Environ. Sci. Hokkaido	5 (2)	131~150	Dec. 1982

A Study on Regional Income Disparity Arising from Regional Allocation of Investments in Continuous Space

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連続空間における地域投資配分に基づく地域所得格差に関する研究

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1. Introduction

This paper aims to find a policy of the regional investment allocation to maximize the national income under the condition where there is no wide regional income disparity at the end of the planned period.

It was M. A. Rahman[1,2] who first oriented the study but he treated a centralized economy and did not consider the equalization of regional income. Later, N. Sakashita[3] modified Rahman model into a mixed economy which consists of private and public sectors. He treated the case of two regions as M. A. Rahman did, and his result was extented into n-region model by Y. Ohtsuki.[4] However, in both two researches, the detailed research of the regional income disparities has not been made. On the other hand, E. Yamamura[5,6] investigated the regional development model under the condition that any wide regional income disparity do not result at the end of the planned period. In his study, he represented the regional economic growth model as a system of difference equations and analyzed them by numerical methods in discrete space. Furthermore, he proved the existence theorem of the solution in the case where the minimum proportion of regional investment was variable.

In this paper, we shall consider the regional economic growth model that we

Received 12 Jul. 1982
1982年7月12日受理

reframe the Yamamura formulation a continuous space format.

2. Mathematical Formulation

In this chapter, we shall represent the mathematical formulation of two region growth model as follows[5,6] :

$$\frac{dx_1}{dt}=P_1u_1(1-r)(S_1x_1+S_2x_2)+P_1rS_1x_1 \quad (1)$$

$$\frac{dx_2}{dt}=P_2u_2(1-r)(S_1x_1+S_2x_2)+P_2rS_2x_2 \quad (2)$$

$$u_1+u_2=1 \quad (3)$$

$$D_1\leqq u_1\leqq 1-D_2 \quad (4)$$

$$D_2\leqq u_2\leqq 1-D_1 \quad (5)$$

$$x_1(T)=x_2(T) \quad (6)$$

$$0\leqq t\leqq T \quad (7)$$

Where,

x_i : the regional income of region i.
P_i : the productivity of investment of region i.
S_i : the saving ratio of region i $(0<S_i<1)$
u_i : the investment allocation rate to region i.
D_i : the minimum proportion of investment of region i.
r : the local autonomy rate.
$x_i(T)$: the regional income of region i at the end of the planned period T.

This model seems to be clear that the variations of the regional incomes are made by the both investments of the central and local authority through the productivity of investment. And at the end of the planned period, there is no wide regional income disparity. We also assume that the extreme investment in specific regions is not made.

Furthermore, we can get the equation such as total investments=total savings by adding (1) and (2).

$$\sum_{i=1}^{2}\frac{1}{P_i}\frac{dx_i}{dt}=\sum_{i=1}^{2}S_ix_i \quad (8)$$

It is the main purpose to maximize the national income at the end of the planned period such as $x_1(T)+x_2(T)$ under these conditions. This purpose can be written as follows :

$$x_1(T)+x_2(T)=\int_0^T\left(\frac{dx_1}{dt}+\frac{dx_2}{dt}\right)dt+x_1(0)+x_2(0)\rightarrow\max \quad (9)$$

Subject to (1)～(7)

In the following context, let P_i, S_i be constant for simplicity.

3. Analysis in Phase Plane

In this chapter, we shall analyze how the solutions of (1)~(7) will behave by using a phase plane. The phase plane represents the orbit of the regional income growth as the point (x_1, x_2) in Cartesian Coordinates.[7,8]

Now, we rewrite the equations (1) and (2) as follows:

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 \tag{1'}$$

$$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 \tag{2'}$$

In vector representation:

$$\frac{dx}{dt} = Ax, \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{10}$$

Where:

$$\begin{aligned} a_{11} &= P_1u_1(1-r)S_1 + P_1rS_1 \qquad & a_{12} &= P_1u_1(1-r)S_2 \\ a_{21} &= P_2u_2(1-r)S_1 & a_{22} &= P_2u_2(1-r)S_2 + P_2rS_2 \end{aligned} \tag{11}$$

Though u_i are constrained by (4), (5) and (6), u_i can move in the interval $0 \leqq u_i \leqq 1$ corresponding to the variation of D_i. We put $u_2 = 1 - u_1$ in order to show the changes of a_{ij} as in Table 1.

Table 1. The changes of a_{ij} corresponding to u_1

a_{ij} \ u_1	0	1
a_{11}	P_1rS_1	P_1S_1
a_{12}	0	$P_1(1-r)S_1$
a_{21}	$P_2(1-r)S_1$	0
a_{22}	P_2S_2	P_2rS_2

Next, we consider the regional incomes x_1 and x_2 when $\frac{dx_1}{dt} = 0$ as follows:

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 = 0 \qquad \therefore \quad x_2 = -\frac{a_{11}}{a_{12}}x_1$$

$$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 = 0 \qquad \therefore \quad x_2 = -\frac{a_{21}}{a_{22}}x_1 \qquad (a_{12}, a_{22} \neq 0)$$

And using the relation $\frac{dx_2}{dx_1} = \frac{dx_2}{dt} \Big/ \frac{dx_1}{dt}$, the changes $\frac{dx_2}{dx_1}$ can be shown as in Table 2.

Table 2. The changes of $\frac{dx_2}{dx_1}$

$\frac{dx_2}{dx_1}$ \ $\frac{dx_1}{dt}$	$\frac{dx_1}{dt}>0$	$\frac{dx_1}{dt}<0$
$\frac{dx_2}{dx_1}\geqq 1$	$\frac{dx_2}{dt}\geqq\frac{dx_1}{dt}$	$\frac{dx_2}{dt}\leqq\frac{dx_1}{dt}$
$0\leqq\frac{dx_2}{dx_1}<1$	$0\leqq\frac{dx_2}{dt}<\frac{dx_1}{dt}$	$0\geqq\frac{dx_2}{dt}>\frac{dx_1}{dt}$
$-1\leqq\frac{dx_2}{dx_1}<0$	$-\frac{dx_1}{dt}\leqq\frac{dx_2}{dt}<0$	$-\frac{dx_1}{dt}\geqq\frac{dx_2}{dt}>0$
$\frac{dx_2}{dx_1}<-1$	$\frac{dx_2}{dt}<-\frac{dx_1}{dt}$	$\frac{dx_2}{dt}>-\frac{dx_1}{dt}$

And using the notations (1′) and (2′), the inequalities in Table 2 are represented as in Table 3. Here, we assume $a_{22}-a_{12}>0$.

Table 3. The relation between $\frac{dx_2}{dx_1}$ and x_i

$\frac{dx_2}{dx_1}$ \ x_1, x_2	$x_2>-\frac{a_{11}}{a_{12}}x_1$	$x_2<-\frac{a_{11}}{a_{12}}x_1$
$\frac{dx_2}{dx_1}\geqq 1$	$x_2\geqq\frac{a_{11}-a_{21}}{a_{22}-a_{12}}x_1$	$x_2\leqq\frac{a_{11}-a_{21}}{a_{22}-a_{12}}x_1$
$0\leqq\frac{dx_2}{dx_1}<1$	$x_2<\frac{a_{11}-a_{21}}{a_{22}-a_{12}}x_1$	$x_2>\frac{a_{11}-a_{21}}{a_{22}-a_{12}}x_1$
$-1\leqq\frac{dx_2}{dx_1}<0$	$x_2\geqq-\frac{a_{11}+a_{21}}{a_{22}+a_{12}}x_1$	$x_2\leqq-\frac{a_{11}+a_{21}}{a_{22}+a_{12}}x_1$
$\frac{dx_2}{dx_1}<-1$	$x_2<-\frac{a_{11}+a_{21}}{a_{22}+a_{12}}x_1$	$x_2>-\frac{a_{11}+a_{21}}{a_{22}+a_{12}}x_1$

We can obtain the situation of (x_1, x_2) and its direction of growth path in the phase plane by the result in Table 3. Consequently x_1-x_2 phase plane is divided into subplanes having the following boundaries:

$$x_2=-\frac{a_{11}}{a_{12}}x_1 \tag{12}$$

$$x_2=-\frac{a_{21}}{a_{22}}x_1 \tag{13}$$

$$x_2=\frac{a_{11}-a_{21}}{a_{22}-a_{12}}x_1 \tag{14}$$

$$x_2=-\frac{a_{11}+a_{21}}{a_{22}+a_{12}}x_1 \tag{15}$$

Furthermore, we shall indicate the changes of boundaries (12)~(15) in Table 4 corresponding to the changes of a_{ij} in Table 1.

