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Quantum mean-field decoding algorithm for error-correcting codes

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Abstract. We numerically examine a quantum version of TAP (Thouless-Anderson-Palmer)-like mean-field algorithm for the problem of error-correcting codes. For a class of the so-called Sourlas error-correcting codes, we check the usefulness to retrieve the original bit-sequence (message) with a finite length. The decoding dynamics is derived explicitly and we evaluate the average-case performance through the bit-error rate (BER).

1. Introduction

Statistical mechanics of information has been applied to a lot of problems in various research fields of information science and technology [1, 2]. Among them, error-correcting code is one of the most developed subjects. In the research field of error-correcting codes, Sourlas showed that the convolutional codes can be constructed by spin glass with infinite range p-body interactions and the decoded message should be corresponded to the ground state of the Hamiltonian [3]. Ruján suggested that the bit error can be suppressed if one uses finite temperature equilibrium states as the decoding result, instead of the ground state [4], and the so-called Bayes-optimal decoding at some specific condition was proved by Nishimori [5] and Nishimori and Wong [6]. Kabashima and Saad succeeded in constructing more practical codes, namely, low density parity check (LDPC) codes by using the infinite range spin glass model with finite connectivities [7]. They used the so-called TAP (Thouless-Anderson-Palmer) equations to decode the original message for a given parity check.

As we shall see later on, an essential key point to obtain the Bayes-optimal solution is controlling the 'thermal fluctuation' in order to satisfy the condition on the Nishimori line (the so-called Nishimori-Wong condition [6]). Then, a simple question is arisen, namely, is it possible to obtain the Bayes-optimal solution by means of the 'quantum fluctuation' induced by tunneling effects? or what is condition for the optimal control of the fluctuation?

To answer these questions, Tanaka and Horiguchi introduced a quantum fluctuation into the mean-field annealing algorithm and showed that performance of image restoration is improved by controlling the quantum fluctuation appropriately during its annealing process [8, 9]. The average-case performance is evaluated analytically by one of the present authors [10]. However,

there are few studies concerning such a quantum mean-field algorithm for information processing described by spin glasses.

In this paper, we examine a quantum version of TAP-like mean-field algorithm for the problem of error-correcting codes. For a class of the so-called Sourlas error-correcting codes, we check the usefulness to retrieve the original message with a finite length. The decoding dynamics is derived explicitly and we evaluate the average-case performance numerically through the bit-error rate. We find that TAP-like mean-field approach examined here is useful to decode the original message with a low BER for a relatively large signal-to-noise ratio.

This paper is organized as follows. In the next section, we explain our model system and comment on the Shannon's bound. In section 3, the Bayesian approach to the problem is introduced. Then, quantum Sourlas codes and the preliminary analysis for the case of $p \to \infty$ (p is the number of bit products in the parity check) are reported. In the next section 4, we show the bit-error rate performance at the zero temperature for finite p. In section 5, we construct the TAP-like mean-field decoding algorithm for the Sourlas codes with finite p and examine the average-case performance. The last section is a concluding remark.

2. The model system and the Shannon's bound

In this section, we introduce our model system of error-correcting codes and mention the Shannon's bound. In our error-correcting codes, in order to transmit the original message $\{\xi\} \equiv (\xi_1, \cdots, \xi_N), \xi_i \in \{-1, 1\}$ through some noisy channel, we send all possible combinations $_NC_p$ of the products of p-components in the N-dimensional vector $\{\xi\}$ such as $J^0_{i1,\cdots,ip} = \xi_{i1}\xi_{i2}\cdots\xi_{ip}$ as 'parity'. Therefore, the rate of the transmission is now evaluated as

$$R = \frac{N}{_NC_p} \simeq \frac{p!}{N^{p-1}} \tag{1}$$

in the limit of $N \to \infty$ keeping the p finite.

On the other hand, when we assume the additive white Gaussian noise (AWGN) channel with mean $(J_0 p!/N^{p-1})J^0_{i1i2\cdots ip}$ and variance $\{J\sqrt{p!/2N^{p-1}}\}^2$, that is, when the output of the channel $J_{i1i2\cdots ip}$ is given by

$$J_{i1i2\cdots ip} = \left(\frac{J_0 p!}{N^{p-1}}\right) J^0_{i1i2\cdots ip} + J\sqrt{\frac{p!}{2N^{p-1}}}\, \eta, \quad \eta = \mathcal{N}(0,1), \tag{2}$$

the channel capacity C leads to

$$C = \frac{1}{2}\log_2\left(1 + \frac{\{(J_0 p!/N^{p-1})J_{i1\cdots ip}\}^2}{J^2 p!/2N^{p-1}}\right) \simeq \frac{J_0^2 p!}{J^2 N^{p-1}\log 2} \tag{3}$$

in the same limit as in the derivation (1) (we also used the fact $(J_{i1\cdots ip})^2 = 1$). The factors $p!/N^{p-1}$ or $\sqrt{p!/2N^{p-1}}$ appearing in (2) are needed to take a proper thermodynamic limit (to make the energy of order 1 object) as will be explained in the next section.

