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A Study on Model Reference Adaptive Control in Economic Development (III)

—Model Reference Adaptive I–O Analysis—

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Abstract

The principal aim of this paper is to introduce the application of the Model Reference Adaptive Control technique to the Dynamic I–O Analysis, and attention is also given to the basic theory of Model Reference Adaptive I–O Analysis. The basic theoritical analysis of adaptive principle and stability of two models such as the continuous model and the discrete model are explored using Model Reference Adaptive Control technique.

Key Words: Dynamic Input-Output model, Model Reference Adaptive Control, Equivalent Feedback System.

1. Introduction

The desirable industrial structure of region is one of the most important of the problems of regional planning. Though many attempts have been made to give a development object of a region and to consider the optimal industrial development, however, there are some restrictions sorrounding those optimal growth approaches, e. g., a strict policy in the middle stage of industial development. Therefore, some evidence showed that the optimal policy could not be done successfully.

Because one and of some stated reasons we adopt the difference philosophy from the common meaning of optimal solution. We shall clarify then, how the actual regional industrial structure would converge to the reference structure when the reference regional industrial structure is looked over. For this aim we shall apply the theory of model reference adaptive control refered in [1]～[4] to dynamic input-output analysis, and we shall propose the new method named "Model Reference Adaptive I–O Analysis".

2. Dynamic Input-Output Model

We use very often input-output table in order to grasp the industrial structure of region. In this chapter we will explain the dynamic input-output model in order to consider the dynamic industrial development [5].

Let n be a number of the economic activities, then the equilibrium equation of input-output model is as follows,

$$X(t) = A(t)\, X(t) + I(t) + H(t) + E(t) - M(t) \tag{2. 1}$$

Where, $X(t)$: output vector at t-th period $(n \times 1)$
$A(t)$: input-output coefficient matrix at t-th period $(n \times n)$
$I(t)$: private investment vector at t-th period $(n \times 1)$
$H(t)$: public expenditure and other final demand vector at t-th period $(n \times 1)$
$E(t)$: export vector at t-th period $(n \times 1)$
$M(t)$: import vector at t-th period $(n \times 1)$

Next we will define $S_{ij}(t)$ as the amount of the capital good of the i-th sector hold by the j-th sector.

And we shall construct the capital structure table $S(t)$.

$$S(t) = \begin{pmatrix} S_{11}(t) \cdots\cdots S_{1n}(t) \\ S_{n1}(t) \cdots\cdots S_{nn}(t) \end{pmatrix}$$

Let $I(t)$ be net investment and $I_i(t)$ be a component of the i-th sector, then the next equation is found to be ;

$$I_i(t) = d/dt \sum_{j=1}^{n} S_{ij}(t) \tag{2. 2}$$

And we shall introduce capital coefficient $b_{ij}(t)$, then $S_{ij}(t)$ is represented as follows.

$$S_{ij}(t) = b_{ij}(t)\, X_j(t) \tag{2. 3}$$

Therefore,

$$\begin{aligned} I_i(t) &= d/dt \sum_{j=1}^{n} S_{ij}(t) \\ &= d/dt \sum_{j=1}^{n} b_{ij}(t)\, X_j(t) \\ &= \sum_{j=1}^{n} \left\{ db_{ij}(t)/dt\, X_j(t) + b_{ij}(t)\, dX_i(t)/dt \right\} \end{aligned} \tag{2. 4}$$

From the above equations, we put capital coefficient matrix $B(t)$ as $B(t) = (b_{ij}(t))$, then (2. 1) is written in the following.

$$X(t) = A(t)\, X(t) + dB(t)/dt\, X(t) + B(t)\, dX(t)/dt + H(t) + E(t) - M(t) \tag{2. 5}$$

Moreover we shall define import coefficient matrix $\hat{M}(t)$ as

$$\hat{M}(t) = \begin{pmatrix} \hat{m}_{11}(t) & & 0 \\ & \ddots & \\ 0 & & \hat{m}_{nn}(t) \end{pmatrix}$$

$$\hat{m}_{ii}(t) = M_i(t) / \left(\sum_{j=1}^{n} a_{ij}(t)\, X_j(t) + H_i(t) \right)$$

Then (2. 5) is transformed as follows.

$$X(t) = A(t)\,X(t) + dB(t)/dt\,X(t) + B(t)\,dX(t)/dt + H(t) + E(t) - \hat{M}(t)\left(A(t)\,X(t) + H(t)\right)$$

$$\therefore\quad \left(I - \left(I - \hat{M}(t)\right) A(t) - dB(t)/dt\right) X(t) = B(t)\,dX(t)/dt + \left(I - \hat{M}(t)\right) H(t) + E(t)$$

$$dX(t)/dt = B^{-1}(t)\left(I - \left(I - \hat{M}(t)\right) A(t) - dB(t)/dt\right) X(t) - B^{-1}(t)\left(\left(I - \hat{M}(t)\right) H(t) + E(t)\right) \tag{2. 6}$$

Equation (2. 6) is the system equation which represent the time variant of output vector $X(t)$. In this study we shall call (2. 6) as "Dynamic Input-Output Model in the continuous time variable".

When we assume that $dB(t)/dt = 0$ in (2. 6), then the equation (2. 6) is as follows.

$$dX(t)/dt = B^{-1}\left(I - \left(I - \hat{M}(t)\right) A(t)\,X(t)\right) - B^{-1}\left(\left(I - \hat{M}(t)\right) H(t) + E(t)\right) \tag{2. 7}$$

Next we consider (2. 6) in terms of discrete time. By replacing $dX(t)/dt$ to $X(t+1) - X(t)$ and $dB(t)/dt$ to $B(t+1) - B(t)$, then the dynamic discrete input-output model can be found as follows.

$$X(t+1) = B^{-1}(t+1)\left(I - \left(I - \hat{M}(t)\right) A(t) + B(t)\right) X(t) - B^{-1}(t+1)\left(\left(I - \hat{M}(t)\right) H(t) + E(t)\right) \tag{2. 8}$$

The following is the investigation on analysis of (2. 6) and (2. 8) as the main object.