As the continuity of a_{ij} is hold with respect to u_1, the boundaries in x_1-x_2 plane change continuously within the limit expressed in Table 4. Under these conditions, we show the boundaries corresponding to $u_1=0$ as solid lines and $u_1=1$ as dotted line in Fig. 1. Here, we set r as :

Table 4. The changes of the boundaries due to u_1

Coefficients of x_1 \ u_1	0	1
$-\frac{a_{11}}{a_{12}}$	$-\infty$	$-\frac{S_1}{(1-r)S_2}$
$-\frac{a_{21}}{a_{22}}$	$-\frac{(1-r)S_1}{S_2}$	0
$\frac{a_{11}-a_{21}}{a_{22}-a_{12}}$	$\frac{P_1rS_1-P_2(1-r)S_1}{P_2S_2}$	$\frac{P_1S_1}{P_2rS_2-P_1(1-r)S_2}$
$-\frac{a_{11}+a_{21}}{a_{22}+a_{12}}$	$-\frac{P_1rS_1+P_2(1-r)S_1}{P_2S_2}$	$-\frac{P_1S_1}{P_2rS_2+P_1(1-r)S_2}$

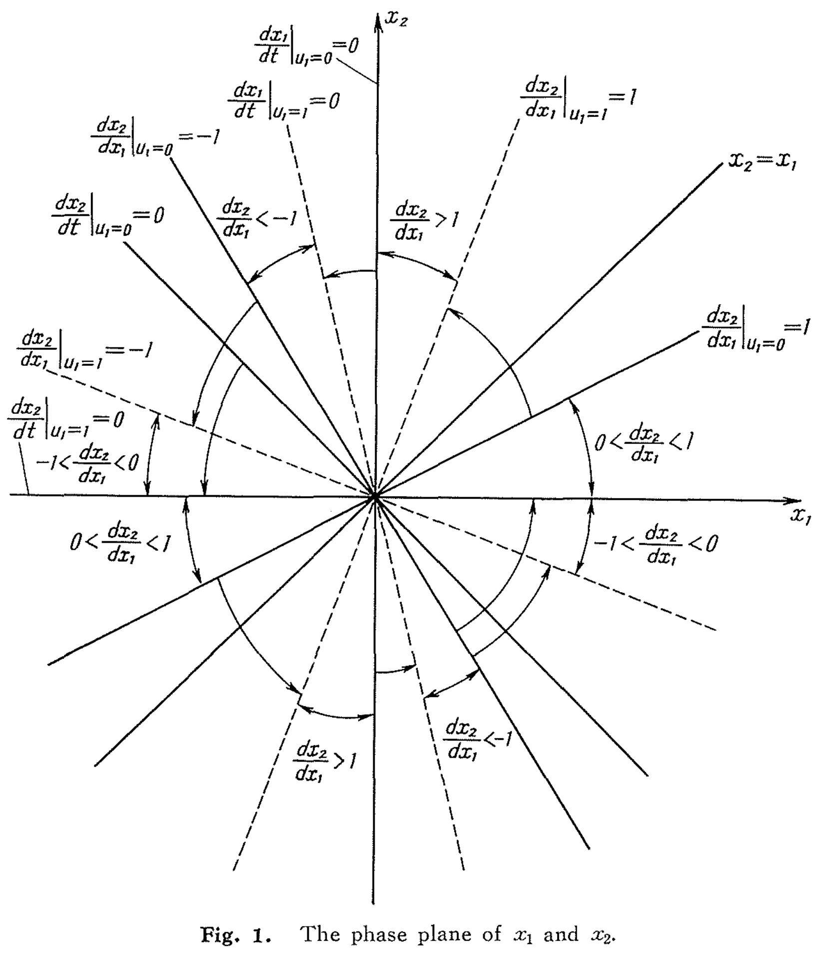

Fig. 1. The phase plane of x_1 and x_2.

$$0<\frac{P_1 r S_1 - P_2(1-r)S_1}{P_2 S_2}<1<\frac{P_1 S_1}{P_2 r S_2 - P_1(1-r)S_1}<\infty$$

In Fig. 1, we indicate the changes of boundaries as arrows and the value of corresponding to (x_1, x_2). And we show the only first quadrant of Fig. 1 in Fig. 2, because we shall only investigate the case where x_1 and x_2 are nonnegative.

Each point of ①, ② and ③ represent the initial value $(x_1(0), x_2(0))$ at $t=0$. The arrows show the direction of the orbit of the regional income growth, and this corresponds to the differential coefficient $\frac{dx_2}{dx_1}$ at each point.

In this study, we shall find some existence theorem of the solution which starts from the initial value shown in Fig. 2 and satisfies the boundary condition $x_1(T)=x_2(T)$ at $t=T$, namely is on the line $x_2=x_1$. And we shall investigate the solution which maximizes the distance d from the origin. The orbit ① and ③ shown in Fig. 2 are the examples that do not fulfill the boundary condition.

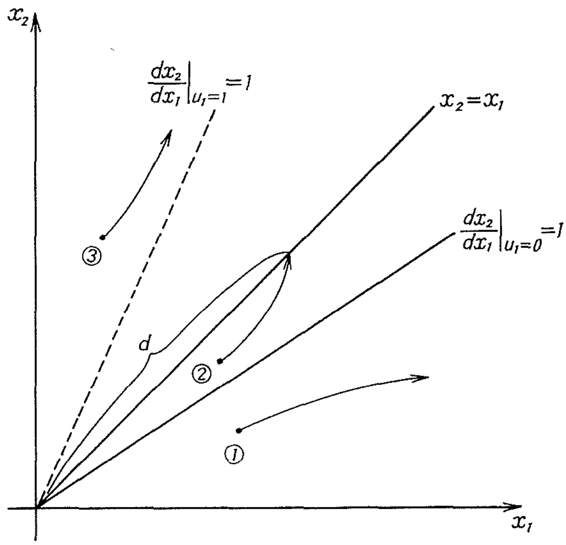

Fig. 2. The orbit of regional income growth.

4. Hamiltonian Approach Analysis

We analyzed the case that the system of the differential equations (1)~(7) has no solution by means of the phase plane in chapter 3.