Then, the channel coding theorem tells us that zero-error transmission is achieved if the condition $R \leq C$ is satisfied. For the above case, we have $R/C = (J/J_0)^2\log 2 \leq 1$, that is,

$$\frac{J_0}{J} \geq \sqrt{\log 2}. \tag{4}$$

The above inequality means that if the signal-to-noise ratio J_0/J is greater than or equal to $\sqrt{\log 2}$, the error probability of decoding behaves as $P_e \simeq 2^{-N(C-R)} \to 0$ in the thermodynamic limit $N \to \infty$. In this sense, we might say that the zero-error transmission is achieved asymptotically in the limit $N \to \infty, C, R \to 0$ keeping $R/C = \mathcal{O}(1) \leq 1$ for the above what we call *Sourlas codes*.

3. The Bayesian approach

For the error-correcting codes mentioned in the previous section, Sourlas pointed out that there exists a close relationship between the error-correcting codes and an Ising spin glass model with infinite range p-body interactions [3]. In this section, we briefly show the relationship for the classical system and then we shall extend the system to the quantum version.

3.1. Classical system

To decode the original message $\{\xi\}$, we construct the posterior distribution:

$$
P(\{\sigma\}|\{J\}) \quad \propto \quad P(\{J\}|\{\sigma\})P(\{\sigma\}) = \frac{\exp\left[-\frac{N^{p-1}}{2a^2 p!}\sum_{i1,\cdots,ip}\left(J_{i1\cdots ip} - \frac{a_0 p!}{N^{p-1}}\sigma_{i1}\cdots\sigma_{ip}\right)^2\right]}{2^N(a^2\pi p!/N^{p-1})^{1/2}} \tag{5}
$$

where $\{\sigma\} = (\sigma_1,\cdots,\sigma_N)$ denotes an estimate of the original message $\{\xi\}$ and a and a_0 are the so-called hyperparameters corresponding to the J_0 and J, respectively. It should be noted that we assumed that the prior $P(\{\sigma\})$ is uniform such as $P(\{\sigma\}) = 2^{-N}$. For the above posterior distribution, the MAP (maximum a posterior) estimate is obtained as the ground state of the following Hamiltonian:

$$
H(\{\sigma\}|\{J\}) \quad = \quad \frac{N^{p-1}}{2a^2 p!}\sum_{i1,\cdots,ip}\left(J_{i1\cdots ip} - \frac{a_0 p!}{N^{p-1}}\sigma_{i1}\cdots\sigma_{ip}\right)^2 \tag{6}
$$

It is obvious that the system $\{\sigma\}$ described by the above Hamiltonian is an Ising spin glass with infinite range p-body interactions. Therefore, the decoding is achieved by finding the ground state of (6) via, for instance, simulated annealing.

In the context of the MPM (maximizar of the posterior marginal) estimate instead of the MAP, the Bayes-optimal solution is obtained for each bit as a simple majority vote:

$$
\overline{\xi}_i \quad = \quad P(\sigma_i = +1|\{J\}) - P(\sigma_i = -1|\{J\}) = \text{sgn}\left(\sum_{\sigma_i=\pm 1}\sigma_i P(\sigma_i|\{J\})\right) \equiv \text{sgn}(\langle\sigma_i\rangle), \tag{7}
$$

where $P(\sigma_i|\{J\})$ is a posterior marginal calculated as

$$
P(\sigma_i|\{J\}) \quad = \quad \text{tr}_{\{\sigma\}\neq\sigma_i}P(\{\sigma\}|\{J\}). \tag{8}
$$

It might be convenient for physicists to rewrite the above estimate $\overline{\xi}_i$ in terms of the local magnetization of the system described by the Hamiltonian (6) as

$$
\overline{\xi}_i \quad = \quad \text{sgn}\left(\frac{\text{tr}_{\{\sigma\}}\sigma_i\exp[-H(\{\sigma\}|\{J\})]}{\text{tr}_{\{\sigma\}}\exp[-H(\{\sigma\}|\{J\})]}\right). \tag{9}
$$

In the classical system specified by a given finite temperature $T = 1$, the Bayes-optimal solution $\overline{\xi}_i = \text{sgn}(\langle\sigma_i\rangle)$ minimizes the following BER:

$$
p_{\text{B}} \quad = \quad \frac{1}{2}\left(1 - \frac{1}{N}\sum_i \xi_i\overline{\xi}_i\right) = \frac{1}{2}(1 - [\xi\overline{\xi}]_{\{\xi\},\{J\}}) \tag{10}
$$

$$
[\cdots]_{\{\xi\},\{J\}} \quad \equiv \quad \text{tr}_{\{\xi\}}\text{tr}_{\{J\}}(\cdots)P(\{J\}|\{\xi\})P(\{\xi\}) \tag{11}
$$

on the Nishimori line $a_0/a^2 = J_0/J^2$ [6].

