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A Study on Model Reference Adaptive Control in Economic Development (I)

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経済発展におけるモデル規範適応制御に関する研究 (I)

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1. Introduction

There has been evident in the history of methodological advancement of economic analysis, an interaction between economic science and control science.[1~7]

Economic development, a branch of economics, may be defined as the quantitative analysis in its relation to control science. Although the analytical framework of control theory is of relatively recent vintage, the scope of applications of it has broadened greatly to the economic development.[8~16]

However, only a relatively small amount of research has been made on the use of adaptive control techniques. It was R. E. Murphy, Jr.[17] who first oriented the study on adaptive processes in economic systems. But he treated one of the adaptive techniques such as stochastic information control techniques.

Among various alternative methods for adaptive control techniques, the use of the technique known as model reference adaptive technique seems to be one of the most feasible approaches possible for the economic development analysis.

In this paper, we shall briefly review the model reference adaptive technique and consider one of the adaptive processes of the economic development policy model by using the model reference adaptive technique.

2. The Need for Model Reference Adaptive Technique

Among the various types of adaptive system configuration, model reference adaptive techniques are important since they lead to relatively easy-to-implement systems with a high speed of adaptation which can be used in a variety of situations.

The basic scheme of a model reference adaptive system is given in Fig. 1.

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One of the most important advantages of this type of adaptive system is its high speed of adaptation. This is due to the fact that a measure of the difference between the given index of performance specified by the reference model and the index of performance of the adjustable system is obtained directly by the comparison of states of the model with those of the adjustable model.

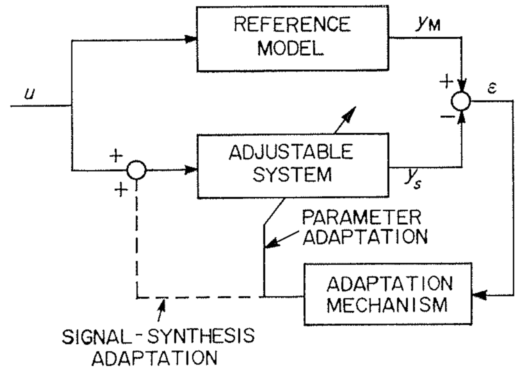

Fig. 1. Basic configuration of a model referance adaptive system.

The basic scheme of the model reference adaptive system given in Fig. 1 is called a parallel model reference adaptive system in order to differentiate it from other model reference adaptive model configurations where the relative placement of the reference model and of the adjustable system is not the same.

Consider the basic scheme of a model reference adaptive system, represented in Fig. 1, and a state-space representation for the reference model and the adjustable model.

The reference model is described by:

$$\frac{dx_M}{dt} = A_M x_M + B_M u \tag{2.1}$$

$$y_M = C \cdot x_M \tag{2.2}$$

The parallel adjustable model is described by:

$$\frac{dx_s}{dt} = A_s(t)\, x_s + B_s(t)\, u \tag{2.3}$$

$$y_s = C \cdot x_s \tag{2.4}$$

Where x_M and x_s are state vectors (n-dimensional), u is the input vector (m-dimensional), A_M and B_M are constant matrices, $A_s(t)$ and $B_s(t)$ are time-varying matrices (all of appropriate dimension), y_M and y_s are output vectors (r-dimensional), and C is the output matrix of appropriate dimension.

Note that the same matrix C is considered for both the model and the adjustable model without loss of generality, but it fixes the definition of the state vector of one system relative to that of the other.

We now introduce some important definitions as follows.

Definition 1.

State generalized error(e); the variable vector that represents the difference between the state vector of the model (x_M) and the state vector of the adjustable model (x_s).

$$e = x_M - x_s \tag{2.5}$$

Definition 2.

Output generalized error(e) ; the variable vector that represents the difference between the output of the model (y_M) and the output of the adjustable model (y_s).

$$e = y_M - y_s \tag{2.6}$$

Definition 3.

State distance ; any norm of the difference between the state vector of the reference model (x_M) and the state vector of the adjustable model (x_s).

Definition 4.

Parameter distance ; any norm of the difference between the parameter vector (or matrix) of reference model and the parameter vector (or matrix) of the adjustable model.

Definition 5.

Adaptive law (algorithm) ; the relation between the generalized error and the corresponding modifications of the parameters or of the output to the adjustable model.

Definition 6.

Adaptive mechanism ; set of interconnected linear, nonlinear, or time-varying blocks used to implement the adaptation law.

Definition 7.

Model reference adaptive system ;

Reference model

$$y_M = f_M(u, P_M, x_M, t) \tag{2.7}$$

Where P_M are parameters of the reference model.

$$y_s = f_s(u, P_s, x_s, t) \tag{2.8}$$

Where P_s are parameters of the adjustable model.

Index of performance which expresses the difference between the given index of performance specified by the model and that of the adjustable model.

$$J_{IP} = F(\varepsilon, P_M - P_s, e, t) \tag{2.9}$$

Model reference adaptive system minimizes the J_{IP} defined above by parameter adaptation or by signal-synthesis adaptation via the adaptation mechanism that has the generalized error as one of its inputs.

In this section, we shall briefly present some application of model reference adaptive techniques to solve the specific economic control problem.

Consider the typical economic development system such as the reference model specified by the developed country (region) economy and the developing country (region) economy specified by the adjustable model.

The aim of this economic development system is how to develop the economic level from the developing country (region) economy to the developed country (region) economy.

Namely, the economic development system minimizes the J_{IP} defined above by parameter adaptation or by signal-synthesis adaptation via the adaptation mechanism.

The parameter adaptation is represented in an industrial structure change to develop the developing country (region) industrial structure from the primary industrial structure to high industrial structure such as the developed country (region) economy.

The signal-synthesis adaptation is represented in an allocation changes of investments between industrial sectors to develop the output of developing country (region) economy to the same level of developed country (region) economy.

Consider the another typical economic policy system such as the real economic system specified by the reference model and the economic planning model specified by the adjustable model and also the reverse specification.

The aim of this economic policy system is how to control the index of performance which expresses the difference between the given index of performance specified by real economic system and that of the economic planning model.

Namely, the economic policy system minimizes the J_{IP} defined above by parameter adaptation or by signal-synthesis adaptation via the adaptation mechanism.

