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Strong Interactions among Particlelike Solutions in Dissipative Systems

Research Institute for Electronic Science
Hokkaido University

Yasumasa Nishiura

Abstract

Three typical transient dynamics in dissipative systems are discussed, namely self-replication, self-destruction, and scattering among spatially localized patterns. The difficulty lies in the fact that patterns are deformed a lot and the associated orbits behave globally in the phase space. A conventional treatment in general does not work and needs a new viewpoint to describe them. Our strategy is not to trace the orbital behavior iteself, but to search a global structure of network of bifurcating solutions which drives such transient dynamics. A hierarchy structure of saddle-node bifurcation points plays a crucial role for self-replication and self-destruction processes, and unstable patterns called *scattors* for scattering process. The aim of this note is to clarify what are the issues and convey the basic ideas to overcome them.

1 Introduction

Scattering of particle-like patterns in dissipative systems is studied, especially we focus on the issue how the input-output relation is controlled at a head-on collision where traveling pulses or spots interact strongly. It remains an open problem due to the large deformation of patterns at a colliding point. We found that special type of unstable steady or time-periodic solutions called scattors and their stable and unstable manifolds direct the traffic flow of orbits. Such scattors are in general highly unstable even in 1D case which causes a variety of input-output relations through the scattering process. We

illustrate the ubiquity of scattors by using the complex Ginzburg-Landau equation, the Gray-Scott model and a three-component reaction diffusion model arising in gas-discharge phenomena.

We focus on the dynamic patterns of spatially localized patterns such as pulses and spots in dissipative systems, especially we are interested in the following transient type of dynamics

"Self-replication, Self-destruction, and Scattering".

The reason is that complicated dynamics as in Figs 1 and 2 could be obtained by combination of those dynamics. For definiteness, we employ the following Gray-Scott model (see for instance, [2], [15]), although the strategy below seems to work also for other model systems. Here we only describe the outline of the results, the readers are encouraged to consult the reference such as [10], [11], [12], [13], and [14]. See also the related reference [7], [8], [9].

$$\begin{cases} \dfrac{\partial u}{\partial t} = & D_u \Delta u - uv^2 + F(1-u) \\ \\ \dfrac{\partial v}{\partial t} = & D_v \Delta v + uv^2 - (F+k)v, \end{cases} \tag{1}$$

The simulations depicted in Figs 1 and 2 show some aspects of the dynamics built in this system. The issues are

1. Self-replication and self-destruction and their relevance to understand complex spatio-temporal dynamics.

2. Scattering phenomena through collisions among particlelike patterns.

In Fig. 1 ordered states are observed locally in space and time, however those are transient patterns and the orbits itinerate several different states through splitting and destruction. Scattering phenomena like Fig.7 and Fig.8 show peculiar type of solutions called *scattors* to which the orbits come close near transition points of input-output relations. However from mathematical point of view there are a couple of difficulties to overcome, namely

- The patterns are deformed to a large extent.

- The orbits itinerate globally in the phase space.

- It is not a proiori clear how to reduce the dynamics into finite dimensional space.

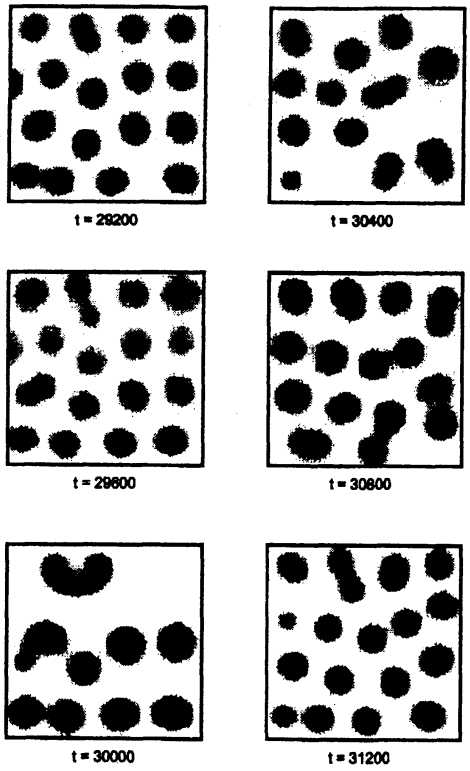

Figure 1: 2D Saptio-temporal chaos for the Gray-Scott model.