3.2. Quantum system

Apparently, essential key point to obtain the Bayes-optimal solution is controlling the 'thermal fluctuation' in order to satisfy the condition on the Nishimori line $T = 1, a_0/a^2 = J_0/J^2$. Then, a simple question arises, namely, is it possible to obtain the Bayes-optimal solution by means of the 'quantum fluctuation' induced by tunneling effects? or what is condition for the optimal control of the fluctuation? However, in the corresponding quantum system, the condition is not yet clarified. In our preliminary study [11], we considered the quantum version of the posterior by modifying the Hamiltonian as

$$
\begin{aligned}
\hat{H}(\{\sigma\}|\{J\}) &= \frac{N^{p-1}}{a^2 p!} \sum_{i1,\cdots,ip} \left(J_{i1\cdots ip} - \frac{a_0 p!}{N^{p-1}} \hat{\sigma}_{i1}^z \cdots \hat{\sigma}_{ip}^z \right)^2 - \gamma \sum_i \hat{\sigma}_i^x \\
\hat{\sigma}_i^{z,x} &\equiv \boldsymbol{I}_{(1)} \otimes \cdots \otimes \boldsymbol{\sigma}_{(i)}^{z,x} \otimes \cdots \otimes \boldsymbol{I}_{(N)} \\
\boldsymbol{I} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \boldsymbol{\sigma}^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \boldsymbol{\sigma}^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = |+\rangle\langle-| + |-\rangle\langle+|
\end{aligned}
$$

where the subscript such as $\{\cdots\}_{(i)}$ of each matrix denotes the order in the tensor product. Then, a single bit flip: $|+\rangle \equiv {}^t(1,0) \rightarrow |-\rangle \equiv {}^t(0,1)$ or $|-\rangle \rightarrow |+\rangle$ is caused due to the existence
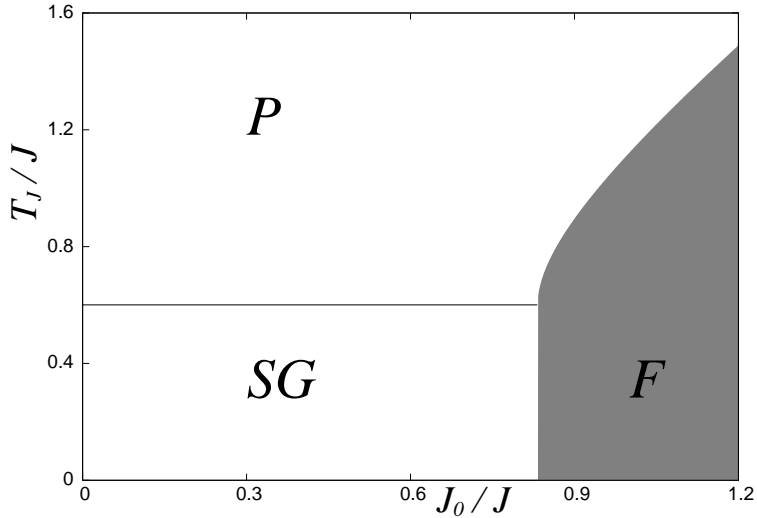
Figure 1. Phase diagram of the Sourlas codes for $p \rightarrow \infty$. In the shaded area (F), zero-error transmission is achieved. The area P denotes the para-magnetic phase and the area SG is the spin glass phase. For instance, at the ground state, the critical signal-to-noise ratio is $(J_0/J)_c = \sqrt{\log 2} = 0.8326$. We set $T_J \equiv \beta_J^{-1}$.

of the second term in the Hamiltonian $\hat{H}$. As the result, the Bayes-optimal solution

$$
\hat{\xi}_i = \mathrm{sgn}[\mathrm{tr}(\hat{\sigma}_i^z \hat{\rho})] \tag{12}
$$

with the density matrix $\hat{\rho} \equiv \mathrm{e}^{-\hat{H}(\{\sigma\}|\{J\})}/\mathrm{tr}\,\mathrm{e}^{-\hat{H}(\{\sigma\}|\{J\})}$ could be constructed by the quantum fluctuation (which is controlled by the amplitude γ) even at zero temperature. With the assistance of the replica method combining the static approximation in the Suzuki-Trottter formula, the phase diagram for the case of $p \rightarrow \infty$ is easily obtained within one step replica symmetry breaking scheme as shown in Figure 1. At the ground state, the Ferromagnet-SpinGlass transition takes place at the critical signal-noise ratio $(J_0/J)_c = \sqrt{\log 2} \simeq 0.8326$.