3. Model Reference Adaptive Regional Input-Output Model (Continuous Format)

Let us assume that desirable industrial structure and capital equipment of region were given. In this chapter, we suppose that those desirable structure are reflected in input-output coefficient and capital coefficient. Then we can formulate as follows.

$$dX_m(t)/dt = B_m^{-1}\left(I - (I - \hat{M}_m)\,A_m\right) X_m(t) - B_m^{-1}\left((I - \hat{M}_m)\,H(t) + E(t)\right) \tag{3. 1}$$

Where, $X_m(t)$:reference output vector at t-th period
A_m : reference input-output coefficient matrix
B_m : reference capital coefficient matrix
$\hat{M}_m$: reference import matrix

In this study we call this model as reference regional input-output model. On the other side, we consider that actual economic development model at t-th period is represented in the next form.

(3. 2) $$dX(t)/dt = B^{-1}(t)\Big(I-(I-\hat{M}(t)\Big)A(t)-dB(t)/dt\ X(t)$$
$$-B^{-1}(t)\Big((I-\hat{M}_m)\ H(t)+E(t)\Big)$$

Here we shall consider how we make the actual economic development model converge to the model reference adaptive regional input-output model when the final demand and export vector $(I-\hat{M}_m)\ H(t)+E(t)$ are given. This can be formed as follows.

$$\| X_m(t)-X(t) \| \longrightarrow 0 \ \text{as} \ t\rightarrow\infty$$
$$\| A_m-A(t) \| \longrightarrow 0 \ \text{as} \ t\rightarrow\infty$$
$$\| \hat{M}_m-\hat{M}(t) \| \longrightarrow 0 \ \text{as} \ t\rightarrow\infty$$
$$\| B_m-B(t) \| \longrightarrow 0 \ \text{as} \ t\rightarrow\infty$$

Where $\|\cdot\|$ stands for a norm which is homeomorphism to Euclidian norm.

In this study we call the model mentioned above as model reference adaptive regional input-output model.

3. 1. *Model Reference Adaptive Control*

For bullding up the model reference adaptive regional input-output model, we shall introduce the notion of model reference adaptive control. Here we shall give the outline along [1], [2] and [3] in the references mentioned after. First of all, we shall represent (3. 1) and (3. 2) as follows.

(3. 3) $$dX_m(t)/dt = C_m X_m(t)+D_m U(t)$$

(3. 4) $$dX(t)/dt = C(t)\ X(t)+D(t)\ U(t)$$

Where, $C_m = B_m^{-1}(I-(I-\hat{M}_m)\ A_m)$
$D_m = B_m^{-1}$
$U(t) = (I-\hat{M}_m)\ H(t)+E(t)$
$C(t) = B^{-1}(t)\ (I-(I-\hat{M}(t))\ A(t)-dB(t)/dt)$
$D(t) = B^{-1}(t)$

We call (3. 3) as reference model and (3. 4) as adaptive model conformed to the theory of model reference adaptive control.

Substructing (3. 4) from (3. 3), then

$$d\varepsilon(t)/dt = C_m\varepsilon(t)+\Big(C_m-C(t)\Big)\ X(t)+\Big(D_m-D(t)\Big)\ U(t)$$

Where, $\varepsilon(t) = X_m(t)-X(t)$

Further we put $W(t) = (C_m-C(t))\ X(t)+(D_m-D(t))\ U(t)$ then

(3. 5) $$d\varepsilon(t)/dt = C_m\varepsilon(t)+W(t)$$

(3. 6) $$W(t) = \Big(C_m-C(t)\Big)\ X(t)+\Big(D_m-D(t)\Big)\ U(t)$$

Figure 1 showed that (3. 5) and (3. 6) are transformed into the equivalent feedback system consisted of the linear and nonlinear block.

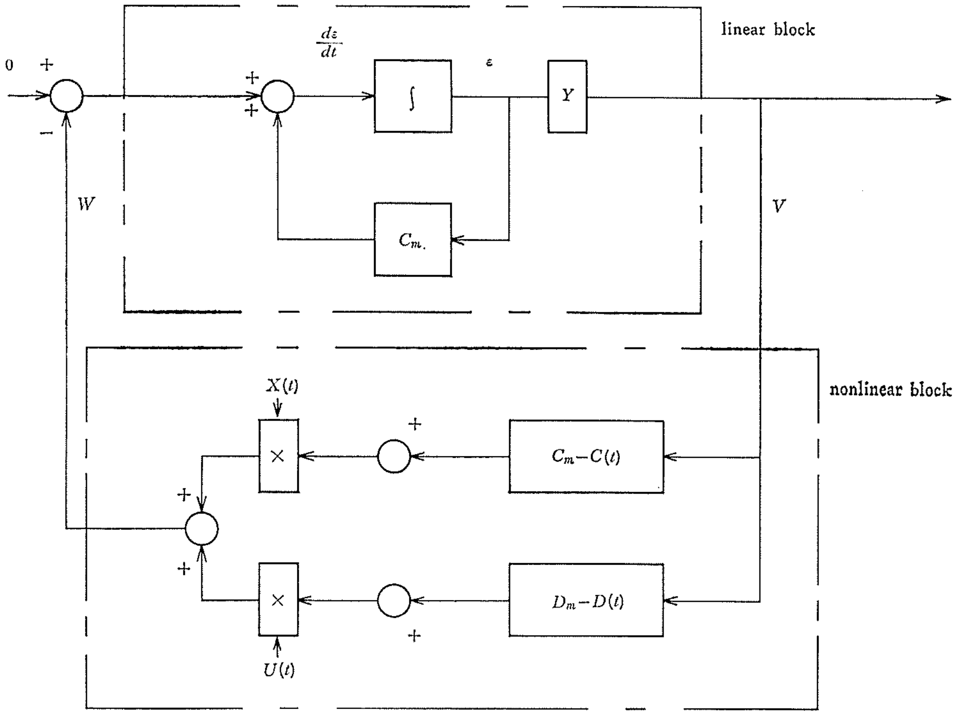

Figure 1. Equivalent Feedback System.

