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DESTABILIZATION OF FRONTS IN A CLASS OF BISTABLE SYSTEMS*

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Abstract. In this article, we consider a class of bistable reaction-diffusion equations in two components on the real line. We assume that the system is singularly perturbed, i.e., that the ratio of the diffusion coefficients is (asymptotically) small. This class admits front solutions that are asymptotically close to the (stable) front solution of the "trivial" scalar bistable limit system $u_t = u_{xx}+u(1-u^2)$. However, in the system these fronts can become unstable by varying parameters. This destabilization is caused by either the essential spectrum associated to the linearized stability problem or by an eigenvalue that exists near the essential spectrum. We use the Evans function to study the various bifurcation mechanisms and establish an explicit connection between the character of the destabilization and the possible appearance of saddle-node bifurcations of heteroclinic orbits in the existence problem.

Key words. pattern formation, bistable systems, geometric singular perturbation theory, stability analysis, Evans functions

AMS subject classifications. 35B25, 35B32, 35B35, 35K57, 35P20, 34A26, 34C37

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1. Introduction.

The class of bistable reaction-diffusion equations we consider in this paper is given by

$$\left\{\begin{array}{rcrcl} U_t & = & \varepsilon^2 U_{xx} & + & (1+V-U^2)U, \\ \tau V_t & = & V_{xx} & + & F(U^2,V;\varepsilon), \end{array}\right. \tag{1.1}$$

where $F(U^2,V;\varepsilon)$ is a smooth function of U^2, V, and ε such that $F(1,0;\varepsilon)\equiv 0$ and $\lim_{\varepsilon\to 0} F(U^2,V;\varepsilon)$ exists; $\tau>0$ is a parameter. Thus, the system is such that the background state $(U,V)\equiv(\pm 1,0)$ is always a solution. We furthermore assume that the ratio of the two diffusion coefficients, ε^2, is asymptotically small; thus, the problem has a singularly perturbed nature. We consider the system on the (unbounded) line, i.e., $(U,V)=(U(x,t),V(x,t))$ with $(x,t)\in\mathbb{R}\times\mathbb{R}^+$. Note that (1.1) is (by construction) symmetric under

$$U\to -U. \tag{1.2}$$

To motivate the structure of (1.1) we introduce the fast variable

$$\xi=\frac{x}{\varepsilon} \tag{1.3}$$

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so that (1.1) can be written in its equivalent "fast" form,

$$\left\{ \begin{array}{rcll} U_t & = & U_{\xi\xi} & + \quad (1+V-U^2)U, \\ \varepsilon^2 \tau V_t & = & V_{\xi\xi} & + \quad \varepsilon^2 F(U^2, V; \varepsilon). \end{array} \right. \tag{1.4}$$

Since $U(x,t)$ and $V(x,t)$ are a priori supposed to be bounded on the entire domain $\mathbb{R} \times \mathbb{R}^+$, we find in the natural (fast reduced) limit, i.e., $\varepsilon \to 0$ in (1.4), that $V \equiv V_0$ and that U is a solution of the well-studied, scalar (standard) bistable or Nagumo equation,

$$U_t = U_{\xi\xi} + (1 + V_0 - U^2)U. \tag{1.5}$$

In this paper we interpret the original system, (1.1) or (1.4), as a scalar bistable Nagumo equation (1.5) in which the coefficient of the linear term is allowed to evolve by reaction and diffusion on a long, or slow, spatial scale. Note that the (slow) dynamics of the V-component are allowed to be completely general, except that it is assumed that the full system conserves the symmetry (1.2) and the background states $U \equiv \pm 1$, at $V \equiv 0$, of the scalar limit (see also Remark 1.1). A priori, one expects that the V-component of front-like solutions will remain small ($\mathcal{O}(\varepsilon)$) due to the "boundary conditions" $V = 0$ at $\pm\infty$ so that the effect of the slowly varying $V(x,t)$-component cannot have a significant influence on the (well-understood) dynamics of the scalar Nagumo equation. An important motivation of the research in this paper is to find out whether or not this intuition is correct.

We will focus completely on the existence and stability issues associated to the persistence of the asymptotically stable stationary front solutions of the bistable equation (1.5) with $V_0 = 0$. In fact, this paper can also be seen as a first step towards analyzing the dynamics (and possibly defects) of striped patterns in a class of relatively simple bistable reaction-diffusion equations, i.e., (1.1) for $(U,V) = (U(x,y,t), V(x,y,t))$ with $(x,y) \in \mathbb{R}^2$. The methods and techniques developed in this paper are supposed to carry over to the analysis of the existence and stability of spatially periodic solutions of (1.1) and their two-dimensional counterparts (the planar fronts and the stripe patterns). See also section 5.

The problem of the persistence of the stable front solution of the scalar bistable equation (1.5) is quite subtle, as can be expected in light of recent results on the stability of pulses in singularly perturbed reaction-diffusion equations of the Gray–Scott and Gierer–Meinhardt type [4, 5]. Such systems can also be written in the form (1.4); however, the scalar limit systems are monostable, i.e., in essence of the form $U_t = U_{\xi\xi} - U + U^2$. The pulses correspond in this (fast reduced) limit to the stationary homoclinic solution of $u_{\xi\xi} - u + u^2 = 0$. Thus, one would expect that the pulses of the full system cannot be stable, since the stability problem associated to the homoclinic solution has an $\mathcal{O}(1)$ unstable eigenvalue. Nevertheless, stable pulses of this type do exist in the Gray–Scott and the Gierer–Meinhardt equation [4, 5]. On the other hand, the stability of the pulses in these monostable equations is strongly related to the freedom one has in these systems to scale the magnitude of the pulses; i.e., the amplitude of the stable pulses is asymptotically large in ε in these monostable systems. Such scalings are not possible for the fronts in the bistable case, since the background states $(\pm 1, 0)$ are fixed (and $\mathcal{O}(1)$).

In the analysis of the front solutions, we will find that it is natural to decompose $F(U^2, V; \varepsilon)$ into a component that has a factor of $(1+V-U^2)$ and a rest term $G(V;\varepsilon)$

that does not depend on U^2. Hence, we write (1.1) as

$$\left\{\begin{array}{rcrcl} U_t & = & \varepsilon^2 U_{xx} & + & (1+V-U^2)U, \\ \tau V_t & = & V_{xx} & + & (1+V-U^2)H(U^2,V;\varepsilon)+G(V;\varepsilon), \end{array}\right. \tag{1.6}$$

with $G(0,\varepsilon)\equiv 0$. Note that this decomposition induces no restriction on $F(U^2,V;\varepsilon)$ since we have assumed that F is smooth. In fact,

$$G(V)=F(1+V,V) \text{ and } (1+V-U^2)H(U^2,V)=F(U^2,V)-F(1+V,V).$$

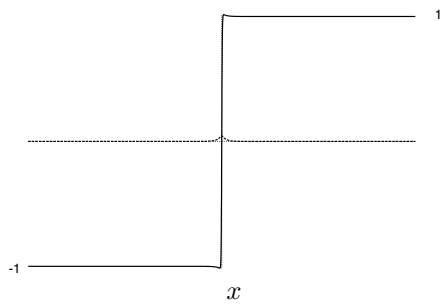
We will find that the quantities $\frac{\partial G}{\partial V}(0;\varepsilon)$ and $H(1,0;\varepsilon)$ have a crucial impact on the structure and the dynamics of the front-like solutions. Therefore, we define

$$G_1(\varepsilon)=\frac{\partial G}{\partial V}(0;\varepsilon) \text{ and } H_0=H(1,0;\varepsilon); \tag{1.7}$$

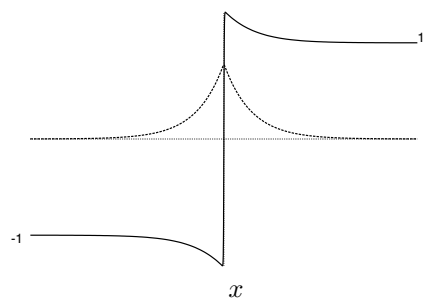
G_1 is the main bifurcation parameter used in this paper. Throughout this paper we assume that $H(U^2,V)$ is nondegenerate, i.e., that $H(1+V,V)$ is not identically 0, and that $\tau=\mathcal{O}(1)$ (see Remark 4.13).

In section 2 we will show that as long as $G_1<0$ and $\mathcal{O}(1)$, the front solutions of (1.5) with $V_0=0$ persist in a regular fashion, in the sense that the system (1.1) has a front solution with U-components that are asymptotically and uniformly close to a front in (1.5) with $V_0=0$ and with V-components that are asymptotically and uniformly small (Theorem 2.1). However, if G_1 becomes $O(\varepsilon^2)$, these fronts become truly singular, in the sense that V becomes $\mathcal{O}(1)$, while the U-component is close to a front of (1.5) with $V_0\neq 0$ on the fast spatial scale (and it converges to $U=\pm 1$ on the slow spatial scale). Moreover, the front solutions are no longer uniquely determined; there can be several types of heteroclinic front solutions if $G_1=\mathcal{O}(\varepsilon^2)$ that may or may not merge in saddle-node bifurcations of heteroclinic orbits when G_1 is varied (Theorems 2.3 and 2.5). It should be noted here that for simplicity we consider $G(V)=-\varepsilon^2\gamma V$ in (1.1) in the singular limit $G_1=\mathcal{O}(\varepsilon^2)$ throughout this paper—see Remark 2.4. We refer to Figure 1.1 for a numerical representation of a regular front (Figure 1.1(a)) and a singular front (Figure 1.1(b)). The magnitude of G_1 is also extremely relevant in the stability analysis. It can be shown that the (regular) front solutions are asymptotically stable as long as $G_1<0$ and $\mathcal{O}(1)$ and $H_0+G_1-2\tau<0$ and $\mathcal{O}(1)$—see Theorem 4.3. It seems, at leading order, that the destabilization of the front is caused by the essential spectrum, σ_{ess}, associated to the stability of the front (σ_{ess} reaches the imaginary axis exactly at $G_1=0$ or at $H_0+G_1-2\tau=0$—see Lemma 3.1). However, the analysis also shows that there can be eigenvalues near the "tips" of σ_{ess} and that it is possible that the destabilization is caused by such an eigenvalue, i.e., by an element of the discrete spectrum and not by σ_{ess}. These "new" eigenvalues do not have counterparts in the (scalar) fast reduced limit problem; they have a singular slow-fast nature and may appear through edge bifurcations from the essential spectrum.

In section 4 we study in detail the nature of the destabilization as $G_1<0$ increases towards 0. In this section it becomes clear that there is an intimate relation between the geometrical character of the singularly perturbed existence problem and the character of the destabilization of the front. This is a natural and frequently encountered relation—see, for instance, [14] and the references therein. We establish that a front solution destabilizes at a critical value of $G_1=-\varepsilon^2\gamma_{\text{double}}<0$ by an eigenvalue if and only if it merges with another front solution in a saddle-node bifurcation of heteroclinic orbits. Moreover, we are able to determine the explicit value



(a) Regular front, $G_1 = -1.0$.



(b) Singular front, $G_1 = -2\varepsilon^2$.

FIG. 1.1. *Two stable front solutions of* (1.1)/(1.6) *plotted on the slow spatial scale x (by a numerical simulation). Here $H(U^2, V) = H_0U^2$, $G(V) = G_1V$, $\varepsilon = 0.1$, and $H_0 = 1$. The solid curves represent the U-coordinates, and the dotted curves represent the V-coordinates.*

of this bifurcation value to be $\gamma_{\text{double}} > 0$. If the front does not "encounter" such a saddle-node as G_1 increases to 0, the front will be destabilized by σ_{ess} at $G_1 = 0$—see Theorems 4.6 and 4.10.

Another way to motivate the analysis of this paper is as follows. In this paper we show that the technique of decomposing the Evans function associated to the stability of a "localized structure" (a (traveling) pulse or front) into the product of an analytic "fast" and a meromorphic "slow" transmission function [4, 5] can be extended to a class of bistable equations. We show that the slow transmission function ($t_2(\lambda, \varepsilon)$) is a natural tool for analyzing the existence or appearance of eigenvalues near or from the essential spectrum and that such eigenvalues play a crucial role in the stability of the front. Note that in this sense the theme of this paper is similar to that of [15], where Evans function techniques are developed to study eigenvalues near σ_{ess} in a class of nearly integrable systems.

The paper is organized as follows. The existence problem is studied in section 2. In section 3 the basic properties of the linearized stability problem are studied and (the decomposition of) the Evans function is introduced. Section 4 is the main section of the paper; in it we develop an approach by which the (possible) location and existence of "slow-fast eigenvalues" near the essential spectrum can be studied. This section is split into three parts: a subsection on the regular problem, a subsection in

which we study an explicit example ($G(V) = -\varepsilon^2\gamma$, $H(U^2, V) = H_0U^2$) in full detail, and a subsection in which we study the "fate" of the regular front as G_1 approaches 0 in the general case. In section 5 we present simulations which clearly exhibit the impact of the distinction between a destabilization by the discrete or by the essential spectrum. Moreover, we discuss some related issues and topics of future research.

Remark 1.1. Large parts of the theory developed in this paper can be generalized to systems of the type (1.1)/(1.4) in which the fast reduced limit system is of the type $U_t = U_{\xi\xi} + B(U^2; V_0)U$ for some function B, i.e., to bistable systems of a more general nature. We focused on the standard case, i.e., $B = 1 + V_0 - U^2$, since the analysis is more transparent. If one drops the condition on the symmetry (1.2), the fronts will, in general, travel with a certain (nonzero) speed. Although the symmetry is used throughout this paper, there is no reason to expect that such asymmetric systems cannot be studied along the lines of the methods presented here.

Notation and definitions. Let $\rho(\varepsilon)$ be a function of $\varepsilon \geq 0$ that is smooth and positive for $\varepsilon > 0$ such that $\lim_{\varepsilon\downarrow 0} \rho(\varepsilon) = 0$ ($\rho(\varepsilon)$ is called an order function [8]). Let $R(z;\varepsilon) \in \mathbb{R}^m$ or $\in \mathbb{C}^m$ ($m \geq 1$) be a certain expression that depends on ε (among other variables or parameters $z \in \mathbb{R}^p, \mathbb{C}^p$ for some $p \geq 0$) such that the limit $\varepsilon \downarrow 0$ exists, i.e., $\lim_{\varepsilon\downarrow 0} R(z;\varepsilon) \stackrel{\text{def}}{=} R_0(z)$. Throughout this paper, the following notation will be used to describe the rate of convergence of $R(z;\varepsilon)$ to $R_0(z)$:

$$R(z;\varepsilon) = R_0(z) + \mathcal{O}(\rho(\varepsilon)).$$

By definition, this is equivalent to the statement that there exists a constant $C > 0$, which is independent of ε, and an $\varepsilon_0 > 0$ such that $\|R(z;\varepsilon) - R_0(z)\| < C\rho(\varepsilon)$ for $0 < \varepsilon < \varepsilon_0$ (here $\|.\|$ is the standard Euclidean norm). Note that both C and ε_0 may be z-dependent. As is usual in (singular) perturbation theory, the precise structure of $\rho(\varepsilon)$ is crucial at many steps in the forthcoming analysis. If this is not the case, $R(z;\varepsilon)$ is often said to be "asymptotically close" to $R_0(z)$, or, equivalently, $\|R(z;\varepsilon) - R_0(z)\| \ll 1$, which implies only that there exists an (unspecified) $\rho(\varepsilon)$ such that $R(z;\varepsilon) = R_0(z) + \mathcal{O}(\rho(\varepsilon))$, i.e., that $\lim_{\varepsilon\downarrow 0} R(z;\varepsilon) = R_0(z)$.

In this paper, the expression $R(z;\varepsilon)$ is said to be $\mathcal{O}(1)$ with respect to ε if there are constants $C^{\pm} > 0$ and $\varepsilon_0 > 0$ such that $C^- < \|R(z;\varepsilon)\| < C^+$ for $0 < \varepsilon < \varepsilon_0$. We refer the reader to [8] for more details on order functions (including the definitions of "$\gg$" and $\mathcal{O}(\frac{1}{\rho(\varepsilon)})$).

2. The existence problem. We analyze the existence of stationary one-dimensional patterns through geometric singular perturbation theory [9, 12] using the methods developed in [6, 5]. Therefore, we write the ODE associated to (1.6) as a dynamical system in $\mathbb{R}^4$,

$$\left\{\begin{array}{lcl} \dot{u} & = & p, \\ \dot{p} & = & -(1+v-u^2)u, \\ \dot{v} & = & \varepsilon q, \\ \dot{q} & = & \varepsilon\left[-(1+v-u^2)H(u^2,v;\varepsilon) - G(v;\varepsilon)\right], \end{array}\right. \tag{2.1}$$

where $\dot{}$ denotes the derivative with respect to the spatial variable ξ (1.3) (i.e., ξ "plays the role of time"). Note that this system inherits two symmetries of (1.6),

$$\xi \to -\xi, \quad p \to -p, \quad q \to -q \ \text{ and } \ u \to -u, \quad p \to -p. \tag{2.2}$$

We consider the "superslow" case in which $G_1(\varepsilon) = \mathcal{O}(\varepsilon^2)$ separately in sections 2.2 and 2.3. Note that in the fast reduced limit, i.e., $\varepsilon \to 0$ in (2.1), the monotonically increasing heteroclinic front solution is given by (u_0, p_0, v_0, q_0), where

(2.3)
$$(u_0(\xi; v_0), p_0(\xi; v_0)) = \left(\sqrt{1+v_0}\tanh\left(\sqrt{\frac{1+v_0}{2}}\xi\right), \frac{1+v_0}{\sqrt{2}}\mathrm{sech}^2\left(\sqrt{\frac{1+v_0}{2}}\xi\right)\right),$$

and v_0 and q_0 are constants.