As the result, we find that $R \leq C$, namely, zero-error transmission $p_B = 0$ is achieved beyond the $(J_0/J)_c$. It should be noted that the critical behavior is independent of the amplitude γ. However, for finite p, the minimum BER state is dependent on the γ and we should control it when we construct the algorithm based on the TAP-like mean-field approximation. It is our main issue in this article.

4. Replica analysis for finite p at zero temperature

Before we provide such a decoding algorithm, we show the performance of the MPM estimate at zero temperature for finite p case.

By using the Suzuki-Trotter decomposition, the replicated partition function is given by

$$Z^n = \mathrm{tr}_{\{\sigma\}}\exp\left[\frac{\beta_J}{M}\sum_{i1,\cdots,ip}\sum_{\alpha=1}^{n}\sum_{t=1}^{M}J_{i1\cdots ip}\,\sigma_{i1}^{\alpha}(t)\cdots\sigma_{ip}^{\alpha}(t) + B\sum_{i}\sum_{t=1}^{M}\sigma_i^{\alpha}(t)\sigma_i^{\alpha}(t+1)\right] \quad (13)$$

with $B \equiv (1/2)\log\coth(\gamma/M)$ and $\beta_J \equiv a_0/a^2$.

Using the replica symmetric and the static approximations, we have the average

$$[Z^n]_{\{\xi\},\{J\}} = \prod_{tt',\alpha\beta}\int_{-\infty}^{\infty}dQ_{\alpha\beta}(t,t')\int_{-\infty}^{\infty}d\lambda_{\alpha\beta}(t,t')\int_{-\infty}^{\infty}dm_{\alpha}(t)\int_{-\infty}^{\infty}d\hat{m}_{\alpha}(t)\exp\left[-Nf_{RS}\right] \quad (14)$$

in terms of the following order parameters.

$$m_{\alpha}(t) = \frac{1}{N}\sum_{i}\sigma_i^{\alpha}(t) = m, \quad \hat{m}_{\alpha}(t) = \hat{m} \quad (15)$$

$$Q_{\alpha\beta}(t,t') = \frac{1}{N}\sum_{i}\sigma_i^{\alpha}(t)\sigma_i^{\beta}(t') = \begin{cases} \chi & (\alpha = \beta) \\ q & (\alpha \neq \beta) \end{cases}, \quad \lambda_{\alpha\beta}(t,t') = \begin{cases} \lambda_1 & (\alpha = \beta) \\ \lambda_2 & (\alpha \neq \beta) \end{cases} \quad (16)$$

where $\hat{m}_{\alpha}(t)$ and $\lambda_{\alpha\beta}(t,t')$ are the conjugate order parameters for the $m_{\alpha}(t)$ and the $Q_{\alpha\beta}(t,t')$, respectively. Those are defined by

$$\int_{-\infty}^{\infty}dm_{\alpha}(t)\int_{-\infty}^{\infty}d\hat{m}_{\alpha}(t)\exp\left[i\hat{m}_{\alpha}(t)\left(m_{\alpha}(t) - \frac{1}{N}\sum_{i}\sigma_i^{\alpha}(t)\right)\right] = 1 \quad (17)$$

$$\int_{-\infty}^{\infty}dQ_{\alpha\beta}(t,t')\int_{-\infty}^{\infty}d\lambda_{\alpha\beta}(t)\exp\left[i\lambda_{\alpha\beta}(t,t')\left(Q_{\alpha\beta}(t,t') - \frac{1}{N}\sum_{i}\sigma_i^{\alpha}(t)\sigma_i^{\beta}(t')\right)\right] = 1. \quad (18)$$

Then, we obtain the free energy density f^{RS} explicitly in terms of the above order parameters as

$$f^{RS}(m,\chi,q) = (p-1)J_0 m^p + \frac{(p-1)}{4}\beta_J J^2(\chi^p - q^p) - \beta_J^{-1}\int_{-\infty}^{\infty}Dw\log\int_{-\infty}^{\infty}Dz\,2\cosh\Xi \quad (19)$$

where we used the saddle point equations with respect to $\hat{m}, \lambda_1, \lambda_2$, namely, $\hat{m} = p\beta_J J_0 m^{p-1}$ and $\lambda_1 = p(\beta_J J)^2\chi^{p-1}/2, \lambda_2 = p(\beta_J J)^2 q^{p-1}/2$. Then, the saddle point equations that determine the equilibrium state are derived as follows.

