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A Study on Model Reference Adaptive Control In Economic Development (V)

—Model Reference Adaptive Turnpike Theorem (I)—

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Abstract

The pricipal aim of this paper is to present "Model Reference Adaptive Turnpike Theorem". Model Reference Adaptive Turnpike Theorem shows a path of economic development which converges to the Turnpike and the adaptation processes of technological system and its stability. Model Reference Adaptive Turnpike Theorem is obtained by extending the theorems in our previous papers 1) and 2) using the projective transformation. Model Reference Adaptive Turnpike Theorem, it is hoped, would play an important role in regional economic planning policy formulation.

Key Words: Model reference adaptive turnpike theorem, Model reference adaptive system, Turnpike theorem, Dynamic input-output model, Relatively stable, Projective transformation.

1. Introduction

In our previous papers 1), 2), we applied a model reference adaptive system to a dynamic input-output model and considered some basic theories for Model Reference Adaptive I-O System. From the consideration, we could get some results about the adaptation principle and the asymptotical stability. On the other hand, however, there were two major problems in these previous papers. The first was with a construction of a reference model, and the second was how to make the reference model stable when the transfer function is strictly positive real. This article aims at resolving these afore-mentioned problems.

The theory of dynamic imput-output analysis has a long history since W. Leontief first introduced it. It is also generally known that Turnpike Theorem is one of the most important results in the theory. Turnpike Theorem is applied here to lead us to a solution to the first problem.

On the point of the stability of the reference model, the necessary condition that the system representation matrix must be stable is too strict because the conventional dynamic economic model takes into account the shift of economic equilibrium during the process of adjustment of market economy and the representation matrix of the model is usually unstable. In this paper, therefore, we shall introduce a concept of the projective transformation and it is hoped that this would act as a solution to this problem of unstability in the reference model.

2. Turnpike Theorem

Turnpike Theorem was conjectured first by Dorfmann, Samuelson, and Solow (DOSSO) 3), and mathematically proved by Morishima 4) and Radner 5) in n-sector economic growth model. Later many mathematical economists have proved and extended various types of the theorem. Prominent amongst there are T. Watanabe and J. Tsukui 6) whose results are introduced here after and which in our aim would help charifying issues in the subsequent chapters.

Now let the notations be as follows;

A: input-output coefficient matrix $(n\times n)$
B: capital coefficient matrix $(n\times n)$
$X(t)$: output vector at t th period $(n\times 1)$
$H(t)$: final demand vector at t th period ($n\times 1$, except a private investment and assumed constant)

Then dynamic input-output model is as follows.

$$X(t)=AX(t)+B\big(X(t+1)-X(t)\big)+H(t) \qquad (2.1)$$

Further when we set a stationary equilibrium solution as $X=(I-A)^{-1}H(t)$ and define $x(t)$ as $x(t)\equiv X(t)-X$, (2. 1) is written in the form (2. 2).

$$x(t+1)=\big(I+B^{-1}(I-A)\big)\,x(t) \qquad (2.2)$$

Here let us put three assumptions.

Assumption 1. Each element of $(I-A)^{-1}$ is positive.
Assumption 2. Determinant of B is not zero. i.e. $|B|\neq 0$
Assumption 3. For the eigen values μ_i of $I+B^{-1}(I-A)$ $(i=1,\cdots,n)$, $\mu_1\neq|\mu_i|$ $(i\neq 1)$ holds.

For the assumption 3. Let γ be Frobenius root of $(I-A)^{-1}B$ then we can see that $\mu_1=1+\dfrac{1}{\gamma}>1$ from the fact that $(I-A)^{-1}B$ is a positive matrix.

Under the formulations stated above, let us consider the next programming problems.

Programming Problem (P. P.)

$$\max P^T Bx(N)$$

$$\text{subject to } (I-A+B)\,x(t)\geqq Bx(t+1)$$

$$x(t)\geqq 0 \quad (t=0,1,\cdots,N)$$

Dual Problem (D. P.)

$$\min u^T(1)(I-A+B)\,x(0)$$

$$\text{subject to } u^T(t+1)(I-A+B)-u^T(t)B \leqq 0 \qquad (t=1, \cdots, N-1)$$
$$u^T(N) \geqq P^T B$$
$$u(t) \geqq 0 \qquad (t=1, \cdots, N)$$

where, P: given price vector of capital goods at planned period N $(n \times 1)$

$u(t)$: price vector of $x(t)$ $(n \times 1)$

Using the relationship between (P. P.) and (D. P.), we can get

$$\begin{aligned} u^T(1)(I-A+B)\,x(0) &\geqq u^T(1)\,Bx(1) \geqq u^T(2)(I-A+B)\,x(1) \geqq \cdots \\ &\geqq u^T(N-1)\,Bx(N-1) \geqq u^T(N)(I-A+B)x(N-1) \\ &\geqq u^T(N)\,Bx(N) \geqq P^T Bx(N) \end{aligned} \tag{2. 3}$$

