sum_{i=1}^{N_l} Q_{li} \log \left(\frac{\sin \{(z - \zeta_{l0})\pi/\omega_1\}}{\sin \{(z + \overline{\zeta_{l0}})\pi/\omega_1\}} \right) \tag{10}$$

from the function (9), in which ζ_{l0} and $-\overline{\zeta_{l0}}$ are the positions of additional charge points, which leads us to

$$H(z) = Q_0 + \sum_{l=1}^{n} \sum_{i=1}^{N_l} Q_{li} \left\{ \log \left(\frac{\sin \{(z - \zeta_{li})\pi/\omega_1\}}{\sin \{(z - \zeta_{l0})\pi/\omega_1\}} \right) - \log \left(\frac{\sin \{(z + \overline{\zeta_{li}})\pi/\omega_1\}}{\sin \{(z + \overline{\zeta_{l0}})\pi/\omega_1\}} \right) \right\}. \tag{11}$$

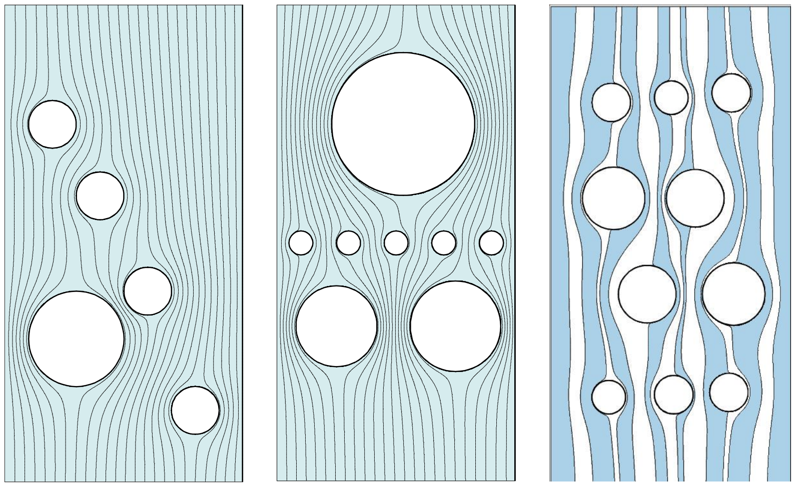

Figure 4: Computational results.

At the infinity, the flow is uniform because the flow is not affected by islands. Thus, we have $Q_0 = 0$. As a result we can compute the approximating conformal mapping by solving the following linear equation for Q_{li} and U_{mj}:

$$\sum_{l=1}^{n}\sum_{i=1}^{N_l} Q_{li}\left\{\log\left|\frac{\sin\{(z_{mj}-\zeta_{li})\pi/\omega_1\}}{\sin\{(z_{mj}-\zeta_{l0})\pi/\omega_1\}}\right| - \log\left|\frac{\sin\{(z_{mj}+\overline{\zeta_{li}})\pi/\omega_1\}}{\sin\{(z_{mj}+\overline{\zeta_{l0}})\pi/\omega_1\}}\right|\right\} - U_{mj} = -\mathrm{Re}(z_{mj})$$
$$\text{for}\quad m = 1,\ldots,n, \quad\text{and}\quad j = 1,\ldots,N_l. \tag{12}$$
$$\sum_{i=1}^{N_l} Q_{li} = 0, \qquad \text{for}\quad l = 1,\ldots,n.$$

in which z_{mj} is the position of the j-th collocation point on the boundary of the island C_m. We give three comptational results in Figure 4. The number of the collocation points and the charge points is $N_l = 64$. Let r_m and $\delta_m \in \mathbb{C}$ denote the radius and the center of C_m. Then, the positions of the collocation points inside C_m and the charge points on C_m are given by $z_{mj} = \delta_m + r_m \exp(2\pi ij/N_l)$ and $\zeta_{mj} = \delta_m + 0.7r_m \exp(2\pi ij/N_l)$ for $j = 1,\ldots,N_l$. The additional charge points for each island C_m are put at $\zeta_{m0} = \delta_m$.

References

[1] K. Amano, D. Okano, H. Ogata, H. Shimohira and M. Sugihara, “A systematic scheme of numerical conformal mappings of unbounded multiply-connected domains by the charge simulation method”, Information Processing Society of Japan Journal, **42** (3) (2001), 385-395.

[2] H. Ogata, “Numerical conformal mapping of periodic structure domains”, Japan J. Indust. Appl. Math., **19** (2002), 257-275.

[3] H. Okamoto and H. Katsurada, “A mathematical study of the charge simulation method 1”, J. Fac. Sci. Univ. Tokyo, Sect.IA, Math., **35** (3) (1998), 507-518.

[4] E.R. Johnson and N.R. McDonald, “The motion of a vortex near two circular sylinders”, Proc. Royal Soc. A, **460** (2004), 939-954.

[5] A. Hurwitz, and R. Courant, “Vorlesungen über allgemeine Funktionentheorie und elliptische Funktionen”, Springer-Verlag (1964).

[6] Z. Nehari, “Conformal Mapping”, Dover, New York (1975), General Pub., Tronrto (1952).