$$m = \int_{-\infty}^{\infty}D\omega\int_{-\infty}^{\infty}Dz\left(\frac{\Phi\sinh\Xi}{\Xi\Omega}\right), \quad q = \int_{-\infty}^{\infty}D\omega\left[\int_{-\infty}^{\infty}Dz\left(\frac{\Phi\sinh\Xi}{\Xi\Omega}\right)\right]^2 \quad (20)$$

$$\chi = \int_{-\infty}^{\infty}\frac{D\omega}{\Omega}\int_{-\infty}^{\infty}Dz\left[\left(\frac{\Phi}{\Xi}\right)^2\cosh\Xi + \gamma^2\left(\frac{\sinh\Xi}{\Xi^3}\right)\right] \quad (21)$$

where we defined $\Phi \equiv \omega\sqrt{p(\beta_J J)^2 q^{p-1}/w} + z\sqrt{p(\beta_J J)^2(\chi^{p-1} - q^{p-1})/2} + p\beta_J J_0 m^{p-1}$ and $\Xi \equiv \sqrt{\Phi^2 + \gamma^2}$, $\Omega \equiv \int_{-\infty}^{\infty} Dz \cosh \Xi$ with $Dz \equiv (dz/\sqrt{2\pi})\,e^{-z^2/2}$. For the solution of the saddle point equations, the BER leads to

$$P_B = \int_{-\infty}^{\infty} Dw\, H(-z_p) \qquad (22)$$

where we defined $z_p \equiv -(p\beta_J J_0 m^{p-1} + w\sqrt{p(\beta_J J)^2 q^{p-1}/2})/\sqrt{p(\beta_J J)^2(\chi^{p-1} - q^{p-1})/2}$. The error function $H(x)$ is defined as $H(x) = \int_x^{\infty} Dz$. We find that the above p_B depends on γ through the order parameters χ, q and m. At finite temperature, the phase diagrams obtained by solving the above saddle point equations numerically were reported in our previous article [11]. However, our interest here is rather zero temperature properties.

In order to investigate 'pure' quantum effects on the decoding performance of the Sourlas codes for a finite number of the bit-products p, we here derive the saddle point equations for quantum Sourlas codes at zero temperature, namely, $T_J \equiv \beta_J^{-1} \to 0$ in the above replica symmetric saddle point equations. To do this, we take the limit $\beta_J, \gamma \to \infty$ keeping $\Gamma = \gamma/\beta_J$ finite and find the relevant solution so as to satisfy $\chi - q \to 0$ and $\beta_J(\chi - q) = t = \mathcal{O}(1)$. We should notice that the parameter that controls the quantum fluctuation at zero temperature is not γ but Γ. Then, we have immediately as

$$m = \int_{-\infty}^{\infty} \frac{\phi\, Dw}{\sqrt{\phi^2 + \Gamma^2}}, \quad q = \int_{-\infty}^{\infty} \frac{\phi^2\, Dw}{\phi^2 + \Gamma^2}, \quad t = \Gamma^2 \int_{-\infty}^{\infty} \frac{Dw}{(\phi^2 + \Gamma^2)^{3/2}} \qquad (23)$$

with $\phi = wJ\sqrt{pq^{p-1}/2} + \phi^2 J^2 p(p-1)q^{p-2}t/2\sqrt{\phi^2 + \Gamma^2} + pJ_0 m^{p-1}$. For the solution for the above saddle point equations (23), the BER leads to $p_B = H(pJ_0 m^{p-1}/J\sqrt{pq^{p-1}/2})$. We show the results for $p = 2$ and 3 cases in Figure 2. The left panel shows the behavior of magnetization
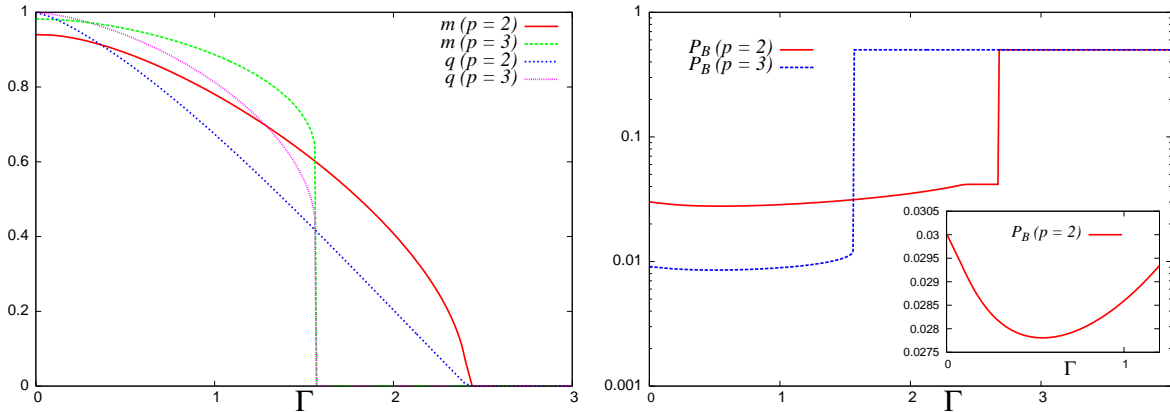
Figure 2. The behavior of order parameters m, q (left) and the BER (right). The inset of the right panel shows the behavior around the optimal amplitude of the transverse field. We set $J_0 = J = 1$.