Beside in (3. 5) system, stability of (3. 5) depends on only C_m, and for the many times the stability is not guaranteed. Then, we shall consider a linear compensator matrix $Y(m \times n)$ for $\varepsilon(t)$ and investigate the stability of $V(t) = Y \cdot \varepsilon(t)$.

From the formulation mentioned above, we can make stability of $\varepsilon(t)$ in the following sequence.

1) Give an adaptation algorithm for $C(t)$ and $D(t)$ in order to establish the next Popov's integral inequality.

(3. 7) $$\eta(0, t_1) \equiv \int_0^{t_1} V^T(t) \left(-W(t)\right) dt \geqq -r_0^2$$

2) Select Y in order that the transfer function $H(s) = Y(sI - C_m)^{-1}$ of linear block is strictly positive real.

(3. 8) $$d\varepsilon(t)/dt = C_m \varepsilon(t) + W(t)$$

(3. 9) $$V(t) = Y\varepsilon(t)$$

3) From the above operation, it could be seen that the equivalent feedback system will be globably asymptotically hyperstable arising from Popov's theorem and $||\varepsilon(t)|| \to 0 \; (t \to \infty)$

3. 2. *Stability of Model Reference Adaptive Regional Input-Output Model*

From design method of model reference adaptive control, we shall investigate adaptation principle and stability of (3. 5) and (3. 6).

First we shall define the notation of the elements of the matrixes as follows.

$$C(t)=\left(c_{ij}(t)\right),\quad D(t)=\left(d_{ij}(t)\right),\quad X(t)=\left(x_j(t)\right),$$
$$U(t)=\left(u_j(t)\right),\quad V(t)=\left(v_j(t)\right)$$

Then we can get the following theorem on the Popov's integral inequality.

Theorem 1. When we define adaptation principles of $C(t)$, $D(t)$ as

$$c_{ij}(t)=K_{cij}\int_0^t v_i(t)\,x_j(t)\,dt+L_{cij}\,v_i(t)\,x_j(t)+c_{ij}(0) \tag{3.10}$$

$$d_{ij}(t)=K_{dij}\int_0^t v_i(t)\,x_j(t)\,dt+L_{dij}\,v_i(t)\,x_j(t)+d_{ij}(0) \tag{3.11}$$

then the next Popov's integral inequality is realized.

$$\eta(0,t_1)=\int_0^{t_1} V^T(t)\left(-W(t)\right)dt\geqq -r_0^2 \tag{3.12}$$

Where K_{cij}, L_{cij}, K_{dij}, L_{dij}, stand for positive real numbers.

Proof: We can represent the integrated function in (3.9) as follows.

$$\eta(0,t_1)=\int_0^{t_1}\sum_{i=1}^n v_i(t)\left(-w_i(t)\right)dt$$

As $w_i(t)=\sum_{j=1}^n(c_{mij}-c_{ij}(t))\,x_j(t)+\sum_{j=1}^n(d_{mij}-d_{ij}(t))\,u_j(t)$
substitute (3.10), (3.11) into $c_{ij}(t)$, $d_{ij}(t)$ yield

$$\begin{aligned}-w_i(t)=&\sum_{j=1}^n\Big\{\left(c_{ij}(0)-c_{mij}\right)+K_{cij}\int_0^t v_i(t)\,x_j(t)\,dt\\&+L_{cij}\,v_i(t)\Big\}\,x_j(t)\,x_j(t)+\sum_{j=1}^n\Big\{\left(d_{ij}(0)-d_{mij}\right)\\&+K_{dij}\int_0^t v_i(t)\,x_j(t)\,dt+L_{dij}\,v_i(t)\,x_j(t)\Big\}\,u_j(t)\end{aligned}$$

$$\begin{aligned}\therefore\quad \eta(0,t_1)=&\int_0^{t_1}\sum_{i=1}^n v_i(t)\left(-w_i(t)\right)dt\\=&\int_0^{t_1}\sum_{i=1}^n v_i(t)\Big[\sum_{j=1}^n\Big\{\left(c_{ij}(0)-c_{mij}\right)+K_{cij}\int_0^t v_i(t)\,x_j(t)\,dt\\&+L_{cij}\,v_i(t)\,x_j(t)\Big\}\,x_j(t)\\&+\sum_{j=1}^n\Big\{\left(d_{ij}(0)-d_{mij}\right)+K_{dij}\int_0^t v_i(t)\,x_j(t)dt\\&+L_{dij}\,v_i(t)\,x_j(t)\Big\}\,u_j(t)\Big]\,dt\\=&\sum_{i,j=1}^n\int_0^{t_1}v_i(t)\Big[\Big\{\left(c_{ij}(0)-c_{mij}\right)+K_{cij}\int_0^t v_i(t)\,x_j(t)\,dt\\&+L_{cij}\,v_i(t)\,x_j(t)\Big\}\,x_j(t)\\&+\Big\{\left(d_{ij}(0)-d_{mij}\right)+K_{dij}\int_0^t v_i(t)\,x_j(t)\,dt\\&+L_{dij}\,v_i(t)x_j(t)\Big\}\,u_j(t)\Big]\,dt\end{aligned}$$