In this chapter, we shall consider the existence and the behavior of the system of the equations (1)~(7) by the Hamiltonian approach analysis. We shall introduce Pontrijagin maximum principle[9], because this study is an optimization problem of the system of the differential equations. The Hamiltonian is defined as follows:

$$\begin{aligned} &H(x_1, x_2, \psi_1, \psi_2, u_1, u_2) \\ &= (1+\psi_1)\left\{(P_1 u_1(1-r)S_1 + P_1 r S_1)x_1 + P_1 u_1(1-r)S_2 x_2\right\} \\ &+ (1+\psi_2)\left\{P_2 u_2(1-r)S_1 x_1 + (P_2 u_2(1-r)S_2 + P_2 r S_2)x_2\right\} \\ &= (1+\psi_1)(a_{11}x_1 + a_{12}x_2) + (1+\psi_2)(a_{21}x_1 + a_{22}x_2) \end{aligned} \tag{16}$$

A neccessary condition such that $\hat{u}_1$ and $\hat{u}_2$ maximize $x_1(T)+x_2(T)$ is to exist $\psi_1(T)$ and $\psi_2(T)$ satisfied following equation:

$$H(x_1, x_2, \psi_1, \psi_2, \hat{u}_1, \hat{u}_2) \geqq H(x_1, x_2, \psi_1, \psi_2, u_1, u_2) \tag{17}$$

Where, ψ_1 and ψ_2 are the auxiable variables satisfied following equations:

$$\frac{d\psi_1}{dt}=-\frac{\partial H}{\partial x_1}=-\left\{a_{11}(1+\psi_1)+a_{21}(1+\psi_2)\right\} \tag{18}$$

$$\frac{d\psi_2}{dt}=-\frac{\partial H}{\partial x_2}=-\left\{a_{12}(1+\psi_1)+a_{22}(1+\psi_2)\right\} \tag{19}$$

And the transversality condition is as follows:

$$\psi_1(T)\,x_1(T)+\psi_2(T)\,x_2(T)=0 \tag{20}$$

The transversality condition is equivalent to the following equation corresponding to $x_1(T)=x_2(T)\neq 0$.

$$\psi_1(T)+\psi_2(T)=0 \tag{21}$$

The equation (17) shows that $\hat{u}_1$ and $\hat{u}_2$ maximize H at each time t, but they are constrained by (3)~(5). H can be written as follows:

$$H=\left\{(1+\psi_1)\,P_1u_1+(1+\psi_2)\,P_2u_2\right\}(1-r)\,(S_1x_1+S_2x_2)+P_1rS_1x_1+P_2rS_2x_2$$

We notice that H is a linear function with respect to u_1, and u_2. Then, we can get u_1 and u_2 which maximize H from following lemma 1.

Lemma 1

Let us consider the following optimization problem.

$$\underset{(u_1,\cdots,u_n)}{\text{maximize}}\qquad F(u_1,\cdots,u_n)=\sum_{i=1}^{n}a_iu_i$$

subject to

$$a_1>a_2>\cdots\cdots>a_n$$

$$\sum_{i=1}^{n}u_i=1\,,\qquad \alpha_i\leqq u_i\leqq\beta_i\,,\qquad 0\leqq\alpha_i\leqq\frac{1}{n}\,,\qquad \beta_i=1-\sum_{j\neq i}\alpha_j$$

Then, the solutions of this problem are given by the following equations.

$$u_1=\beta_1\,,\qquad u_i=\alpha_i\qquad(i\geqq 2)$$

Proof

We shall introduce the Lagrangian L from Kuhn-Tucker theorem, because this problem is the optimization problem under the equality and inequality conditions.

$$L\equiv-\sum_{i=1}^{n}a_iu_i+\lambda\left(\sum_{i=1}^{n}u_i-1\right)+\sum_{i=1}^{n}\gamma_i(\alpha_i-u_i)+\sum_{i=1}^{n}\delta_i(u_i-\beta_i)$$

A neccessary condition of this problem is to exist u_i, λ, γ_i, δ_i which satisfy the following system of equations.

$$\frac{\partial L}{\partial u_i}=0$$

$$\frac{\partial L}{\partial\lambda}=0$$

$$\frac{\partial L}{\partial\gamma_i}\leqq 0\,,\qquad \frac{\partial L}{\partial\gamma_i}\gamma_i=0\qquad \gamma_i\geqq 0$$

$$\frac{\partial L}{\partial \delta_i} \leqq 0, \qquad \frac{\partial L}{\partial \delta_i}\delta_i = 0 \qquad \delta_i \geqq 0 \quad (i=1, 2, \cdots, n)$$

We obtain the next equations which are equivalent to the equations mentioned above.

$$-a_i+\lambda-\gamma_i+\delta_i=0$$

$$\sum_{j=1}^{n} u_j = 1$$

$$\alpha_i - u_i \leqq 0, \qquad \gamma_i(\alpha_i-u_i)=0 \qquad \gamma_i \geqq 0$$

$$u_i-\beta_i \leqq 0, \qquad \delta_i(u_i-\beta_i)=0 \qquad \delta_i \geqq 0$$

If $i>j$, then $-a_i+\lambda-\gamma_i+\delta_i=-a_j+\lambda-\gamma_j+\delta_j$

$\therefore \quad a_i-a_j=\gamma_j-\gamma_i+\delta_i-\delta_j$

$\therefore \quad \gamma_j$ or δ_i is positive,

$\gamma_j>0 \longrightarrow u_j=\alpha_j, \qquad \delta_j=0$

$\delta_i>0 \longrightarrow u_i=\beta_i, \qquad \gamma_i=0$

If $\forall \gamma_j>0$, then $u_j=\alpha_j$ and $\sum_{j=1}^{n} u_j = \sum_{j=1}^{n} \alpha_j \leqq 1$

$\therefore \quad u_j=\alpha_j$ satisfies the condition only for $\sum_{j=1}^{n} \alpha_j = 1$

This means that $\alpha_i=\frac{1}{n}$ and $u_j=\beta_j$ by the relation $\beta_i=1-\sum_{j\neq i} \alpha_j=\frac{1}{n}$

when $\sum_{j=1}^{n} \alpha_j<1$, $\exists \gamma_j$ $s.\ t.$ $\gamma_j=0$. Then $\delta_j>0$ and $u_j=\beta_j$

As $\sum_{i\neq j} u_i=1-\beta_j$ and $\sum_{i\neq j} \alpha_i=1-\beta_i$

we obtain $u_i=\alpha_i$ and $\delta_i=0$.

If $i<j$, then $a_i-a_j=0-\gamma_i+0-\delta_j<0$

This shows contradiction $\quad \therefore \ j=1$

This means that $u_1=\beta_1$, $u_i=\alpha_i$ $(i\geqq 2)$

It is clear that these solutions maximize the objective function of the problem.

Q. E. D.

From this Lemma 1 we can see that the maximization of H is made by the following equations.

$$\hat{u}_1=1-D_2, \qquad \hat{u}_2=D_2 \qquad \text{if} \quad (1+\psi_1)P_1>(1+\psi_2)P_2 \tag{22}$$

$$\hat{u}_1=D_1, \qquad \hat{u}_2=1-D_1 \qquad \text{if} \quad (1+\psi_1)P_1<(1+\psi_2)P_2 \tag{23}$$

Because, ψ_1 and ψ_2 are the solutions of the system of the differential equations (18) and (19), and they are continuous with respect to t. The set of t satisfied (22) or (23) has its inner point. Then, $\hat{u}_1$ and $\hat{u}_2$ are piecewise constant, and we can develop Lemma 2 as follows:

Lemma 2

Coefficients a_{ij} of the system of the differential equations (10) satisfied $\frac{da_{ij}}{dt}=0$ at almost everywhere. In other words, a_{ij} are a piecewise constant.

From this Lemma 2, we can treat the system of the differential equations (10) as one of constant coefficient.

5. The Optimal Regional Investment Allocation Policy

5.1 Phase plane approach

We assume that the system of the differential equations (1)~(7) has solutions, and we shall consider how the regional investment allocation would be decided to maximize the national income. Later, we will investigate the existence of the solution of the system.

In the chapter 4, we could see that the optimal rate of the investment allocation were able to be obtained by (22) and (23). In this chapter, we shall consider when the policy should be done in time interval $0 \leqq t \leqq T$. Let ϕ_1 and ϕ_2 be the solutions of the differential equations (18) and (19). We set $\varphi_i = 1 + \phi_i$.