and spin glass order parameter for $p = 2$ and 3, that is, the number of spin products in the parity is 2 and 3. We find that at the critical point, the second order phase transition takes place for $p = 2$, whereas the first order phase transition occurs for $p = 3$. The right panel is showing the BER as a function of the amplitude Γ. Thus, we find that there exists a close relationship between the bit-error performance and quantum phase transitions [12, 13]. We also find that there exists an optimal amplitude Γ and the BER is minimized at the value.

5. TAP-like mean-field decoding

In the previous section, we evaluated the performance of the Bayes-optimal decoding in the Sourlas codes with finite p. Within the replica symmetric theory, we found that there exists the optimal value of the Γ. In practice, for a given set of the parity $\{J\}$, we should calculate the estimate $\hat{\xi}_i = \lim_{\beta \to \infty} \text{sgn}[\text{tr}(\hat{\sigma}_i^z \hat{\rho}_\beta)]$, $\hat{\rho}_\beta = e^{-\beta\hat{H}}/\text{tr}\, e^{-\beta\hat{H}}$ for each bit. To calculate the trace effectively by sampling, the quantum Monte Carlo method (QMCM) might be applicable and useful [14]. However, unfortunately, the QMCM approach encounters several crucial difficulties. First, it takes quite long time for us to simulate the quantum states for large number of the Trotter slices. Second, in general, it is technically quite hard to simulate the quantum states at zero temperature. Thus, we are now stuck for the computational cost problem.

Nevertheless, as an alternative to decode the original message practically, we here examine a TAP (Thouless-Anderson-Palmer)-like mean-field algorithm which has a lot of the variants applying to various information processing [15, 16]. In this paper, we shall provide a simple attempt to apply the mean-field equations to the Sourlas error-correcting codes for the case of $p = 2$.

In following, the derivation of the equations is briefly explained.

We shall start the Hamiltonian:

$$\hat{H} = -\sum_{ij} J_{ij}\hat{\sigma}_i^z\hat{\sigma}_j^z - \Gamma\sum_i \hat{\sigma}_i^x, \quad J_{ij} = \left(\frac{2J_0}{N}\right)J_{ij}^0 + \frac{J}{\sqrt{N}}\eta, \quad J_{ij}^0 = \xi_i\xi_j, \quad \eta = \mathcal{N}(0,1) \quad (24)$$

Then, we rewrite the above Hamiltonian as follows.

$$\hat{H} = -\sum_i (\Gamma\hat{\sigma}_i^x + h_i\hat{\sigma}_i^z) + \sum_{ij} J_{ij}(m_i\hat{I}_i)(m_j\hat{I}_j) - \sum_{ij} J_{ij}(\hat{\sigma}_i^z - m_i\hat{I}_i)(\hat{\sigma}_j^z - m_j\hat{I}_j) \equiv \hat{H}^{(0)} + \hat{V}$$

$$(25)$$

$$\hat{H}^{(0)} \equiv -\sum_i (\Gamma\hat{\sigma}_i^x + h_i\hat{\sigma}_i^z) + \sum_{ij} J_{ij}(m_i\hat{I}_i)(m_j\hat{I}_j) \quad (26)$$

$$\hat{V} \equiv -\sum_{ij} J_{ij}(\hat{\sigma}_i^z - m_i\hat{I}_i)(\hat{\sigma}_j^z - m_j\hat{I}_j), \quad h_i \equiv 2\sum_j J_{ij}m_j \quad (27)$$

where we defined the $2^N \times 2^N$ identity matrix $\hat{I}_i$, which is formally defined by $\hat{I}_i \equiv \boldsymbol{I}_{(1)} \otimes \cdots \otimes \boldsymbol{I}_{(i)} \otimes \cdots \otimes \boldsymbol{I}_{(N)}$. m_i is the local magnetization for the system described by the mean-field Hamiltonian $\hat{H}^{(0)}$, that is,

$$m_i \equiv m_i^z = \lim_{\beta \to \infty} \text{tr}(\hat{\sigma}_i^z \hat{\rho}_\beta^{(0)}), \quad \hat{\rho}_\beta^{(0)} \equiv \frac{\exp(-\beta\hat{H}^{(0)})}{\text{tr}\exp(-\beta\hat{H}^{(0)})}. \quad (28)$$