The parameter adaptation is represented in an minimization of any norm of the difference between the state vector of real economic system and the economic planning model.

The signal-synthesis adaptation is represented in an input control that minimizes the difference between the output of the real economic system and the economic planning model.

There are also many economic control studies for various applications of model reference adaptive techniques.

3. The Design for Model Reference Adaptive System

In this chapter, we shall make precise the analytical design problem for model reference adaptive systems refered to Landau's book.[18]

There are many types of model reference adaptive systems, but it will be easy to relate the specific configuration to some typical types by considering several criteria for classification and applying these criteria to a given configuration.

The most popular structure is the parallel configuration, often called the output error method. The series-parallel configuration, often called the equation error method. The series configuration is also used for identification, often called the input error method.

The classification of indices of performance specified such as minimization of a norm of the output generalized error and its derivatives, minimization of the state distance and minimization of the parameter distance.

The classification of types of application specified such as adaptive model-following control system, identification with an adjustable model, state observation, adaptive regulation and extremal control.

3.1 Mathematical Description of Model Reference Adaptive Systems

We shall present the mathematical description of the model reference adaptive systems such as parallel, series-parallel, and series types.

1) Parallel Model Reference Adaptive Systems

Consider again Fig. 1. To describe the reference model, we shall consider the following two formats ; (a) state-variable equations and (b) input/output relations in terms of differential operators.

(a) state-variable equations

Reference model

$$\frac{dx_M}{dt}=A_M x_M+B_M u \qquad x_M(t_0)=x_M \tag{3.1}$$

Where,

x_M : the model state vector (n-dimensional)

u : the vector input (m-dimensional) belonging to the case of piecewise continous vector functions.

A_M : the constant matrices $(n\times n)$-dimensional.

B_M : the constant matrices $(n\times m)$-dimensional.

The reference model is taken to be stable and completely controllable.

Adjustable model with adjustable parameters

$$\begin{aligned}&\frac{dx_s}{dt}=A_s(e,t)\,x_s+B_s(e,t)\,u\\&x_s(t_0)=x_{s_0},\ A_s(t_0)=A_{s_0},\ B_s(t_0)=B_{s_0}\,.\end{aligned} \tag{3.2}$$

x_s : the adjustable model state vector (n-dimensional)

A_s : the time-varying matrices $(n\times n)$-dimensional.

B_s : the time-varying matrices $(n\times m)$-dimensional.

A_s and B_s depend (at least) on the state generalized error vector e through the adaptation law.

$$e=x_M-x_s \tag{3.3}$$

The adaptation law will be defined by ;

$$A_s(e,t)=F(e,\tau,t)+A_s(t_0) \qquad t_0\le\tau\le t \tag{3.4}$$

$$B_s(e,t)=G(e,\tau,t)+B_s(t_0) \qquad t_0\le\tau\le t \tag{3.5}$$

In the case of signal-synthesis adaptation,

$$\begin{aligned}&\frac{dx_s}{dt}=A_s x_s+B_s\big[u(t)+u_a(e,t)\big]\\&x_s(t_0)=x_{s_0},\ u_a(t_0)=u_{a_0}\end{aligned} \tag{3.6}$$

The adaptation law will be defined by,

$$u_a(e,t)=h(e,\tau,t)+u_a(0) \tag{3.7}$$

h denotes a function relation between $u_a(e,t)$ and the values of the vector e in the interval $0\le\tau\le t$.

(b) input-output relations in terms of differential operators.

Reference model

$$A(D)\, y_M = B(D)\, u \tag{3.8}$$

Where,

$$D = \frac{\Delta d}{dt},\ A(D) = \sum_{i=0}^{n} a_i D^i,\ B(D) = \sum_{i=0}^{m} b_i D^i. \tag{3.9}$$

u: the scalar input.
y_M: the scalar output of the model.
a_i, b_i: the constant coefficients.

In the case of parameter adaptation, the adjustable model is described by,

$$\hat{A}(t, D)\, y_s = \hat{B}(t, D)\, u \tag{3.10}$$

Where,

$$\hat{A}(t, D) = \sum_{i=0}^{n} \hat{a}_i(\varepsilon, t)\, D^i \tag{3.11}$$

$$B(t, D) = \sum_{i=0}^{m} \hat{b}_i(\varepsilon, t)\, D^i \tag{3.12}$$

y_s: the scalar output of adjustable model.
$\hat{a}_i(\varepsilon, t)$, $\hat{b}_i(\varepsilon, t)$: the time-varying coefficients of the differential operators.

The generalized error is defined as,

$$\varepsilon = y_M - y_s \tag{3.13}$$

The adaptation law takes the following form:

$$\hat{a}_i(\varepsilon, t) = f_i(\varepsilon, \tau, t) + \hat{a}_i(t_0) \qquad t_0 \leq \tau \leq t \tag{3.14}$$

$$\hat{b}_i(\varepsilon_i\, t) = g_i(\varepsilon, \tau, t) + \hat{b}_i(t_0) \qquad t_0 \leq \tau \leq t \tag{3.15}$$

In the case of a signal-synthesis adaptation, the adjustable model is described by,

$$\hat{A}(D)\, y_s = \hat{B}(D)\big[u + u_a(\varepsilon, t)\big] \tag{3.16}$$

Where,

$$\hat{A}(D) = \sum_{i=0}^{n} \hat{a}_i D^i \tag{3.17}$$

$$\hat{B}(D) = \sum_{i=0}^{m} \hat{b}_i D^i \tag{3.18}$$

$$u_a(\varepsilon, t) = h(\varepsilon, \tau, t) + u_a(t_0) \qquad t_0 \leq \tau \leq t \tag{3.19}$$

2) Series-Parallel Model Reference Adaptive Systems

(a) state-variable equations

The reference model is described by Eq. (3. 1), and the adjustable model is given by,

$$\begin{aligned} \frac{dx_s}{dt} &= A_s(\varepsilon, t)\, x_M + B_s(\varepsilon, t)\, u \\ x_s(t_0) &= x_{s_0},\ A_s(t_0) = A_{s_0},\ B(t_0) = B_{s_0} \end{aligned} \tag{3.20}$$

Where the state generalized error vector e is given by Eq. (3. 3).