2.1. The regular case. The main result of this section is the following theorem.

THEOREM 2.1. *Let $G_1(\varepsilon)$ (1.7) be $\mathcal{O}(1)$ and negative. Then, for $\varepsilon > 0$ small enough, system* (2.1) *has a symmetric pair of heteroclinic orbits: $\Gamma_h^+(\xi;\varepsilon) = (u_h(\xi;\varepsilon), p_h(\xi;\varepsilon), v_h(\xi;\varepsilon), q_h(\xi;\varepsilon))$ and $\Gamma_h^-(\xi;\varepsilon) = (-u_h(\xi;\varepsilon), -p_h(\xi;\varepsilon), v_h(\xi;\varepsilon), q_h(\xi;\varepsilon))$, with $\lim_{\xi\to\pm\infty}\Gamma_h^+(\xi;\varepsilon) = (\pm 1, 0, 0, 0)$ and $\lim_{\xi\to\pm\infty}\Gamma_h^-(\xi;\varepsilon) = (\mp 1, 0, 0, 0)$; $u_h(\xi;\varepsilon)$ and $q_h(\xi;\varepsilon)$ are odd and monotonic as functions of ξ, and $v_h(\xi;\varepsilon)$ and $p_h(\xi;\varepsilon)$ are even. Moreover, $|u_h(\xi;\varepsilon) - u_0(\xi;0)| = \mathcal{O}(\varepsilon)$ (2.3) and $|v_h(\xi;\varepsilon)|, |q_h(\xi;\varepsilon)| = \mathcal{O}(\varepsilon)$, both uniformly on $\mathbb{R}$; $v_h(0;\varepsilon)$ is the extremal value of $v_h(\xi;\varepsilon)$, with*

$$v_h(0;\varepsilon) = \frac{\varepsilon}{2\sqrt{-G_1(0)}}\int_{-\infty}^{\infty}\left(1 - u_0^2(\xi;0)\right)H(u_0^2(\xi;0), 0)d\xi + \mathcal{O}(\varepsilon^2). \tag{2.4}$$

The orbits $\Gamma^\pm(\xi;\varepsilon)$ correspond to the (stationary) front patterns $(\pm U_h(\xi;\varepsilon), V_h(\xi;\varepsilon))$ of (1.6) *with $U_h(\xi;\varepsilon) = u_h(\xi;\varepsilon)$ odd as a function of ξ, $V_h(\xi;\varepsilon) = v_h(\xi;\varepsilon) = \mathcal{O}(\varepsilon)$ even, $\lim_{\xi\to\pm\infty} U_h(\xi;\varepsilon) = \pm 1$, and $\lim_{\xi\to\pm\infty} V_h(\xi;\varepsilon) = 0$.*

Proof. As the system is singularly perturbed, we also consider (2.1) with the slow scaling $x = \varepsilon\xi$; equation (2.1) is given by

$$\begin{cases} \varepsilon u' &= p, \\ \varepsilon p' &= -(1+v-u^2)u, \\ v' &= q, \\ q' &= \left[-(1+v-u^2)H(u^2, v;\varepsilon) - G(v;\varepsilon)\right], \end{cases} \tag{2.5}$$

where $'$ refers to differentiation with respect to x. System (2.5) is referred to as the slow system. We begin by finding the locally invariant manifolds of (2.5) in the limit $\varepsilon \to 0$. In this limit, the first two equations of (2.5) will reduce to

$$p = 0\,, \quad -(1+v-u^2)u = 0\,. \tag{2.6}$$

The manifold given by $(u, p, v, q) = (0, 0, v, q)$ is not normally hyperbolic and will not be considered. However, the manifolds, denoted $\mathcal{M}_0^\pm$, determined by $(u, p, v, q) = (\pm\sqrt{1+v}, 0, v, q)$ are normally hyperbolic and thus, by [9, 12], equation (2.5) possesses locally invariant manifolds $\mathcal{M}_\varepsilon^\pm$, which are $\mathcal{O}(\varepsilon)$ close to $\mathcal{M}_0^\pm$. We now determine the leading order correction to these manifolds. Let the manifold $\mathcal{M}_\varepsilon^\pm$ be given by

$$\mathcal{M}_\varepsilon^\pm = \{u = \pm\sqrt{1+v} + \varepsilon U^\pm(v, q;\varepsilon), p = \varepsilon P^\pm(v, q;\varepsilon), v, q\}\,. \tag{2.7}$$

To obtain successive approximations of $\mathcal{M}_\varepsilon^\pm$, we can expand $U^\pm = u_1^\pm + \varepsilon u_2^\pm + \cdots$, and $P^\pm = p_1^\pm + \varepsilon p_2^\pm + \cdots$. Using the first two lines of (2.5) we find

$$\begin{aligned} p_1^\pm &= \frac{q}{2\sqrt{1+v}}, \quad p_2^\pm = \frac{\partial u_1^\pm}{\partial v}q - \frac{\partial u_1^\pm}{\partial q}G(v;\varepsilon), \\ u_1^\pm &= 0, \quad u_2^\pm = \mp\frac{q^2}{4(1+v)^{5/2}} \mp \frac{G(v;\varepsilon)}{(1+v)^{3/2}}. \end{aligned} \tag{2.8}$$

Hence, the (slow) flow on the slow manifold is given by

$$v'' = -G(v;\varepsilon) + \mathcal{O}(\varepsilon^2)\,. \tag{2.9}$$

To leading order, this flow is integrable. The point $(v,q) = (0,0)$, which corresponds to $(\pm 1, 0, 0, 0)$, is a critical point on $\mathcal{M}_\varepsilon^\pm$. Since $G_1 < 0$, $(0,0)$ is a saddle on $\mathcal{M}_\varepsilon^\pm$ with unstable direction $(1, \sqrt{-G_1})$ and stable direction $(-1, \sqrt{-G_1})$.

A heteroclinic orbit $\Gamma_h^\pm$ from $(\mp 1, 0, 0, 0)$ to $(\pm 1, 0, 0, 0)$ is both an element of $W^u(\mathcal{M}_\varepsilon^\mp)$ and of $W^s(\mathcal{M}_\varepsilon^\pm)$. Here we will consider only Γ_h^+. The existence of Γ_h^- follows from the symmetry (2.2). The orbit Γ_h^+ remains exponentially close to $W^u(-1,0,0,0)|_{\mathcal{M}_\varepsilon^-}$ before it "takes off" and makes a "jump" through the fast field, i.e., the region in phase space in which $(u_\xi, p_\xi) = \mathcal{O}(1)$. After that, it "touches down" on $\mathcal{M}_\varepsilon^+$ and remains exponentially close to it (and to $W^u(1,0,0,0)|_{\mathcal{M}_\varepsilon^+}$)—see Figure 2.1. The change in q by the passage through the fast field is $\mathcal{O}(\varepsilon)$ (2.1); therefore Γ_h^+ must take off from $\mathcal{M}_\varepsilon^-$ and touch down on $\mathcal{M}_\varepsilon^+$ with a q-coordinate that is $\mathcal{O}(\varepsilon)$. Since Γ_h^+ is asymptotic to the saddle points $(0,0) \in \mathcal{M}_\varepsilon^\pm$, it follows that the v-coordinate of Γ_h^+ must also be $\mathcal{O}(\varepsilon)$. Note that we have used here implicitly that $G_1 = \mathcal{O}(1)$.

We will determine whether such a trajectory, as Γ_h^+, is possible using a Melnikov method. Both $W^u(\mathcal{M}_\varepsilon^-)$ and $W^s(\mathcal{M}_\varepsilon^+)$ are $\mathcal{O}(\varepsilon)$ close to the family of heteroclinic orbits in the fast reduced limit of (2.1) given in (2.3). The leading order distance between $W^u(\mathcal{M}_\varepsilon^-)$ and $W^s(\mathcal{M}_\varepsilon^+)$ can be determined by a Melnikov function for slowly varying systems [19]. Both $W^u(\mathcal{M}_\varepsilon^-)$ and $W^s(\mathcal{M}_\varepsilon^+)$ intersect the hyperplane $\{u = 0\}$ transversally. Note that $W^{s,u}(\mathcal{M}_\varepsilon^\pm) \cap \{u = 0\}$ is two-dimensional; thus, since $\{u = 0\}$ is three-dimensional, one expects a one-dimensional intersection $W^u(\mathcal{M}_\varepsilon^-) \cap W^s(\mathcal{M}_\varepsilon^+) \cap \{u = 0\}$. The separation between $W^u(\mathcal{M}_\varepsilon^-)$ and $W^s(\mathcal{M}_\varepsilon^+)$ is, at leading order, measured by the integral,

$$\Delta = \int_{-\infty}^{\infty} \begin{pmatrix} p(\xi) \\ u(\xi) + u^3(\xi) - u(\xi)v_0 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ -u(\xi)\frac{\partial q}{\partial \delta}(\xi) \end{pmatrix} d\xi\,. \tag{2.10}$$

Here the wedge product refers to the scalar cross product, and $\frac{\partial q}{\partial \delta}$ solves the differential equation, $\frac{d}{d\xi}(\frac{\partial q}{\partial \delta}) = q_0\xi$, $\frac{\partial q}{\partial \delta}(0) = 0$. Substituting (2.3) into (2.10) results in the following expression for the leading order splitting distance:

$$\Delta = -\int_{-\infty}^{\infty} \frac{q\xi}{\sqrt{2}} \tanh\left(\frac{\xi}{\sqrt{2}}\right) \operatorname{sech}^2\left(\frac{\xi}{\sqrt{2}}\right) d\xi \quad = -q_0\sqrt{2}\,.$$

Thus, $W^u(\mathcal{M}_\varepsilon^-) \cap W^s(\mathcal{M}_\varepsilon^+) \cap \{u = 0\}$ must be $\mathcal{O}(\varepsilon)$ close to $q = 0$. By the symmetries (2.2), we conclude that $W^u(\mathcal{M}_\varepsilon^-) \cap W^s(\mathcal{M}_\varepsilon^+) \cap \{u = 0\}$ must be identically $q = 0$. Hence, again by (2.2), any solution that connects $\mathcal{M}_\varepsilon^-$ to $\mathcal{M}_\varepsilon^+$ must have a u component that is odd with respect to ξ and a v component that is even with respect to ξ.

We are now ready to determine the take off (touch down) curves $T_o^- \subset \mathcal{M}_\varepsilon^-$ ($T_d^+ \subset \mathcal{M}_\varepsilon^+$) [6, 5]. These curves represent the points at which the one-dimensional family of orbits in $W^u(\mathcal{M}_\varepsilon^-) \cap W^s(\mathcal{M}_\varepsilon^+)$ leave (land on) $\mathcal{M}_\varepsilon^\pm$. Let the elements of this family be denoted $\gamma(\xi;p)$, where the parameter $p > 0$ corresponds to the p-component of $\gamma(\xi;p)$ as it crosses through $\{u = q = 0\}$. Note that the γ-family forms the Fenichel fibering of $W^u(\mathcal{M}_\varepsilon^-) \cap W^s(\mathcal{M}_\varepsilon^+)$ [9] and that each $\gamma(\xi;p)$ is asymptotically close to an unperturbed orbit given in (2.3). To each $\gamma(\xi;p)$ we associate two orbits, $\gamma_{\mathcal{M}_\varepsilon^-}(\xi;p) \subset \mathcal{M}_\varepsilon^-$ and $\gamma_{\mathcal{M}_\varepsilon^+}(\xi;p) \subset \mathcal{M}_\varepsilon^+$, by the fact that $||\gamma(\xi;p) - \gamma_{\mathcal{M}_\varepsilon^\pm}(\xi;p)||$ is

exponentially small if $\pm\xi > \mathcal{O}(\varepsilon^{-1})$. We define T_o^- and T_d^+ as the collections of base points of the Fenichel fibers on $\mathcal{M}_\varepsilon^-$ and on $\mathcal{M}_\varepsilon^-$,

$$T_o^- = \bigcup_{p>0} \gamma_{\mathcal{M}_\varepsilon^-}(0;p)\,, \quad T_d^+ = \bigcup_{p>0} \gamma_{\mathcal{M}_\varepsilon^+}(0;p)\,. \tag{2.11}$$

We can compute the leading order structure of T_o^- and T_d^+ by considering the effect of the journey through the fast field on the slow variables v and q. Since $v_\xi = \varepsilon q$ and $q = \mathcal{O}(\varepsilon)$ it follows that the change in v through the fast field is of higher order, i.e., $\mathcal{O}(\varepsilon^2)$. By construction, q will be an odd function of ξ; thus, the value of q for a given v on T_o^- must be $-\frac{1}{2}\Delta q(v)$, where $\Delta q(v)$ is the change in q due to one full pass through the fast field (during which v remains (at leading order) constant, $v = v_0$). Similarly, the value of q on T_d^- must be $\frac{1}{2}\Delta q(v)$. Since we already know that both v and q must be $\mathcal{O}(\varepsilon)$ in this regular case, we compute $\Delta q(0)$ (by (2.1), (2.3)):

$$\Delta q(0) = \int_{-\infty}^{\infty} \dot{q}|_{v=0}\, d\xi = -\varepsilon \int_{-\infty}^{\infty} \left[1 - \tanh^2\left(\frac{\xi}{\sqrt{2}}\right)\right] H\left(\tanh^2\left(\frac{\xi}{\sqrt{2}}\right), 0\right) d\xi + \mathcal{O}(\varepsilon^2).$$

To establish the existence of the heteroclinic orbit $\Gamma_h^+(\xi)$, we consider the intersection $T_o^- \cap W^u(-1,0,0,0)|_{\mathcal{M}_\varepsilon^-}$ on $\mathcal{M}_\varepsilon^-$, $\mathcal{O}(\varepsilon)$ close to $(-1,0,0,0)$. Thus, T_o^- and $W^u(-1,0,0,0)|_{\mathcal{M}_\varepsilon^-}$ are given by $\{q = -\frac{1}{2}\Delta q(0) + \mathcal{O}(\varepsilon^2)\}$ and $\{q = \sqrt{-G_1}v + \mathcal{O}(\varepsilon^2)\}$. Figure 2.1 shows the superposition of T_o^- with $W^u(-1,0,0,0)|_{\mathcal{M}_\varepsilon^-}$ and of T_d^+ with $W^s(1,0,0,0)|_{\mathcal{M}_\varepsilon^+}$. The v-coordinate of $T_o^- \cap W^u(-1,0,0,0)|_{\mathcal{M}_\varepsilon^-}$ is given in (2.4). Thus, we have established the existence of an orbit $\Gamma_h^+ \in W^u(\mathcal{M}_\varepsilon^-) \cap W^s(\mathcal{M}_\varepsilon^+)$ that is asymptotic to $(-1,0,0,0) \in \mathcal{M}_\varepsilon^-$. Since Γ_h^+ passes through $\{u = 0, q = 0\}$ during its jump through the fast field, it follows by the symmetries (2.2) that Γ_h^+ is indeed the orbit described in the statement of the theorem. As already mentioned, the existence of Γ_h^- also follows immediately from (2.2). ∎

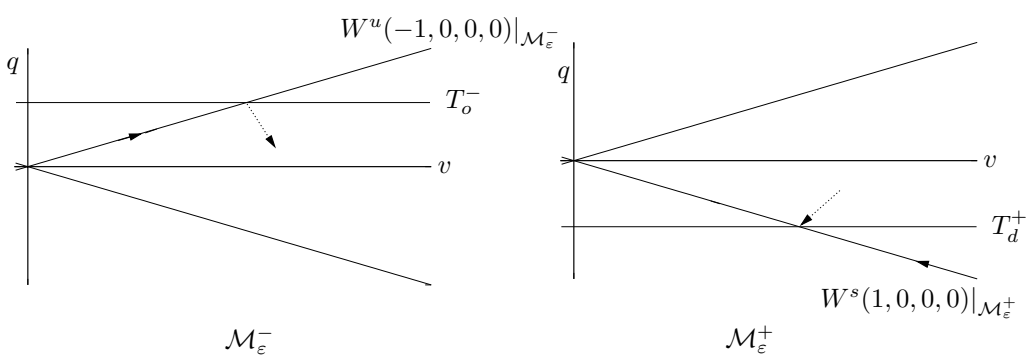


FIG. 2.1. *Superposition of the take off and touch down curves $T_{o,d}^\pm$ with $W^{s,u}(\pm1,0,0,0)|_{\mathcal{M}_\varepsilon^\pm}$. The intersections $T_o^- \cap W^u(-1,0,0,0)|_{\mathcal{M}_\varepsilon^-}$ and $T_d^+ \cap W^s(1,0,0,0)|_{\mathcal{M}_\varepsilon^+}$ determine the heteroclinic front solution $\Gamma_h^+(\xi;\varepsilon)$. The dotted arrows indicate the orbit $\Gamma_h^+(\xi)$ "taking off" and "touching down."*

Remark 2.2. We note that if $G_1 = \mathcal{O}(\varepsilon^\sigma)$ for some $\sigma \in [0,2)$, then the intersection of T_o^- and $W^u(-1,0,0,0)|_{\mathcal{M}_\varepsilon^-}$ will result in a value of v_0 of $\mathcal{O}(\varepsilon^{1-\frac{1}{2}\sigma}) \ll 1$ (2.4); thus $\Gamma_h^+(\xi)$ will still be a regular perturbation of the orbit in the scalar limit. Moreover, this argument also shows that singular orbits may exist for $G_1 = \mathcal{O}(\varepsilon^2)$.

2.2. The superslow limit: An example. In this section we consider the "significant degeneration" $G_1(\varepsilon) = \mathcal{O}(\varepsilon^2)$. For simplicity, we consider only the case in which the flow on the slow manifolds $\mathcal{M}_\varepsilon^\pm$ is linear, i.e., $G(v;\varepsilon) = \varepsilon^2 G_1(\varepsilon) v \stackrel{\text{def}}{=} -\varepsilon^2\gamma v$, where γ does not depend on ε. Moreover, we first consider an explicit expression for $H(u^2, v;\varepsilon)$, $H(u^2, v;\varepsilon) = H_0 u^2$. The case of a general $H(U^2, V)$ will be considered in the next subsection. We refer the reader to Remark 2.4 for a brief discussion of the case of a general function $G(V)$. System (2.1) reduces to

$$\begin{cases} \dot{u} &= p, \\ \dot{p} &= -(1+v-u^2)u, \\ \dot{v} &= \varepsilon q, \\ \dot{q} &= \varepsilon\left[-(1+v-u^2)H_0 u^2 + \varepsilon^2\gamma v\right]. \end{cases} \tag{2.12}$$

This system has various types of (singular) heteroclinic orbits.