We put, $f_{ij}(t) \equiv v_i(t) x_j(t)$

$c_{ij} \equiv c_{ij}(0) - c_{mij}$ then ,

$$\int_0^{t_1} v_i(t) \left\{ \left(c_{ij}(0) - c_{mij}\right) + K_{cij} \int_0^t v_i(t)\, x_j(t)\, dt + L_{cij}\, v_i(t) \right\} x_j(t)\, x_j(t)\, dt \tag{3.13}$$

$$= \int_0^{t_1} f_{ij}(t)\, c_{ij} + K_{cij} \int_0^t f_{ij}(t)\, dt + L_{cij} f_{ij}(t)\, dt$$

If we put $F_{ij}(t) = \int_0^t f_{ij}(t)\, dt$ then,

$$\int_0^{t_1} f_{ij}(t) \int_0^{t_1} f_{ij}(\xi)\, d\xi dt$$

$$= \int_0^{t_1} F'_{ij}(t)\, F_{ij}(t)\, dt$$

$$= \left[F_{ij}(t)^2 \right]_0^{t_1} - \int_0^{t_1} F_{ij}(t)\, F'_{ij}(t)\, dt$$

$$\therefore \quad \int_0^{t_1} f_{ij}(t) \int_0^t f_{ij}(\xi)\, d\xi dt = 1/2 \left[F_{ij}(t)^2 \right]_0^{t_1} = 1/2\, F_{ij}(t_1)^2$$

$$\therefore \quad (3.13) = c_{ij} F_{ij}(t_1) + 1/2\, K_{cij} F_{ij}(t_1)^2 + L_{cij} \int_0^{t_1} f_{ij}(t)^2\, dt$$

$$\geqq 1/2\, K_{cij} \left(F_{ij}(t_1)^2 + 2 c_{ij}/K_{cij}\, F_{ij}(t_1) \right)$$

$$= 1/2\, K_{cij} \left\{ \left(F_{ij}(t_1) + c_{ij}/K_{cij} \right)^2 - (c_{ij}/K_{cij})^2 \right\}$$

$$\geqq 1/2\, c_{ij}^2/K_{cij}$$

Similary if we put $d_{ij} = d_{ij}(0) - d_{mij}$ then,

$$\eta(0, t_1) \geqq \sum_{i,j=1}^{n} \left(-1/2\, c_i^{j2}/K_{cij} - 1/2\, d_{ij}^2/K_{dij} \right)$$

$$= \sum_{i,j=1}^{n} -1/2 \left(c_{ij}^2/K_{cij} + d_{ij}^2/K_{dij} \right)$$

Popov's integral inequality is realized. Q. E. D.

Next strictly positive real of linear blocks (3. 8) and (3. 9) can be assured using the next lemma 1 and theorem 2.

Lemma 1, Linear system

$$dX(t)/dt = AX(t) + BU(t) \tag{3.14}$$

$$Y(t) = CX(t) + DU(t) \tag{3.15}$$

Where, $X(t)$: n dimentional state vector

$U(t)$: n dimentional input vector

$Y(t)$: n dimentional output vector

A, B, C, D : $n \times n$ constant matrix

In the following linear system the transfer function is represented as $H(s)=C(sI-A)^{-1}B+D$. Then the necessary and sufficient condition for $H(s)$ being strictly positive real is as follows. There exist some matrixes L, W, $P=P^T>0$, $Q=Q^T>0$ which satisfy the next system of equations.

$$\begin{cases} A^TP+PA=-LL^T-Q \\ B^TP+W^TL^T=C \\ D+D^T=W^TW \end{cases}$$

Proof: For instance, Cf. reference [4].

Lemma 2. In (3. 14) and (3. 15) systems, next condition is a sufficient for that $H(s)$ is strictly positive real.

$^{\exists}W$, $^{\exists}P=P^T>0$, $^{\exists}Q=Q^T>0$ such that

$$\begin{cases} A^TP+PA=-Q \\ B^TP=C \\ D+D^T=W^TW \end{cases}$$

Proof: In lemma 1, put $L=0$ Q. E. D.

Theorem 2. In linear blocks (3. 8) and (3. 9), if linear compensation matrix Y satisfy Lyapunov's equation.

$$C_m^TY+YC_m=-I \tag{3.16}$$

The linear blocks are strictly positive real.

Proof: From lemma 2 when we set $A=C_m$, $B=I$, $C=Y$ and $D=0$, transfer function $H(s)=Y(sI-C_m)^{-1}$ is strictly positive real if $^{\exists}P=P^T>0$, $^{\exists}Q=Q^T>0$ such that

$$\begin{cases} C_m^TP+PC_m=-Q \\ P=Y \end{cases}$$

In this place if we put $Q=I$, we can see that Y satisfy Lyapunov's equation (3. 16). Since Y of (3. 16) gives n^2 homogeneous linear system of equation for elements y_{ij}, we can decide Y from (3. 16). Q. E. D.

From the above theorem 1 and 2, the equivalent feedback system (3. 8), (3. 9) and (3. 6) are globaly asymptotically hyperstable from Popov's theorem.

From these theorem, we can get next theorem 3.

Theorem 3. (The fundamental theorem of model reference adaptive regional input-output model)

We shall represent the model reference adaptive regional input-output model as follows.