The parallel part of the adjustable model is given the term $B_s(e, t)u$ and the series part by $A_s(\varepsilon, t)\, x$ in Eq. (3. 20).

(b) differential operators

In the first case,

$$y_s = \sum_{i=0}^{n} a_i(\varepsilon, t)\, D^i y_M \tag{3. 21}$$

$$y_p = \sum_{i=0}^{m} b_i(\varepsilon, t)\, D^i u \tag{3. 22}$$

$$\varepsilon = y_s - y_p \tag{3. 23}$$

In the second case,

$$y_s = -\sum_{i=0}^{n} \hat{a}_i(\varepsilon, t)\, D^i y_M + \sum_{i=0}^{m} b_i(\varepsilon, t)\, D^i u \tag{3. 24}$$

$$\varepsilon = y_M - y_s \tag{2. 25}$$

The two representations are equivalent if $a_0 = \hat{a}_0 = 1$.

3) Series Model Reference Adaptation Systems

The realization of series model reference adaptive is conditioned by the invertibility properties of the reference model. This kind of problem is relatively simple in the case of single input/single output systems but much more complicated in the

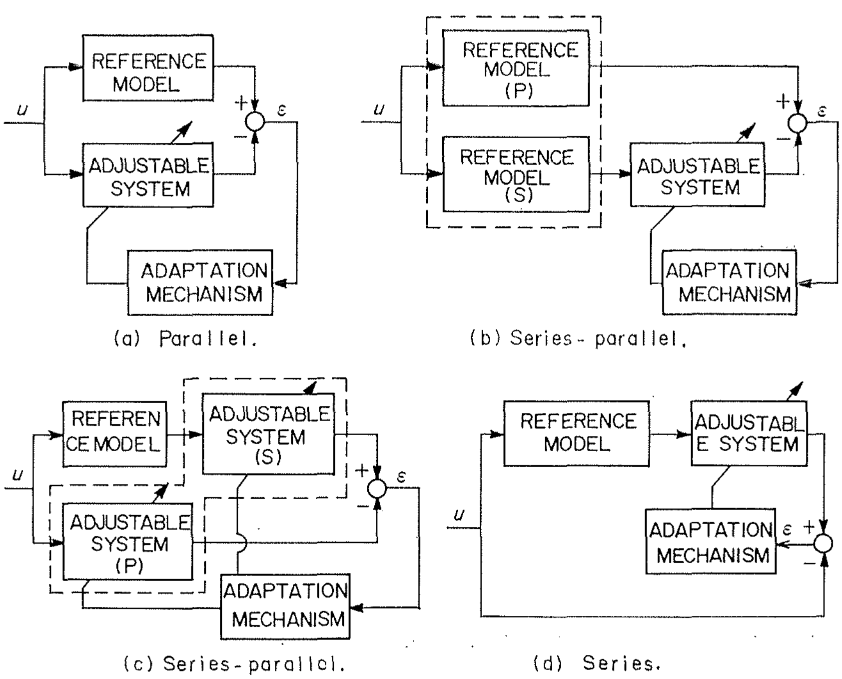

Fig. 2. Basic structures of model reference adaptive systems.

case of multivariable systems. In the following, we shall restrict ourselves to the case of single input/single output systems.

In this case, the reference model is described by Eq. (3. 8), the series adjustable system by,

$$\sum_{i=0}^{m} \hat{b}_i(\varepsilon, t)\, y_s^i = \sum_{i=0}^{n} \hat{a}_i(\varepsilon, t)\, D^i y_M \tag{3. 26}$$

$$\varepsilon = y_s - u \tag{3. 27}$$

To avoid the necessity of pure derivatives in the implementation of this type of model reference adaptive system, state-variable filters are introduced in both paths similar to the series-parallel case, as shown in Fig. 2.

Equation (3. 26) and (3. 27) remain of the same form, but y_M and u are replaced by their filtered values y_{M_f} and u_f, respectively, while the new output of the series adjustable system is u_{s_f}.

3. 2 Design Hypotheses

The various representations of model reference adaptive systems given in section 3. 1 correspond to the following basic hypotheses.

(1) The reference model is a time-invariant linear system.

(2) The reference model and the adjustable system are of the same dimension.

(3) In the case of parameter adaptation, all the parameters of the adjustable system are accessible for adaptation.

(4) During the adaptation process, the parameters of the adjustable system depend only on the adaptation mechanism.

(5) Except for the input vector u, there are no other external signals acting on the system.

(6) The initial difference between the parameters of the model and those of the adjustable system is unknown.

(7) The state generalized error vector and the output generalized error vector are measurable.

This set of hypotheses is called the ideal or basic case because it allows a straight forward analytical treatment of the model refernce adaptive system design. Many real situations closely fit the basic case.

4. Model Reference Adaptive Process of Economic Development Model

In this chapter, we shall represent the economic development model in the state-space form, and model reference adaptive process descibed by discrete-type.

4. 1 Economic development model in the state-space form

We shall represent the typical economic development model in the state-space form.

Consider the saving and investment behavior given by,

$$C(t) = \alpha Y(t-1) \tag{4. 1}$$

$$I(t)=\beta\big(C(t)-C(t-1)\big) \tag{4. 2}$$

Where,

$$Y(t)=C(t)+I(t)+g(t) \tag{4. 3}$$

The consumption $C(t)$ during the t th period depends on the output of the $(t-1)$, $Y(t-1)$. The investment during the t th period $I(t)$ is given by the behavioral equation (4. 2), and the output $Y(t)$ is as given by (4. 3) where $g(t)$ is the government expenditure on goods and services during the t th period.