THEOREM 2.3. *Assume that $G(V) = -\varepsilon^2\gamma V$, that $H(U^2, V) = H_0 U^2$, and that ε is small enough.*

(i) $H_0 > 0$. *If $\gamma > \gamma_{\text{double}}$, where $\gamma_{\text{double}} = \frac{3}{2}H_0^2 + \mathcal{O}(\varepsilon)$, equation* (2.12) *has two pairs of heteroclinic orbits, $\Gamma_h^{+,j}(\xi;\varepsilon) = (u_h^j(\xi), p_h^j(\xi), v_h^j(\xi), q_h^j(\xi))$, $j = 1, 2$, and their symmetrical counterparts $\Gamma_h^{-,j}(\xi;\varepsilon) = (-u_h^j(\xi), -p_h^j(\xi), v_h^j(\xi), q_h^j(\xi))$, with $\lim_{\xi\to\pm\infty}\Gamma_h^{+,j}(\xi;\varepsilon) = (\pm 1, 0, 0, 0)$. In the fast field $u_h^j(\xi)$ (resp., $v_h^j(\xi)$) is asymptotically and uniformly close to $u_0(\xi; v_j)$* (2.3) *(resp., v_j); the constants v_j are the zeros of $\sqrt{\gamma}v = \frac{2}{3}\sqrt{2}H_0(v+1)^{3/2}$ so that $0 < v_1 < 2 < v_2$ (at leading order). In the slow field, $\Gamma_h^{+,j}(\xi;\varepsilon)$ is exponentially close to $W^{u,s}(\pm 1, 0, 0, 0)|_{\mathcal{M}_\varepsilon^\pm} \subset \mathcal{M}_\varepsilon^\pm$. The orbits $\Gamma_h^{\pm,1}(\xi;\varepsilon)$ and $\Gamma_h^{\pm,2}(\xi;\varepsilon)$ merge in a saddle-node bifurcation of heteroclinic orbits as $\gamma \downarrow \gamma_{\text{double}}$. There are no heteroclinic orbits for $\gamma < \gamma_{\text{double}}$.*

(ii) $H_0 < 0$. *The relation $\sqrt{\gamma}v = \frac{2}{3}\sqrt{2}H_0(v+1)^{3/2}$ has a unique zero for all $\gamma > 0$, and there is one pair of heteroclinic orbits $\Gamma_h^\pm(\xi;\varepsilon)$ for all $\gamma > 0$. These orbits have the same structure as described in* (i).

The orbits $\Gamma_h^{\pm(,j)}(\xi;\varepsilon)$ correspond to the front solutions $(U_h^{\pm(,j)}(\xi;\varepsilon), V_h^{\pm(,j)}(\xi;\varepsilon))$ of (1.6) *with $U_h^{\pm(,j)}(\xi;\varepsilon) = \pm u_h^j(\xi;\varepsilon)$ odd and $V_h^{\pm(,j)}(\xi;\varepsilon) = v_h^j(\xi;\varepsilon)$ even as functions of ξ.*

Proof. The essence of the analysis of the superslow system is similar to that of the regular case. The important difference is that, although the change in q by a "jump" through the fast field is still $\mathcal{O}(\varepsilon)$, the v-coordinate of the heteroclinic orbit may now be $\mathcal{O}(1)$, due to the superslow character of the flow on $\mathcal{M}_\varepsilon^\pm$. It is this difference that will cause the bifurcation and the formation of the second orbit in case (i). The flow on the slow manifold is now $\mathcal{O}(\varepsilon^2)$, i.e., superslow, and is at leading order governed by

$$v'' = \varepsilon^2\gamma v\,. \tag{2.13}$$

Since the right-hand side of this equation is $\mathcal{O}(\varepsilon^2)$, one might expect that one needs to incorporate the higher order corrections to the approximation of $\mathcal{M}_\varepsilon^\pm$ (2.8) to determine the leading order flow on $\mathcal{M}_\varepsilon^\pm$. However, the $\mathcal{O}(\varepsilon^2)$ correction contains a term with a q^2 factor and a term with $G(v)$ (2.8). Since we consider $q = \mathcal{O}(\varepsilon)$ on $\mathcal{M}_\varepsilon^\pm$ and since $G(v) = \mathcal{O}(\varepsilon^2)$, the resulting correction will not be of leading order.

Again the equilibria on $\mathcal{M}_\varepsilon^\pm$ are saddles, with stable and unstable directions, $(\pm 1, \varepsilon\sqrt{\gamma})$. As in Theorem 2.1 we consider only the orbit that jumps from $\mathcal{M}_\varepsilon^-$ to $\mathcal{M}_\varepsilon^+$ (the others follows from the symmetry (2.2). We repeat the Melnikov calculations and again conclude that $W^u(\mathcal{M}_\varepsilon^-) \cap W^s(\mathcal{M}_\varepsilon^+) \cap \{u = 0\}$ must be identically $q = 0$. Hence,

again by (2.2), any solution that connects $\mathcal{M}_\varepsilon^-$ to $\mathcal{M}_\varepsilon^+$ must have a u component that is odd with respect to ξ and a v component that is even with respect to ξ.

We define the take off, T_o^-, and touch down, T_d^+, curves as in (2.11). We find the leading order behavior of T_d^+ and T_o^- by calculating the change in q as we traverse the fast field. As in the regular case, v remains a constant up to $\mathcal{O}(\varepsilon^2)$, and the value of q on the take off (touch down) curve must be $-\frac{1}{2}\Delta q(v_0)$ ($\frac{1}{2}\Delta q(v_0)$), where v_0 is the (leading order) constant value of the v-coordinate of the orbit that is heteroclinic to $\mathcal{M}_\varepsilon^+$ in the fast field. The calculation of the change in q is similar to that of the regular case except that v_0 now effects the leading order term (2.3),

$$\begin{aligned}\Delta q(v_0) &= -\varepsilon H_0(1+v_0)^2\int_{-\infty}^{\infty}\left[1-\tanh^2\left(\sqrt{\tfrac{v_0+1}{2}}\xi\right)\right]\tanh^2\left(\sqrt{\tfrac{v_0+1}{2}}\xi\right)\,d\xi+\mathcal{O}(\varepsilon^2)\\ &= -\varepsilon\tfrac{2\sqrt{2}}{3}H_0(v_0+1)^{3/2}+\mathcal{O}(\varepsilon^2)\,.\end{aligned}$$

The heteroclinic orbits are again determined by $T_o^-\cap W^u(-1,0,0,0)|_{\mathcal{M}_\varepsilon^-}$, where $T_o^- = \{q=-\frac{1}{2}\Delta q(v_0)+\mathcal{O}(\varepsilon^2)\}$ and $W^u(-1,0,0,0)|_{\mathcal{M}_\varepsilon^-}=\{q=\varepsilon\sqrt{\gamma}v+\mathcal{O}(\varepsilon^2)\}$,

$$\frac{1}{3}\sqrt{2}H_0(v_0+1)^{3/2}=\sqrt{\gamma}v_0\,; \tag{2.14}$$

see Figure 2.2. Thus, in the superslow case, a heteroclinic orbit may leave $\mathcal{M}_\varepsilon^-$ with a v-coordinate of $\mathcal{O}(1)$. Now if $H_0>0$ and $\gamma>\gamma_{\text{double}}=\frac{3}{2}H_0^2+\mathcal{O}(\varepsilon)$, (2.14) has two possible solutions, $v_0=v_j$, $j=1,2$, with $0<v_1<2<v_2$ (at leading order). These intersections correspond to the heteroclinic orbits $\Gamma_h^{+,j}(\xi)$. For $\gamma<\gamma_{\text{double}}$, there are no solutions to (2.14), and thus no heteroclinic connections exist: the orbits $\Gamma_h^{+,1}(\xi)$ and $\Gamma_h^{+,2}(\xi)$ have coalesced at $\gamma=\gamma_{\text{double}}$. In the case that $H_0<0$, (2.14) has a unique solution for all values of $\gamma>0$; there is only one pair of heteroclinic orbits. ∎

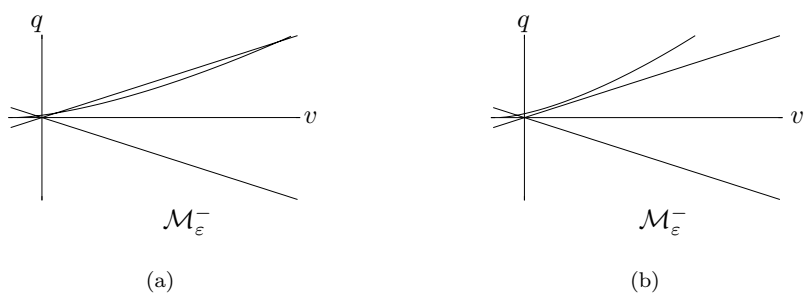


FIG. 2.2. *Superposition of T_o^- with $W^u(-1,0,0,0)|_{\mathcal{M}_\varepsilon^-}$ in the superslow case with $H_0>0$ for $\gamma>\gamma_{\text{double}}$* (a) *and $\gamma<\gamma_{\text{double}}$* (b).

Remark 2.4. If $G(V)$ is not linear in the singular limit (i.e., $G_1=\mathcal{O}(\varepsilon^2)$), then the analysis becomes more involved, but there are no essentially new phenomena. In this case, the magnitude (with respect to ε) of the second derivative of $G(v)$ at $v=0$ will start to play a role comparable to G_1. Moreover, the flow on $\mathcal{M}_\varepsilon^\pm$ is nonlinear so that $W^{u,s}(\pm1,0,0,0)|_{\mathcal{M}_\varepsilon^-}$ is no longer a straight line (at leading order); therefore, many "new" intersections of $T_o^-\cap W^u(-1,0,0,0)|_{\mathcal{M}_\varepsilon^-}$, and thus "new" heteroclinic orbits, may appear.

2.3. The superslow limit: The general case. We now consider the general superslow problem; i.e., $H(U^2,V)$ is a general (smooth) function of U^2 and V in this

section. As in section 2.2 and motivated in Remark 2.4, we choose to consider only the case of $G(V;\varepsilon)$ linear; i.e., $G(v;\varepsilon) = -\varepsilon^2\gamma v$ in (2.1). The treatment of the general superslow case and (2.12) is in essence identical to that of the previous section. However, the statement of the main results cannot be formulated as explicitly as in Theorem 2.3, as long as there is no explicit expression given for $H(U^2, V)$. Nevertheless, the character of the existence result is similar to that of Theorem 2.3; there can be various kinds of heteroclinic orbits that might coalesce in saddle-node bifurcations.

As in the proofs of Theorems 2.1 and 2.3, the existence of the heteroclinic orbits is established by the intersection of T_o^- and $W^u(-1,0,0,0)|_{\mathcal{M}_\varepsilon^-}$, i.e., by the solution v_0 of

$$\sqrt{\gamma}v_0 = \frac{1}{2}\int_{-\infty}^{\infty}\left[1 + v_0 - u_0^2(\xi;v_0)\right] H(u_0^2(\xi;v_0), v_0)d\xi, \tag{2.15}$$

at leading order. Note that the right-hand side equals $-\frac{1}{2}\Delta q(v_0)$, i.e., half the accumulated change in q during a circuit through the fast field, and that we have used (2.3).

THEOREM 2.5. *Assume that $G(V) = -\varepsilon^2\gamma V$ and that ε is small enough. System* (2.1) *has $n \geq 0$ pairs of heteroclinic orbits, $\Gamma_h^{\pm,j}(\xi;\varepsilon) = (\pm u_h^{\pm,j}(\xi), \pm p_h^{\pm,j}(\xi), v_h^{\pm,j}(\xi), q_h^{\pm,j}(\xi))$, where $j = 1,\dots,n$, with $\lim_{\xi\to\pm\infty}\Gamma_h^{+,j}(\xi;\varepsilon) = (\pm 1,0,0,0)$. The number $n = n(\gamma)$ is given by the number of solutions v_j of* (2.15). *In the fast field $u_h^j(\xi)$ (resp., $v_h^j(\xi)$) is asymptotically and uniformly close to $u_0(\xi;v_j)$* (2.3) *(resp., v_j), where the constant v_j is the jth zero of* (2.15). *In the slow field, $\Gamma_h^{+,j}(\xi;\varepsilon)$ is exponentially close to $W^{u,s}(\pm 1,0,0,0)|_{\mathcal{M}_\varepsilon^\pm} \subset \mathcal{M}_\varepsilon^\pm$.*

Two orbits $\Gamma_h^{\pm,j}(\xi;\varepsilon)$ and $\Gamma_h^{\pm,j+1}(\xi;\varepsilon)$ coalesce in a saddle-node bifurcation of heteroclinic orbits at a certain value $\gamma = \gamma_{\text{double}}^j$ if the zeros $v_j \leq v_{j+1}$ of (2.15) *merge, i.e., if the intersection $T_o^- \cap W^u(-1,0,0,0)|_{\mathcal{M}_\varepsilon^-}$ is nontransversal.*

The orbits $\Gamma_h^{\pm,j}(\xi;\varepsilon)$ correspond to the front solutions $(U_h^{\pm,j}(\xi;\varepsilon), V_h^{\pm,j}(\xi;\varepsilon))$ of (1.6) *with $U_h^{\pm,j}(\xi;\varepsilon) = \pm u_h^j(\xi;\varepsilon)$ odd and $V_h^{\pm,j}(\xi;\varepsilon) = v_h^j(\xi;\varepsilon)$ even as functions of ξ.*

The proof of this result is in essence identical to that of Theorem 2.3. In Figure 2.3 two examples of the possible richness of the intersection $T_o^- \cap W^u(-1,0,0,0)|_{\mathcal{M}_\varepsilon^-}$ are given.

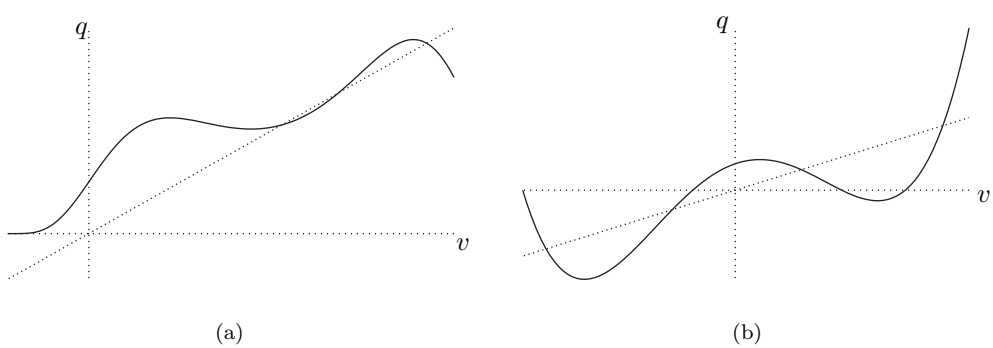


FIG. 2.3. *Two examples of the possible character of the intersection $T_o^- \cap W^u(-1,0,0,0)|_{\mathcal{M}_\varepsilon^-}$ for a given $H(U^2,V)$;* (a) *there are* 3 *different singular heteroclinic orbits and* (b) 4 *heteroclinic orbits.*

3. The stability of fronts.

3.1. The essential spectrum. The essential spectrum associated to the stability of the front patterns $(U,V) = (U_h(\xi;\varepsilon), V_h(\xi;\varepsilon))$ is fully determined by the spectrum of the linear stability problem for the (trivial) background states (at $\pm\infty$) $(U,V) \equiv (\pm 1, 0)$ [11]. Therefore, we introduce $k \in \mathbb{R}$ and $\alpha, \beta, \lambda \in \mathbb{C}$ by

$$U(x,t) = \pm 1 + \alpha e^{ik\xi+\lambda t}, \quad V(x,t) = \beta e^{ik\xi+\lambda t}$$

and substitute this expression into (1.6) (using (1.3)). This yields the matrix equation

$$\begin{pmatrix} -k^2-2 & \pm 1 \\ \mp 2\varepsilon^2 H_0 & -k^2+\varepsilon^2(H_0+G_1) \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \lambda \begin{pmatrix} \alpha \\ \varepsilon^2 \tau \beta \end{pmatrix},$$

where G_1 and H_0 have been introduced in (1.7). Thus, $\lambda = \lambda(k^2)$ is a solution of the characteristic equation

$$Q(\lambda,k) = (\lambda + k^2 + 2)(\varepsilon^2\tau\lambda + k^2 - \varepsilon^2(H_0+G_1)) + 2\varepsilon^2 H_0 = 0. \tag{3.1}$$

Note that this equation holds for both background states $(\pm 1, 0)$, due to the symmetry (1.2). We may conclude the following lemma.

LEMMA 3.1. *The essential spectrum σ_{ess} associated to* (3.3) *is given by the solutions $\lambda = \lambda(k^2)$ of* (3.1) *with $k \in \mathbb{R}$; σ_{ess} is stable, i.e., $\sigma_{\text{ess}} \in \{\text{Re}(\lambda) < 0\}$, if $G_1 < 0$ and $H_0 + G_1 - 2\tau < 0$.*

Proof. The two conditions in this lemma are obtained directly from

$$\begin{array}{rclcl} \lambda_1 + \lambda_2 & = & \frac{1}{\varepsilon^2\tau}\left[\varepsilon^2(H_0+G_1-2\tau) - k^2(1+\varepsilon^2\tau)\right] & < & 0 \quad \forall\, k, \\ \lambda_1\lambda_2 & = & \frac{1}{\varepsilon^2\tau}\left[k^4 + k^2(2-\varepsilon^2(H_0+G_1)) - 2\varepsilon^2 G_1\right] & > & 0 \quad \forall\, k. \end{array} \tag{3.2}$$

Both relations attain their extremal value at $k = 0$. ∎

However, we need to have more information on the essential spectrum than just this stability result. In section 4 we will see that the appearance of edge bifurcations is closely related to the structure of σ_{ess}. We focus on the stable case $G_1 < 0$ and $H_0 + G_1 - 2\tau < 0$. It is straightforward to check that (3.1) has two solutions $\lambda_{1,2}(k) \in \mathbb{R}$ for all $k \in \mathbb{R}$ if $H_0 < 0$. As H_0 passes through zero two k-intervals, $(-k^+, -k^-)$ and (k^-, k^+), $(0 < k^- < k^+)$ appear in which $\lambda_{1,2}(k)$ are complex valued. These intervals merge (i.e., $k^- \downarrow 0$) as H_0 approaches $(\sqrt{2\tau} - \sqrt{-G_1})^2$. For $(\sqrt{2\tau} - \sqrt{-G_1})^2 < H_0 < 2\tau - G_1$ (which is a nonempty region), $\lambda_{1,2}(k) \in \mathbb{C}$ if $-k^+ < k < k^+$. See Figure 3.1.