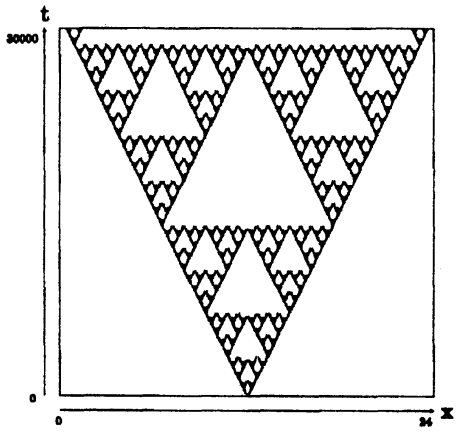

Figure 2: Spatio-temporal self-similar pattern for the Gray-Scott model. Self-replication and annhilation are combined together to create a Sierpinski gasket-like patterns. This was first found by Hayase and Ohta [6] for the reaction-diffusion system of Bonhoeffer-van der Pol type.

It seems an appropriate computational approach is indispensable to find mechanisms which drive those dynamics, and then set up a framework for rigorous analysis. Conventional techniques such as bifurcation analysis, singular perturbation, and some asymptotic methods are useful to some extent, however those are not sufficient to clarify the most essential aspect. It turns out that a set of bifurcating branches form a network in an organized way and it becomes a hidden mechanism driving the orbits in various ways. This note is based on the joint works with D.Ueyama, K.-I.Ueda, S.-I.Ei and T.Teramoto.

2 Weak and strong interactions among particlelike patterns

A typical example of strong interaction is the annihilation of traveling pulses of the FitzHugh-Nagumo equations at head-on collision, which still remains open problem in rigorous sense. Self-replication and self-destruction as in Fig.1 belong to another category of strong interaction, namely the pattern splits or destructs by iteself. The following two complimentary approaches may work well

- **Pulse interaction approach with instabilities.**
- **To find a geometric characterization for the set of global solution branches which drives such dynamics.**

The former one is a perturbative method and was initiated by physists, however rigorous analysis is rather recent (see. for instance, [3], [4], [5]). The latter one developped by [10], [11], [14] owes very much to the recent development of fast computer and path-tracking software like AUTO([1]).

3 Self-replication and self-destruction

Unlike soliton pulses the localized pulses in dissipative systems can duplicate and annihilate spontaneously. One may think that the dynamics of self-replication has nothing to do with the destruction process, however it turns out that they have basically the same structure from bifurcational view point, namely they share the common structure called "hierarchy structure of saddle-node points". See more precisely [14]. Here we briefly describe the

destruction process of multiple-pulses with oscillatory tails for the FitzHugh-Nagumo (FHN) equations. It is known that FHN has a stable multiple-traveling pulse in some parameter region if it has an oscillatory tail, however such a multiple-pulse destruct by itself when the threshold parameter a is increased like Fig.3. The interesting thing is that it does not collapse at once, but destruct successively. In other words, the top one first dies out, then it travels for a while like a two-pulse, but again the first hump disappear and it eventually collapses after traveling as a single pulse for certain time. The associated bifurcation diagram is depicted as in Fig.4 in which a hierarchy structure of saddle-node points is clearly visible and self-destruction process is observed slightly off the location of it. It should be remarked that dynamics of self-destruction is observed in a region where there are no branches corresponding to traveling pulses, nevertheless it is clear that the above hierarchy structure drives such a dynamics behind the scenes. Also such a structure tells you about the onset of such a destruction, i.e., location of saddle-node point.

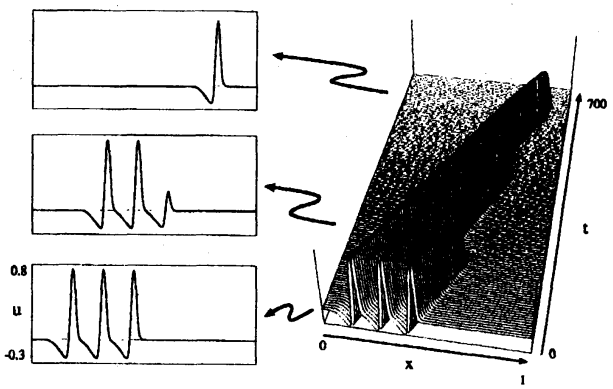

Figure 3: Self-destruction of triple-pulse with oscillatory tail.