Then:

$$\frac{d\varphi_1}{dt} = -a_{11}\varphi_1 - a_{21}\varphi_2 \tag{18'}$$

$$\frac{d\varphi_2}{dt} = -a_{12}\varphi_1 - a_{22}\varphi_2 \tag{19'}$$

$$\varphi_1(T) + \varphi_2(T) = 2 \tag{21'}$$

And in vector representation we have:

$$\frac{d\varphi}{dt} = -A^*\varphi, \qquad \varphi = \begin{pmatrix}\varphi_1 \\ \varphi_2\end{pmatrix} \tag{24}$$

Where A^* means a conjugate matrix of A. Let us depict a phase plane of φ_1 and φ_2 likewise x_1 and x_2. The variations of $\frac{d\varphi_2}{d\varphi_1}$ can be shown in the following Table 5 as $\frac{d\varphi_2}{d\varphi_1} = \frac{a_{12}\varphi_1 + a_{22}\varphi_2}{a_{11}\varphi_1 + a_{21}\varphi_2}$, $\frac{d\varphi_1}{dt} = 0 \Leftrightarrow \varphi_2 = -\frac{a_{11}}{a_{21}}\varphi_1$, $\frac{d\varphi_2}{dt} = 0 \Leftrightarrow \varphi_2 = -\frac{a_{12}}{a_{22}}\varphi_1$.

Table 5. The relation between φ_i and $\frac{d\varphi_2}{d\varphi_1}$

$\frac{d\varphi_2}{d\varphi_1}$ \ φ_1, φ_2	$\varphi_2 > -\frac{a_{11}}{a_{21}}\varphi_1$	$\varphi_2 < -\frac{a_{11}}{a_{21}}\varphi_1$
$\frac{d\varphi_2}{d\varphi_1} \geqq \frac{P_1}{P_2}$	$\varphi_2 \geqq \frac{P_1a_{11} - P_2a_{12}}{P_2a_{22} - P_1a_{21}}\varphi_1$	$\varphi_2 \leqq \frac{P_1a_{11} - P_2a_{12}}{P_2a_{22} - P_1a_{21}}\varphi_1$
$0 \leqq \frac{d\varphi_2}{d\varphi_1} < \frac{P_1}{P_2}$	$\varphi_2 < \frac{P_1a_{11} - P_2a_{12}}{P_2a_{22} - P_1a_{21}}\varphi_1$	$\varphi_2 > \frac{P_1a_{11} - P_2a_{12}}{P_2a_{22} - P_1a_{21}}\varphi_1$
$-\frac{P_1}{P_2} \leqq \frac{d\varphi_2}{d\varphi_1} < 0$	$\varphi_2 \geqq -\frac{P_1a_{11} + P_2a_{12}}{P_1a_{21} + P_2a_{22}}\varphi_1$	$\varphi_2 \leqq -\frac{P_1a_{11} + P_2a_{12}}{P_1a_{21} + P_2a_{22}}\varphi_1$
$\frac{d\varphi_2}{d\varphi_1} < -\frac{P_1}{P_2}$	$\varphi_2 < -\frac{P_1a_{11} + P_2a_{12}}{P_1a_{21} + P_2a_{22}}\varphi_1$	$\varphi_2 > -\frac{P_1a_{11} + P_2a_{12}}{P_1a_{21} + P_2a_{22}}\varphi_1$

Therefore, because of the situation (φ_1, φ_2) and its moving direction $\frac{d\varphi_2}{d\varphi_1}$, $\varphi_1-\varphi_2$ plane is devided into subplanes which have the following boundaries:

$$\varphi_2=-\frac{a_{11}}{a_{21}}\varphi_1 \tag{25}$$

$$\varphi_2=-\frac{a_{12}}{a_{22}}\varphi_1 \tag{26}$$

$$\varphi_2=\frac{P_1a_{11}-P_2a_{12}}{P_2a_{22}-P_1a_{21}} \tag{27}$$

$$\varphi_2=-\frac{P_1a_{11}+P_2a_{12}}{P_1a_{21}+P_2a_{22}} \tag{28}$$

Next, we shall show the change of the boundaries mentioned above in Table 6 corresponding to a_{ij} in Table 1.

Table 6. The changes of the boundaries in $\varphi_1-\varphi_2$ plane

Coefficient of φ_1 \ u_1	0	1
$-\frac{a_{11}}{a_{21}}$	$-\frac{P_1r}{P_2(1-r)}$	$-\infty$
$-\frac{a_{12}}{a_{22}}$	0	$-\frac{P_1(1-r)}{P_2r}$
$\frac{P_1a_{11}-P_2a_{12}}{P_2a_{22}-P_1a_{21}}$	$\frac{P_1^2rS_1}{P_2(P_2S_2-P_1(1-r)S_1)}$	$\frac{P_1(P_1S_1-P_2(1-r)S_2)}{P_2^2rS_2}$
$-\frac{P_1a_{11}+P_2a_{12}}{P_1a_{21}+P_2a_{22}}$	$-\frac{P_1^2rS_1}{P_2(P_2S_2+P_1(1-r)S_1)}$	$-\frac{P_1(P_1S_1+P_2(1-r)S_2)}{P_2^2rS_2}$

Fig. 3 shows the phase plane of φ_1 and φ_2 due to the result of Table 6.

We obtain the orbit of φ_1 and φ_2 where $\varphi_1(T)>\frac{2P_2}{P_1+P_2}$ or $\varphi_1(T)<\frac{2P_2}{P_1+P_2}$, because the interaction of the transversality condition $\varphi_1(T)+\varphi_2(T)=2$ and the switching point of controls $P_1\varphi_1(T)=P_2\varphi_2(T)$ at $t=T$ is $\left(\frac{2P_2}{P_1+P_2}, \frac{2P_1}{P_1+P_2}\right)$. The orbit is depicted in Fig. 4.

There are three possible cases for the orbit of φ_1 and φ_2 due to the variations of the minimum proportion rate D_1 and D_2. Namely, as we denote the line as $\varphi_2=\alpha(u_1)\varphi_1$ such that $\frac{d\varphi_2}{d\varphi_1}=\frac{P_1}{P_2}$ is held, one of the following cases is realized.

(1) $\alpha(D_1)$ and $\alpha(1-D_2)\leqq\frac{P_1}{P_2}$

(2) $\alpha(D_1)>\frac{P_1}{P_2}>\alpha(1-D_2)$

(3) $\alpha(D_1)$ and $\alpha(1-D_2)\geqq\frac{P_1}{P_2}$

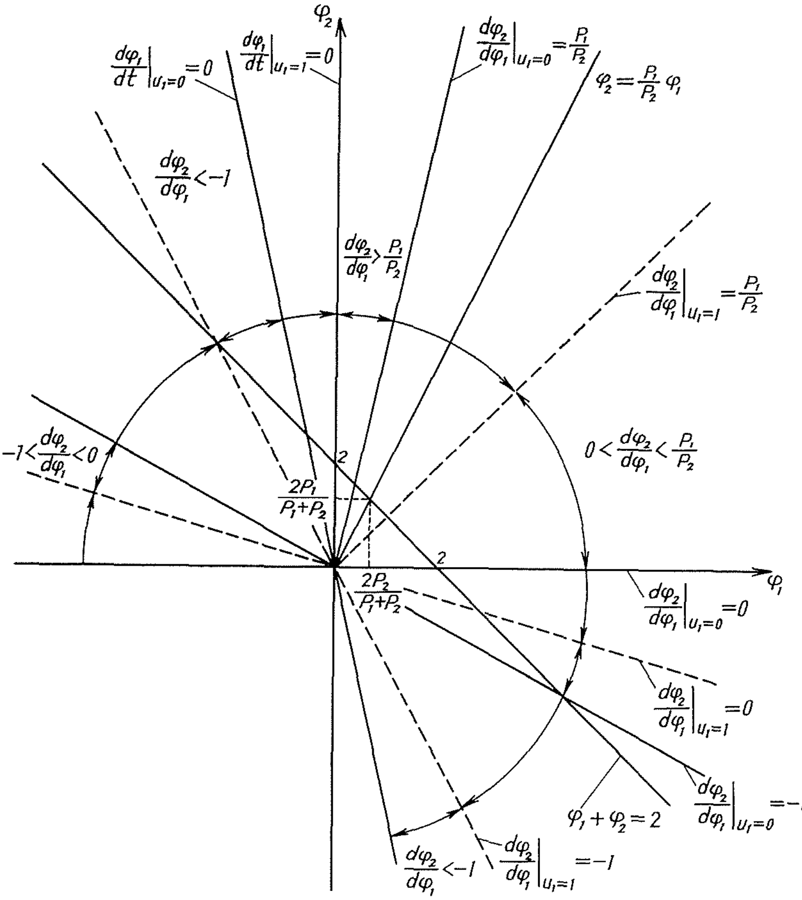

Fig. 3. The phase plane of φ_1 and φ_2.