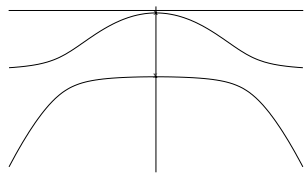
3.2. The linearized stability problem. With a slight abuse of notation we (re-)introduce $u(\xi)$ and $v(\xi)$ by

$$U(\xi,t) = U_h(\xi;\varepsilon) + u(\xi)e^{\lambda t}, \quad V(\xi,t) = V_h(\xi;\varepsilon) + v(\xi)e^{\lambda t},$$

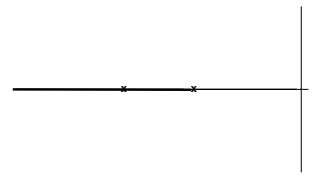
substitute this into (1.6), and linearize

$$\begin{array}{rcl} u_{\xi\xi} & + & (1 + V_h - 3U_h^2 - \lambda)u = -U_h v \\ v_{\xi\xi} & = & \varepsilon^2 \left\{ 2\left[H(U_h^2,V_h) - (1+V_h-U_h^2)\frac{\partial H}{\partial U^2}(U_h^2,V_h)\right] U_h u \right. \\ & & \left. - \left[H(U_h^2,V_h) - (1+V_h-U_h^2)\frac{\partial H}{\partial V}(U_h^2,V_h) + \frac{\partial G}{\partial V}(V_h) - \tau\lambda\right] v \right\}. \end{array} \tag{3.3}$$

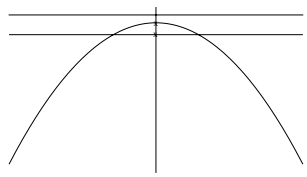
Note that the front pattern $(U_h(\xi), V_h(\xi))$ corresponds to any of the regular or singular heteroclinic orbits $\Gamma_h^{\pm,j}(\xi)$ of Theorems 2.1, 2.3, and 2.5. In the stability analysis of forthcoming sections we will consider only the front patterns of $+$-type, i.e., those



(a) $\mathrm{Re}(\lambda)$ vs. k with $H_0 < 0$.

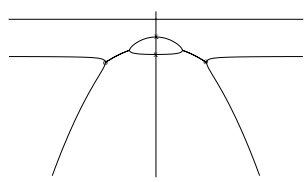


(b) $\mathrm{Re}(\lambda)$ vs. $\mathrm{Im}(\lambda)$ with $H_0 < 0$.

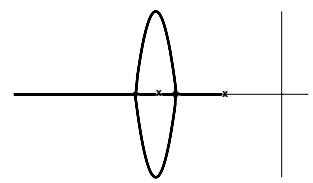


(c) $H_0 = 0$.

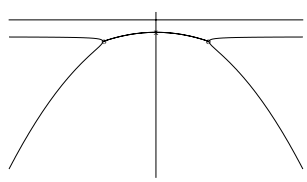
(d) $H_0 = 0$.



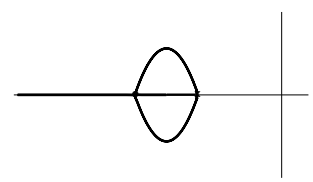
(e) $H_0 \in (0, (\sqrt{2\tau} - \sqrt{-G_1})^2)$.



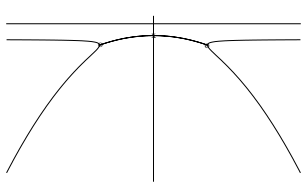
(f) $H_0 \in (0, (\sqrt{2\tau} - \sqrt{-G_1})^2)$.



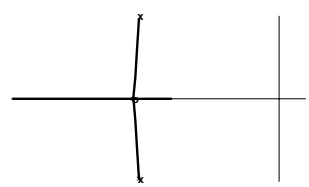
(g) $H_0 = (\sqrt{2\tau} - \sqrt{-G_1})^2$.



(h) $H_0 = (\sqrt{2\tau} - \sqrt{-G_1})^2$.



(i) $H_0 \in ((\sqrt{2\tau} - \sqrt{-G_1})^2, 2\tau - G_1)$.



(j) $H_0 \in ((\sqrt{2\tau} - \sqrt{-G_1})^2, 2\tau - G_1)$.

FIG. 3.1. *The five possible different structures of the stable essential spectrum. On the left we plot* $\mathrm{Re}(\lambda)$ *vs.* k *and on the right* $\mathrm{Re}(\lambda)$ *vs.* $\mathrm{Im}(\lambda)$.

fronts for which $\lim_{\xi\to\pm\infty} U_h(\xi;\varepsilon) = \pm 1$. Thus, we do not explicitly consider their symmetric counterparts. Due to the symmetry (1.2) this is, of course, also not necessary. The coupled system of second order equations (3.3) is equivalent to a linear system in $\mathbb{C}^4$,

$$\phi_\xi = A(\xi;\lambda,\varepsilon)\phi \ \text{ with } \ \phi(\xi) = (u(\xi), p(\xi), v(\xi), q(\xi)), \tag{3.4}$$

where $A(\xi;\lambda,\varepsilon)$ is a 4×4 matrix with $\mathrm{Tr}(A(\xi;\lambda,\varepsilon) \equiv 0$, and $u_\xi = p$, $v_\xi = \varepsilon q$. It follows that

$$\lim_{\xi\to\pm\infty} A(\xi;\lambda,\varepsilon) \stackrel{\text{def}}{=} A_\infty^\pm(\lambda,\varepsilon) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 2+\lambda & 0 & \mp 1 & 0 \\ 0 & 0 & 0 & \varepsilon \\ \pm 2\varepsilon H_0 & 0 & -\varepsilon(H_0+G_1-\tau\lambda) & 0 \end{pmatrix}; \tag{3.5}$$

see (1.7). The matrices $A_\infty^\pm$ have the same set of eigenvalues $\Lambda_i(\lambda,\varepsilon)$, $i = 1,2,3,4$,

$$\Lambda_{1,4}^2(\lambda,\varepsilon) = \lambda + 2 + \mathcal{O}(\varepsilon^2), \quad \Lambda_{2,3}^2(\lambda,\varepsilon) = \varepsilon^2 \frac{\tau\lambda^2 - \lambda(G_1+H_0-2\tau) - 2G_1}{\lambda+2} + \mathcal{O}(\varepsilon^4). \tag{3.6}$$

Note that both expansions break down as λ approaches -2 (see Remark 3.2). We define a branch cut such that for $z \in \mathbb{C}$ $\arg(\sqrt{z}) \in (-\frac{1}{2}\pi, \frac{1}{2}\pi]$ so that the Λ_i's can be ordered

$$\mathrm{Re}(\Lambda_4(\lambda,\varepsilon)) < \mathrm{Re}(\Lambda_3(\lambda,\varepsilon)) < 0 < \mathrm{Re}(\Lambda_2(\lambda,\varepsilon)) < \mathrm{Re}(\Lambda_1(\lambda,\varepsilon)). \tag{3.7}$$

This ordering, of course, breaks down if $\lambda \in \sigma_{\text{ess}}$, the essential spectrum associated to (3.3)/(3.4), since σ_{ess} coincides with values of λ for which either $\mathrm{Re}(\Lambda_{1,4}(\lambda,\varepsilon)) = 0$ or $\mathrm{Re}(\Lambda_{2,3}(\lambda,\varepsilon)) = 0$ [11]; see also section 3.1. The eigenvectors $E_i^\pm(\varepsilon,\lambda)$ of the matrices $A_\infty^\pm(\varepsilon,\lambda)$ associated to $\Lambda_i(\lambda,\varepsilon)$ are given by

$$E_{1,4}^\pm(\varepsilon,\lambda) = \begin{pmatrix} 1 \\ \Lambda_{1,4}(\lambda,\varepsilon) \\ \mathcal{O}(\varepsilon^2) \\ \pm\frac{2H_0}{\Lambda_{1,4}(\lambda,\varepsilon)}\varepsilon + \mathcal{O}(\varepsilon^3) \end{pmatrix}, \quad E_{2,3}^\pm(\varepsilon,\lambda) = \begin{pmatrix} \pm\frac{1}{\lambda+2} + \mathcal{O}(\varepsilon^2) \\ \mathcal{O}(\varepsilon^2) \\ 1 \\ \frac{1}{\varepsilon}\Lambda_{2,3}(\lambda,\varepsilon) \end{pmatrix} \tag{3.8}$$

(for $\lambda + 2 \gg \varepsilon$—see Remark 3.2).

Remark 3.2. The expansions (3.6) and (3.8) are valid only for $\lambda + 2 \gg \varepsilon$. It is straightforward to check that $\Lambda_{1,4}^2(\lambda,\varepsilon) = \mathcal{O}(\varepsilon) = \Lambda_{2,3}^2(\lambda,\varepsilon)$ if $\lambda + 2 = \mathcal{O}(\varepsilon)$ and that, in general, when $\lambda + 2 = \mathcal{O}(\varepsilon^\sigma)$ for some $\sigma \in [0,1]$, $\Lambda_{1,4}^2(\lambda,\varepsilon) = \mathcal{O}(\varepsilon^\sigma)$ and $\Lambda_{2,3}^2(\lambda,\varepsilon) = \mathcal{O}(\varepsilon^{2-\sigma})$. Thus, $\Lambda_{1,4}$ cannot be assumed to be large/fast compared to $\Lambda_{2,3}$ if $\lambda+2 = \mathcal{O}(\varepsilon)$. Since $\lambda = -2+\mathcal{O}(\varepsilon)$ is way into the stable region, we do not consider this degeneration further and assume throughout this paper that $|\Lambda_{2,3}| \ll |\Lambda_{1,4}|$.

3.3. The Evans function. The use of the Evans function in the analysis of linear systems associated to the stability of traveling waves is by now well established. Here, we give a brief exposition of the characteristics of the Evans function in reaction-diffusion systems. We refer the reader to [1, 18, 10, 4, 5] for the full analytic details of the statements in this section.

We define the complement of the essential spectrum by

$$\mathcal{C}_e = \mathbb{C}\backslash\sigma_{\text{ess}}. \tag{3.9}$$

For $\lambda \in \mathcal{C}_e$ the ordering (3.7) holds so that we have the following lemma.

LEMMA 3.3. *For all $\lambda \in \mathcal{C}_e$ there exist two two-dimensional families of solutions $\Phi_-(\xi;\lambda,\varepsilon)$ and $\Phi_+(\xi;\lambda,\varepsilon)$ to* (3.4) *such that* $\lim_{\xi\to\pm\infty}\phi_\pm(\xi;\lambda,\varepsilon) = (0,0,0,0)^t$ *for all* $\phi_\pm(\xi;\lambda,\varepsilon) \in \Phi_\pm(\xi;\lambda,\varepsilon)$*;* $\Phi_\pm(\xi;\lambda,\varepsilon)$ *depend analytically on* λ*.*

An eigenfunction of (3.4) must be in the intersection of $\Phi_-(\xi;\lambda,\varepsilon)$ and $\Phi_+(\xi;\lambda,\varepsilon)$. We define the Evans function $\mathcal{D}(\lambda,\varepsilon)$ by

$$\mathcal{D}(\lambda,\varepsilon) = \det[\phi_1(\xi;\lambda,\varepsilon), \phi_2(\xi;\lambda,\varepsilon), \phi_3(\xi;\lambda,\varepsilon), \phi_4(\xi;\lambda,\varepsilon)], \tag{3.10}$$

where $\{\phi_1,\phi_2\}$ (resp., $\{\phi_3,\phi_4\}$) span the space $\Phi_-(\xi;\lambda,\varepsilon)$ (resp., $\Phi_+(\xi;\lambda,\varepsilon)$). Since $\mathrm{Tr}(A) \equiv 0$, it follows by Abel's theorem that $\mathcal{D}(\lambda,\varepsilon)$ is independent of ξ. Moreover, $\mathcal{D}(\lambda,\varepsilon) = 0$, by construction, at an eigenvalue, since an eigenfunction must be in $\Phi_+(\xi;\lambda,\varepsilon)\cap\Phi_+(\xi;\lambda,\varepsilon)$. The Evans function is analytic in $\lambda \in \mathcal{C}_e$, and its zeros correspond one-to-one with eigenvalues of (3.4), counting multiplicities [1, 18]. Of course, this definition does not determine $\mathcal{D}(\lambda)$ uniquely. However, this can be achieved by choosing $\phi_1(\xi)$ and $\phi_2(\xi)$ as follows.

LEMMA 3.4. *For all $\lambda \in \mathcal{C}_e$ there is a unique solution $\phi_1(\xi;\lambda,\varepsilon) \in \Phi_-(\xi;\lambda,\varepsilon)$ of* (3.4) *such that*

$$\lim_{\xi\to-\infty}\phi_1(\xi;\lambda,\varepsilon)e^{-\Lambda_1(\lambda,\varepsilon)\xi} = E_1^-(\lambda,\varepsilon);$$

see (3.6), (3.8)*. There exists an analytic transmission function $t_1(\lambda,\varepsilon)$ such that*

$$\lim_{\xi\to\infty}\phi_1(\xi;\lambda,\varepsilon)e^{-\Lambda_1(\lambda,\varepsilon)\xi} = t_1(\lambda,\varepsilon)E_1^+(\lambda,\varepsilon).$$

For $\lambda \in \mathcal{C}_e$ such that $t_1(\lambda,\varepsilon) \neq 0$ there is a unique solution $\phi_2(\xi;\lambda,\varepsilon) \in \Phi_-(\xi;\lambda,\varepsilon)$ of (3.4)*, which is independent of $\phi_1(\xi;\lambda,\varepsilon)$, that satisfies*

$$\lim_{\xi\to-\infty}\phi_2(\xi;\lambda,\varepsilon)e^{-\Lambda_2(\lambda,\varepsilon)\xi} = E_2^-(\lambda,\varepsilon) \quad\text{and}\quad \lim_{\xi\to\infty}\phi_2(\xi;\lambda,\varepsilon)e^{-\Lambda_1(\lambda,\varepsilon)\xi} = (0,0,0,0)^t.$$

There exists a second meromorphic transmission function $t_2(\lambda,\varepsilon)$ that is determined by

$$\lim_{\xi\to\infty}\phi_2(\xi;\lambda,\varepsilon)e^{-\Lambda_2(\lambda,\varepsilon)\xi} = t_2(\lambda,\varepsilon)E_2^+(\lambda,\varepsilon).$$

The solutions $\phi_{3,4}(\xi;\lambda,\varepsilon) \in \Phi_+(\xi;\lambda,\varepsilon)$ of (3.4) can be defined likewise. Since $\sum_{i=1}^4 \Lambda_i(\lambda,\varepsilon) \equiv 0$ (3.6),

$$\mathcal{D}(\lambda,\varepsilon) = \det[\phi_1(\xi)e^{-\Lambda_1\xi}, \phi_2(\xi)e^{-\Lambda_2\xi}, \phi_3(\xi)e^{-\Lambda_3\xi}, \phi_4(\xi)e^{-\Lambda_4\xi}]$$

so that $\mathcal{D}(\lambda,\varepsilon)$ can be decomposed into a product of $t_1(\lambda,\varepsilon)$ and $t_2(\lambda,\varepsilon)$ by taking the limit $\xi \to +\infty$.

LEMMA 3.5. *Let $\lambda \in \mathcal{C}_e$; then*

$$\mathcal{D}(\lambda,\varepsilon) = t_1(\lambda,\varepsilon)t_2(\lambda,\varepsilon)\det\left[E_1^+(\lambda,\varepsilon), E_2^+(\lambda,\varepsilon), E_3^+(\lambda,\varepsilon), E_4^+(\lambda,\varepsilon)\right]. \tag{3.11}$$

We conclude that the eigenvalues of (3.4) correspond to zeros of the transmission functions $t_1(\lambda,\varepsilon)$ and $t_2(\lambda,\varepsilon)$. However, we will see that a zero of $t_1(\lambda,\varepsilon)$ does not necessarily correspond to a zero of $\mathcal{D}(\lambda,\varepsilon)$, since $t_2(\lambda,\varepsilon)$ can have poles (see section 4.1 and [4, 5]).

3.4. The fast eigenvalues. The next section will be devoted to the analysis of (the zeros of) $t_2(\lambda,\varepsilon)$; here we consider the zeros of the fast transmission function $t_1(\lambda,\varepsilon)$. In order to do so, we first consider the stability problem associated to the front solution $U_f(\xi;V_0)$, with $U_f(\xi;V_0)\to\pm\sqrt{1+V_0}$ as $\xi\to\pm\infty$, of the scalar fast reduced limit problem (1.5),

$$w_{\xi\xi}+(1+V_0-3u_0^2(\xi;V_0)-\lambda)w=0, \tag{3.12}$$

since $U_f(\xi;V_0)=u_0(\xi;V_0)$ (2.3). This system can be written as a linear system in $\mathbb{C}^2$,

$$\psi_\xi=B(\xi;\lambda)\psi \ \text{ with } \ \psi(\xi)=(u(\xi),p(\xi)), \tag{3.13}$$

where $B(\xi;\lambda)$ is a 2×2 matrix of which the coefficients are by construction $\mathcal{O}(\varepsilon)$ close (uniformly in ξ) to those of the 2×2 block in the upper left corner of the 4×4 matrix $A^{\pm}(\xi;\lambda,\varepsilon)$ defined in (3.4), if we set $V_0=V_h(0)$. The Evans function associated to this problem can be written as $\mathcal{D}_f(\lambda)=\det[\psi_1(\xi,\lambda),\psi_4(\xi,\lambda)]$, in which $\psi_1(\xi)$ and $\psi_4(\xi)$ are solutions of (3.4) determined by $\lim_{\xi\to-\infty}\psi_1(\xi)e^{-\sqrt{\lambda+2}\xi}=(1,\sqrt{\lambda+2})^t$ and $\lim_{\xi\to\infty}\psi_4(\xi)e^{\sqrt{\lambda+2}\xi}=(1,-\sqrt{\lambda+2})^t$ (where $\pm\sqrt{\lambda+2}$ and $(1,\pm\sqrt{\lambda+2})^t$ are the eigenvalues and eigenvectors of the matrix $B_\infty(\lambda)=\lim_{\xi\to\pm\infty}B(\xi;\lambda)$ (compare to (3.6), (3.8))). As for the full system, we can define an analytic fast reduced transmission function $t_f(\lambda)$ by $\lim_{\xi\to\infty}\psi_1(\xi)e^{-\sqrt{\lambda+2}\xi}=t_f(\lambda)(1,\sqrt{\lambda+2})^t$ so that

$$\mathcal{D}_f(\lambda)=\lim_{\xi\to\infty}\det[\psi_1(\xi),\psi_4(\xi)]=\det[t_f(\lambda)(1,\sqrt{\lambda+2})^t,(1,-\sqrt{\lambda+2})^t]=-2t_f(\lambda)\sqrt{\lambda+2}.$$

The transmission function $t_1(\lambda)$ is, by construction, asymptotically close to its fast reduced limit $t_f(\lambda)$.