Self-replication is an inverse process of the above destruction dynamics, namely the unstable manifold of say, one-hump pattern is connected to two-hump pattern, i.e., the increasing direction. Here we only show a global bifurcation diagram for the Gierer-Meinhardt model arising in the morphogenesis as in Fig.5. It is clearly seen that hierarchy structure of saddle-node points drives self-replication dynamics as in the above evolution numerics of Fig.5 where pulses repel each other and its number increases not like $\rightarrow 2 \rightarrow 4 \rightarrow 8$ proportional to 2^n, but $1 \rightarrow 2 \rightarrow 4 \rightarrow 6$. This is because only pulses located at the edge are able to split. We can prove the edge-splitting phenomena rigorously by using pulse-interaction method [4]. The principal part

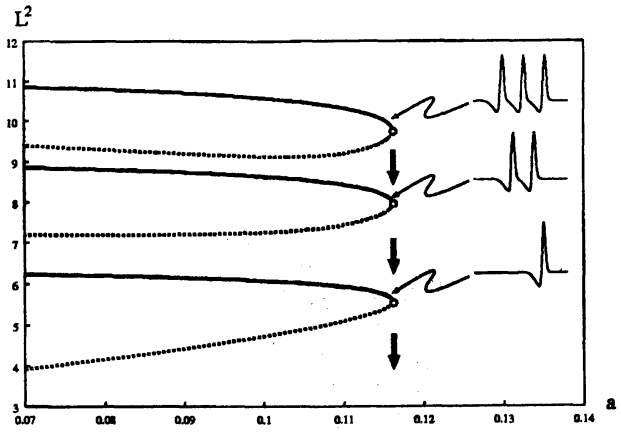

Figure 4: Global bifurcation diagram for multiple-pulseswith oscillatory tails. The abscissa is the threshold parameter a and the ordinate shows a norm of the pulses. The location of each saddle-node point coincide almost perfectly and self-destruction occurs when a exceeds this critical point.

of the resulting equations read

$$\begin{aligned}
\dot{h}_1 &= -M_0(e^{-\alpha h_2} - 2e^{-\alpha h_1}) \\
\dot{h}_j &= -M_0(e^{-\alpha h_{j-1}} - 2e^{-\alpha h_j} + e^{-\alpha h_{j+1}}) \\
\dot{h}_N &= -M_0(e^{-\alpha h_{N-1}} - 2e^{-\alpha h_N}) \\
\dot{r}_0 &= M_1 r_0^2 - \epsilon M_2 - M_3 e^{-\alpha h_1} \\
\dot{r}_j &= M_1 r_j^2 - \epsilon M_2 - M_3(e^{-\alpha h_{j+1}} + e^{-\alpha h_j}) \\
\dot{r}_N &= M_1 r_N^2 - \epsilon M_2 - M_3 e^{-\alpha h_N}
\end{aligned} \tag{2}$$

where r_j denotes the depth of dimple of each pulse, h_j the distance between pulses, ϵ signed distance from the saddle-node point and all the coefficients M_j are positive. Note that the equations for h_j decouple the equations for r_j, therefore it can be analysed independently. In fact h_j basically increases monotonically due to the repulsion, on the other hand, whether a concerning pulse splits or not is determined by whether r_j starts to increase rapidly or remains finite. In view of the right-hand side of r_j, the intercept of the parabola is crucial for this. It turms out that the pulse with the largest h_j primarily starts to split, and such pulses should be located at edge, i.e., edge-splitting ([4]), which is consistent with Fig.5.