Shortly, we derive closed equations to determine m_i. It is very tough problem for us to diagonalize the $2^N \times 2^N$ matrix $\hat{H}$, whereas it is rather easy to diagonalize the mean-field Hamiltonian $\hat{H}^{(0)}$. Actually, we immediately obtain the ground state internal energy as

$$E^{(0)} = -\sum_i E_i + \frac{1}{2}\sum_i h_i m_i, \quad E_i \equiv \sqrt{\Gamma^2 + h_i^2}. \quad (29)$$

Then, taking the derivative of the $E^{(0)}$ with respect to m_i and setting it to zero, namely, $\partial E^{(0)}/\partial m_i = \sum_k (\partial h_k/\partial m_i)\{h_k/\sqrt{\Gamma^2 + h_k^2} - m_k\} = 0$, we have

$$(\forall i) \quad m_i = \frac{h_i}{\sqrt{\Gamma^2 + h_i^2}}, \quad h_i = 2\sum_j J_{ij}m_j. \quad (30)$$

The above equations are nothing but the so-called *naive mean-field equations* for the Ising spin glass (the Sherrington-Kirkpatrick model [17]) in a transverse field. It should be noted that the equations are reduced to $(\forall_i)$ $m_i = h_i/|h_i| = \mathrm{sgn}(h_i) = \lim_{\beta \to \infty} \tanh(\beta h_i)$ which is naive mean-field equations at the ground state for the corresponding classical system.

To improve the approximation, according to [18, 19], we introduce the reaction term R_i for each pixel i and rewrite the local field h_i such as $2\sum_j J_{ij}m_j - R_i$. Then, the naive mean-field equations (30) are rewritten as

$$(\forall_i) \quad m_i = \frac{2\sum_j J_{ij}m_j - R_i}{\sqrt{\Gamma^2 + (2\sum_j J_{ij}m_j - R_i)^2}} \simeq \frac{h_i}{(\Gamma^2 + h_i^2)^{3/2}}\left[1 - \frac{\Gamma^2}{\Gamma^2 + h_i^2}\left(\frac{R_i}{h_i}\right)\right]. \quad (31)$$

In the last line of the above equation, we expanded the equation with respect to R_i up to the first order. We next evaluate the expectation of the Hamiltonian $\hat{H}$ by using the eigenvector that diagonalizes the mean-field Hamiltonian $\hat{H}^{(0)} = -\sum_i(\Gamma\hat{\sigma}_i^x + h_i\hat{\sigma}_i^z) + \sum_{ij} J_{ij}(m_i\hat{I}_i)(m_j\hat{I}_j)$. We obtain

$$E_g = E^{(0)} - \Gamma^4 \sum_{ij}\left(\frac{J_{ij}^2}{2E_i^2 E_j^2(E_i + E_j)}\right). \quad (32)$$

Then, $(\partial E_g/\partial m_i) = 0$ gives

$$m_i = \frac{h_i}{(\Gamma^2 + h_i^2)^{3/2}}\left[1 - \frac{\Gamma^2}{\Gamma^2 + h_i^2}\left(\frac{1}{h_i}\right)\sum_j \frac{J_{ij}^2 m_i[2(1-m_i^2)(1-m_j^2)^{\frac{3}{2}} + 3(1-m_i^2)^{\frac{1}{2}}(1-m_j^2)^2]}{2\Gamma[(1-m_i^2)^{\frac{1}{2}} + (1-m_j^2)^{\frac{1}{2}}]^2}\right]. \quad (33)$$

By comparing (31) and (33), we might choose the reaction term R_i for each bit i consistently as

$$R_i = \sum_j \frac{J_{ij}^2 m_i[2(1-m_i^2)(1-m_j^2)^{\frac{3}{2}} + 3(1-m_i^2)^{\frac{1}{2}}(1-m_j^2)^2]}{2\Gamma[(1-m_i^2)^{\frac{1}{2}} + (1-m_j^2)^{\frac{1}{2}}]^2}. \quad (34)$$