LEMMA 3.6. *For all $\lambda_i^f\in\mathcal{C}_e$ such that $t_f(\lambda_i^f)=0$, there is a uniquely determined $\lambda_i(\varepsilon)$ with $\lim_{\varepsilon\to 0}\lambda_i(\varepsilon)=\lambda_i^f$ such that $t_1(\lambda_i(\varepsilon),\varepsilon)=0$; $t_1(\lambda,\varepsilon)\neq 0$ for $\lambda\neq\lambda_i(\varepsilon)$.*

The (quite technical) proof of this lemma is analogous to the proofs of similar statements in [1, 10, 4, 5] and is based on the "elephant trunk" procedure [1, 10]—see especially the proof of Corollary 3.9 (and thus of Theorem 3.7) in [4] for a complete and detailed analysis.

Thus, by Lemma 3.6, we can find (the leading order behavior of) the zeros of $t_1(\lambda,\varepsilon)$ by computing the spectrum of (3.12). By (2.3) and by introducing $\eta=\sqrt{\frac{1}{2}(1+V_0)}$ we can write (3.12) as

$$w_{\eta\eta}+\left(\frac{6}{\cosh^2\eta}-P^2\right)w=0 \ \text{ with } \ P^2=\frac{2\lambda}{1+V_0}+4,$$

which is a well-studied problem of Schrödinger/Sturm–Liouville type (see, for instance, [21, 5]). It has discrete spectrum at $P=1$ and $P=4$ and essential spectrum for $P\in i\mathbb{R}$. We conclude that the eigenvalues of (3.12), and thus the leading order approximations of the zeros of $t_1(\lambda)$, are given by

$$\lambda_1^f=0,\quad \lambda_2^f=-\frac{3}{2}(1+V_0)<0. \tag{3.14}$$

The essential spectrum of (3.12) is given by

$$\sigma_{\text{ess}}^f=\{\lambda\le -2(1+V_0)\}. \tag{3.15}$$

We conclude this subsection by stating two simple, but useful results.

LEMMA 3.7. *Let $(u(\xi;\varepsilon), v(\xi;\varepsilon))$ be a pair of eigenfunction solutions of* (3.3) *associated to a simple eigenvalue $\lambda(\varepsilon)$; then either $u(\xi)$ is even as a function of ξ and $v(\xi)$ odd, or $u(\xi)$ is odd and $v(\xi)$ even.*

Proof. We write (3.3) in the following way:

$$v_{\xi\xi} = \varepsilon^2[F_o(\xi)u + F_e(\xi)v]\,. \tag{3.16}$$

By construction, U_h is an odd function of ξ and V_h is an even function of ξ. It thus follows that the above functions, F_o and F_e, must be odd and even functions of ξ, respectively. Let (u, v) be an eigenfunction associated to the eigenvalue λ. We decompose (u, v) into odd and even components, $u = u_o + u_e$, $v = v_o + v_e$, where u_o, v_o are odd and u_e, v_e are even. By the parity of the functions U_h, V_h, F_o, and F_e it is clear that (u_o, v_e) and (u_e, v_o) form two independent solutions of the eigenvalue problem associated to the eigenvalue λ. Since we have assumed that λ is simple, we have a contradiction. ∎

LEMMA 3.8. *Assume that the eigenfunction solution $v(\xi)$ of* (3.3) *with eigenvalue $\lambda(\varepsilon)$ is odd; then $\lambda(\varepsilon) \equiv 0$ so that $(u(\xi), v(\xi)) = (U_{h,\xi}(\xi;\varepsilon), V_{h,\xi}(\xi;\varepsilon))$.*

We will see in section 4 that there can be several eigenvalues for which $u(\xi)$ is odd and $v(\xi)$ even.

Proof. It is clear that there is an eigenvalue $\lambda = 0$ associated to the derivative of the front $(u(\xi), v(\xi)) = (U_{h,\xi}(\xi;\varepsilon), V_{h,\xi}(\xi;\varepsilon))$. We assume there is another eigenfunction with v odd. Since $v_{\xi\xi}$ is $\mathcal{O}(\varepsilon^2)$ and v is odd, it follows that $|v| \ll 1$ on the fast spatial scale. Hence, the equation for the u-component is to leading order homogeneous and given by (3.12) (with w replaced by u). Lemma 3.7 implies that u is even. Since the only even eigenfunction of (3.12) is $U_{h,\xi}$ with eigenvalue 0, it follows that the leading order behavior of u is given by $U_{h,\xi}$ and that λ is asymptotically close to 0. We thus write

$$u = U_{h,\xi} + \delta(\varepsilon)u_1\,, \quad v = V_{h,\xi} + \delta(\varepsilon)v_1\,, \quad \lambda = \delta(\varepsilon)\hat{\lambda}(\epsilon)\,, \tag{3.17}$$

where $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$ and $\hat{\lambda}(0) \neq 0$ (i.e., $\hat{\lambda}(\varepsilon) = \mathcal{O}(1)$, $\delta(\varepsilon)$ represents the leading order magnitude of λ). We substitute (3.17) into (3.3) to get the following equation for u_1:

$$u_{1,\xi\xi} + (1 - 3U_h^2)u_1 = \hat{\lambda}U_{h,\xi} - U_h v_1\,.$$

Note that this equation implies that u_1 is $\mathcal{O}(1)$, i.e., that the scaling chosen for $u - U_{h,\xi}$ in (3.17) is indeed the correct one. This equation has the solvability condition, $\int_{-\infty}^{\infty}(\hat{\lambda}U_{h,\xi} - U_h v_1)U_{h,\xi}\,dy = 0$. Now, since also $v_{1,\xi\xi}$ is $\mathcal{O}(\varepsilon^2)$ (3.3) and odd, it again follows that $|v_1| \ll 1$ on the fast spatial scale. Thus, we observe by the fast exponential decay of $U_h(\xi)$ that $\int_{-\infty}^{\infty} U_h U_{h,\xi} v_1\,dy \to 0$ as $\varepsilon \to 0$. Hence, we conclude from the solvability condition that $\lim_{\varepsilon\to 0}\hat{\lambda}(\varepsilon) = 0$, which contradicts the assumption that $\hat{\lambda}(\varepsilon) = \mathcal{O}(1)$. So the only possible eigenfunctions with v odd must correspond to a 0 eigenvalue, and hence $(u, v) = (U_{h,\xi}, V_{h,\xi})$. ∎

4. Slow-fast eigenvalues and edge bifurcations. The "slow-fast eigenvalues" are the eigenvalues that exist due to the interaction of the fast U-equation and the slow V-equation in (1.6); thus, these eigenvalues do not have a counterpart in the fast reduced scalar limit problem (1.5). The slow-fast eigenvalues correspond to the zeros of the $t_2(\lambda;\varepsilon)$, since this transmission function is based on a balance between slow and fast effects. See also Remark 4.5.

In order to study the combined effect of slow and fast terms, we need to define the region in which the fast ξ-jump takes place more accurately:

$$I_f = \left\{ \xi \in \left(-\frac{1}{\sqrt{\varepsilon}}, \frac{1}{\sqrt{\varepsilon}} \right) \right\} \text{ or } \{x \in (-\sqrt{\varepsilon}, \sqrt{\varepsilon})\}; \tag{4.1}$$

see (1.3). Note that the exact choice of the boundaries of I_f is not relevant; any choice will be suitable as long as it is in the transition zone between x and ξ (i.e., on the boundary of I_f we must have $|x| \ll 1$ and $|\xi| \gg 1$).

4.1. The regular case. Again, we first consider the case $G_1 = \mathcal{O}(1)$ (1.7). In the slow coordinate x (1.3), i.e., outside the region I_f, the equation for u reads

$$(1 - 3U_h^2 - \lambda + \mathcal{O}(\varepsilon))u = -U_h v + \mathcal{O}(\varepsilon^2 u_{xx}) \tag{4.2}$$

(see (3.3)), since $V_h(\xi) = \mathcal{O}(\varepsilon)$ on $\mathbb{R}$ (Theorem 2.1). Thus, u can be expressed in terms of v outside the fast ξ-region I_f (4.1). Using that $U_h^2(\xi;\varepsilon) = 1 + \mathcal{O}(\varepsilon)$ outside I_f (Theorem 2.1), we find for the v-equation of (3.3) on the slow x-scale

$$\begin{aligned} v_{xx} &= \left[2H(1,0)U_h + \mathcal{O}(\varepsilon)\right] u - \left[H(1,0) + \tfrac{\partial G}{\partial V}(0) - \tau\lambda + \mathcal{O}(\varepsilon)\right] v \\ &= \left[\tfrac{2H_0}{\lambda+2} - H_0 - G_1 + \tau\lambda + \mathcal{O}(\varepsilon)\right] v + \mathcal{O}(\varepsilon^2 v_{xx}); \end{aligned}$$

see (1.7). Hence, outside I_f,

$$v_{xx} = \left[\frac{-H_0\lambda + \lambda(\lambda+2)\tau - G_1(\lambda+2)}{\lambda+2} + \mathcal{O}(\varepsilon) \right] v, \tag{4.3}$$

uniformly in ξ. The v-equation is thus at leading order of constant coefficients type. By (3.1) and (3.6) we have on the ξ-scale

$$v_{\xi\xi} = \left[\frac{Q(\lambda;0)}{\lambda+2} + \mathcal{O}(\varepsilon^3) \right], \quad v = \left[\Lambda_{2,3}^2(\lambda,\varepsilon) + \mathcal{O}(\varepsilon^3)\right] v. \tag{4.4}$$

In order to determine an expression for $t_2(\lambda,\varepsilon)$, we need to control the solution $\phi_2(\xi;\lambda,\varepsilon)$ (Lemma 3.4) of (3.4). This is done in the following lemma.

LEMMA 4.1. *For all $\lambda \in \mathcal{C}_e$ such that $t_1(\lambda,\varepsilon) \neq 0$ there exist $\mathcal{O}(1)$ constants $C_-, C_+ > 0$ and a third meromorphic transmission function $t_3(\lambda,\varepsilon)$ such that*

$$\phi_2(\xi;\lambda,\varepsilon) = \begin{cases} \left[E_2^-(\lambda) + \mathcal{O}(\varepsilon)\right] e^{\Lambda_2(\lambda)\xi} + \mathcal{O}(e^{C_-\xi}) & \text{for} \quad \xi < -\frac{1}{\sqrt{\varepsilon}}, \\ t_2(\lambda)E_2^+(\lambda)e^{\Lambda_2(\lambda)\xi} + t_3(\lambda)E_3^+(\lambda)e^{\Lambda_3(\lambda)\xi} + \mathcal{O}(e^{-C_+\xi}) & \text{for} \quad \xi > \frac{1}{\sqrt{\varepsilon}}. \end{cases} \tag{4.5}$$

Moreover, there exists an $\mathcal{O}(1)$ constant C_f such that $||\phi_2(\xi)|| \leq C_f$ for $\xi \in I_f$. The v-coordinate of $\phi_2(\xi)$ satisfies $v(\xi) = 1 + \mathcal{O}(\sqrt{\varepsilon})$ on I_f so that

$$t_2(\lambda,\varepsilon) + t_3(\lambda,\varepsilon) = 1 + \mathcal{O}(\sqrt{\varepsilon}). \tag{4.6}$$

Proof. It follows from the above analysis that the (leading order) behavior of $\phi_2(\xi)$ outside I_f is determined by equations with constant coefficients. In other words, outside I_f, the matrix $A(\xi;\lambda,\varepsilon)$ of (3.4) can be approximated the constant coefficients matrix $A_\infty^\pm(\lambda,\varepsilon)$ of (3.5). Thus, the approximation (4.5) for $\xi < -1/\sqrt{\varepsilon}$ follows from the definition of $\phi_2(\xi)$ (i.e., the (boundary) conditions on $\phi_2(\xi)$ as $\xi \to -\infty$; see

Lemma 3.4). This same lemma establishes the leading order term in (4.5) for $\xi \to \infty$. The transmission function $t_3(\lambda, \varepsilon)$ measures the component of $\phi_2(\xi)$ that decays on the slow spatial scale x. Inside I_f, $v_{\xi\xi} = \mathcal{O}(\varepsilon^2)$ (3.3) and $\Lambda^2_{2,3}(\lambda, \varepsilon) = \mathcal{O}(\varepsilon^2)$ (3.6) so that (4.6) follows. As in section 3.3 we refrain from giving the full analytic details of this result, since these are essentially the same as in [10, 4, 5]. $\square$

The transmission function $t_2(\lambda, \varepsilon)$ can be determined by the methods originally developed in [3]. We deduce from Lemma 4.1 and (4.4) that the total change in v_ξ over I_f is given by

$$\Delta_{\text{slow}} v_\xi = 2\varepsilon(t_2(\lambda) - 1)\sqrt{\frac{\tilde{Q}(\lambda; 0)}{\lambda + 2}} + \mathcal{O}(\varepsilon\sqrt{\varepsilon}), \tag{4.7}$$

where $\tilde{Q}(\lambda; 0) = \mathcal{O}(1)$ is defined by $Q(\lambda; 0) = \varepsilon\tilde{Q}(\lambda; 0)$ (3.1). This change in v_ξ must be an effect of the evolution on the fast ξ-scale, that is, given by

$$\Delta_{\text{fast}} v_\xi = \int_{-\frac{1}{\sqrt{\varepsilon}}}^{\frac{1}{\sqrt{\varepsilon}}} v_{\xi\xi}|_{\{u=u_{\text{in}}, v=1\}} d\xi + \mathcal{O}(\varepsilon^2\sqrt{\varepsilon}), \tag{4.8}$$

where $u_{\text{in}}(\xi; \lambda)$ is a bounded solution of the inhomogeneous problem

$$u_{\xi\xi} + (1 - 3U_h^2(\xi; 0) - \lambda)u = -U_h(\xi; 0) \tag{4.9}$$

(recall that $v(\xi) = 1 + \mathcal{O}(\sqrt{\varepsilon})$ in I_f). The transmission function $t_2(\lambda; \varepsilon)$ is determined by combining (4.7) and (4.8). Since, a priori $\Delta_{\text{slow}} v_\xi = \mathcal{O}(\varepsilon)$ and $\Delta_{\text{fast}} v_\xi = \mathcal{O}(\varepsilon\sqrt{\varepsilon})$ we are led to the conclusion that $t_2(\lambda) - 1$ must be $\mathcal{O}(\sqrt{\varepsilon})$ for λ not close to a zero of $\Delta_{\text{slow}} v_\xi$ or a singularity of $\Delta_{\text{fast}} v_\xi$.

LEMMA 4.2. *Consider $\lambda \in \mathcal{C}_e \cap \{\text{Re}(\lambda) > -2 + \delta\}$ for some $\delta > 0$ independent of ε. Let $\lambda_2^f = -\frac{3}{2}$ be the second eigenvalue of the limit system* (3.12) *with $V_0 = 0$* (3.14), *and let $\lambda^+(0)$ and $\lambda^-(0)$ be the solutions of $Q(\lambda, 0) = 0$* (3.1). *Then $t_2(\lambda) = 1 + \mathcal{O}(\sqrt{\varepsilon})$ if $|\lambda - \lambda_2^f|, |\lambda - \lambda^+(0)|, |\lambda - \lambda^-(0)| = \mathcal{O}(1)$; $t_2(\lambda) = 1 + \mathcal{O}(\varepsilon^{\frac{1}{2}-\sigma})$ if $|\lambda - \lambda_2^f| = \mathcal{O}(\varepsilon^\sigma)$, $|\lambda - \lambda^+(0)| = \mathcal{O}(\varepsilon^{2\sigma})$, or $|\lambda - \lambda^-(0)| = \mathcal{O}(\varepsilon^{2\sigma})$ for some $\sigma \in (0, \frac{1}{2})$.*

Thus, this lemma establishes that $t_2(\lambda, \varepsilon)$ can only be zero in $\{\text{Re}(\lambda) > -2\}$ if $\lambda \in \mathcal{C}_e$ is $\mathcal{O}(\sqrt{\varepsilon})$ close to λ_2^f or $\mathcal{O}(\varepsilon)$ close to $\lambda^+(0)$ or $\lambda^-(0)$, so we have to study only λ near these three points to determine the slow-fast eigenvalues of (3.4). Note that the fast reduced (scalar) limit problem has an eigenvalue $\lambda_2^f = -\frac{3}{2}$ ((3.14), since $V_0 = V_h(0) \to 0$ as $\varepsilon \to 0$ (Theorem 2.1)). We will prove below that $t_2(\lambda)$ has a (simple) zero close to λ_2^f, i.e., that the fast reduced eigenvalue λ_2^f persists.

However, before going further into the details of the (possible) existence of eigenvalues near λ_2^f, $\lambda^+(0)$ or $\lambda^-(0)$, we formulate a result that is an immediate consequence of Lemma 4.2 and that establishes the stability of the wave for values of G_1 and H_0 such that the essential spectrum, and hence $\lambda^+(0)$ and $\lambda^-(0)$, is in the negative half-plane and not too close to the imaginary axis (see Lemma 3.1).