(3. 17) reference model
$$dX_m(t)/dt=B_m^{-1}\left(I-(I-\hat{M}_m)\,A_m\right)X_m(t) - B_m^{-1}\left((I-\hat{M}_m)\,H(t)+E(t)\right)$$

(3. 18) adaptive model
$$dX(t)/dt=B^{-1}(t)\left(I-\left(I-\hat{M}(t)\right)A(t)\right.$$

$$-dB(t)/dt\Big)\ X(t)-B^{-1}(t)\ \Big((I-\hat{M}_m)\ H(t)+E(t)\Big)$$

Now, we make equivalent feedback systems as next equations.

$$d\varepsilon(t)/dt=C_m\varepsilon(t)+W(t) \tag{3.19}$$

$$V(t)=Y\varepsilon(t) \tag{3.20}$$

$$W(t)=\Big(C_m-C(t)\Big)\ X(t)+\Big(D_m-D(t)\Big)\ U(t) \tag{3.21}$$

Where we represent as follows.

$$\varepsilon(t)=X_m-X(t)$$

$$C_m=B_m^{-1}\Big(I-(I-\hat{M}_m)\Big)\ A_m$$

$$D_m=-B_m^{-1}$$

$$C(t)=B^{-1}(t)\ \Big(I-(I-\hat{M}(t)\Big)\ A(t)-dB(t)/dt\Big)$$

$$D(t)=-B^{-1}(t)$$

$$U(t)=(I-\hat{M}_m)\ H(t)+E(t)$$

When Y is a solution of $C_m^T Y+YC_m=I$ and adaptation priniciples of $C(t)=(c_{ij}(t),\ D(t)=(d_{ij}(t))$ are defined.

$$c_{ij}(t)=K_{cij}\int_0^t v_i(t)\ x_j(t)\ dt+L_{cij}v_i(t)\ x_j(t)+c_{ij}(0)$$

$$d_{ij}(t)=K_{dij}\int_0^t v_i(t)\ x_j(t)\ dt+L_{dij}v_i(t)\ x_j(t)+d_{ij}(0)$$

$$K_{cij},\ L_{cij},\ K_{dij},\ L_{dij}>0$$

Then the equivalent feedback systems will be globably asymptotically hyperstable.

4. Model Reference Adaptive Regional input-Output Model (Discrete Format)

In dynamic input-output model, it is more convenient for the operation of the real data if we proceed by considering the discrete time terms. For this purpose we shall mention model reference adaptive regional input-output discrete model.

And in the following context, as far as we do not especially refer, the definition of the variables is same to mentioned above.

4.1. *Model Reference Adaptive Control*

In the first model reference adaptive regional input-output model is represented as follows.

(4.1) reference model $$X_m(t+1)=B_m^{-1}\Big(I-\Big(I-\hat{M}_m)\ A_m+\mathrm{B}_m\Big)\ X_m(t)$$
$$-B_m^{-1}\Big((I-\hat{M}_m)\ H(t)+E(t)\Big)$$

(4. 2) adaptive model $X(t+1)=B^{-1}(t+1)\Big(I-\big(I-\hat{M}(t)\big)A(t)$
$+B(t)\Big)X(t)-B^{-1}(t+1)\Big((I-\hat{M}_m)H(t)+E(t)\Big)$

Equivalent feedback system is represented as follows.

(4. 3) $\varepsilon(t+1)=C_m\varepsilon(t)+W(t+1)$

(4. 4) $V(t+1)=Y\varepsilon(t)+LW(t+1)$

(4. 5) $W(t+1)=\big(C_m-C(t+1)\big)X(t)+\big(D_m-D(t+1)\big)U(t)$

Where, $\varepsilon(t)=X_m(t)-X(t)$
$C_m=B_m^{-1}(I-(I-\hat{M}_m)A_m+B_m)$
$D_m=-B_m^{-1}$
$C(t+1)=B^{-1}(t+1)(I-(I-\hat{M}(t))A(t)+B(t))$
$D(t+1)=-B^{-1}(t+1)$
$U(t)=(I-\hat{M}_m)H(t)+E(t)$

In this equivalent feedback system, we can also see that it is constituted by

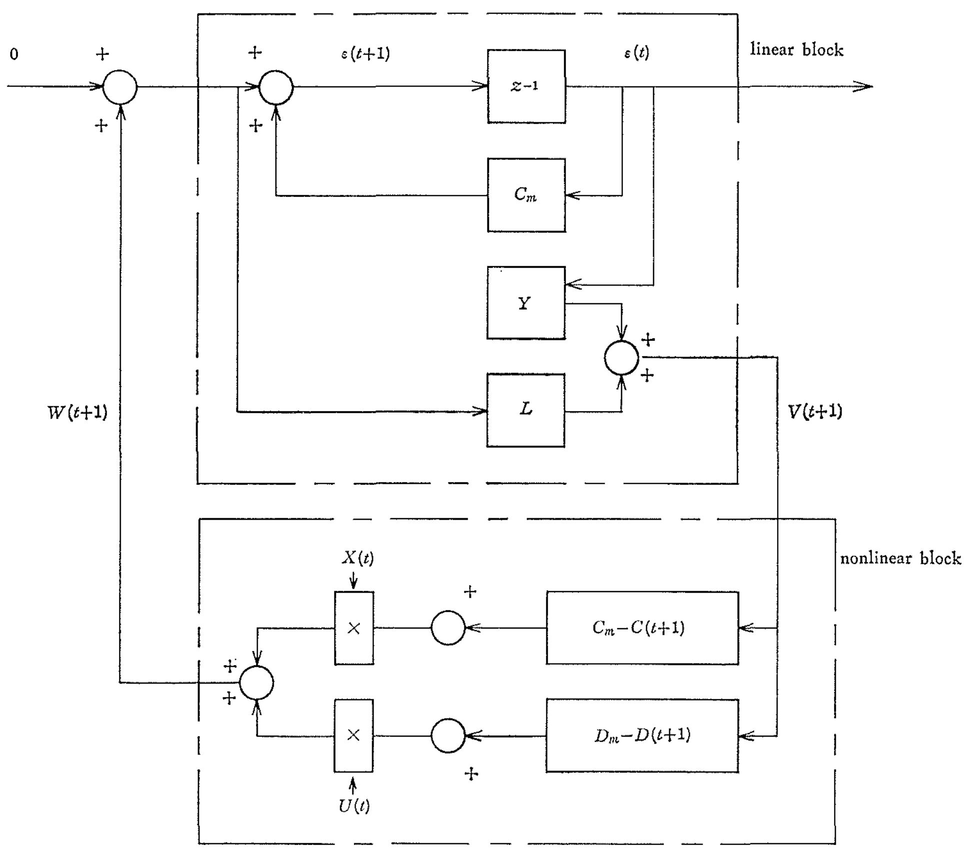

Figure 2. Discrete Equivalent Feedback System.

linear and nonlinear block depicted in the figure 2.