Corresponding to these cases, the behaviors of $\varphi_1(t)$ and $\varphi_2(t)$ are classified in the following :

①	$P_1\varphi_1(t) > P_2\varphi_2(t)$	$0 \leqq t \leqq T$
②	$P_1\varphi_1(t) < P_2\varphi_2(t)$	$^{\exists}t^* < t \leqq T$
	$P_1\varphi_1(t) = P_2\varphi_2(t)$	$t = t^*$
	$P_1\varphi_1(t) > P_2\varphi_2(t)$	$^{\exists}t^* < t \leqq T$
③	$P_1\varphi_1(t) > P_2\varphi_2(t)$	$0 \leqq t < {}^{\exists}t^* < T$
	$P_1\varphi_1(t) = P_2\varphi_2(t)$	$t = t^*$
	$P_1\varphi_1(t) < P_2\varphi_2(t)$	$^{\exists}t^* < t \leqq T$
④	$P_1\varphi_1(t) < P_2\varphi_2(t)$	$0 \leqq t \leqq T$

Then we can get the following Theorem 1.

Theorem 1

Assume that the solution of the regional growth model with (1)~(7) exists, the

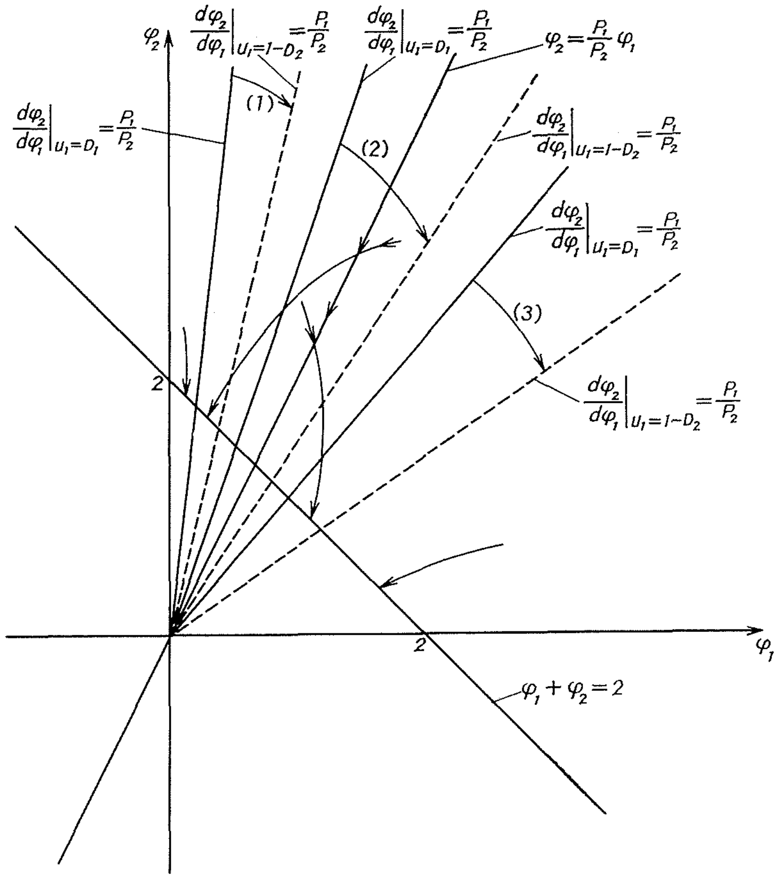

Fig. 4. The behaviors of φ_1 and φ_2.

Table 7. The optimal regional allocation policy

case \ interval	$0 \leqq t \leqq t^*$	$t^* < t \leqq T$
i	$u_1=1-D_2$, $u_2=D_2$	$u_1=1-D_2$, $u_2=D_2$
ii	$u_1=D_2$, $u_2=1-D_2$	$u_1=1-D_2$, $u_2=D_2$
iii	$u_1=1-D_2$ $u_2=D_2$	$u_1=D_1$, $u_2=1-D_1$
iv	$u_1=u_1^*$, $u_2=1-u_1^*$	$u_1=1-D_2$, $u_2=D_2$
v	$u_1=u_1^*$, $u_2=1-u_1^*$	$u_1=D_1$, $u_2=1-D_1$
vi	$u_1=D_1$, $u_2=1-D_1$	$u_1=D_1$, $u_2=1-D_1$

policy of regional investment allocation which maximizes the sum of two regional incomes is any one of the cases shown in Table 7.

Here, u_1^* means the value of u_1 such that $P_2\varphi_2=P_1\varphi_1$ and $\frac{d\varphi_2}{d\varphi_1}=\frac{P_1}{P_2}$.

Proof

From Lemma 1 and the linearity of Hamiltonian H with respect to u_i, the optimal policy of regional investment allocation i, ii, iii and vi is corresponding to ①~④ in (1) and (3) mentioned above. And so it is sufficient to prove iv and v.

We shall depict case (2) in the phase plane Fig. 5 of φ_1 and φ_2. In Fig. 5, there are five points from Q_1 to Q_5 gratifying the boundary condition $\varphi_1(T)+\varphi_2(T)=2$. Q_1 and Q_5 are on the orbit corresponding to case i and case ii. When we choose Q_2 for the boundary condition, the optimal policy of regional allocation is $u_1=D_1$, $u_2=1-D_2$ at $t^*<t\leqq T$ due to $^{\exists}t^*<t\leqq T$ such that $P_1\varphi_1(t)<P_2\varphi_2(t)$.

When we had performed a policy such as $u_1=D_1$, $u_2=1-D_1$ at $t<t^*$ assuming $P_1\varphi_1(t^*)=P_2\varphi_2(t^*)$ at $t=t^*$, $\dfrac{d\varphi_2}{d\varphi_1}<\dfrac{P_1}{P_2}$ holds and $\varphi_2=\varphi_2(\varphi_1)$ curve is under $\varphi_2=\dfrac{P_1}{P_2}\varphi_1$ (see dotted line). Therefore $P_1\varphi_1>P_2\varphi_2$ is held and the optimal policy should be $u_1=1-D_2$ and $u_2=D_2$. This means contradiction. And when we had carried out a policy that $u_1=1-D_2$ and $u_2=D_2$ at $t<t^*$, $\dfrac{d\varphi_2}{d\varphi_1}>\dfrac{P_1}{P_2}$ holds and $\varphi_2=\varphi_2(\varphi_1)$ curve is above $\varphi_2=\dfrac{P_1}{P_2}\varphi_1$. So the optimal policy sholuld be $u_1=D_1$ and $u_2=1-D_1$, and this shows contradiction. Consequently, the optimal policy at $t\leqq t^*$ must be performed so that $\varphi_1(t)$ and $\varphi_2(t)$ are on the line $\varphi_2=\dfrac{P_1}{P_2}\varphi_1$. When we denote the optimal policy as $u_1=u_1^*$, $u_2=1-u_1^*$. We have:

$$\varphi_2\bigg|_{u_1=u_1^*}=\frac{P_1}{P_2}\varphi_1\bigg|_{u_1=u_1^*},\qquad \frac{d\varphi_2}{d\varphi_1}\bigg|_{u_1=u_1^*}=\frac{P_1}{P_2}$$