Substituting (4. 1) and (4. 2) into (4. 3), we obtain the second order linear difference equation of output.

$$Y(t)=(1+\beta)\ Y(t-1)-\alpha\beta Y(t-2)+g(t) \tag{4. 4}$$

In generally, the state-space form is described by,

$$y(t)=\sum_{i=1}^{n} a_i y(t-i)+\sum_{i=0}^{m} b_i u(t-i) \tag{4. 5}$$

With the aid of the backward shift operator z^{-1}, defined by $y(t)\,z^{\pm i}=y(t\pm i)$, we can rewrite the relation in (4. 5) as,

$$a(z^{-1})\,y(t)=b(z^{-1})\,u(t) \tag{4. 6}$$

Where,

$$a(z^{-1})=I-a_1 z^{-1}-a_2 z^{-2}- \qquad -a_n z^{-n} \tag{4. 7}$$

$$b(z^{-1})=b_0+b_1 z^{-1}+b_2 z^{-2}+ \qquad +b_m z^{-m} \tag{4. 8}$$

Either (4. 5) or (4. 6) can be expressed in the final form

$$y(t)=a(z^{-1})^{-1} b(z^{-1})\,u(t) \tag{4. 9}$$

$$=T(z^{-1})\,u(t) \tag{4. 10}$$

$$=\sum_{i=0}^{\infty} T_i z^{-1} u(t-i) \tag{4. 11}$$

which corresponds to the transfer function refered to as the system control literature, where

$$T(z^{-1})=\sum_{i=0}^{\infty} T_i z^{-1} \tag{4. 12}$$

It follows from (4. 9) and (4. 10) that

$$a(z^{-1})\ T(z^{-1})=b(z^{-1}) \tag{4. 13}$$

Substitution of (4. 7), (4. 8) and (4. 12) into (4. 13) yields

$$T_t=b_t+\sum_{i=1}^{\infty} a_i T_{t-i} \tag{4. 14}$$

Suppose that the (4. 5) or (4. 9) can be converted to the state-space representation of the form.

$$x(t) = Ax(t-1) + Bu(t) \tag{4.15}$$

$$y(t) = C \cdot x(t) \tag{4.16}$$

Where $x(t)$ is defined as the state vector of dimension n.

The transfer function matrices of (4. 9) can be written as,

$$T(z^{-1}) = zC(zI-A)^{-1}B \tag{4.17}$$

$$= CB + CABz^{-1} + CA^2Bz^{-2} + \cdots\cdots\cdots \tag{4.18}$$

Compare (4. 12) with (4. 18), then it will be shown to be,

$$T_i = CA^iB \qquad i = 0, 1, 2, \tag{4.19}$$

Where T are the matrix multipliers of the system referred to as the Markov matrices in the control system.

4. 2 Model reference adaptive process described by discrete type

The practical implementation of model reference adaptive technique in economic development model requires the derivation of discrete time adaptation.

Consider the parallel model reference adaptive system described in chapter 3. 1 and restated here for the discrete time case.

The reference model

$$\begin{aligned} y_M(t) &= \sum_{i=1}^{n} a_i y_M(t-i) + \sum_{i=0}^{m} b_i u(t-i) \\ &= P^T \phi(t-1) \end{aligned} \tag{4.20}$$

Where,

$$P^T = [a_1, a_2, \cdots, a_n, \ b_1, b_2, \cdots, b_m] \tag{4.21}$$

$$\phi^T(t-1) = \left[y_M(t-1), \cdots, y_M(t-n), \ u(t), \cdots, u(t-m)\right] \tag{4.22}$$

P: the parameter vector.

$y_M(t)$: the model output at instant t.

$u(t)$: the model input at instant t.

The reference model is characterized by the discrete transfer function:

$$h_M(z) = \left(\sum_{i=0}^{m} b_i z^i\right) \Big/ \left(1 - \sum_{i=1}^{n} a_i z^{-i}\right) \tag{4.23}$$

The parallel adjustable model.

$$\begin{aligned} y_s(t) &= \sum_{i=1}^{n} \hat{a}_i(t)\, y_s(t-i) + \sum_{i=0}^{m} \hat{b}_i(t)\, u(t-i) \\ &= \hat{P}^T(t)\, \phi(t-1) = \left[\hat{P}^I(t) + \hat{P}^P(t)\right]^T \phi(t-1) \end{aligned} \tag{4.24}$$

$$y^0(t) = \left[\hat{P}^I(t-1)\right]^T \phi(t-1) \tag{4.25}$$

Where,

$$\hat{P}^T(t) = \left[\hat{a}_1(t), \cdots, \hat{a}_n(t), \ \hat{b}_0(t), \cdots, \hat{b}_m(t)\right] \tag{3.26}$$

$$\phi^T(t-1)=\left[y(t-1),\ \cdots,\ y(t-n),\ u(t),\ \cdots,\ u(t-m)\right] \tag{4.27}$$

$y_s(t)$: the posterior outputs at instant t.
$y_s^0(t)$: the prior output at instant t.
The generalized error:

$$\text{a prior:}\quad \varepsilon_t^0=y_M(t)-y_s^0(t) \tag{4.28}$$

$$\text{a posterior:}\quad \varepsilon_t=y_M(t)-y_s(t) \tag{4.29}$$

The adaptation algorithm:

$$\nu(t)=\varepsilon(t)+\sum_{i=1}^{n} c_i\varepsilon(t-i) \tag{4.30}$$

$$\nu^0(t)=\varepsilon^0(t)+\sum_{i=1}^{n} c_i\varepsilon(t-i) \tag{4.31}$$

$$\hat{P}(t)=\hat{P}^I(t)+\hat{P}^P(t) \tag{4.32}$$

$$\hat{P}^I(t)=\hat{P}^I(t-1)+\Gamma_1\left(\nu^0(t)\right)=\sum_{l=0}^{t}\left(\nu^0(l)\right)+\hat{P}^I(-1) \tag{4.33}$$

$$\hat{P}^P(t)=\Gamma_2\left(\nu^0(t)\right) \tag{4.34}$$

Where,
$\hat{P}^I(t)$: the part of the adaptation algorithm which provides the memory of the adaptation mechanism.
$\hat{P}^P(t)$: a transient term which vanishes when $\nu_t^0=0$.
We shall use a modified adaptation algorithm of the form.