We can secure the stability of equivalent feedback system according to the following algorithm like the case of continuous form.

1) Give adaptation principles of C(t), D(t) such that next Popov's inequality is realized.

$$\eta(0, t_1)=\sum_{t=0}^{t_1-1}\left(V^T(t+1)\left(-W(t+1)\right)\geqq -r_0^2\right.$$

2) linear block

$$\begin{cases}\varepsilon(t+1)=C_m\varepsilon(t)+W(t+1)\\ V(t+1)=Y\varepsilon(t)+LW(t+1)\end{cases}$$

We shall select Y and L in order that pulse transfer function will be strictly positive real.

3) From the above operation, we can see that the equivalent feedback system will be globaly asymptocally hyperstable arrising from Popov's theorem for the discrete format, and $||\varepsilon(t)||\rightarrow 0\ (t\rightarrow\infty)$.

4. 2. *Stability of Model Reference Adaptive Regional Input-Output Model*

Similar to the case of continuous form, the next theorem is established of about adaptation algorithm of $C(t)$, $D(t)$.

Theorem 4. We shall represent adaptation algorithm of $C(t)$, $D(t)$ as follows.

$$c_{ij}(t+1)=K_{dij}\sum_{k=0}^{t} v_i(k+1)\, x_j(k)+L_{cij}v_i(t+1)\, x_j(t)+c_{ij}(0) \tag{4. 6}$$

$$d_{ij}(t+1)=K_{dij}\sum_{k=1}^{t} v_i(k+1)\, u_j(k)+L_{dij}v_i(t+1)\, u_j(t)+d_{ij}(0) \tag{4. 7}$$

Then Popov's inequality is existed as follows.

$$\eta(0, t_1)=\sum_{k=0}^{t_1-1} V^T(k+1)\left(-W(k+1)\right)\geqq -r_0^2 \tag{4. 8}$$

In this place K_{cij}, L_{cij}, K_{dij} and L_{dij} are positive constants.

Proof: Because the adaptation principles (4. 7) and (4. 8) include $v_i(t+1)$ in the right side, the right sides of (4. 7) and (4. 8) depend on $c_{ij}(t+1)$ and $d_{ij}(t+1)$. Therefore we can not apply the adaptation principles in this formulation. We shall refer to this point after completing the proof of the theorem 4.

$$w_i(k+1)=\sum_{j=1}^{n}\left(c_{mij}-c_{ij}(k+1)\, x_j(k)+\sum_{j=1}^{n}\left(d_{mij}-d_{ij}(k+1)\right) u_j(k)\right.$$

Then we substitute (4. 1) and (4. 2) into $c_{ij}(k+1)$, $d_{ij}(k+1)$.

$$\begin{aligned}-w_i(k+1)=&\sum_{j=1}^{n}\left\{\left(c_{ij}(0)-c_{mij}\right)+K_{cij}\sum_{l=0}^{k} v_i(l+1)\, x_j(l)\right.\\ &\left.+L_{cij}v_i(k+1)\, x_j(k)\right\} x_j(k)+\sum_{j=1}^{n}\left\{\left(d_{ij}(0)-d_{mij}\right)\right.\\ &\left.+K_{dij}\sum_{l=0}^{k} v_i(l+1)\, x_j(l)+L_{dij}v_i(k+1)\, x_j(k)\right\} u_j(k)\end{aligned}$$

$$\begin{aligned}\therefore\quad \eta(0,t_1) &= \sum_{k=0}^{t_1-1}\sum_{i=1}^{n} v_i(k+1)\left(-w_i(k+1)\right)\\ &= \sum_{k=0}^{t_1-1}\sum_{i=1}^{n} v_i(k+1)\Bigg[\sum_{j=1}^{n}\Big\{\left(c_{ij}(0)-c_{mij}\right)+K_{cij}\sum_{l=0}^{k} v_i(l+1)\,x_j(l)\\ &\quad + L_{cij}v_i(k+1)\,x_j(k)\Big\}\,x_j(k)\\ &\quad +\sum_{j=1}^{n}\Big\{\left(d_{ij}(0)-d_{mij}\right)+K_{dij}\sum_{l=0}^{k} v_i(l+1)\,x_j(l)\\ &\quad + L_{dij}v_i(k+1)\,x_j(k)\Big\}\,u_j(k)\Bigg]\\ &= \sum_{i,j=1}^{n}\sum_{k=0}^{t_1-1} v_i(k+1)\Bigg[\Big\{\left(c_{ij}(0)-c_{mij}\right)+K_{cij}\sum_{l=0}^{k} v_i(l+1)\,x_j(l)\\ &\quad + L_{cij}v_i(k+1)x_j(k)\Big\}\,x_j(k)\\ &\quad +\Big\{\left(d_{ij}(0)-d_{mij}\right)+K_{dij}\sum_{l=2}^{k} v_i(1+1)\,x_j(1)\\ &\quad + L_{dij}v_i(k+1)x_j(k)\Big\}\,u_j(k)\Bigg]\end{aligned}$$

Where we put $f_{ij}(k)=v_i(k+1)x_j(k)$ and $c_{ij}=c_{ij}(0)-c_{mij}$ then Popov's inequality is built as follows.