When the boundary condition is Q_3 or Q_4, the proof is similar to the analysis

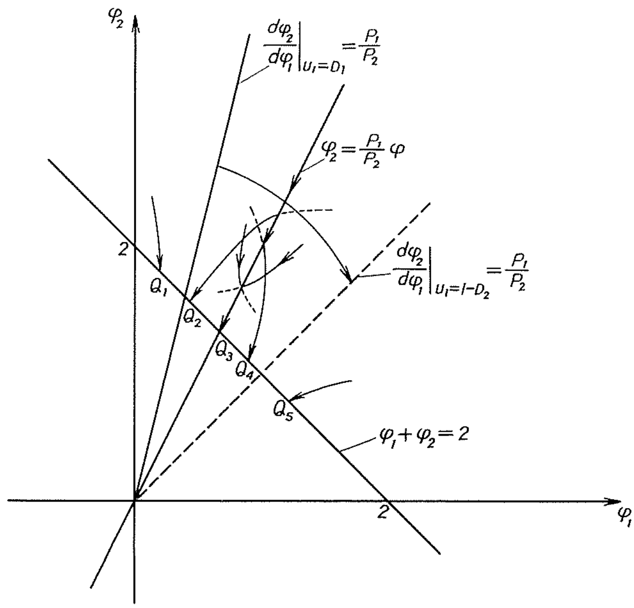

Fig. 5. The behaviors φ_1 and φ_2 (case of $\alpha(1-D_2)<\dfrac{p_1}{p_2}<\alpha(D_1)$).

mentioned above. Q. E. D.

Corollary 1-1

u_1^* and u_2^* are the time invariant constants.

Proof

From the definition of u_1^*, $P_1\varphi_1=P_2\varphi_2$ and $\frac{d\varphi_2}{d\varphi_1}=\frac{P_1}{P_2}$ are realized at $u_1=u_1^*$.

$$\frac{d\varphi_2}{d\varphi_1}=\frac{a_{12}\varphi_1+a_{22}\varphi_2}{a_{11}\varphi_1+a_{21}\varphi_2}$$

$$=\frac{a_{12}P_2\varphi_1+a_{22}P_2\varphi_2}{a_{11}P_2\varphi_1+a_{21}P_2\varphi_2}$$

$$=\frac{a_{12}P_2\varphi_1+a_{22}P_1\varphi_1}{a_{11}P_2\varphi_1+a_{21}P_1\varphi_1}$$

$$=\frac{a_{12}P_2+a_{22}P_1}{a_{11}P_2+a_{21}P_1}=\frac{P_1}{P_2}$$

Because this relation is not dependent on time, u_1^* and u_2^* are independent of time vaiable t. Q. E. D.

We have a question of how we choose an allocation policy in the cases from i to iv when the planned period T and the values of $\varphi_1(T)$ and $\varphi_2(T)$ are given. For this question we must consult the concrete behaviors of $\varphi_1(t)$ and $\varphi_2(t)$. Next, we shall try to investigate the behaviors of $\varphi_1(t)$ and $\varphi_2(t)$.

5.2 Behaviors of $\varphi_1(t)$ and $\varphi_2(t)$

Let $\varphi_1(t)$ and $\varphi_2(t)$ be solutions of the system of differential equations (24). We assume that the system (24) is transformed into (30) by the linear transformation (29).

$$B\cdot\begin{pmatrix}\varphi_1(t)\\ \varphi_2(t)\end{pmatrix}=\begin{pmatrix}\eta_1(t)\\ \eta_2(t)\end{pmatrix}\qquad B=\begin{pmatrix}b_{11}, & b_{12}\\ b_{21}, & b_{22}\end{pmatrix} \tag{29}$$

$$\frac{d}{dt}\begin{pmatrix}\eta_1(t)\\ \eta_2(t)\end{pmatrix}=\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}\begin{pmatrix}\eta_1(t)\\ \eta_2(t)\end{pmatrix} \tag{30}$$

i. e. $\frac{d\eta_1}{dt}=\lambda_1\eta_1$, $\quad\frac{d\eta_2}{dt}=\lambda_2\eta_2$

Substituting (29) into (30):

$$B\cdot\frac{d}{dt}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}=-BA^*\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}=\frac{d}{dt}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}=\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}B\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}$$

From the arbitarity of $\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}$, $-BA^*=\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}B$

$$\therefore\quad\begin{pmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix}\begin{pmatrix}-a_{11}, & -a_{21}\\ -a_{12}, & -a_{22}\end{pmatrix}=\begin{pmatrix}\lambda_1 b_{11}, & \lambda_1 b_{12}\\ \lambda_2 b_{21}, & \lambda_2 b_{22}\end{pmatrix} \tag{31}$$

Consequently, we can see that λ_1 and λ_2 are eigen values of $-A$ and (b_{11},b_{12}), (b_{21}, b_{22}) are eigen vectors corresponding to λ_1 and λ_2. Therefore, if we find eigen values and eigen vectors of $-A$, we can obtain the linear transformation (29).

The eigen equation of A is $\det(A-\lambda E)=0$.

Where, E means 2×2 unit matrix.

$$\therefore \quad \lambda^2-(a_{11}+a_{22})\,\lambda+a_{11}a_{22}-a_{12}a_{21}=0$$

$$\therefore \quad \lambda=\frac{1}{2}\left\{a_{11}+a_{22}\pm\sqrt{(a_{11}+a_{22})^2-4\,(a_{11}a_{22}-a_{12}a_{21})}\right\}$$

$$=\frac{1}{2}\left\{a_{11}+a_{22}\pm\sqrt{(a_{11}-a_{22})^2+4a_{12}a_{21}}\right\}$$

λ is a real number because of $(a_{11}-a_{22})^2+4a_{12}a_{21}>0$. And set $f(\lambda)\equiv\lambda^2-(a_{11}+a_{22})\lambda_1+a_{11}a_{22}-a_{12}a_{21}$. Then $f(0)=a_{11}a_{22}-a_{12}a_{21}\geqq 0$ by the result in Table 1. The graph of $f(\lambda)$ as a function of λ has a positive axis and its value at $\lambda=0$ is nonnegative. Then we can see that two real roots of $f(\lambda)=0$ are nonnegative. Let us denote the roots $0\leqq\lambda_1<\lambda_2$.