$$\hat{P}^I(t)=\hat{P}^I(t-1)+\Gamma_1'\left(\nu(t)\right) \tag{4.35}$$

$$\hat{P}^P(t)=\Gamma_2'\left(\nu(t)\right) \tag{4.36}$$

Theorem 1

The parallel model reference adaptive system described by Eqs, (4. 20) to (4. 34) is globally asymptotically stable if the following adaptation algorithm is used:

$$\hat{P}(t)=\hat{P}^I(t)+\hat{P}^P(t) \tag{4.37}$$

$$\hat{P}^I(t)=\hat{P}^I(t-1)+\frac{F\phi(t-1)}{1+\phi^T(t-1)\,[F+F'(t-1)]\,\phi(t-1)}\,\nu^0(t) \tag{4.38}$$

$$\hat{P}^P(t)=\frac{F'\,t(-1)\,\phi(t-1)}{1+\phi^T(t-1)\,[F+F'(t-1)]\,\phi(t-1)}\,\nu^0(t) \tag{4.39}$$

$$\nu^0(t)=y_M(t)-\left[\hat{P}^I(t-1)\right]^T\phi(t-1)+\sum_{i=1}^{n} c_i(t-i) \tag{4.40}$$

F: an arbitrary positive definite matrix.
$F'(t)$: a constant or time-varying matrix satisfying condition (4. 41) and c are selected a strictly positive real satisfying condition (4. 42).

$$F'(t)+(1/2)\cdot F\geq 0 \tag{4.41}$$

$$h(z)=\frac{1+\sum_{i=1}^{n} c_i z^{-i}}{1-\sum_{i=1}^{n} a_i z^{-i}} \tag{4.42}$$

Theorem 2

The parallel model reference adaptive system described by Eqs. (4. 20) to (4. 34) is globally asymptotically stable if the following integral adaptation argorithm is used :

$$\hat{P}(t)=\hat{P}^I(t) \tag{4.43}$$

$$\hat{P}(t)=\hat{P}^I(t-1)+\frac{F(t-1)\,\phi(t-1)}{1+\phi^T(t-1)\,F(t-1)\,\phi(t-1)}\,\nu^0(t) \tag{4.44}$$

$$F^{-1}(t)=\lambda_1(t)\,F^{-1}(t-1)+\lambda_2(t)\,\phi(t-1)\,\phi^T(t-1) \tag{4.45}$$

$$F(0)>0,\ 0<\lambda(t)\leq 1,\ 0\leq\lambda_2(t)<2$$

$$\nu^0(t)=y_M(t)-\left[\hat{P}^I(t-1)\right]^T\phi(t-1)+\sum_{i=1}^{n} c_i(t-i) \tag{4.46}$$

Where, $F(0)$: an arbitary positive definite matrix. c are selected a strictly positive real satisfying condition (4. 47).

$$\begin{aligned} h'(z)&=\frac{1+\sum_{i=1}^{n} c_i z^{-i}}{1-\sum_{i=1}^{n} a_i z^{-i}}-\frac{\lambda}{2} \\ &=h(z)-\frac{\lambda}{2} \qquad \lambda=\max_{0\leqq t<\infty}\left[\lambda_2(t)\right]<2 \end{aligned} \tag{4.47}$$

Next, we shall consider the following simple economic development model with single-input and single-output parallel model reference adaptive system.

The developed regional economic model :

$$y_M(t)=a_1 y_M(t-1)+a_2 y_M(t-2)+b_1 u(t-1) \tag{4.48}$$

$y_M(t)$: the regional output at the instant t per regional resident.

$u(t)$: the government expenditure on goods and services per regional resident.

a_1, a_2, b_1 : the parameters of developed regional economic model.

The developing regional economic model :

$$y_s^0(t)=\hat{a}_1(t-1)\,y_s(t-1)+\hat{a}_2(t-1)\,y_s(t-2)+\hat{b}_1(t-1)\,u(t-1) \tag{4.49}$$

$$y_s(t)=\hat{a}_1(t)\,y_s(t-1)+\hat{a}_2(t)\,y_s(t-2)+\hat{b}_1(t)\,u(t-1) \tag{4.50}$$

Where,

$y_s(t)$: the regional output at the instant t per regional resident.

$y_s^0(t)$: the prior regional output at the instant t per regional resident using the value of the adjustable parameters at the instant $t-1$.

$\hat{a}_1$, $\hat{a}_2$, $\hat{b}_1$: the parameters of developing regional economic model.

The generalized output difference.

$$\varepsilon^0(t) = y_M(t) - y_s^0(t) \tag{4.51}$$

$$\varepsilon(t) = y_M(t) - y_s(t) \tag{4.52}$$

The adaptation mechanism will contain a linear compensator generating a signal $\nu(t)$:

$$\text{a prior} \qquad \nu^0(t) = \varepsilon^0(t) + \sum_{i=1}^{r} c_i \varepsilon(t-i) \tag{4.53}$$

$$\text{a posterior} \qquad \nu(t) = \varepsilon(t) + \sum_{i=1}^{r} c_i \varepsilon(t-i) \tag{4.54}$$

The degree r and the coefficients c_i will be determined as part of the design. The signal $\nu^0(t)$ will be used to implement the adaptation algorithms, which for this example will be chosen in the form.

$$\hat{a}_i(t) = \hat{a}_i(t-1) + \Gamma a_i\big(\nu^0(t)\big) = \sum_{l=0}^{t} \Gamma_{a_i}\big(\nu^0(l)\big) + \hat{a}_i(-1) \qquad i = 1, 2 \tag{4.55}$$

$$\hat{b}_1(t) = \hat{b}_1(t-1) + \Gamma b_1\big(\nu^0(t)\big) = \sum_{l=0}^{t} \Gamma_{b_1}\big(\nu^0(l)\big) + \hat{b}_1(-1) \tag{4.56}$$

In developing the design, we shall use modified adaptation algorithms of the from.

$$\hat{a}_i(t) = \hat{a}_i(t-1) + \Gamma_{a_i}\big(\nu(t)\big) \qquad i = 1, 2 \tag{4.57}$$

$$\hat{b}_1(t) = \hat{b}_1(t-1) + \Gamma'_{b_1}\big(\nu(t)\big) \tag{4.58}$$

First step

Subtracting Eq. (4.50) from (4.48) and also using Eq. (4.52), we obtain

$$\begin{aligned} \varepsilon(t) = {} & a_1 \varepsilon(t-1) + a_2 \varepsilon(t-2) + \big[a_1 - \hat{a}_1(t)\big] y_s(t-1) + \big[a_2 - \hat{a}_2(t)\big] y_s(t-2) \\ & + \big[b_1 - \hat{b}_1(t)\big] u(t-1) \end{aligned} \tag{4.59}$$