Therefore, we now have a decoding dynamics described by

$$m_i(t+1) = \frac{2\sum_j J_{ij}m_j(t) - R_i(t)}{\sqrt{\Gamma^2 + \{2\sum_j J_{ij}m_j(t) - R_i(t)\}^2}} \quad (35)$$

$$R_i(t) = \sum_j \frac{J_{ij}^2 m_i(t)[2(1-m_i(t)^2)(1-m_j(t)^2)^{\frac{3}{2}} + 3(1-m_i(t)^2)^{\frac{1}{2}}(1-m_j(t)^2)^2]}{2\Gamma[(1-m_i(t)^2)^{\frac{1}{2}} + (1-m_j(t)^2)^{\frac{1}{2}}]^2} \quad (36)$$

for each bit i. Then, the MPM estimate is given as a function of time t as $\overline{\xi}_i(t) = \mathrm{sgn}[m_i(t)]$ and the BER is evaluated at each time step through the following expression

$$p_B(t) = \frac{1}{2}\left(1 - \frac{1}{N}\sum_i \xi_i\overline{\xi}_i(t)\right). \quad (37)$$

We should notice that the naive mean-field equations are always retrieved by setting $R_i = 0$ for all i. The naive mean-field equations were applied to image restoration by Tanaka and Horiguchi [8].
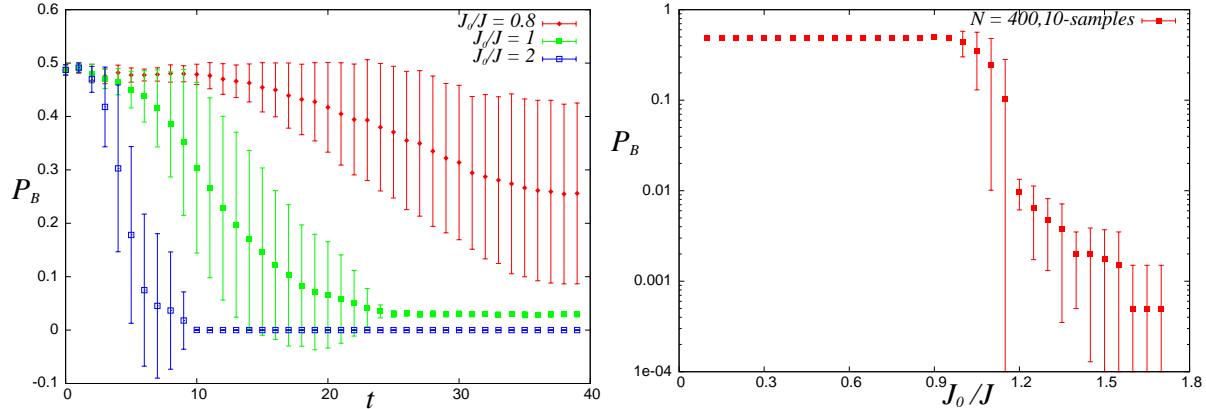
Figure 3. The dynamics of the TAP-like mean-field decoding (left, $N = 1000$, the error-bars are evaluated by 10-samples). We set $p = 2, \Gamma_0 = 0.5$ and $J/J_0 = 0.8, 2$ and 1. The horizontal axis t in the left panel denotes the number of time step in the TAP-like update described by (35)(36). The right panel shows the signal-to-noise ratio dependence of the BER. We set $p = 2$ and $\Gamma_0 = 0.5$.

5.1. Preliminary results

We plot several results in Figure 3. In the left panel of this figure, we plot the dynamics of mean-field decoding. We plot them for several cases of the signal-to-noise ratio. During the decoding dynamics, we control the Γ by means of

$$\Gamma(t) \quad = \quad \Gamma_0 \left(1 + \frac{c}{t + 1} \right) \qquad (38)$$

where t denotes the number of time step in the TAP-like update described by (35)(36). In the Figure 3, we set $\Gamma_0 = 0.5$. From this figure, we find that the BER drops monotonically as the number of iterations increases. We also find in the right panel that beyond the SN ratio $J_0/J \simeq 1$, the BER drops. Although the above results are still at preliminary level, however, from these limited results, we might confirm that TAP-like mean-field approach examined here is useful to decode the original message with a low BER for relatively large SN ratio. It might be important for us to consider the relationship between the performance of the TAP-like mean-field algorithm and the averaged case performance predicted by the replica symmetric theory under the static approximation. However, to clarify this issue, we need more careful and extended numerical studies.

6. Concluding remark

We examined a quantum version of TAP (Thouless-Anderson-Palmer)-like mean-field algorithm at zero temperature for the problem of error-correcting codes. Although the presented results are still at preliminary level and we should be careful to conclude, the algorithm seems to work well for our decoding problem. Of course, much more extended studies are needed. For instance, we have problems to be clarified such as the structure of basin (the initial condition dependence of the decoding dynamics), studies for the case of $p \geq 3$, a comparison of the results with those obtained by the QMCM, the relationship between the convergence of the algorithm and the Almeida-Thouless instability which was investigated for the case of the LDPC (Low Density Parity Check) codes [20]. Some of these issues will be investigated extensively in our future studies.

Acknowledgments

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