The eigen equation of $-A$ is $\det(A+\lambda E)=0$.

$$\therefore \quad \lambda^2+(a_{11}+a_{22})\,\lambda+a_{11}a_{22}-a_{12}a_{21}=0$$

Set $g(\lambda)\equiv\lambda^2+(a_{11}+a_{22})\,\lambda+a_{11}a_{22}-a_{12}a_{21}$. Then $g(-\lambda)=f(\lambda)$. Accordingly the graph of $g(\lambda)$ is symmetric to $f(\lambda)$ at $\lambda=0$. The eigen values of $-A$ are both nonpositive and they are given by $-\lambda_2<-\lambda_1\leqq 0$. And the eigen vectors are as follows:

$$\begin{cases}(\lambda_1-a_{11})\,b_{11}-a_{12}b_{12}=0\\ -a_{21}b_{11}+(\lambda_1-a_{22})\,b_{12}=0\end{cases}\qquad \begin{cases}(\lambda_2-a_{11})\,b_{21}-a_{12}b_{22}=0\\ -a_{21}b_{21}+(\lambda_2-a_{22})\,b_{22}=0\end{cases}$$

$$\therefore \quad b_{11}=a_{12}, \qquad b_{12}=\lambda_1-a_{11}, \qquad b_{21}=\lambda_2-a_{22}, \qquad b_{22}=a_{21}$$

$$\therefore \quad B=\begin{pmatrix}a_{12} & \lambda_1-a_{11}\\ \lambda_2-a_{22} & a_{21}\end{pmatrix}$$

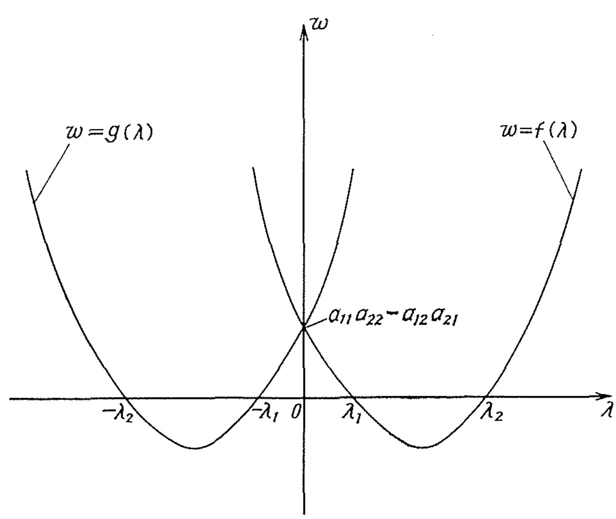

Fig. 6. Graphs of $f(\lambda)$ and $g(\lambda)$.

Through the linear transformation we have:

$$\frac{d}{dt}\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = -\begin{pmatrix} -\lambda_1 \eta_1 \\ -\lambda_2 \eta_2 \end{pmatrix}$$

$$\therefore \quad \eta_1 = c_1 e^{-\lambda_1 t}, \qquad \eta_2 = c_2 e^{-\lambda_2 t}$$

By the way, u_1 and u_2 have a possibility that they change their values at $t=t^*$. In such a case the coefficients of (29) must change and the eigen values λ_1 and λ_2 also change. And let us represent the changes of the coefficients and the eigen values as follows:

$$0 \leqq t \leqq t^* \Rightarrow b_{ij},\ \lambda_i$$

$$t^* < t \leqq T \Rightarrow b'_{ij},\ \lambda'_i$$

Consequently, we obtain the exact solutions η_1 and η_2 with the boundary condition $\varphi_1(T)+\varphi_2(T)=2$ in the following:

$$\eta_1(t) = \begin{cases} \left(b_{11}\varphi_1(t^*)+b_{12}\varphi_2(t^*)\right) e^{-\lambda_1(t-t^*)} & 0 \leqq t \leqq t^* \\ \left(b'_{11}\varphi_1(T)+b'_{12}(2-\varphi_1(T)\right) e^{-\lambda'_1(t-T)} & t^* \leqq t \leqq T \end{cases}$$

$$\eta_2(t) = \begin{cases} \left(b_{21}\varphi_1(t^*)+b_{22}\varphi_2(t^*)\right) e^{-\lambda_2(t-t^*)} & 0 \leqq t \leqq t^* \\ \left(b'_{21}\varphi_1(T)+b'_{22}(2-\varphi_1(T)\right) e^{-\lambda'_2(t-T)} & t^* < t \leqq T \end{cases}$$

When $\lambda_i \neq \lambda'_i$ i.e. the policy of the investment allocation change at $t=t^*$, $P_1\varphi_1(t^*)=P_2\varphi_2(t^*)$ holds and, η_1 and η_2 must be continuous at $t=t^*$. Then we get:

$$\left(b_{11}P_2\varphi_1(t^*)+b_{12}P_1\varphi_1(t^*)\right) = P_2\left(b'_{11}\varphi_1(T)+b'_{12}(2-\varphi_1(T)\right) e^{-\lambda'_1(t^*-T)}$$

$$\left(b_{21}P_2\varphi_1(t^*)+b_{22}P_1\varphi_1(t^*)\right) = P_2\left(b'_{21}\varphi_1(T)+b'_{22}(2-\varphi_1(T)\right) e^{-\lambda'_2(t^*-T)}$$

The ratio of the both equations is as follows:

$$\frac{b_{11}P_2+b_{12}P_1}{b_{21}P_2+b_{22}P_1} = \frac{2b'_{12}+(b'_{11}-b'_{12})\,\varphi_1(T)}{2b'_{22}+(b'_{21}-b'_{22})\,\varphi_1(T)}\, e^{(\lambda'_1-\lambda'_2)(T-t^*)}$$

Then, we obtain the following theorem.

Theorem 2

If there exists some t^* such that $0<t^*<T$ and

$$\frac{b_{11}P_2+b_{12}P_1}{b_{21}P_2+b_{22}P_1} = \frac{2b'_{12}+(b'_{11}-b'_{12})\,\varphi_1(T)}{2b'_{22}+(b'_{21}-b'_{22})\,\varphi_1(T)}\, e^{(\lambda'_1-\lambda'_2)(T-t^*)} \tag{32}$$

is realized for the optimal policy of investment allocation $u_i(T)$ and the eigen values corresponding to $u_i(T)$ which are decided the relation $P_1\varphi_1(T) \gtreqless P_2\varphi_2(T)$ at $t=T$, then $t=t^*$ is a switching time of optimal policy.

Corollary 2-1

When the planned period T is sufficiently short and

$$\frac{b_{11}P_2+b_{12}P_1}{b_{21}P_2+b_{22}P_1} \neq \frac{2b'_{12}+(b'_{11}-b'_{12})\varphi_1(T)}{2b'_{22}+(b'_{21}-b'_{22})\varphi_1(T)}$$

holds, the optimal policy of the investment allocation is concentrated to the either of two regions.

Proof

As $e^{(\lambda'_1-\lambda'_2)T}$ is a continuous function with respect to T, there exists some positive number δ such that $0<{}^\forall T<\delta$,

$$\frac{b_{11}P_2+b_{12}P_1}{b_{21}P_2+b_{22}P_1} \neq \frac{2b_{12}+(b_{11}-b_{12})\varphi_1(T)}{2b_{22}+(b_{21}-b_{22})\varphi_1(T)}\, e^{(\lambda'_1-\lambda'_2)T}$$

$$\therefore\quad 0<{}^\forall t^*<T<\delta,\qquad \frac{b_{11}P_2+b_{12}P_1}{b_{21}P_2+b_{22}P_1} \neq \frac{2b_{12}+(b_{11}-b_{12})\varphi_1(T)}{2b_{22}+(b_{21}-b_{22})\varphi_1(T)}\, e^{(\lambda'_1-\lambda'_2)(T-t^*)}$$

This shows that there is no switching time in the optimal policy. Q. E. D.

5.3 Existence Theorem

Up to the present, we have discussed the analysis under assuming that the system of the differential equations (1)~(7) had the solutions. Now, we must investigate the existence of the solutions of (1)~(7).

If we assume that (10) is transformed into:

$$\frac{d}{dt}\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$$

by a linear transformation $\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}=\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$.