Also using Eqs. (4.54), (4.57) and (4.58), one obtains the following equivalent feedback system:

$$\varepsilon(t) = a_1 \varepsilon(t-1) + a_2 \varepsilon(t-2) + w_1(t) \tag{4.60}$$

$$\nu(t) = \varepsilon(t) + \sum_{i=1}^{r} c_i \varepsilon(t-i) \tag{4.61}$$

$$\begin{aligned} m(t) = -w_1(t) = {} & \sum_{i=1}^{2} \left[\sum_{l=0}^{t} \Gamma'_{a_i}\big(\nu(l)\big) + \hat{a}_i(-1) - a_i \right] y(t-i) \\ & + \left[\sum_{l=0}^{t} \Gamma'_{b_1}\big(\nu(l)\big) + P_1(-1) - b_1 \right] u(t-1) \end{aligned} \tag{4.62}$$

Where Eqs. (4.60) and (4.61) define a linear time invariant feedforward block and Eq. (4.62) defines a nonlinear time varying feedback block.

Second step

To be able to apply the hyperstability theorem to the equivalent feedback system of Eqs. (4. 60) to (4. 62) one should first determine $\Gamma'_i(\nu(l))$ and $\Gamma'_{b_1}(\nu(l))$ such that the equivalent feedback block defined by Eq. (4. 62).

$$\eta(0, t_1)=\sum_{t=0}^{t_1}\nu(t)\,\omega(t)\geq\gamma_0^2 \tag{4. 63}$$

By using Eq. (4. 62) the inequlity of Eq. (4. 63) becomes

$$\begin{aligned}\eta(0, t_1)&=\sum_{i=1}^{2}\sum_{t=0}^{t_1}\nu(t)\,y(t-i)\left[\sum_{l=0}^{t}\Gamma_{a_i}\big(\nu(l)\big)+\hat{a}_i(-1)-a_i\right]\\&+\sum_{t=0}^{t_1}\nu(t)\,u(t-1)\left[\sum_{l=0}^{t}\Gamma'_{b_1}\big(\nu(l)\big)+\hat{b}_1(-1)-b_1\right]\geq-\gamma_0^2\end{aligned} \tag{4. 64}$$

By the use of the following relation :

$$\sum_{t=0}^{t_1}x(t)\left[\sum_{l=0}^{t}\{ax(l)\}+c\right]=\frac{a}{2}\left[\sum_{t=0}^{t_1}x(t)+\frac{c}{a}\right]+\frac{a}{2}\sum_{t=0}^{t_1}x^2(t)-\frac{c}{2a}\geq-\frac{c}{2a}$$

$$a>0 \tag{4. 65}$$

Using the relation given by Eq. (4. 65), one obtains the following particular solution Γ'_{a_i} and Γ'_{b_1} :

$$\Gamma'_{a_i}\big(\nu(t)\big)=\alpha_i\nu(t)\,y_s\,(t-i)\qquad \alpha_i>0\;\; i=1,2 \tag{4. 66}$$

$$\Gamma'_{b_1}\big(\nu(t)\big)=\beta_1\nu(t)\,u(t)\qquad \beta_1>0 \tag{4. 67}$$

Third step

To have $\lim\varepsilon(t)=0$ for all ε_0, $\hat{a}_1(-1)-a_i$, $\hat{b}_1(-1)-b_1$ and all bounded input sequences $u(t)$, the equivalent feedback system defined by Eqs. (4. 60), (4. 61), and (4. 62) should be asymptotically hyperstable with the feedback block of Eq. (4. 62) satisfying inequality (4. 63) one can then apply the Popov hyperstability theorem for discrete systems.

Therefore, the discrete transfer function of the equivalent feed forward block defined by Eqs. (4. 60) and (4. 61), which is,

$$h(z)=\frac{1+\sum_{i=1}^{r}c_iz^{-1}}{1-a_1z^{-1}-a_2z^{-2}} \tag{4. 68}$$

should be strictly positive real.

So that the transfer function of Eq. (4. 68) will be strictly positive real, the poles of $h(z)$ should lie in $|z|<1$, considering now the parameter plane a_1-a_2, we note that this condition is satisfied if a_1 and a_2 lie within the triangle shown in Fig. 3. This stability domain is defined by the inequlities.

$$\begin{aligned}1+a_1-a_2&>0\\1-a_1-a_2&>0\\1+a_2&>0\end{aligned} \tag{4. 69}$$

Applying the transformation $z=(1+s)/(1-s)$ to Eq. (4. 68) with $r=0$ (i. e., $c_i=0$), one obtains,

$$h'(s)=\frac{1+2s+s^2}{(1-a_1-a_2)+s(2+2a_2)+s(1+a_1-a_2)} \tag{4.70}$$

and the real part of $h'(s)$ for $s=jw$ is given by,

$$\mathrm{Re}\left\{h(jw)\right\}=\frac{(1-a_1-a_2)+2(1+3a_2)\,w^2+(1+a_1-a_2)\,w^4}{[1-a_1-a_2-w^2(1+a_1-a_2)]+4w^2(1+a_2)} \tag{4.71}$$

The real part of $h'(jw)$ will be strictly positive real for any real w, and therefore $h'(s)$ will be strictly positive real if, in addition to the condition of Eq. (4.69), a and a satisfy one of the following two conditions:

$$\begin{aligned}&1+3a_2\geq 0\\&(1+3a_2)^2-(1+a_1-a_2)(1-a_1-a_2)<0\end{aligned} \tag{4.72}$$

The corresponding domain in rhe a_1-a_2 plane for which strictly real positivity is assured with $r=0$ is shown in Fig. 3 by the cross hatched area. we note that this domain is smaller than the stability domain.