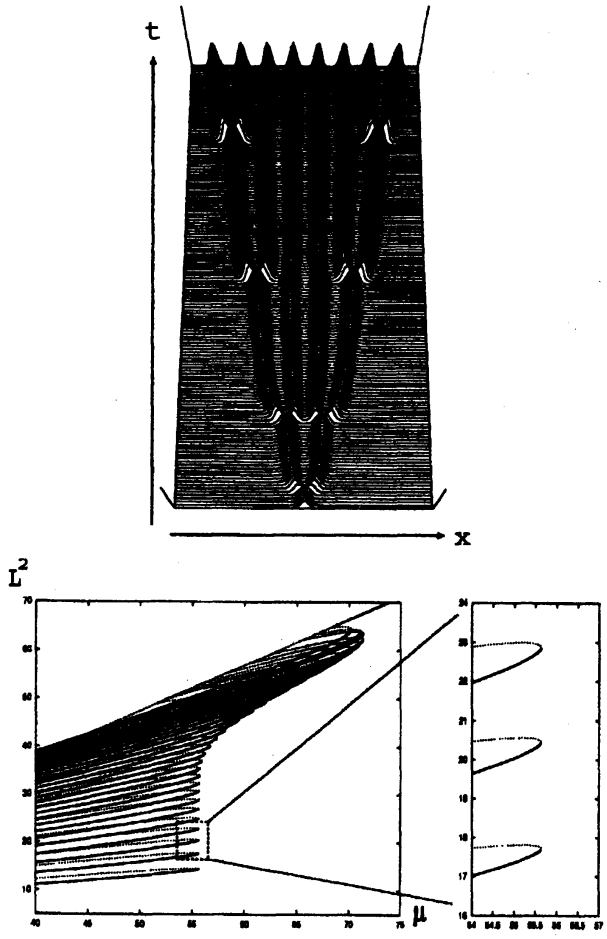

Figure 5: Self-replication process for the Gierer-Meinhardt model. The upper one shows the evolution in 1D case. The lower one is the associated global bifurcation diagram for the set of N-hump standing pulses on a finite interval. The bifurcation parameter μ is the decay-rate for the inhibitor.

4 Spatio-temporal chaos with replication and destruction

It may be a challenge to construct a exotic transient dynamics, say spatio-temporal chaos by piecing replication and destruction together. One of the difficulties is how to tune parameters not to fall in a attractor but to keep each characteristic dynamics. Recall that the destruction process of muti-pulses in Fig.3 ends up with the stable rest state and nothing happens after that. We need to somehow destabilize the final state and activate it to re-enter into a nontrivial state. This is a real global problem and computational approach is indispensable to clarify the situation. One of the good candidates for mechanisms creating such a dynamics is *generalized heteroclinic cycle* in infinite dimensional space (see Fig.6). See [11] for details.

5 Scattering among particlelike patterns

So far we consider the destabilizations of localized patterns by themselves, i.e., without interaction with other patterns. The other important class of strong interactions is *scattering* among moving localized objects. A variety of different types of input-output relations are reported in experiments and numerics (see the reference in [12], [13]). Generically in dissipative systems not only the velocity and the profile of the traveling objects but also input-output relation depends on the parameters. The detailed process connecting input to output during the collision is quite hard to describe even numerically, therefore it is necessary to introduce a new viewpoint. What we present here is the special type of unstable patterns called *scattors* which control the flows near the transition points. Originally this was strongly motivated by the numerics on the head-on collision of 1D pulses for the Gray-Scott model when the parameters are chosen to be close to the transition point from annihilation to repulsion. The stable manifold of the scattor is loosely speaking an infinite-dimensional version of separatrix in ODE, however it is much more subtle and difficult to find it even by numerics. Here we only present just one example of such a scattor arsing in the Gray-Scott model. For more details and applications to other model systems, we refer to [12] and [13].

From repulsion to annihilation

First F is fixed to be 0.0198 and study a symmetric collision. When k is increased and exceeds $k_c \approx 0.0497859$, the input-output relation changes from annihilation (A) to repulsion (B) as in Fig.7. The input-output relation depends on the initial condition, therefore in order to make the transition