Then, we can see that λ_1 and λ_2 are the eigen values of A and (c_{11}, c_{12}), (c_{21}, c_{22}) are the eigen vectors of A through the same discussions in 5.1.

$$\therefore\quad 0\leqq\lambda_1<\lambda_2 \text{ and}$$

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \lambda_1 c_{11} & \lambda_1 c_{12} \\ \lambda_2 c_{21} & \lambda_2 c_{22} \end{pmatrix}$$

$$\therefore\quad \begin{cases} (a_{11}-\lambda_1)\,c_{11}+a_{21}c_{12}=0 \\ a_{12}c_{11}+(a_{22}-\lambda_1)\,c_{12}=0 \end{cases} \qquad \begin{cases} (a_{11}-\lambda_2)\,c_2+a_{21}c_{22}=0 \\ a_{12}c_{21}+(a_{22}-\lambda_2)\,c_{22}=0 \end{cases}$$

$$\therefore\quad c_{11}=a_{21},\qquad c_{12}=\lambda_1-a_{11},\qquad c_{21}=\lambda_2-a_{22},\qquad c_{22}=a_{12}$$

$$\therefore\quad \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} a_{21} & \lambda_1-a_{11} \\ \lambda_2-a_{22} & a_{12} \end{pmatrix} = \begin{pmatrix} b_{22} & b_{12} \\ b_{21} & b_{11} \end{pmatrix}$$

Thus, we can see that c is a matrix which is replaced the diagonal elements of B. Under this linear transformation, (1) and (2) are turned into $\frac{dy_1}{dt}=\lambda_1 y_1$ and $\frac{dy_2}{dt}=\lambda_2 y_2$. And we have general solutions $y_1=d_1e^{\lambda_1 t}$, $y_2=d_2e^{\lambda_2 t}$.

Where, d_1 and d_2 are constant and also decided by the switching time u_1 and u_2 as follows:

$$y_1(t) = \begin{cases} (b_{22}x_1^0 + b_{12}x_2^0)\, e^{\lambda_1 t} & 0 \leqq t \leqq t^* \\ (b'_{22}x_1^T + b'_{12}x_2^T)\, e^{\lambda'_1 (t-T)} & t^* \leqq t \leqq T \end{cases}$$

$$y_2(t) = \begin{cases} (b_{21}x^0 + b_{11}x^0)\, e^{\lambda_2 t} & 0 \leqq t \leqq t^* \\ (b'_{21}x_1^T + b'_{11}x_2^T)\, e^{\lambda'_2 (t-T)} & t^* \leqq t \leqq T \end{cases}$$

Where, $x_i^0 \equiv x_i(0)$ and $x_i^T \equiv x_i(T)$, $i=1,\ 2$.
Because of the continuity of $y_i(t)$ at $t=t^*$:

$$(b_{22}x^0 + b_{12}x_2^0)\, e^{\lambda_1 t^*} = (b'_{22}x_1^T + b'_{12}x_2^T)\, e^{\lambda'_1 (t^*-T)}$$

$$(b_{21}x_1^0 + b_{11}x_2^0)\, e^{\lambda_2 t^*} = (b'_{21}x_1^T + b'_{11}x_1^T)\, e^{\lambda'_2 (t^*-T)}$$

Cancelling x_1^T and x_2^T to use the relation $x_1^T = x_2^T$ in the equations above, we have:

$$e^{(\lambda_1-\lambda_2)t^* - (\lambda'_1-\lambda'_2)t^* + (\lambda'_1-\lambda'_2)T} = \frac{(b_{21}x_1^0 + b_{11}x_2^0)(b'_{22}+b'_{12})}{(b_{22}x_1^0 + b_{12}x_2^0)(b'_{21}+b'_{11})}$$

Then, we can obtain the following theorem 3.

Theorem 3

In the regional growth model represented as (1) and (2), the intial regional incomes are x_1^0 and x_2^0, the terminal regional incomes at the planned period $t=T$ are equalized, and the national income $(x_1(T)+x_2(T))$ is maximized if and only if:

$$e^{(\lambda_1-\lambda_2)t^* - (\lambda'_1-\lambda'_2)t^* + (\lambda'_1-\lambda'_2)T} = \frac{(b_{21}x_1^0 + b_{11}x_2^0)(b'_{22}+b'_{12})}{(b_{22}x_1^0 + b_{12}x_2^0)(b'_{21}+b'_{11})} \tag{34}$$

Proof

From the investigation mentioned above, the necessary condition has already been proved. Then, we may only prove the sufficient condition.

From the general theory of the ordinary differential equation, there exist the unique solutions $x_1(t)$ and $x_2(t)$ for the intial condition $x_1(0)=x_1^0$ and $x_2(0)=x_2^0$, if we do not regard with the boundary condition $x_1(T)=x_2(T)$. Transforming (34) into the equation as follows:

$$x^T \equiv \frac{b_{22}x_1^0 + b_{12}x_2^0}{b'_{22}+b'_{12}}\, e^{(\lambda_1-\lambda'_1)t^* + \lambda'_1 T} = \frac{b_{21}x_1^0 + b_{11}x_2^0}{b'_{21}+b'_{11}}\, e^{(\lambda_2-\lambda'_2)t^* + \lambda'_2 T}$$

and setting $x_1(T) \equiv x_2(T) \equiv x^T$, we can see that this relation satisfies the boundary condition.

Therefore, the terminal regional incomes at $t=T$ are equalized. And in these solutions u_1 and u_2 are selected as the optimal ratio of regional investment allocation. Concequently, the national income at the planned period $t=T$ is maximized. The proof of the sufficient condition is completed. Q. E. D.

Corollary 3-1

Assume that there is a regional income disparity at the initial time and the planned period T is very short. Then, the equalization of the regional incomes at $t=T$ can not be made.

Proof

There comes into being $b_{ij}=b'_{ij}$ $(i, j=1, 2)$ because $x_1^0 \neq x_2^0$, and there is no switching point in the optimal policy from corollary 2-1.

$$\therefore \quad \frac{(b_{21}x_1^0+b_{11}x_2^0)(b'_{21}+b'_{11})}{(b_{22}x_1^0+b_{12}x_2^0)(b'_{22}+b'_{12})}=\frac{(b_{21}x_1^0+b_{11}x_2^0)(b_{21}+b_{11})}{(b_{22}x_1^0+b_{21}x_2^0)(b_{22}+b_{12})}\neq 1$$

On the other hand, $\lambda_i=\lambda'_i\,(i=1, 2)$ holds.

$$\therefore \quad e^{(\lambda_1-\lambda_2)t^*-(\lambda'_1-\lambda'_2)t^*+(\lambda'_1-\lambda'_2)T}=e^{(\lambda_1-\lambda_2)T}\to 1 \text{ as } T\to 0$$

This means that (34) is not satisfied if T is very small. Therefore, from the theorem 3 the equalization of the regional incomes can not be made. Q. E. D.

6. Conclusion

This paper has investigated how we should make a policy of regional investment allocation to maximize the national income in the regional growth model represented by (1)∼(7).

Firstly, we have investigated the optimal policy of the investment allocation assuming the existence of the solutions (1)∼(7). Secondly, we have examined the existence of the solutions corresponding to the optimal policy. Under these procedures we have been able to obtain the existence theorem and some results for the optimal policy. These give the exact theoretical proof to the simulation analysis in the papers 5) and 6).

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Summary

We have investigated the regional economic growth model that we reframe Yamamura formulation a continuous space format. The main results are as follows.

(1) Assume that the solution of the regional growth model with the equations from (1) to (7) exists, the policy of regional investment allocation which maximizes the sum of two regional incomes is any one of the cases shown in Table 7.

(2) If there exists some t^* such that $0<t^*<T$ and the equation (32) is realized for the optimal policy of investment allocation $u_i(T)$ and the eigen values λ_i^t corresponding to $u_i(T)$ which are decided the relation such as $P_1\varphi_1T \gtreqless P_2\varphi_2T$ at $t=T$, $t=t^*$ is a switching time of the optimal policy.

(3) When the planned period T is sufficiently short and the equation (33) holds, the optimal policy of the investment allocation is concentrated to the either of two regions.

(4) In the regional growth model represented as (1) and (2), the initial regional incomes are x_1^0 and x_2^0, the terminal regional incomes at the planned period $t=T$ are equalized, and the national income is maximized if and only if the equation (34) is satisfied.

(5) Assume that there is a regional income disparity at the initial time and the planned period T is very short, the equalization of the regional incomes at $t=T$ cannot be made.