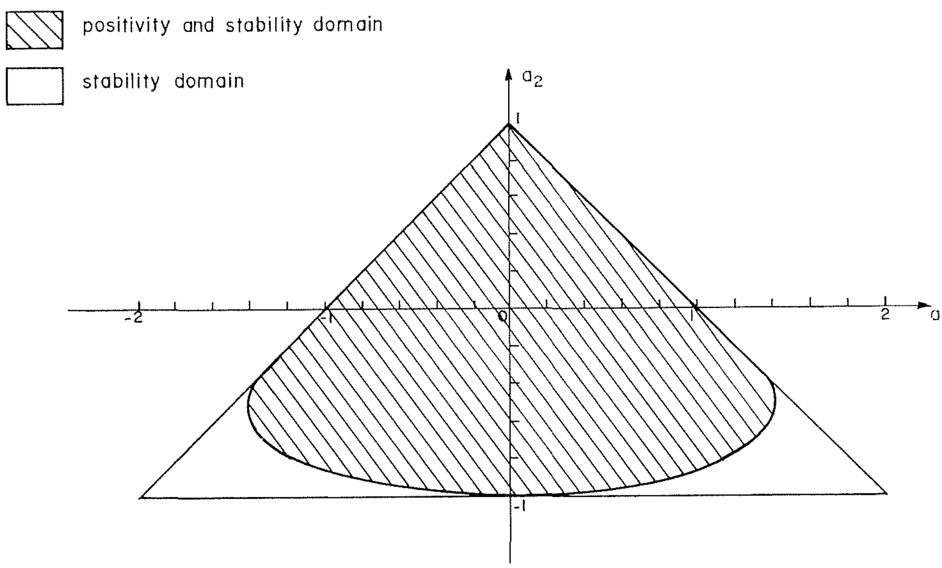

Fig. 3. Stability and positivity domain in the parameter plane for the discrete transfer function.

To obtain a strictly positive real transfer function when a and a are outside of this domain but still in the stability domain, one must consider either $r=1$ or $r=2$.

For $r=1$, applying the same procedure as above and taking into account that $-1<c_1<1$, one finds that c_1, a_1 and a_2 must satisfy either the condition

$$1-c_1a_1+3a_2\geq 0 \tag{4.73}$$

or the condition,

$$(1-c_1a_1+3a_2)^2-(1-c_1^2)(1+a_1-a_2)(1-a_1-a_2)<0 \tag{4.74}$$

The condition of Eq. (4. 74) can always be satisfied by a convenient choice of c_1 and in particular it is satisfied for any values of a_1, and a_2 within the stability domain if one choses.

$$c_1 = -0.5a_1 \tag{4.75}$$

For $r=2$, applying the transformation $z=(1+s)/(1-s)$, one obtains

$$h'(s) = \frac{1+c_1+c_2}{1-a_1-a_2} + \frac{(1+s)\,[2(1-d_2)/(1+d_1d_2)]+s\,[(1+d_2-d_1)/(1+d_1+d_2)]}{(1+s)\,[2(1+a_2)/(1-a_1-a_2])+s\,[(1+a_1-a_2)/(1-a_1-a_2)]} \tag{4. 76}$$

If we choose

$$c_2 = c_1 - 1 \tag{4. 77}$$

the coefficients of s^2 in the numerator of $h'(s)$ becomes null, and $h'(s)$ has the form of a transfer function already discussed. Therefore so that $h'(s)$ will be strictly positive real one should satisfy the inequality

$$2(1-c_2)/(1+c_1+c_2) = (2-c_1)/c_1 \geq (1+a_1-a_2)/2(1+a_2) \tag{4. 78}$$

which becomes

$$c_1 < 4(1+a_2)/(3+a_1+a_2) \tag{4. 79}$$

Fourth step

In this last step, it only remains to determine the adaptation law of the form given by Eqs. (4. 55) and (4. 56) knowing the adaptation laws given by Eqs. (4. 66) and (4. 67) which use $\nu(t)$ instead of $\nu^0(t)$. To do this, we shall try first to find the relation between $\nu(t)$ and $\nu^0(t)$, and then we shall express the adaptations laws of Eqs. (4. 66) and (4. 67) in terms of $\nu^0(t)$ instead of $\nu(t)$.

The explicit expression of $\nu(t)$ is obtained from Eq. (4. 61) using Eq. (4. 59). One obtains

$$\begin{aligned}\nu(t) &= \varepsilon(t)+c_1\varepsilon(t-1)+c_2\varepsilon(t-2) = a_1\varepsilon(t-1)\\ &\quad + a_2\varepsilon(t-2)+\left[a_1-\hat{a}_2(t)\right]y_s(t-1)+\left[a_2-\hat{a}_2(t)\right]y_s(t-2)\\ &\quad +\left[b_1-\hat{b}_1(t)\right]u(t-1)+c_1\varepsilon(t-1)+c_2\varepsilon(t-2)\end{aligned} \tag{4. 80}$$

Replacing $\hat{a}_1(t)$, $\hat{a}_2(t)$, $\hat{b}_1(t)$ in Eq. (4. 80) by their expressions given by Eqs. (4. 57), (4. 58), (4. 66) and (4. 67), one obtains

$$\begin{aligned}\nu(t) &= a_1\varepsilon(t-1)+a_2\varepsilon(t-2)+\left[a_1-\hat{a}_1(t-1)-\alpha_1\nu(t)\,y_s(t-1)\right]y_s(t-1)\\ &\quad +\left[a_2-a_2(t-1)-\alpha_2\nu(t)\,y_s(t-2)\right]y_s(t-2)\\ &\quad +\left[b_1-\hat{b}_1(t-1)-\beta_1\nu(t)\,u(t-1)\right]u(t-1)+c_1\varepsilon(t-1)+c_2\varepsilon(t-2)\end{aligned} \tag{4. 81}$$

The expression for $\nu^0(t)$ is obtain by using Wqs. (4. 48), (4. 49), (4. 51) and (4. 53) :

$$\nu^0(t)=a_1\varepsilon(t-1)+a_2\varepsilon(t-2)+\left[a_1-\hat{a}_1(t-1)\right]y_s(t-1)$$
$$+\left[a_2-\hat{a}_2(t-1)\right]y_s(t-2)+\left[b_1-\hat{b}_1(t-1)\right]u(t-1) \tag{4. 82}$$
$$+c_1\varepsilon(t-1)+c_2\varepsilon(t-2)$$

