Optimal Solutions to the Infinite Horizon Optimization Problems

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For economic growth models with infinite planning horizon, the infinite series of utility sequences may diverge, rendering the "maximization" of which meaningless. The discounting approach, widely used in intertemporal economics, avoids this problem as the discounted integral of utilities becomes finite under the "relentless force of compound interest" (Weitzman, 1998). However, economists have long been scathing about the ethical dimensions and logical difficulties of discounting since Ramsey's time (Ramsey, 1928; Pigou, 1932; Solow, 1974; Heal, 1998).[1]

There has been a large literature that analyzes the undiscounted growth models. In Ramsey's original work, he formulated the problem as minimizing the deviation from a given reference curve, the "bliss level," which avoids the problem that the sum of the objective function may not converge. A more general approach was later introduced by Von Weizsächer (1965) and Atsumi (1965) and further refined by Gale (1967) and Brock (1970). This approach uses the concept of the overtaking criterion, which replaces the comparison of infinite sums with a comparison of finite partial sums. Under the assumption of the existence of such a given reference curve, the existence and dynamical properties of the resultant optimal path have been considered in, for example, Michel (1990), Durán (2000), Kamihigashi (2001), and Le Van and Morhaim (2006). However, few studies have proposed how to explicitly construct the optimal paths for the undiscounted infinite horizon problems. Gale (1967) could be considered as a basis

[1] Stern (2007) argues that when considering climate change, "we [should] treat the welfare of future generations on a par with our own".

for such an approach. However, it involves checking, together with other assumptions, the existence of an optimal stationary path, which may be impossible for most problems.

Cai and Nitta (2007) present an alternative approach that explicitly constructs the optimal paths for the discrete time undiscounted infinite horizon optimization problems. They conclude that under a fairly general condition, the conjecture that "the limit of the solutions for the finite problems is optimal for the infinite horizon problem" is correct for the undiscounted problem, in the sense of a modified overtaking criterion.[2] However, as their analysis focuses on a particular example of the undiscounted infinite horizon problems, their approach is not directly applicable to most other problems. This paper aims at improving the applicability of their approach by extending their model to one in which both the utility and the production functions are allowed to depend on time. As a result, our proof of the conjecture and the analysis cover both the undiscounted and the discounted cases. This enables us to examine the impacts of discounting. We show that the higher the level of the productivity, the higher are the stationary values in the steady state. Moreover, we see that at extremely low levels of productivity, when $t \to \infty$, capital approaches zero at a positive discount rate, and a positive value when the rate is zero. In other words, the effects of discounting may be most disastrous when the level of productivity is extremely low.

Besides the modified overtaking criterion, we also consider another intertemporal optimality criterion: the sum-of-utilities criterion. We clarify the conditions under which the limits of the solutions for the finite horizon problem is optimal among all attainable

[2] The conjecture has been around for a while. The case when the discount factor is less than one has been examined in, for example, Stokey and Lucas (1989). Proving the conjecture involves establishing the legitimacy of interchanging the operators "max" and "$\lim_{T\to\infty}$", which has been "more challenging than one might guess" for the discounted case (Stokey and Lucas, 1989).

paths for the infinite horizon problem in the sense of a modified overtaking criterion, as well as the conditions under which such a limit is the unique optimum under the sum-of-utilities criterion.

For Reference, Refer to Finished Papers

1. Cai, D. and Nitta, T. G. Limit of the Solutions for the Finite Horizon Problems as the Optimal Solution to the Infinite Horizon Optimization Problems, arXiv:0803.4050

2. Cai, D. and Nitta, T. G. Constructing the Optimal Solutions to the Undiscounted Continuous-Time Infinite Horizon Optimization Problems, arXiv:0803.4046

3. Cai, D. and Nitta, T. G. Treating the Future Equally: Solving Undiscounted Infinite Horizon Optimization Problems, arXiv:math/0701371