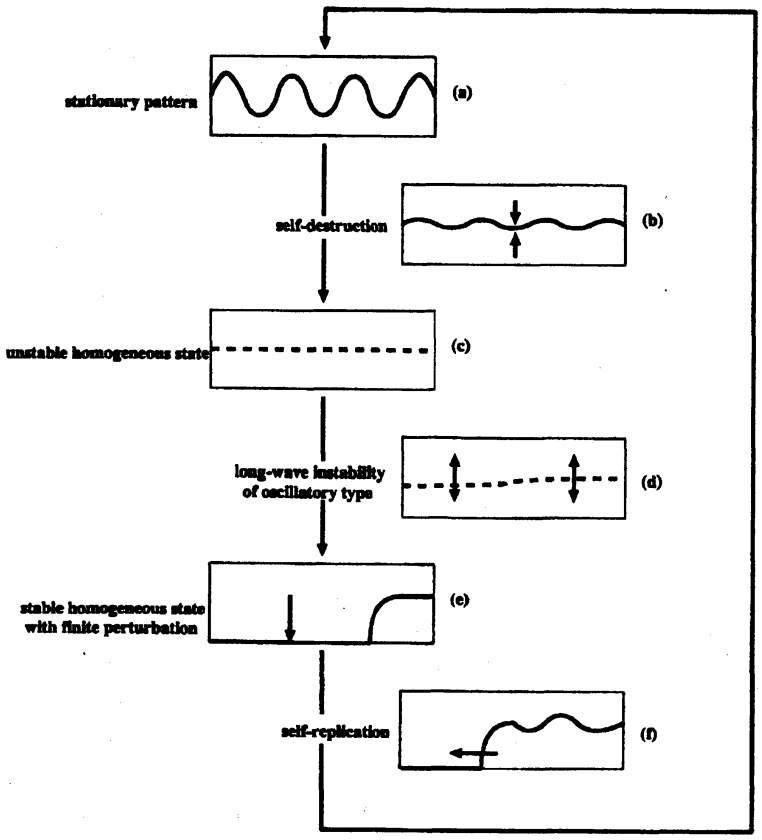

Figure 6: Heteroclinic cycle on the infinite line passing through a unstable homogeneous state P, $(1,0)$, and a spatially periodic pattern for the Gray-Scott model. The homegeneous state P is unstable for large wave-length region originated from Hopf instability. Strictly speaking, such a cycle does not exist on a finite interval, since $(1,0)$ is stable in the PDE sense. However, after replacing $(1,0)$ by $(1,0)$ with trigger (the resulting cycle is called a *generalized* heteroclinic cycle), we can observe an aftereffect of the generalized cycle numerically. Here, $(1,0)$ with trigger means that there is some portion of the interval where (u,v) is not equal to $(1,0)$ and from which a replication wave can start propagating.

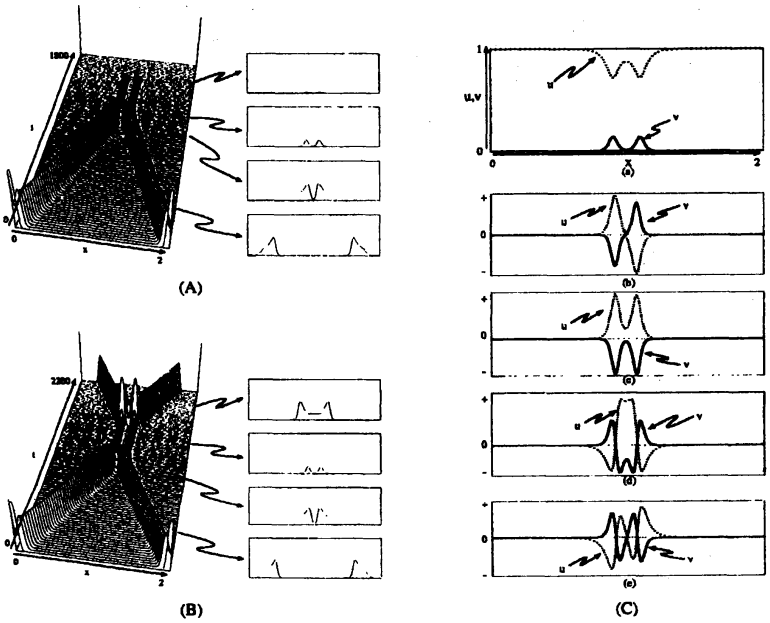

Figure 7: Symmetric collisions for $F = 0.0198$. (A) Annihilation occurs at $(k, F) = (0.0497859, 0.0198)$. (B) As k is slightly increased to 0.0497860, transition from annihilation to repulsion occurs. Note that just before the occurrence of annihilation or creation of counter-propagating pulses, both orbits in (A) and (B) stay very close to the separator depicted in (C)(a). Only v-component is shown in (A) and (B). (C)(a)The profile of the unstable steady state of codim 3 (scattor). Three unstable eigenfunctions ϕ_1, ϕ_2, ϕ_3 are depicted as (b)-(d), and (e) corresponds to the Goldstone mode. The associated eigenvalues are $\lambda_1 = 0.06389 > \lambda_2 = 0.06378 > \lambda_3 = 0.00233$. The first two eigenvalues are much larger than the first one. The solid (gray) line indicates $v(u)$-component.