By the duality theorem of dynamic programming, the necessary and sufficient condition that sequences $\{x(t)\}_{t=0}^{N}$ and $\{\hat{u}(t)\}_{t=1}^{N}$ are the solution sequences of (P. P.) and (D. P.) is that every equality is realized in (2. 3).

Now the general solutions of (2. 1) and its dual equation

$$u^T(t+1) = u^T(t)B(I-A+B)^{-1} \tag{2. 4}$$

are represented as

$$x(t) = \sum_{i=1}^{n} \alpha_i \mu_i^t h_i \tag{2. 5}$$

$$u(t) = \sum_{i=1}^{n} \beta_i \left(\frac{1}{\mu_i}\right)^t k_i \tag{2. 6}$$

where, μ_i : eigen value of $I+B^{-1}(I-A)$

h_i : eigen vector of μ_i and $|h_i|=1$ $(n \times 1)$

α_i : constant determined by the initial value $x(0)$

k_i : eigen vector of $\dfrac{1}{\mu_i}$ which is an eigen value of $B(I-A+B)^{-1}$ and $|k_i| =1$ $(n \times 1)$

β_i : constant determined by the initial value $u(1)$

Accordingly the behaviors of $x(t)$ and $u(t)$ are dominated by μ_i. This leads us to the next two cases,

Case 1. Case of $\mu_1 > |\mu_i|$. In this case $\lim_{t\to\infty} \dfrac{x(t)}{\alpha_1 \mu_1^t h_1} = 1$ holds so the sectorwise share rate of the activity outputs $x(t)$ will asymptotically converge to the sectorwise share rate of h_1. We call this case as relatively stable and the half line $\alpha_1 \mu_1^t h_1 (t \geqq 0)$ which is drawn from the origin with the direction h_1 is the so-called Turnpike or von Neumann Ray.

Case 2. Case that $^{\exists}\mu_i$ s. t. $\mu_1 < |\mu_i|$. In this case if $x(0)$ is not on Turnpike, $^{\exists}t$, $^{\exists}i$ s. t. $x_i(t) < 0$ then economic meaning of $x(t)$ is lost. We call the case as relatively unstable.

For case 1 and case 2 the following two theorems can be obtained.

Theorem 1. If (2. 2) is relatively stable, the solution sequence $\{x(t)\}_{t=0}^{N}$ determined by

$$x(0) \geqq 0 \tag{2. 7}$$

$$x(t+1) = \left(I + B^{-1}(I-A)\right) x(t) \qquad (t=0, \cdots, N-2) \tag{2. 8}$$

$$x(N) = \left\{x(N) \middle| \max P^T Bx(N) \text{ subject to } Bx(N) \leqq (I-A+B)\, x(N-1)\right\} \tag{2. 9}$$

is a solution of (P. P.).

And the solution sequence $\{u(t)\}_{t=1}^{N}$ determined by

$$u^T(N) = \left\{u^T(N) \middle| \min u^T(N)\,(I-A+B)\, x(N-1) \text{ subject to } u^T(N) \geqq P^T B\right\} \tag{2. 10}$$

$$u^T(t) = u^T(t+1)\left(I + (I-A)\, B^{-1}\right) \qquad (t = N-1, N-2, \cdots, 1) \tag{2. 11}$$

is a solution of (D. P.). (see Figures 1 and 2)

(proof) see reference 6)

Theorem 2. If (2. 2) is relatively unstable, the solution sequence $\{\eta^* x^*(t)\}_{t=1}^{N}$ obtained from

$$x^*(N) = \left\{x(N) \middle| \max P^T Bx(N) \text{ subject to } k_1^T\ Bx(N) \leqq 1,\ x(N) \geqq 0\right\} \tag{2. 12}$$