Substracting Wq. (4. 81) from (4. 82), one obtains

$$\nu^0(t)-\nu(t)=\sum_{i=1}^{2}\alpha_i\nu(t)\,y_s^2(t-i)+\beta_1\nu(t)\,u^2(t-1) \tag{4. 83}$$

which yields

$$\nu(t)=\nu^0(t)\Big/\left[1+\sum_{i=1}^{2}\alpha_i y_s^2(t-i)+\beta_1 u^2(t-1)\right] \tag{4. 84}$$

Using Eq. (4. 84), the adaptation law of Eqs. (4. 57), (4. 58), (4. 66) and (4. 67) become

$$\hat{a}_i(t)=\hat{a}_i(t-1)+\left\{\alpha_i y_s(t-i)\Big/\left[1+\sum_{i=1}^{2}\alpha_i y_s^2(t-i)+\beta_2 u^2(t-1)\right]\right\}\nu^0(t) \tag{4. 85}$$

$$\hat{b}(t)=\hat{b}_1(t-1)+\left\{\beta_1 u(t-1)\Big/\left[1+\sum_{i=1}^{2}\alpha_i y_s^2(t-i)+\beta_2 u^2(t-1)\right]\right\}\nu^0(t) \tag{4. 86}$$

Note that $\nu^0(t)$ can be computed using Eqs. (4. 48), (4. 49) and (4. 53). one obtains.

$$\nu^0(t)=y_M(t)-\sum_{i=1}^{2}\hat{a}_i(t-1)\,y_s(t-i)-\hat{b}_1(t-1)\,u(t-1)+\sum_{i=1}^{2}c_i\varepsilon(t-i) \tag{4. 87}$$

The adaptation delay at the instant t is equal to the time necessary for computing $\nu^0(t)$ and for multiplication of $\nu^0(t)$ with the weighting factors appearing in Eqs. (4. 85) and (4. 86), which are already computed at $t-1$.

5. Conclusion

In this paper, we have briefly reviewed the model reference adaptive technique and have investigated one of the adaptive processes of the economic development policy model.

Firstly, we have considered the need for the model reference adaptive technique in the adaptive processes of the economic development policy model.

Secondly, we have considered the design for model reference adaptive systems.

Thirdly, we have investigated the model reference adaptive process of economic development model described by discrete type.

The model reference adaptive technique is conceived to be a useful method in specifying the quantiative analysis of the adaptive processes of the economic development policy model.

In this way, it is to be expected that many economic development studies for various applications of model reference adaptive technique are clarified.

References

1) Tinbergen, J. (1952): On the Theory of Economic Policy: North-Holland Pub. Co.
2) Tustin, A. (1953): The Mechanism of Economic Systems: an Approach to the Problem of Economic Stabilization from the Point of View of Control-System Engineering: Harvard University Press.
3) Phillips, A. W. (1954): Stabilization Policy in a Closed Economy: The Economic Journal.
4) Theil, H. (1964): Optimal Decision Rules for Government and Industry: North-Holland Pub. Co.
5) Chow, G. C. (1975): Analysis and Control of Dynamic Economic Systems: John Wiley & Sons.
6) Aoki, M. (1976): Optimal Control System Theory in Dynamic Analysis: North-Holland pub. Co.
7) Myoken, H. (1980): Dynamic Structures, Optimal Control and Stabilization in multivariable Economic Systems: Bunshindo Pub. Co. Ltd.
8) Rahman, M. A. (1963): Regional Allocation of Investment-an Aggregative Study in the Theory of Development Programming: *Quarterly Journal of Economics,* Vol. **77**, No. 1.
9) Sakashita, N. (1967): Regional Allocation of Public Investment: *Papers of Regional Science Association.* Vol. **19**.
10) Ohtsuki, Y. (1971): Regional Allocation of Public Investment in an n-Regional Economy: *Journal of Regional Science,* Vol. **11**, No. 2.
11) Yamamura, E. (1972): A Basic Study on Regional Income Disparity Arising from Regional Allocation of Public Investment: *Proceedings of the Japan Society Civil Engineers,* No. 203.
12) Yamamura, E. (1975): A Basic Study on the Controllability of the Regional Income Disparity Arising from the Second Optimal Policy: *Proceedings of the Japan Society Civil Engineers,* No. 238.
13) Mera, K. (1975): Income Distribution and Regional Development: University of Tokyo Press.
14) Fujita, M. (1978): Spatial Development Planning: North-Holland Pub. Co.
15) Isard, W. and P. Liossatos (1979): Spatial Dynamics and Optimal Space-Time Development: North-Holland Pub. Co.
16) Yamamura, E. and Y. Miyata (1982): A Study on Regional Income Disparity Arising from Regional Allocation of Investments in Continuous Space: *Environmental Science, Hokkaido* Vol. **5**, No. 2.
17) Murphy, R. E. (1965): Adaptive Processes in Economic Systems: Academic Press.
18) Landau, Y. D. (1979): Adaptive Control-The Model Reference Approach: Marcel Dekker, INC.

Summary

We have briefly reviewed the model reference adaptive technique and investigated one of the adaptive processes of the economic development policy model by using the model reference adaptive technique.

The main results are as follows.

(1) There are many economic control studies for various applications of model reference adaptive techniques such as the economic development system with the reference model specified by the developed country (region) economy and the developing country (region) economy specified by the adjustable model, and also the real economic system specified by the reference model and

the economic planning model specified by the adjustable model and the reverse specification.

(2) The mathematical description of the model reference adaptive systems is divided into three types such as parallel, series-parallel and series types.

(3) The economic development model is typicaly specified by the state-space form such as the equation (4.5).

(4) The practical implementation of model reference adaptive technique in economic development model requires the derivation of discrete time adaptation.

(5) The parallel model adaptive system described by Eqs. (4.20) to (4.34) is globally asmptotically stable if the following adaptation algorithm is used by Eqs. (4.37) to (4.42).

(6) The parallel model adaptive system described by Eqs. (4.20) to (4.34) is globally asmptotically stable of the following integral adaptation algorithm is used by Eqs. (4.43) to (4.47).

(7) The model reference adaptive technique is conceived to be a useful method in specifying the quantitative analysis of the adaptive processes of the economic development policy model such as the results of analysis of economic development model specified by Eqs. (4.48) to (4.52).