THEOREM 4.3. *Let $\varepsilon > 0$ be small enough, and let $G_1 < 0$ and $H_0 + G_1 - 2\tau < 0$ be such that $|G_1|, |H_0 + G_1 - 2\tau| \gg \varepsilon$. The spectrum of the eigenvalue problem* (3.3) *associated to the stability of the solution $(U_h(\xi; \varepsilon), V_h(\xi; \varepsilon))$ consists of a (simple) eigenvalue at $\lambda = 0$ and a part that is embedded in the region $\{\text{Re}(\lambda) < -\varepsilon\}$. Therefore, $(U_h(\xi; \varepsilon), V_h(\xi; \varepsilon))$ is (spectrally) stable.*

Note that the operator defined by (3.3) is clearly sectorial in this case (see section 3.1) so that the nonlinear (orbital) stability of $(U_h(\xi; \varepsilon), V_h(\xi; \varepsilon))$ follows by standard arguments (see, for instance, [11]).

Proof of Lemma 4.2. We first note that indeed $\Delta_{\text{fast}} v_\xi = \mathcal{O}(\varepsilon\sqrt{\varepsilon})$ and $\Delta_{\text{slow}} v_\xi = \mathcal{O}(\varepsilon)$, and thus $t_2(\lambda) = 1 + \mathcal{O}(\sqrt{\varepsilon})$, for $\lambda \in \mathcal{C}_e$ that are not asymptotically close to the possible degenerations of (4.8) and (4.7).

The solution $u_{\text{in}}(\xi;\lambda)$ of the inhomogeneous problem (4.9) may become unbounded as λ approaches an eigenvalue, $\lambda_1^f = 0$ or $\lambda_2^f = -\frac{3}{2}$, or the essential spectrum σ_{ess}^f (3.15) of the linear problem associated to the fast reduced limit (3.12) with $V_0 = 0$ (i.e., the homogeneous part of (4.9)). To avoid irrelevant technicalities near σ_{ess}^f we assume that $\lambda \in \mathcal{C}_e \cap \{\text{Re}(\lambda) > -2 + \delta\}$. The eigenfunction associated to λ_1^f, i.e., $U_{h,\xi}(\xi;0)$, is odd, which implies that the inhomogeneous (and even) term $U_h(\xi;0)$ satisfies the solvability condition associated to (4.9) at $\lambda = 0$. Hence, $u_{\text{in}}(\xi;\lambda)$ remains bounded as $\lambda \to 0$ so that $t_2(\lambda) = 1 + \mathcal{O}(\sqrt{\varepsilon})$ also near $\lambda = 0$ [4, 5]. The eigenfunction associated to the second eigenvalue of the homogeneous part of equation (4.9), λ_2^f, is even; thus, the solution $u_{\text{in}}(\xi;\lambda)$ of (4.9) grows as $1/(\lambda_2^f - \lambda)$ as $\lambda \to \lambda_2^f$ [21, 4, 5]. Since u_{in} appears in $\Delta_{\text{fast}} v_\xi$ (4.8), we conclude that $t_2(\lambda,\varepsilon) - 1 = \mathcal{O}(\varepsilon^{\frac{1}{2}-\sigma})$ if $|\lambda - \lambda_2^f| = \mathcal{O}(\varepsilon^\sigma)$ for some $\sigma \in (0, \frac{1}{2})$.

The behavior of $t_2(\lambda)$ near the degenerations of (4.7), i.e., the zeros $\lambda^\pm(0)$ of $Q(\lambda;0)$, follows from observing that $\Delta_{\text{slow}} v_\xi = (t_2 - 1) \times \mathcal{O}(\varepsilon^{1+\sigma})$ if λ is $\mathcal{O}(\varepsilon^{2\sigma})$ close to $\lambda^+(0)$ or to $\lambda^-(0)$ for some $\sigma \in (0, \frac{1}{2})$. ∎

Although it is not essential for the forthcoming analysis, we note that we may also conclude from the proof of this lemma that $t_2(\lambda)$ indeed has a (simple) pole that approaches λ_2^f as $\varepsilon \to 0$. This pole is generated by the solution $u_{\text{in}}(\xi;\lambda)$ of the inhomogeneous problem (4.9) that appears in the expression for $\Delta_{\text{fast}} v_\xi$ (4.8). This solution necessarily develops a singularity near λ_2^f, an eigenvalue of the homogeneous part of (4.9). Since the Evans function $\mathcal{D}(\lambda)$ is analytic as function of λ, it follows that $t_2(\lambda)$ can only have its pole exactly at the zero $\lambda_2(\varepsilon)$ of $t_1(\lambda)$ (that is also asymptotic to λ_2^f—see Lemma 3.6). Note that this is fully consistent with Lemma 3.4, in which the existence of $t_2(\lambda)$ can only be proved for λ such that $t_1(\lambda) \neq 0$. Nevertheless, it can be shown, by a (standard) winding number argument [1, 4, 5], that the eigenvalue λ_2^f persists as an eigenvalue of the full system (3.3) if it is not embedded in the essential spectrum.

LEMMA 4.4. *Let G_1 and H_0 be such that σ_{ess} does not intersect an $\mathcal{O}(\varepsilon^\sigma)$ neighborhood of λ_2^f for some $\sigma < \frac{1}{2}$. Then there is an eigenvalue $\lambda_2(\varepsilon)$ of* (3.3) *with* $\lim_{\varepsilon\to 0} \lambda_2(\varepsilon) = \lambda_2^f = -\frac{3}{2}$.

Proof. By the assumptions in the lemma, there exists a contour K in the complex λ-plane, which does not intersect σ_{ess}, that encircles an $\mathcal{O}(\varepsilon^\sigma)$ neighborhood of λ_2^f and that is $\mathcal{O}(\varepsilon^\sigma)$ close to λ_2^f. It follows from Lemma 4.2 that $t_2(\lambda) = 1 + \mathcal{O}(\varepsilon^{\frac{1}{2}-\sigma})$ for $\lambda \in K$; thus, the winding number of $t_2(\lambda)$ over K is 0. However, $t_2(\lambda)$ must have a (simple) pole in the interior of K, as observed above. We conclude that $t_2(\lambda)$ must also have a (simple, real) zero in the interior of K. ∎

The possible existence of slow-fast eigenvalues near $\lambda^+(0)$ or $\lambda^-(0)$ is much more subtle. Since such eigenvalues become relevant only to the stability of the solution $(U_h(\xi;\varepsilon), V_h(\xi;\varepsilon))$ as G_1 (or $H_0 + G_1 - 2\tau$) approaches 0 (Theorem 4.3) we will consider this issue in forthcoming sections.

Remark 4.5. The eigenvalues $\lambda_1(\varepsilon) = 0$ and $\lambda_2(\varepsilon) \to -\frac{3}{2}$ as $\varepsilon \to 0$ can be interpreted as "fast" eigenvalues, since they correspond to eigenvalues of the fast reduced limit problem. However, strictly speaking, both eigenvalues also have the slow-fast structure described in the beginning of this section.

First, we of course know that $\lambda_1(\varepsilon) = 0$ is an eigenvalue—see also Lemma 3.8. Thus, it is a zero of $\mathcal{D}(\lambda,\varepsilon)$. Since $t_2(\lambda) = 1 + \mathcal{O}(\sqrt{\varepsilon})$ for λ near 0 (see the proof

of Lemma 4.2), we conclude that $t_1(0;\varepsilon) \equiv 0$ (note that this (in a sense) obvious result does not follow directly from Lemma 3.6). Thus, the solution $\phi_1(\xi;0,\varepsilon)$ of (3.4) that by construction has a purely fast structure for $\xi \ll -1$ does not blow up as $e^{\Lambda_1(0,\varepsilon)\xi}$ as $\xi \to \infty$ (Lemma 3.4). Nevertheless, the eigenfunction associated to $\lambda = 0$, $(U_{h,\xi}(\xi), V_{h,\xi}(\xi))$, has a clear slow-fast structure that it inherits from $(U_h(\xi), V_h(\xi))$ (Theorem 2.1). Hence, $\phi_1(\xi;0,\varepsilon)$ is not the eigenfunction associated to $\lambda = 0$. Neither is $\phi_2(\xi;0,\varepsilon)$, since $t_2(0) \neq 0$. It follows that the eigenfunction associated to $\lambda = 0$ must be a linear combination of $\phi_1(\xi;0,\varepsilon)$ and $\phi_2(\xi;0,\varepsilon)$, and thus that $\phi_1(\xi;0,\varepsilon)$ does not decay as $\xi \to \infty$, but instead grows exponentially (and slowly), as $e^{\Lambda_2(0,\varepsilon)\xi}$ (like $\phi_2(\xi;0,\varepsilon)$). The linear combination is such that the two growth terms $e^{\Lambda_2(0,\varepsilon)\xi}$ (for $\xi \to \infty$) cancel.

Second, $\lambda_2(\varepsilon)$ is not a zero of $t_1(\lambda)$, although it is asymptotically close to such a zero, but it is a zero of $t_2(\lambda)$. Thus, $\phi_2(\xi;\lambda_2(\varepsilon),\varepsilon)$ is the eigenfunction of (3.4) at $\lambda = \lambda_2(\varepsilon)$ (and $\phi_1(\xi;\lambda_2(\varepsilon),\varepsilon)$ blows up fast, as $e^{\Lambda_1(\lambda_2(\varepsilon),\varepsilon)\xi}$).

4.2. The superslow case: An example. In the previous section we have seen that the front might destabilize as G_1 approaches 0 (if we assume that $H_0 + G_1 - 2\tau < 0$). In this case, Theorem 2.1 can no longer be used to establish the existence of the front $(U_h(\xi), V_h(\xi))$. Thus, the question about the stability of the front is closely related to the characteristics of the existence problem (as is usual in the analysis of (traveling) waves; see also [14]). In this section we consider the bifurcation as G_1 approaches 0. Therefore, we assume that $H_0 - 2\tau < 0$ and $\mathcal{O}(1)$ with respect to ε. As in section 2 we consider in the superslow case the simplified system in which the general function $G(V)$ is replaced by a linear expression: $G(V) = G_1 V = -\varepsilon^2\gamma V$ (see Remark 2.4). Note that Theorem 4.3 a priori predicts a possible destabilization as G_1 becomes $\mathcal{O}(\varepsilon)$, i.e., already before $G_1 = -\gamma\varepsilon^2$, but it will be shown in the next section that the estimate in Theorem 4.3 is not sharp, in the sense that a bifurcation occurs only as G_1 decreases to $\mathcal{O}(\varepsilon^2)$.

One of the main differences between the analysis in this section and that of the regular case is the fact that $V_h(\xi)$ is no longer $\mathcal{O}(\varepsilon)$; i.e., $V_h(\xi)$ does not contribute only to the higher order terms in the stability analysis of the front solutions. Nevertheless, we follow the approach of the previous section and express the solution u of (3.3) in terms of v, outside I_f (see (4.2)):

$$u = -\frac{U_h}{1 + V_h - 3U_h^2 - \lambda} v + \mathcal{O}(\varepsilon^2 u_{xx}) = \left[\frac{U_h}{2(1+V_h)+\lambda} + \mathcal{O}(\varepsilon^4)\right] v + \mathcal{O}(\varepsilon^2 v_{xx}) \tag{4.10}$$

since $1 + V_h(\xi;\varepsilon) - U_h^2(\xi;\varepsilon) = \mathcal{O}(\varepsilon^4)$ (see (2.8); recall that q^2 and G are $\mathcal{O}(\varepsilon^2)$ in the superslow case). This yields that

$$\begin{aligned} v_{xx} &= \left\{2\left[H(U_h^2, V_h) + \mathcal{O}(\varepsilon^4)\right] U_h u - \left[H(U_h^2, V_h) + \mathcal{O}(\varepsilon^4) - \varepsilon^2\gamma - \tau\lambda\right] v\right\} \\ &= \left\{\tfrac{2H(U_h^2,V_h)U_h^2}{2(1+V_h)+\lambda} - H(U_h^2, V_h) + \tau\lambda + \varepsilon^2\gamma + \mathcal{O}(\varepsilon^4)\right\} v + \mathcal{O}(\varepsilon^2 v_{xx}) \\ &= \left\{\lambda\left[\tau - \tfrac{H(1+V_h,V_h)}{2(1+V_h)+\lambda}\right] + \varepsilon^2\gamma + \mathcal{O}(\varepsilon^4)\right\} v + \mathcal{O}(\varepsilon^2 v_{xx}). \end{aligned} \tag{4.11}$$

It follows from section 3.1 that one of the "tips" of σ_{ess}, $\lambda^+(0)$, is $\mathcal{O}(\varepsilon^2)$ if $G_1 = \mathcal{O}(\varepsilon^2)$ (and $H_0 - 2\tau < 0$), while the other one, $\lambda^-(0)$, is $\mathcal{O}(1)$ and negative (3.2). Thus, the destabilization of the front will be caused by either σ_{ess} at $G_1 = 0 = \gamma$ or possibly by

a slow-fast eigenvalue λ that is close to $\lambda^+(0)$ (Lemma 4.2). Therefore, we introduce $\tilde{\lambda}$ by

$$\lambda = \varepsilon^2 \tilde{\lambda}, \tag{4.12}$$

which implies that (4.11) can also be written as a superslow system,

$$v_{xx} = \varepsilon^2 \left\{ \tilde{\lambda} \left[\tau - \frac{H(1+V_h, V_h)}{2(1+V_h)} \right] + \gamma + \mathcal{O}(\varepsilon^2) \right\} v. \tag{4.13}$$

As in section 2.2, we first consider the explicit example in which $H(U^2, V) = H_0 U^2$. Thus, the existence of (several kinds of) front solutions is established by Theorem 2.3. In this case, the equation for v is, on the ξ-scale, given by

$$v_{\xi\xi} = \varepsilon^4 \left[\tilde{\lambda} \left(\tau - \frac{1}{2} H_0 \right) + \gamma + \mathcal{O}(\varepsilon^2) \right] v = \left[\Lambda_{2,3}^2(\lambda, \varepsilon) + \mathcal{O}(\varepsilon^6) \right] v; \tag{4.14}$$

see (3.6). Note that this equation is of constant coefficients type, and, at leading order, the same as in the equation for $v_{\xi\xi}$ in the regular case (4.4). Hence, we can copy the arguments leading to Lemma 4.1 and conclude that the fundamental solution $\phi_2(\xi; \varepsilon^2 \tilde{\lambda}, \varepsilon)$ of (3.4) can again be expressed as in (4.5) outside the region I_f. Moreover, as in Lemma 4.1, we may conclude that $t_2(\tilde{\lambda}, \varepsilon) + t_3(\tilde{\lambda}, \varepsilon) = 1 + \mathcal{O}(\sqrt{\varepsilon})$ (4.6).

We may now proceed as in the preceding section (and as in [4, 5]) and determine $t_2(\tilde{\lambda})$ by measuring the change in the $q = v_\xi$-coordinate of $\phi_2(\xi)$ over the fast field. It follows from (4.14) that

$$\Delta_{\text{slow}} v_\xi = 2\varepsilon^2 (t_2(\tilde{\lambda}) - 1) \sqrt{\tilde{\lambda} \left(\tau - \frac{1}{2} H_0 \right) + \gamma} + \mathcal{O}(\varepsilon^2 \sqrt{\varepsilon}). \tag{4.15}$$

Note that we have to assume that $\tilde{\lambda}(\tau - \frac{1}{2} H_0) + \gamma > 0$, i.e., $\Lambda_{2,3}^2(\lambda, \varepsilon) > 0$, which is a natural assumption, since

$$\tilde{\lambda}_{\text{tip}} = \tilde{\lambda}^+(0) = -\frac{2\gamma}{2\tau - H_0} < 0 \tag{4.16}$$

determines the "tip" of σ_{ess} (recall that $H_0 - 2\tau < 0$); i.e., $t_2(\tilde{\lambda})$ is not defined if $\tilde{\lambda} \leq \tilde{\lambda}_{\text{tip}}$. By definition, $\Delta_{\text{fast}} v_\xi$ is given by (4.8). Since, at leading order $V_h(\xi) = V_h(0) = v_0$ and $U_h(\xi) = u_0(\xi; v_0)$ (uniformly) in I_f(Theorem 2.3), and since $u_0(\xi; v_0)$ decays exponentially fast on the (fast) ξ-scale, it follows that

$$\Delta_{\text{fast}} v_\xi = \varepsilon^2 H_0 \int_{-\infty}^{\infty} \left\{ 2 \left[2u_0^2(\xi; v_0) - 1 - v_0 \right] u_0(\xi; v_0) u_{\text{in}}(\xi; v_0) - u_0^2(\xi; v_0) \right\} d\xi + \mathcal{O}(\varepsilon^2 \sqrt{\varepsilon}), \tag{4.17}$$

where $u_{\text{in}}(\xi; v_0) = u_{\text{in}}(\xi; \lambda = 0; v_0)$ is the (uniquely determined) bounded solution of the inhomogeneous problem

$$u_{\xi\xi} + (1 + v_0 - 3u_0^2(\xi; v_0))u = -u_0(\xi; v_0).$$

Since we already know one solution of the homogeneous problem, $u(\xi) = u_{0,\xi}(\xi; v_0)$, we can determine $u_{\text{in}}(\xi; v_0)$ explicitly:

$$u_{\text{in}}(\xi; v_0) = \frac{1}{2(1+v_0)} \left(u_0(\xi; v_0) + \xi u_{0,\xi}(\xi; v_0) \right). \tag{4.18}$$

Thus, by (2.3), $\Delta_{\text{fast}} v_\xi$ can be computed explicitly (at leading order):

$$\Delta_{\text{fast}} v_\xi = -\varepsilon^2 H_0 \sqrt{2}\sqrt{1+v_0} + \mathcal{O}(\varepsilon^2\sqrt{\varepsilon}).$$

Combining this with (4.15) yields an explicit expression for $t_2(\tilde{\lambda})$,

$$t_2(\tilde{\lambda},\varepsilon) = 1 - H_0\sqrt{\frac{2(1+v_0)}{\tilde{\lambda}(\tau - \frac{1}{2}H_0)+\gamma}} + \mathcal{O}(\sqrt{\varepsilon}), \tag{4.19}$$

for $\tilde{\lambda} > \tilde{\lambda}_{\text{tip}}$ (4.16). It follows that $t_2(\tilde{\lambda}) \geq 1+\mathcal{O}(\sqrt{\varepsilon})$ for $H_0 \leq 0$ and $t_2(\tilde{\lambda}) < 1+\mathcal{O}(\sqrt{\varepsilon})$ for $H_0 > 0$. Hence, $t_2(\tilde{\lambda})$ cannot have zeros if $H_0 \leq 0$. In other words, there cannot be an eigenvalue near the tip of the essential spectrum in case (ii) of Theorem 2.3. On the other hand, $t_2(\tilde{\lambda})$ can be 0 for $H_0 > 0$; i.e., in case (i) of Theorem 2.3 there indeed is a "new" slow-fast eigenvalue of (3.3); it is given by

$$\lambda_{\text{edge}} = \varepsilon^2\tilde{\lambda}_{\text{edge}} = \frac{-2\gamma + H_0^2(1+v_0)}{2\tau - H_0}\varepsilon^2 + \mathcal{O}(\varepsilon^2\sqrt{\varepsilon}) > \varepsilon^2\tilde{\lambda}_{\text{tip}} = \lambda_{\text{tip}}; \tag{4.20}$$

see (4.16). Note that the eigenvalue λ_{edge} merges with λ_{tip} and thus with σ_{ess} as $H_0 \downarrow 0$. This is, of course, a leading order result; the accuracy of our analysis allows us only to conclude that $|\lambda_{\text{tip}} - \lambda_{\text{edge}}| \leq \mathcal{O}(\varepsilon^2\sqrt{\varepsilon})$ as $H_0 \downarrow 0$ and that λ_{edge} does not exist for $H_0 < 0$. Nevertheless, we conclude that λ_{edge} appears from the essential spectrum as H_0 increases through 0. In other words, λ_{edge} is created, or annihilated, by an edge bifurcation. Note that the new eigenvalue appears exactly as σ_{ess} becomes complex valued (see Figure 3.1).