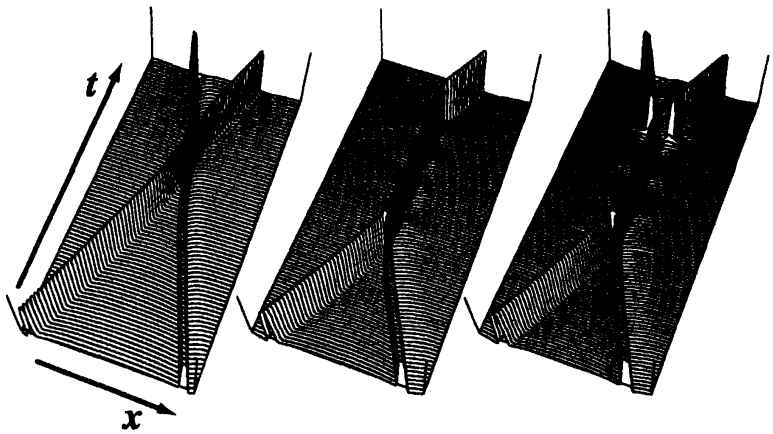

Figure 8: Three different type of input-output relations for the three-component model. Repulsion (left):Two-into-two case. Number of emitting pulses are preserved. Fusion(middle): Two-into-one case. After merging, the fused pattern oscillates for a while, then emits one traveling pulse. Oscillatory repulsion (right): Two-into-two case. Two traveling pulses fused into one localized wave, then oscillates for certain time, and finnaly splits into two counter propagating pulses. See [13] for details.

point k_c to be well-defined, we have to specify the class of initial conditions. Theoretically we employ a symmetric pair of true traveling pulses as an initial condition, which starts initially at $x = \pm\infty$ and collides at the origin. Practically such an initial data and the resulting k_c are well approximated by taking a well-settled symmetric pair of pulses. A remarkable thing is that there appears a quasi-steady state of twin-horn shape right after collision and the orbit approaches it, stays there for certain time, then annihilate or emit two pulses. In fact there exists a real steady state of twin-horn shape, which is numerically confirmed by the Newton method as in Fig.7. This is what we call a *scattor* for the Gray-Scott model. A linearized eigenvalue problem; $L\phi = \lambda\phi$ where L is the linearized operator of the right-hand side of the system (1) around the twin-horn steady state has three unstable eigenvalues $\lambda_1 = 0.06389 > \lambda_2 = 0.06378 > \lambda_3 = 0.00233$ besides the zero eigenvalue λ_4 coming from the translation invariance (see Fig.7(C)). Note that the first two eigenvalues are much larger than the third one, hence the dynamics is basically controlled by λ_1 and λ_2. The associated eigenfunctions are denoted by $\phi_i (i = 1, \cdots, 4)$. The twin-horn scattor plays a role as a traffic controller at collision. In fact, for symmetric head-on collision, the second eigenfunction plays an important role to determine the fate of the orbit, namely, adding

its small constant-multiple perturbation to the twin-horn pattern, then the resulting behavior is either annihilation or emission of two pulses depending on its sign of constant. In other words the output can be classified by looking at the response of the separator along the unstable manifold. In this section we proposed a new viewpoint from scattors. Scattors may be unstable steady states or time-periodic solutions and their codimensions (i.e., the number of unstable eigenvalues) is in general high and the origin of a diversity of input-output relations can be reduced to the local dynamics around scattors. We illustrated this viewpoint by using the Gray-Scott model. The orbit typically approaches a scattor right after collision and is sorted out generically along one of the unstable directions of it. The output can be predictable by using the information on the solution profile right after collision, scattorss and their unstable eigenforms. Scattorss in dissipative systems seems to be ubiquitous and useful to understand the scattering process, in fact even in higher dimensional space such separators are recently found numerically for various models including the 3-component system presented in Fig.8. See[13] for details and forthcoming articles..

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