$$\begin{aligned}&\sum_{k=0}^{t_1-1} v_i(k+1)\Big\{\left(c_{ij}(0)-c_{mij}\right)+K_{cij}\sum_{l=0}^{k} v_i(l+1)\,x_j(l)\\ &\quad + L_{cij}v_i(k+1)\,x_j(k)\Big\}\,x_j(k)\\ &= \sum_{k=0}^{t_1-1} f_{ij}(k)\left(c_{ij}+K_{cij}\sum_{l=0}^{k} f_{ij}(l)+L_{cij}f_{ij}(k)\right)\\ &= K_{cij}/2\left(\sum_{l=0}^{t_1-1} f_{ij}(l)+c_{ij}/K_{cij}\right)^2-c_{ij}{}^2/2K_{cij}+(K_{cij}/2\\ &\quad + L_{cij})\sum_{k=0}^{t_1-1} f_{ij}^2(k)\geqq -c_{ij}^2/2K_{cij}\end{aligned}$$

$$\begin{aligned}\therefore\quad \eta(0,t_1) &\geqq \sum_{i,j=1}^{n}(-c_{ij}^2/2K_{cij}-d_{ij}^2/2K_{dij})\\ &= \sum_{i,j=1}^{n} -1/2(c_{ij}^2/K_{cij}+d_{ij}^2/K_{dij}) \qquad \text{Q. E. D.}\end{aligned}$$

In the above adaptation principles of (4. 6) and (4. 7), we shall deal that the point of right side depend on $c_{ij}(t+1)$, $d_{ij}(t+1)$ as follows.

The prior value of $X(t+1)$ is represent as follows.

$$\text{(4. 9)}\qquad \mathring{X}(t+1)=\sum_{k=0}^{t-1}\Big\{K_c\otimes\left(V(k+1)\,X^T(k)\right)X(t)+Kd\otimes\left(V(k+1)\,U^T(k)\right)U(t)$$

And $\otimes$ represents the operation that $A\otimes B=(a_{ij}\,b_{ij})$ when $A=(a_{ij})$ and $B=(b_{ij})$.

Now the prior value of $\varepsilon(t+1)$ and $V(t+1)$ are represent as follows.

$$\text{(4. 10)}\qquad \mathring{\varepsilon}(t+1)=X_m(t+1)-\mathring{X}(t+1)$$

$$\text{(4. 11)}\qquad \mathring{V}(t+1)=Y\mathring{\varepsilon}(t)+LW(t+1)$$

$\therefore$ (4. 12)

$$
\begin{aligned}
V(t+1)-\mathring{V}(t+1) &= Y\Big(\mathring{X}(t)-X(t)\Big) \\
&= Y\Bigg[\sum_{k=0}^{t-2}\Big\{K_c\otimes\Big(V(k+1)\,X^T(k)\Big)\,X(t-1) \\
&\quad +K_d\otimes\Big(V(k+1)\,U^T(k-1)\Big)\,U(t)\Big\}-C(t)\,X(t-1) \\
&\quad -D(t)\,U(t-1)\Bigg] \\
&= Y\Bigg[\sum_{k=0}^{t-2}\Big\{K_c\otimes\Big(V(k+1)\,X^T(k)\Big)\,X(t-1) \\
&\quad +K_d\otimes\Big(V(k+1)\,U^T(k)\Big)\,U(t-1)\Big\} \\
&\quad -\Big\{K_c\otimes\sum_{k=0}^{t-1}V(k+1)\,X^T(k)+L_c\otimes\Big(V(t)\,X^T(t-1)\Big) \\
&\quad +C(0)\Big\}X(t-1) \\
&\quad -\Big\{K_d\otimes\sum_{k=0}^{t-1}V(k+1)\,U^T(k)+L_d\otimes\Big(V(t)\,U^T(t-1)\Big) \\
&\quad +D(0)\Big\}\,U(t-1)\Bigg] \\
&= -Y\Bigg[\Big\{K_c\otimes\Big(V(t)\,X^T(t-1)\Big)+L_c\otimes\Big(V(t)\,X^T(t-1)\Big) \\
&\quad +C(0)\Big\}\,X(t-1)+\Big\{K_d\otimes\Big(V(t)\,U^t(t-1)\Big) \\
&\quad +L_d\otimes\Big(V(t)\,U^T(t-1)+D(0)\Big\}\,U(t-1)\Bigg] \\
&= -Y\Big\{\Gamma(t-1)\,V(t)+C(0)\,X(t-1)+D(0)\,U(t-1)\Big\}
\end{aligned}
$$

Where $\Gamma(t-1)$ expresses next diagonal matrix.

$$
\Gamma(t-1)=\begin{pmatrix} \sum_{j=1}^{n}\Big\{(K_{cij}+L_{cij})\,x_j^2(t-1)+(K_{dij}+L_{dij})\,u_j^2(t-1)\Big\} & & 0 \\ & \ddots & \\ 0 & & \sum_{j=1}^{n}\Big\{(K_{cnj}+L_{cnj})\,x_j^2(t-1)+(K_{nj}+L_{nj})\,u_j^2(t-1)\Big\} \end{pmatrix}
$$

$\therefore$ (4. 13)

$$
\begin{aligned}
V(t+1) &= \mathring{V}(t+1)-Y\Big\{\Gamma(t-1)\,V(t)+C(0)\,X(t-1) \\
&\quad +D(0)\,U(t-1)\Big\}
\end{aligned}
$$

As we mentioned above, it is possible to solve the value of $V(t+1)$ by the information from the first period untill t-th period and $\mathring{V}(t+1)$.

Next the strictly positive real of (4. 3) and (4. 4) are leaded from the following lemma and theorem.

Lemma 3. linear discrete system

$$
\begin{cases} X(t+1)=AX(t)+BU(t+1) \\ Y(t+1)=CX(t)+DU(t+1) \end{cases}
$$

When we make pulse transfer function of linear discrete system as $H(z)=$

$C(zI-A)^{-1}B+D$, the necesary and sufficient conditions that $H(z)$ becomes strictly positive real is to exist some matrixes L, W, $P=P^T>0$ and $Q=Q^T>0$ which satisfy the next system of equations.

$$\begin{cases} A^TPA-P=-LL^T-Q \\ B^TPA+W^TL^T=C \\ D+D^T=B^TPB+W^TW \end{cases}$$

Proof: For instance, Cf. reference [4].