The existence or nonexistence of λ_{edge} is crucial to the character of the destabilization (see also the numerical simulations in section 5). For $H_0 < 0$, the front solution $(U_h(\xi), V_h(\xi))$ destabilizes as γ, or equivalently G_1, crosses through 0. The destabilization is due to the essential spectrum, which implies that also the "background states" $(U(x,t), V(x,t)) \equiv (\pm 1, 0)$ destabilize at $\gamma = 0$. However, in the case $H_0 > 0$ the eigenvalue is λ_{edge} is $\varepsilon^2 H_0^2(1+v_0)/(2\tau - H_0)$ ahead of σ_{ess} (4.20), in the sense that it reaches the axis $\text{Re}(\lambda) = 0$ before σ_{ess} as $\gamma > 0$ decreases to 0. Thus, if $H_0 > 0$ the front solution $(U_h(\xi), V_h(\xi))$ destabilizes by an element of the discrete spectrum of (3.3) at $\gamma = \gamma_{\text{double}}$, defined as the solution of $\gamma = \frac{1}{2}H_0^2(1+v_0(\gamma)) + \mathcal{O}(\sqrt{\varepsilon})$ (> 0). As a consequence, the background states $(\pm 1, 0)$ remain stable as $(U_h(\xi), V_h(\xi))$ destabilizes for $H_0 > 0$, contrary to the case $H_0 < 0$. The bifurcation at γ_{double} is associated to the saddle-node bifurcation of heteroclinic orbits described in Theorem 2.3.

THEOREM 4.6. *Assume that $G(V) = -\varepsilon^2\gamma V$, that $H(U^2, V) = H_0 U^2$, that $H_0 - 2\tau < 0$ and $\mathcal{O}(1)$, and that $\varepsilon > 0$ is small enough.*

(i) *Let $(U_h^{+,1}(\xi), V_h^{+,1}(\xi))$ and $(U_h^{+,2}(\xi), V_h^{+,2}(\xi))$ be the two types of heteroclinic front solutions that exist for $H_0 > 0$ and $\gamma \geq \gamma_{\text{double}} = \frac{3}{2}H_0^2 + \mathcal{O}(\sqrt{\varepsilon})$ with, at leading order, $0 < V_h^{+,1}(0) = v_1 \leq 2 \leq v_2 = V_h^{+,2}(0)$ (Theorem 2.3). The front solution $(U_h^{+,1}(\xi), V_h^{+,1}(\xi))$ is (nonlinearly) stable for $\gamma > \gamma_{\text{double}}$, and the front $(U_h^{+,2}(\xi), V_h^{+,2}(\xi))$ is unstable; $(U_h^{+,1}(\xi), V_h^{+,1}(\xi))$ destabilizes by an element of the discrete spectrum, λ_{edge}, at $\gamma = \gamma_{\text{double}}$ and merges with $(U_h^{+,2}(\xi), V_h^{+,2}(\xi))$ in a saddle-node bifurcation of heteroclinic orbits.*

(ii) *Let $(U_h^{+}(\xi), V_h^{+}(\xi))$ be a heteroclinic front solution that exists for $H_0 < 0$ and (all) $\gamma > 0$ (Theorem 2.3); $(U_h^{+}(\xi), V_h^{+}(\xi))$ is (nonlinearly) stable for all $\gamma > 0$; it is destabilized at $\gamma = 0$ by the essential spectrum σ_{ess}.*

Remark 4.7. As in the regular case, spectral stability implies nonlinear orbital stability in this superslow case, since the linear operator associated to the stability problem remains sectorial as long as $\varepsilon > 0$.

Proof of Theorem 4.6. We first note that the condition $H_0 - 2\tau < 0$ and $\mathcal{O}(1)$ determines that σ_{ess} can only cross, or come close to, the $\text{Re}(\lambda) = 0$-axis at $\lambda = 0$ (Lemma 3.1 with $G_1 = \mathcal{O}(\varepsilon^2)$).

(i) The eigenvalue "in front of" σ_{ess}, $\lambda^{1,2}_{\text{edge}}(v_{1,2})$, is given by (4.20), where $v_0 > 0$ is a solution of $9\gamma v^2 = 2H_0^2(1+v)^3$, and $v_0 = v_1 \leq 2$ (at leading order) for $(U_h^{+,1}(\xi), V_h^{+,1}(\xi))$, while $v_0 = v_2 \geq 2$ (at leading order) for $(U_h^{+,2}(\xi), V_h^{+,2}(\xi))$ (Theorem 2.3). Thus, by (4.20), $\lambda^1_{\text{edge}}(v_1) < 0$ and $\lambda^2_{\text{edge}}(v_2) > 0$ if $\gamma < \gamma_{\text{double}} = \frac{3}{2}H_0^2 + \mathcal{O}(\sqrt{\varepsilon})$. As a consequence, $\lambda^1_{\text{edge}}(v_1) \uparrow 0$ and $\lambda^2_{\text{edge}}(v_2) \downarrow 0$ as $\gamma \downarrow \gamma_{\text{double}}$, at which the saddle-node bifurcation takes place.

(ii) We have already shown that there can be no eigenvalues in front of the tip of σ_{ess}. Therefore, the statement of the theorem follows. ∎

Remark 4.8. Since $t_2(\lambda) = 0$, the slow-fast eigenfunction associated to the bifurcation at $\gamma = \gamma_{\text{double}}$ is given by $\phi_2(\xi)$. It follows from Lemmas 3.7 and 3.8 that the u-component of ϕ_2 is odd, and the v-component even, as functions of ξ.

4.3. Bifurcations in the general superslow problem. We now consider the stability of a front solution in the general superslow limit. Thus, we assume that we have established the existence of a front $(U_h(\xi), V_h(\xi))$ for a certain given function $H(U^2, V)$ (Theorem 2.5). To analyze its stability, we again try to determine $t_2(\lambda)$ by measuring $\Delta_{\text{fast}} v_\xi$ and $\Delta_{\text{slow}} v_\xi$.

In order to determine $\Delta_{\text{slow}} v_\xi$ we follow the derivation of (4.13) in the previous section. Hence, we again conclude that nontrivial eigenvalues near 0 are possible only for $\lambda = \mathcal{O}(\varepsilon^2)$; thus, we again introduce $\tilde{\lambda}$ (4.12) (see also the proof of Theorem 4.10 for more details on the necessity of this scaling). Note that both G_1 and λ are now $\mathcal{O}(\varepsilon^2)$; thus, we can immediately obtain a leading order expression for $\Delta_{\text{fast}} v_\xi$ in terms of $H(U^2, V)$,

$$\begin{aligned} \Delta_{\text{fast}} v_\xi \quad = \quad & \varepsilon^2 \int_{-\infty}^{\infty} \Big\{ 2\left[H(u_0^2, v_0) - (1 + v_0 - u_0^2)\tfrac{\partial H}{\partial U^2}(u_0^2, v_0)\right] u_0 u_{\text{in}} \\ & - \left[H(u_0^2, v_0) - (1 + v_0 - u_0^2)\tfrac{\partial H}{\partial V}(u_0^2, v_0)\right] \Big\}\, d\xi + \mathcal{O}(\varepsilon^2\sqrt{\varepsilon}) \end{aligned} \tag{4.21}$$

(see (3.3)), where $u_{\text{in}}(\xi)$ is given in (4.18)—recall that $v = 1 + \mathcal{O}(\sqrt{\varepsilon})$ in I_f. As in the previous section, we have approximated $U_h(\xi)$ by $u_0(\xi; v_0)$ (2.3), $V_h(\xi)$ by v_0, and I_f by $\mathbb{R}$ (Theorem 2.5). Note that the integral converges and that $\Delta_{\text{fast}} v_\xi$ is (at leading order) independent of γ and $\tilde{\lambda}$.

It is in principle possible to determine $\Delta_{\text{slow}} v_\xi$ in terms of $t_2(\lambda)$ from (4.13); however, this equation is in general not of constant coefficients type (unlike for the example problem in section 4.2). If we introduce the superslow coordinate X by $X = \varepsilon x = \varepsilon^2 \xi$, we can write (4.13) as

$$v_{XX} = \left\{ \tilde{\lambda} \left[\tau - \frac{H(1 + V_h(X), V_h(X))}{2(1 + V_h(X))} \right] + \gamma + \mathcal{O}(\varepsilon^2) \right\} v; \tag{4.22}$$

i.e., the functions $V_h(X)$ introduce explicit X-dependent terms in the equation (in section 2.3, $V_h(X)$ behaves as $e^{\mp\sqrt{\gamma}X}$ on $\mathcal{M}^{\pm}_{\varepsilon}$). Nevertheless, we can in principle determine the v-components of the solution $\phi_2(\xi)$ of (3.4) outside the fast region I_f. However, the analysis is much less transparent. For instance, the decomposition (4.5) as in Lemma 4.1 now holds only for $X \gg 1$; therefore the relation between $t_3(\tilde{\lambda})$ and

$t_2(\tilde{\lambda})$ that is obtained from the value of v in I_f will in general be more complicated than in (4.6). Moreover, $\tilde{\lambda}[\tau - \frac{H(1+V_h(X),V_h(X))}{2(1+V_h(X))}] + \gamma$ might change sign as a function of X so that the solution $v(X)$ of (4.22) can have oscillatory parts.

Thus, we conclude that it is not a straightforward extension of the approach in the previous section to determine $t_2(\tilde{\lambda})$ for general values of $\tilde{\lambda}$. It should also be noted that a similar problem occurs in the regular case in the study of possible eigenvalues near $\lambda^{\pm}(0)$ (Lemma 4.2). If one introduces $\tilde{\lambda}^{\pm}$ by $\lambda = \lambda^{\pm}(0) + \varepsilon\tilde{\lambda}^{\pm}$ and derives the leading order equation for v_{xx} (4.3) in this case, then one finds an equation like (4.22), i.e., an equation with spatially dependent coefficients (these x-dependent terms originate from the $\mathcal{O}(\varepsilon)$ corrections corresponding to $V_h(x) = \mathcal{O}(\varepsilon)$ in (4.2) and (4.3)). Hence, at this point it is not yet possible to determine in full detail whether or not eigenvalues exist near the tips of σ_{ess} for general nonlinearities $H(U^2, V)$ and general λ. Moreover, it is also not possible to explicitly describe how and when eigenvalues appear from, or disappear into, σ_{ess}. On the other hand, it is clear from (4.21) and (4.22) that the number of zeros of $t_2(\tilde{\lambda})$ depends (for instance) on H_0. It thus follows that eigenvalues will be created/annihilated near the tip of σ_{ess} in the general case (as in the example system considered in the previous section). The analysis of eigenvalues near the tip of σ_{ess} is therefore a continuing subject of research in progress; see also section 5.

Nevertheless, the value $\lambda = \tilde{\lambda} = 0$ is, of course, especially relevant for the stability analysis of the front, and (4.22) is again of constant coefficients type at leading order for this special value of λ. Hence, for $\lambda = 0$ we can obtain the equivalent of Lemma 4.1 so that it follows that

$$\Delta_{\text{slow}} v_\xi|_{\lambda=0} = 2\varepsilon^2(t_2(0) - 1)\sqrt{\gamma} + \mathcal{O}(\varepsilon^2\sqrt{\varepsilon}). \tag{4.23}$$

Note that eventually it becomes clear at this point why the choice $G_1 = -\varepsilon^2\gamma$ is the most relevant scaling of G_1. With this scaling the "jumps" $\Delta_{\text{slow}} v_\xi$ and $\Delta_{\text{fast}} v_\xi$ (4.21) are of the same magnitude in ε at $\lambda = 0$. Therefore, $t_2(0, \varepsilon)$ is asymptotically close to 1 for all G_1 with $|G_1| \gg \varepsilon^2$—see Lemma 4.2 and its proof. Thus, the stability problem (3.3) can only have a double eigenvalue at 0 if $G_1 = \mathcal{O}(\varepsilon^2)$. This establishes a significant link between the stability analysis and the existence analysis of section 2, since it is clear from the analysis there that the scaling $G_1 = \mathcal{O}(\varepsilon^2)$ is also the most relevant scaling for the (superslow) existence problem (Remark 2.2). Moreover, this link is even much more explicit.

THEOREM 4.9. *Assume that $G(V) = -\varepsilon^2\gamma V$, that $H_0 - 2\tau < 0$ and $\mathcal{O}(1)$, and that $\varepsilon > 0$ is small enough. Let the front solution $(U_h(\xi;\varepsilon), V_h(\xi;\varepsilon))$ be a heteroclinic solution that corresponds to an intersection $T_o^- \cap W^u(-1,0,0,0)|_{\mathcal{M}_\varepsilon^-}$ as described in Theorem 2.5. The stability problem associated to the front solution has a double eigenvalue at $\lambda = 0$ if and only if the intersection $T_o^- \cap W^u(-1,0,0,0)|_{\mathcal{M}_\varepsilon^-}$ is non-transversal. If the intersection $T_o^- \cap W^u(-1,0,0,0)|_{\mathcal{M}_\varepsilon^-}$ is a second order contact, then the front bifurcates at*

$$0 < \gamma_{\text{double}} = \frac{1}{4(1+v_0)^2}\left[\int_{-\infty}^{\infty}(1 + v_0 - u_0^2)H(u_0^2, v_0)d\xi + 2\int_{-\infty}^{\infty}(1 + v_0 - u_0^2)[u_0^2\frac{\partial H}{\partial U^2}(u_0^2, v_0) + (1 + v_0)\frac{\partial H}{\partial V}(u_0^2, v_0)]d\xi\right]^2 \tag{4.24}$$

by merging with another front solution in a saddle-node bifurcation of heteroclinic orbits.

Proof. First, we recall from section 2.3 that a heteroclinic connection that corresponds to the intersection of $W^u(-1,0,0,0)|_{\mathcal{M}_\varepsilon^-} = \{q = \varepsilon\sqrt{\gamma}v\}$ and T_o^- is determined by (2.15). This is, of course, a leading order approximation. In the proof of this theorem we refrain from mentioning this obvious fact at several places. To determine the v_0-dependence of the right-hand side of this relation, we define $w_0(\xi)$ as the (monotonically increasing) heteroclinic solution of $\ddot{w} + (1-w^2)w = 0$. It follows that

$$u_0(\xi; v_0) = \sqrt{1+v_0}\, w_0(\sqrt{1+v_0}\,\xi), \quad w_0(t) = \tanh\sqrt{\frac{1}{2}}t; \tag{4.25}$$

see (2.3). Replacing $u_0(\xi; v_0)$ by $w_0(t)$ in (2.15) yields

$$\sqrt{\gamma}v_0 = \frac{1}{2}\sqrt{1+v_0}\int_{-\infty}^{\infty}(1-w_0^2)H((1+v_0)w_0^2, v_0)dt. \tag{4.26}$$

Thus, $T_o^- \cap W^u(-1,0,0,0)|_{\mathcal{M}_\varepsilon^-}$ is nontransversal if (2.15) holds and

(4.27)

$$\begin{aligned}
\sqrt{\gamma} = \;& \tfrac{1}{2}\tfrac{\partial}{\partial v_0}\left\{\sqrt{1+v_0}\int_{-\infty}^{\infty}(1-w_0^2)H((1+v_0)w_0^2, v_0)dt\right\} \\
= \;& \tfrac{1}{4\sqrt{1+v_0}}\int_{-\infty}^{\infty}(1-w_0^2)H((1+v_0)w_0^2, v_0)dt \\
& + \tfrac{1}{2}\sqrt{1+v_0}\int_{-\infty}^{\infty}(1-w_0^2)[w_0^2\tfrac{\partial H}{\partial U^2}((1+v_0)w_0^2, v_0) + \tfrac{\partial H}{\partial V}((1+v_0)w_0^2, v_0)]dt \\
= \;& \tfrac{1}{2(1+v_0)}\int_{-\infty}^{\infty}(1+v_0-u_0^2)H(u_0^2, v_0)d\xi \\
& \tfrac{1}{1+v_0}\int_{-\infty}^{\infty}(1+v_0-u_0^2)[u_0^2\tfrac{\partial H}{\partial U^2}(u_0^2, v_0) + (1+v_0)\tfrac{\partial H}{\partial V}(u_0^2, v_0)]d\xi
\end{aligned}$$

by reintroducing $u_0(\xi; v_0)$. Note that (4.24) follows from this equation. The expression for $t_2(0,\varepsilon)$ is determined by (4.21), (4.23), and (4.18):

$$t_2(0,\varepsilon) = 1 - \frac{\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3}{2\sqrt{\gamma}(1+v_0)} + \mathcal{O}(\sqrt{\varepsilon}),$$

where

$$\begin{aligned}
\mathcal{I}_1 &= \int_{-\infty}^{\infty}(1+v_0-u_0^2)H(u_0^2, v_0)d\xi, \\
\mathcal{I}_2 &= \int_{-\infty}^{\infty}(1+v_0-u_0^2)[u_0^2\tfrac{\partial H}{\partial U^2}(u_0^2, v_0) + (1+v_0)\tfrac{\partial H}{\partial V}(u_0^2, v_0)]d\xi, \\
\mathcal{I}_3 &= \int_{-\infty}^{\infty}[(1+v_0-u_0^2)\tfrac{\partial H}{\partial U^2}(u_0^2, v_0) - H(u_0^2, v_0)]\xi u_0 u_{0,\xi}d\xi.
\end{aligned} \tag{4.28}$$