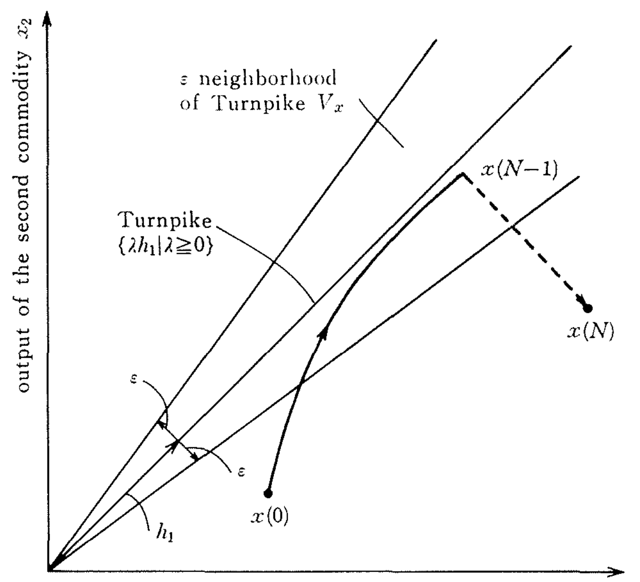

Figure 1. Optimal Path of Outputs in Relatively Stable Case.

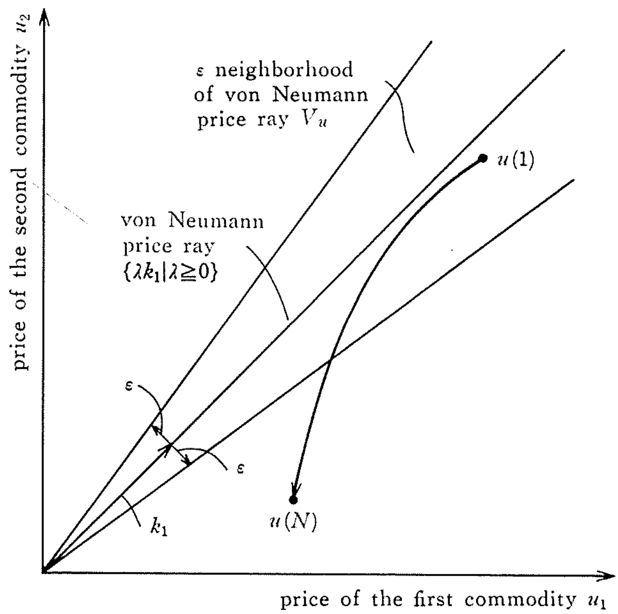

Figure 2. Optimal Path of Prices in Relatively Stable Case.

$$x^*(t) = \sum_{i=1}^{n} \left(\frac{1}{\mu_i}\right)^{N-t} \mu_i^N \alpha_i h_i \qquad (t = N-1, \cdots, 1) \tag{2. 13}$$

$$\eta^* = \left\{\eta \middle| \max \eta \text{ subject to } \eta B x^*(1) \leqq (I - A + B)\, x(0),\ \eta \geqq 0\right\} \tag{2. 14}$$

is a solution of (P. P.).

And the solution sequence $\{\xi^* \eta^*(t)\}_{t=1}^{N}$ obtained from

$$u^*(1) = \left\{u(1) \middle| \min u^T(1)\,(I - A + B)\, x(0) \text{ subject to } u^T(1)\, Bx^*(1) \geqq 1,\ u(1) \geqq 0\right\} \tag{2. 15}$$

$$u(t) = \sum_{i=1}^{n} \beta_i \left(\frac{1}{\mu_i}\right)^t k_i \qquad (t = 1, \cdots, N) \tag{2. 16}$$

$$\xi^* = \left\{\xi \middle| \min \xi \text{ subject to } \xi u^*(N)\, B \geqq PB,\ \xi \geqq 0\right\} \tag{2. 17}$$

is a solution of (D. P.). (see Figures 3 and 4)

(proof) see reference 6).

From the two theorems stated above, we can derive the so-called Turnpike Theorem as follows ;

Theorem 3. (Turnpike Theorem)

For a properly choosen neighbourhood Vx of Turnpike, there exists some finite $t > 0$ which is independ to the planned period N, then the solution sequence $\{x(t)\}_{t=0}^{N}$ of (P. P.) stays in Vx at least during the interval $(N - t - 1)$.

(proof) see reference 6).