Lemma 4. In system of lemma 3, it is the sufficient condition that $H(z)$ is strictly positive real that is formed as the next equations for $^{\exists}P=P^T>0$ and $^{\exists}Q=Q^T>0$

$$\begin{cases} A^TPA-P=-Q \\ B^TPA=C \\ D+D^T=B^TPB \end{cases}$$

Proof: in lemma 3, put $L=0$ and $W=0$. Q. E. D.

Theorem 5. In linear block (3. 19) and (3. 20), if we put the linear compensator matrix Y and L as in next equations, then pulse transfer function $H(z)=Y(zI-C_m)^{-1}+L$ will be strictly positive real.

$$Y=\sum_{k=0}^{\infty} C_m^{Tk}\, C_m^{k+1}$$

$$L=1/2\sum_{k=0}^{\infty} C_m^{Tk}\, C_m^{k}$$

Proof: In lemma 4, when we put $A=C_m$, $-Q=-I$, $B=I$, $C=Y$, $D=L$, a sufficient condition for strictly positive real is as follows.

$$\begin{cases} C_m^TPC_m-P=-I \\ PC_m=Y \\ L+L^T=P \end{cases}$$

And now we shall make $P=\sum_{k=0}^{\infty} C_m^{Tk}\, C_m^k$, then $C_m^T\, PC_m-P=-I$ is realized.

The strictly positive real of $H(z)$ is proved by the following equations.

$$Y=PC_m=\sum_{k=0}^{\infty} C_m^{Tk}\, C_m^{k+1}$$

$$L=1/2\sum_{k=0}^{\infty} C_m^{kT}\, C_m^k$$ Q. E. D.

From the above theorem 3 and 4, the equivalent feedback systems of (4. 3) (4. 4) and (4. 5) will be globaly asymptoticaly hyperstable by Popov's theorem.

Therefore we get next theorem 6.

Theorem 6. (The fundamental theorem of discrete model reference adaptive regional input-output model)

We shall represent the discrete model reference adaptive regional input-output model as follows.

(4. 14) reference model $$X_m(t+1) = B_m^{-1}\Big(I-(I-\hat{M}_m)\ A_m+B_m\Big)\ X_m(t) - B_m^{-1}\Big((I-\hat{M}_m)\ H(t)+E(t)\Big)$$

(4. 15) adaptive model $$X(t+1) = B^{-1}(t+1)\ \Big(I-\Big(I-\hat{M}(t)\Big)\ A(t) + B(t)\Big)\ X(t) - B^{-1}(t+1)\ \Big((I-\hat{M}_m)\ H(t)+E(t)\Big)$$

We shall make equivalent feedback system as follows.

(4. 16) $$\varepsilon(t+1) = C_m\varepsilon(t) + W(t+1)$$

(4. 17) $$V(t+1) = Y\varepsilon(t) + LW(t+1)$$

(4. 18) $$W(t+1) = \Big(C_m - C(t+1)\Big)\ X(t) + \Big(D_m - D(t+1)\Big)\ U(t)$$

Where we shall represent as follows.

$$\varepsilon(t+1) = X_m(t+1) - X(t+1)$$

$$C_m = B_m^{-1}\Big(I-(I-\hat{M}_m)\ A_m+B_m\Big)$$

$$D_m = -B_m^{-1}$$

$$C(t+1) = B^{-1}(t+1)\ \Big(I-\Big(I-\hat{M}(t)\Big)\ A(t) + B^{-1}(t)\Big)$$

$$D(t+1) = -B^{-1}(t+1)$$

$$U(t) = (I-\hat{M}_m)\ H(t)+E(t)$$

And we put $Y=\sum_{k=0}^{\infty} C_m^{Tk}\ C_m^{k+1}$, $L=1/2\sum_{k=0}^{\infty} C_m^{Tk}\ C_m^{k}$.

When we shall define the adaptation algorithm of $C(t)$, $D(t)$ as the next equations, the equivalent feedback systems from (4. 16) to (4. 18) are globaly asymptoticaly hyperstable.

(4. 19) $$C(t+1) = K_c \otimes \sum_{k=0}^{t} V(k+1)\ X^T(k) + L_c \otimes V(t+1)\ X^T(t) + C(0)$$

(4. 20) $$D(t+1) = K_d \otimes \sum_{k=0}^{t} V(k+1)\ U^T(k) + L_d \otimes V(t+1)\ U^T(t) + D(0)$$

$K_c = (k_{cij})$, $L_c = (L_{cij})$, $K_d = (K_{dij})$, $L_d = (L_{dij})$
(elements of each matrix are all positive numbers)

5. Conclusion

In this study, we applied properly the theory of the adaptive reference control to the dynamic input-output analysis, and considered the basic theory about Model Reference Adaptive I-O Analysis. We are confident that this study has proposed the basic theoretical analysis about both adaptation principle and stability of two models, which are the continuous model that is provided for the theoritical development in the near future and the discrete model that will be used for the actual metrical analysis. We believe that Model Reference Adaptive I-O Analysis is developed in this study for the first time. And we are expecting that this theory will be greatly developed and enhanced for ever. From the result, of this study, we propose some of futher problems as follows.

1) Numerical simulating by actual input-output model.
2) Inquiring economic and political meaning of the adaptation principle.

Besides, the theoritical development in the conbination of the optimal control theory and Model Reference Adaptive I-O Analysis is also a interesting problem [14]~[16]. Finally we believe that we must develop this fascinating theory as a theoritical support in the regional planning studies.

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