We find by partial integration that

$$\mathcal{I}_3 = \int_{-\infty}^{\infty}\frac{1}{2}\xi\frac{\partial}{\partial\xi}[(1+v_0-u_0^2)H(u_0^2, v_0)]d\xi = -\frac{1}{2}\mathcal{I}_1,$$

which implies that

$$t_2(0,\varepsilon) = 1 - \frac{\mathcal{I}_1 + 2\mathcal{I}_2}{4\sqrt{\gamma}(1+v_0)} + \mathcal{O}(\sqrt{\varepsilon}),$$

so that we can conclude by (4.28) that $t_2(0,\varepsilon)=0$ is equivalent to the nontransversality condition (4.27). Hence, a double eigenvalue of (3.3) coincides with a saddle-node bifurcation of heteroclinic orbits, unless the tangency between T_o^- and $W^u(-1,0,0,0)|_{\mathcal{M}_\varepsilon^-}$ is degenerate. $\square$

Finally, we can turn to the question about the character of the destabilization of the regular front solution, which has been studied in sections 2.1 and 4.1, as G_1 approaches 0. In order to do so, we first note that the existence problem for the regular case can be recovered from that of the singular limit by reintroducing $G_1=-\gamma\varepsilon^2$ in the existence condition (2.15). This implies that v_0 must become $\mathcal{O}(\varepsilon)$ and that

$$\sqrt{-G_1}v_0=\varepsilon\frac{1}{2}\int_{-\infty}^{\infty}(1-u_0^2)H(u_0^2,0)d\xi+\mathcal{O}(\varepsilon\sqrt{\varepsilon}), \tag{4.29}$$

which is equivalent to (2.4) in Theorem 2.1. Thus, the structure of the front $(U_h(\xi),V_h(\xi))$ as a function of $G_1\uparrow 0$ can be determined by tracing the intersection $T_o^-\cap W^u(-1,0,0,0)|_{\mathcal{M}_\varepsilon^-}$ in the superslow limit as $W^u(-1,0,0,0)|_{\mathcal{M}_\varepsilon^-}=\{q=\varepsilon\sqrt{\gamma}v\}$ goes down from being almost vertical ($G_1=\mathcal{O}(1)$, $\gamma=\mathcal{O}(1/\varepsilon^2)$) to horizontal ($G_1=\gamma=0$). Note that this process determines a unique "regular" element in the intersection $T_o^-\cap W^u(-1,0,0,0)|_{\mathcal{M}_\varepsilon^-}$; all other elements of $T_o^-\cap W^u(-1,0,0,0)|_{\mathcal{M}_\varepsilon^-}$ do not persist in the regular limit $\gamma=\mathcal{O}(1/\varepsilon^2)$ (here, we do not pay attention to possible heteroclinic connections that have $v_0\gg 1$ as $\gamma\gg 1$). It depends on the sign of $\frac{1}{2}\int_{-\infty}^{\infty}(1-u_0^2)H(u_0^2,0)d\xi$ whether v_0 will be positive or negative (4.29), i.e., whether the regular intersection $T_o^-\cap W^u(-1,0,0,0)|_{\mathcal{M}_\varepsilon^-}$ travels through the first or through the third quadrant of the (v,q)-plane as γ decreases. Since $H(U^2,V)$ is smooth, we can make a distinction between two different types of behavior:

Type D: The regular element of $T_o^-\cap W^u(-1,0,0,0)|_{\mathcal{M}_\varepsilon^-}$ merges at a certain critical value of $G_1=-\varepsilon^2\gamma<0$ with another element of $T_o^-\cap W^u(-1,0,0,0)|_{\mathcal{M}_\varepsilon^-}$ in a saddle-node bifurcation of heteroclinic orbits.

Type E: The regular element of $T_o^-\cap W^u(-1,0,0,0)|_{\mathcal{M}_\varepsilon^-}$ exists up to the limit $G_1=0$.

Note that T_o^- approaches $(-1,0)$ as $v_0\downarrow -1$ (4.26) so that an element of $T_o^-\cap W^u(-1,0,0,0)|_{\mathcal{M}_\varepsilon^-}$ can only reach the singular region $\{v_0\le -1\}$ at $\gamma=0$, which indeed implies that there can only be orbits of Type D and E in the third quadrant. We can now describe the destabilization of the regular fronts as G_1 approaches 0.

THEOREM 4.10. *Assume that $G(V)=-\varepsilon^2\gamma V$, that $H_0-2\tau<0$ and $\mathcal{O}(1)$, and that $\varepsilon>0$ is small enough. Consider the heteroclinic front solution $(U_h(\xi),V_h(\xi))$ determined in Theorem* 2.1 *for $G_1<0$ and $\mathcal{O}(1)$ and in Theorem* 2.5 *for $G_1=\mathcal{O}(\varepsilon^2)$. If the front is of Type D as G_1 becomes $\mathcal{O}(\varepsilon^2)$, then it is asymptotically stable up to $G_1=-\varepsilon^2\gamma_{\text{double}}<0$* (4.24) *and is destabilized by a (discrete) eigenvalue through a saddle-node bifurcation of heteroclinic orbits. A front solution of Type E is stable up to $G_1=0$ and is destabilized by the essential spectrum.*

Thus, the destabilization of a regular front solution in the limit $G_1\uparrow 0$ is completely determined by the geometrical structure of $T_o^-\cap W^u(-1,0,0,0)|_{\mathcal{M}_\varepsilon^-}$ in the superslow limit. Note that Figure 2.3 presents examples of Type D and Type E behavior.

Proof. The proof of this theorem is a bit more subtle than a priori might be expected, since in general we do not have control over the eigenvalues of (3.3) near the tip of σ_{ess} (see also Remark 4.11), except that these eigenvalues must be $\mathcal{O}(\varepsilon^2)$ close to σ_{ess} (see also below). Thus, for instance, the following scenario for a Type D orbit might be possible as γ decreases to γ_{double}: two eigenvalues bifurcate (subsequently)

from σ_{ess} (as real eigenvalues), merge, and become a pair of complex eigenvalues. This pair crosses through the $\text{Re}(\lambda) = 0$ axis at $\gamma_{\text{Hopf}} > \gamma_{\text{double}}$ and touches down again on the real axis. At γ_{double} one of these eigenvalues returns to $\text{Re}(\lambda) = 0$. Thus, in this scenario, there already exists an unstable eigenvalue at $\gamma = \gamma_{\text{double}}$; moreover, the front destabilizes by a Hopf bifurcation at $\gamma_{\text{Hopf}} > \gamma_{\text{double}}$.

Let us first note that a destabilization by a Hopf bifurcation is the only alternative to the statements of the theorem, since eigenvalues move through either 0 or (in pairs) through the $\text{Re}(\lambda) = 0$ axis. If we can show that a Hopf bifurcation cannot occur for $\gamma > \gamma_{\text{double}}$, then it is clear that for Type D orbits $\lambda_{\text{edge}} < 0$ for $\gamma > \gamma_{\text{double}}$ and that there is no unstable spectrum at $\gamma = \gamma_{\text{double}}$ (this follows from Theorem 4.3: if γ is $\gg \mathcal{O}(1/\varepsilon)$, all nontrivial eigenvalues must be in $\{\text{Re}(\lambda) < -\varepsilon\}$; hence, by decreasing γ, there is one eigenvalue, λ_{edge}, that is the first to reach 0; this happens at the saddle-node bifurcation (Theorem 4.9), i.e., at $\gamma = \gamma_{\text{double}}$). Thus, the front is stable for $\gamma > \gamma_{\text{double}}$. The same argument can be used to establish the nonexistence of unstable spectrum for Type E orbits if there are no Hopf bifurcations possible.

To show that there cannot be Hopf bifurcations (for $H_0 - 2\tau < 0$ and $\mathcal{O}(1)$, see section 5), we first ascertain that λ must be $\mathcal{O}(\varepsilon^2)$, i.e., that (4.12) is the correct scaling. This follows by the same arguments as in the proof of Lemma 4.2. If $\Delta_{\text{slow}} v_\xi \gg \Delta_{\text{fast}} v_\xi$, then there cannot be an eigenvalue. Thus, it follows from (4.11) that $|\lambda|$ must indeed be $\mathcal{O}(\varepsilon^2)$ near $\lambda^+(0)$. Hence, even if there is a Hopf bifurcation, it will be $\mathcal{O}(\varepsilon^2)$ close to 0. Next, we realize that this situation is covered by (4.21) for the jump through the fast field; thus, $\Delta_{\text{fast}} v_\xi$ is real (at leading order), independent of $\tilde{\lambda}$. However, it follows from (4.22) that $\Delta_{\text{slow}} v_\xi$ cannot be real if $\tilde{\lambda}$ is complex valued. Hence, there cannot be a Hopf bifurcation $\mathcal{O}(\varepsilon^2)$ close to $\lambda = 0$. $\square$

Remark 4.11. By the same geometrical arguments (that are based on Theorem 4.9) we can describe the character of the bifurcations as function γ in the stability problem associated to a heteroclinic orbit that corresponds to a nonregular element of $T_o^- \cap W^u(-1,0,0,0)|_{\mathcal{M}_\varepsilon^-}$. However, it should be noted that, in general, we do not have enough information on the spectrum of (3.3) to establish the stability of such a front, since we did not determine all possible eigenvalues. In general, we cannot exclude the possibility that various eigenvalues have bifurcated from the essential spectrum for these fronts (in fact, the possible oscillatory character of a solution $v(X)$ of (4.22) strongly suggests that this can happen). Nevertheless, we may, for instance, conclude that if the regular orbit is of Type D, then it merges with a nonregular orbit at γ_{double} that is unstable for any $\gamma > \gamma_{\text{double}}$ for which it exists.

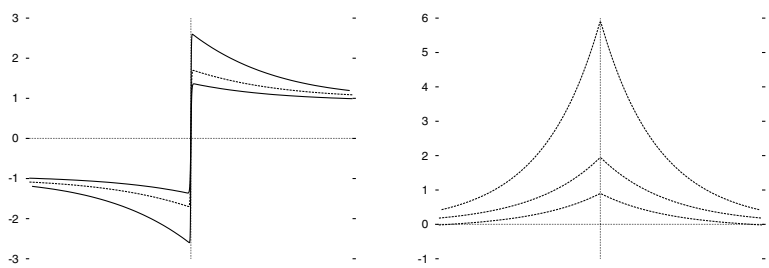
Remark 4.12. The most simple example one can consider is $H(U^2, V) \equiv H_0$. This corresponds to the case in which the function $F(U^2, V)$ in (1.1) is (the most general) linear function of U^2 and V with parameters G_1 and H_0 (i.e., $F(U^2, V) = H_0 + (H_0 + G_1)V - H_0 U^2$; recall that $F(1,0)$ must be 0). In this case, T_o^- is given by $\{q = 2\varepsilon H_0 \sqrt{1 + v_0} + \mathcal{O}(\varepsilon^2)\}$ so that $W^u(-1,0,0,0)|_{\mathcal{M}_\varepsilon^-}$ can never be tangent to T_o^-. Hence, in this case, there is a uniquely determined front solution of Type E for any $H_0 \neq 0$ and $G_1 < 0$; i.e., the front solution is stable up to $G_1 = 0$ and is destabilized by the essential spectrum.

Remark 4.13. We did not consider the degenerate case in which $H(U^2, V)$ is such that $H(1 + V, V) \equiv 0$ (section 1), i.e., functions H such that $H(U^2, V) = (1 + V - U^2)\tilde{H}(U^2, V)$ for some smooth function $\tilde{H}$. In a sense, this is a much more simple problem, for instance, since in the superslow limit, the stability problem in the slow field is automatically of constant coefficients type (at leading order); see (4.11), (4.22). Moreover, it is also clear from these same relations that we can find $\mathcal{O}(1)$ instead of

$\mathcal{O}(\varepsilon^2)$ eigenvalues in this case if $\tau = \mathcal{O}(\varepsilon^2)$. In fact, the situation is very much like the stability analysis of (homoclinic) pulses in monostable systems in [4, 5]. For instance, as in [4, 5], potential eigenvalues are no longer "slaved" to the tips of the essential spectrum or to the eigenvalues of the fast reduced limit (Lemma 4.2). Moreover, the "natural" persistence result of Lemma 4.4 is also not valid in this case, in general.

5. Simulations and discussion.

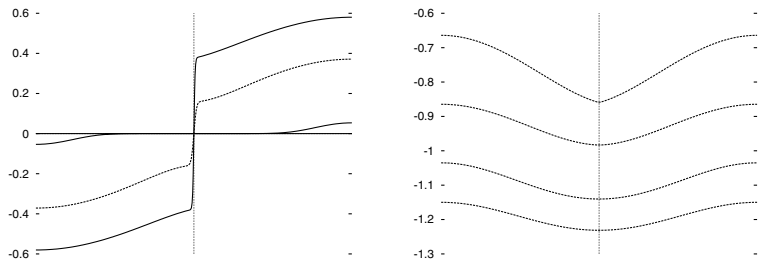
5.1. Simulations. We now examine numerically the difference between the two types of bifurcations discussed in Theorems 4.6 and 4.10. We consider the example system of sections 2.2 and 4.2 for $H_0 > 0$ (case (i), Type D) and $H_0 < 0$ (case (ii), Type E). First, we note that in both cases the simulations confirm that the fronts are asymptotically stable up to the analytically determined bifurcation values. In case (i) the front destabilizes at $\gamma < \gamma_{\text{double}}$ due to an eigenvalue in the discrete spectrum. The eigenfunction associated to this type of destabilization is localized to a neighborhood of the front, as can be seen in Figure 5.1. In this case the front becomes unstable and blows up in finite time, while the background states remain stable. In case (ii), the tip of the essential spectrum becomes positive and the background states become unstable as γ passes through 0. As can be seen in Figure 5.2, this destabilization causes the front to collapse. The U component then tends to 0 on the entire real line, and the V component grows according to $V_t = V_{xx} + \varepsilon^2|\gamma|V$. Thus, we may conclude that Type D or Type E orbits indeed exhibit significantly different behavior at the destabilization. These simulations were performed using SPMDF [2], with Neumann boundary conditions at $x = \pm 50$. The initial conditions used in Figure 5.1 are given by $U(x,0) = u_0(x,\varepsilon;v_1)$ (2.3) and $V(x,0) = v_1 e^{-\varepsilon\sqrt{|\gamma|}|x|}$ (as described in Theorem 2.3).



(a) U-component (b) V-component

FIG. 5.1. *Numerical simulation of destabilization caused by the discrete spectrum; both components blow up in finite time ($H_0 = 1$, $\gamma = 1.4$, $\tau = 1$, and $\varepsilon = 0.1$).*

5.2. Hopf bifurcations. As we have seen in section 4.1, in general there can be (complex) eigenvalues near the endpoints $\lambda^{\pm}(0)$ of σ_{ess}. Thus, if we keep $G_1 < 0$ fixed at an $\mathcal{O}(1)$ value and increase H_0 such that $H_0 + G_1 - 2\tau$ approaches 0, we encounter a similar issue as was studied in the previous section: Will the front be destabilized by σ_{ess} at $H_0 = 2\tau - G_1$, or (just) before that, by an eigenvalue? In this case, the bifurcation is of Hopf type, and it is not associated to the existence problem. This problem can in principle be analyzed by the methods developed here, i.e., by determining $t_2(\lambda,\varepsilon)$ through $\Delta_{\text{slow}} v_\xi$ and $\Delta_{\text{fast}} v_\xi$. We have already mentioned



(a) U-component (b) V-component

FIG. 5.2. *Numerical simulation of destabilization caused by the essential spectrum; $U \to 0$ and $|V|$ grows slowly and exponentially ($H_0 = -1$, $\gamma = -0.1$, $\tau = 1$, and $\varepsilon = 0.1$).*

the new features of the measuring the slow "jump" $\Delta_{\text{slow}} v_\xi$ in section 4.3. Moreover, since the bifurcation does not occur near $\lambda = 0$, we do not have an explicit formula for $u_{\text{in}}(\xi)$, like (4.18), and it is thus not immediately clear whether it is possible to determine $\Delta_{\text{fast}} v_\xi$. Note that this latter issue is solvable with the hypergeometric functions method developed in [3, 5]. Nevertheless, we do not go deeper into this subject here.

5.3. Planar fronts and stripes. A next step in the study of (planar) stripes, as mentioned in the introduction, is the stability analysis of planar fronts, i.e., the analysis of the stability of the fronts $(U_h(\xi), V_h(\xi))$ with respect to two-dimensional perturbations (thus, $(U_h(\xi), V_h(\xi))$ represents a planar front that has a trivial structure in the y-direction). The methods developed here can be used to study this problem (as is also suggested by [7] in which a similar problem has been studied in a monostable Gierer–Meinhardt context). It should be noted here that there are several papers in the literature that consider the question of the (non-)persistence of the stability of one-dimensional fronts as two-dimensional planar fronts (see, for instance, [17, 20, 13, 16]). The analysis in [20, 16] of a class of singularly perturbed bistable systems shows that the planar fronts considered there cannot be stable, while it is shown that planar fronts can be stable in a more regular context in [13]. Thus, this is a nontrivial issue. Preliminary analysis of the front solutions considered in this paper indicates that these solutions remain stable as planar fronts in the regular case (i.e., as long as $G_1 < 0$ and $\mathcal{O}(1)$). The analysis of the planar fronts and their spatially periodic counterparts, the stripe patterns, is the subject of a work in progress.

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