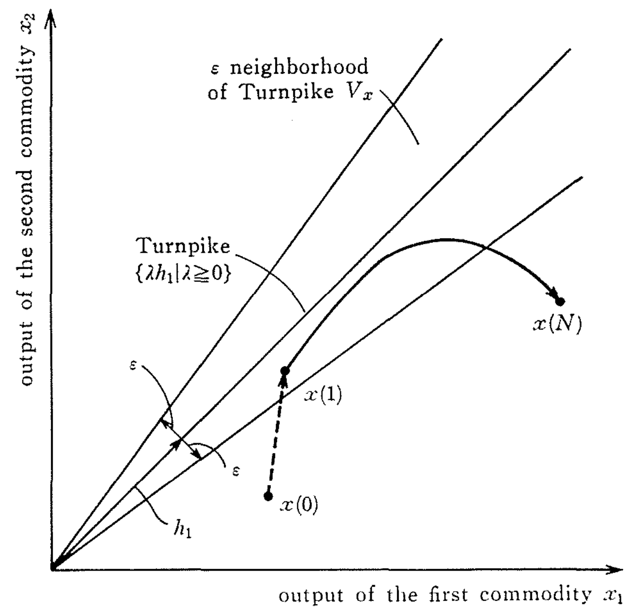

Figure 3. Optimal Path of Outputs in Relatively Unstable Case.

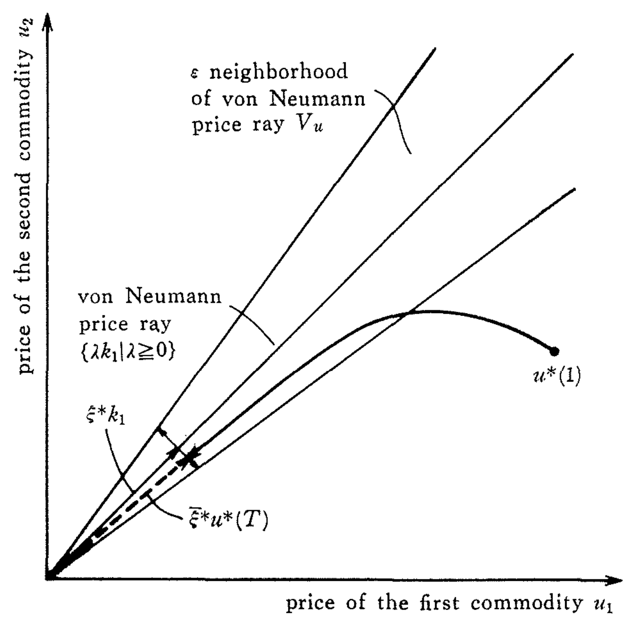

Figure 4. Optimal Path of Prices in Relatively Unstable Case.

3. Model Reference Adaptive I-O System

In this chapter let us simply review Model Reference Adaptive I-O System introduced in our previous papers 1), 2).

Model reference Adaptive I-O System in Continuous Format is defined as follows ;

reference model

$$\frac{dXm(t)}{dt} = Bm^{-1}(I - Am)\, Xm(t) - Bm^{-1}H(t) \tag{3.1}$$

adaptive model

$$\frac{dX(t)}{dt} = B^{-1}(t)\left(I - A(t) - \frac{dB(t)}{dt}\right) X(t) - B^{-1}(t)\, H(t) \tag{3.2}$$

where, $Xm(t)$: reference output vector $(n \times 1)$
Am : reference input-output coefficient matrix $(n \times n)$
Bm : reference capital coefficient matrix $(n \times n)$
$H(t)$: final demand vector ($n \times 1$, except private investment)
$X(t)$: adaptive output vector $(n \times 1)$
$A(t)$: adaptive inout-output coefficient matrix $(n \times n)$
$B(t)$: adaptive capital coefficient matrix $(n \times n)$

And Model Reference Adaptive I-O System in Discrete Format is defined in the next.

reference model

$$Xm(t+1) = Bm^{-1}(I - Am + Bm)\, Xm(t) - Bm^{-1}H(t) \tag{3.3}$$

adaptive model

$$X(t+1) = B^{-1}(t+1)\left(I - A(t) + B(t)\right) X(t) - B^{-1}(t+1)\, H(t) \tag{3.4}$$

For above two formulations, the next fundamental theorems can be obtained as follows ;

Theorem 4. (Fundamental Theorem of Model Reference Adaptive I-O System in Continuous Format)

reference model

$$\frac{dXm(t)}{dt} = Bm^{-1}(I - Am)\, Xm(t) - Bm^{-1}H(t) \tag{3.5}$$

adaptive model

$$\frac{dX(t)}{dt} = B^{-1}(t)\left(I - A(t) - \frac{dB(t)}{dt}\right) X(t) - B^{-1}(t)\, H(t) \tag{3.6}$$

Define the equivalent feedback system as

$$\frac{d\varepsilon(t)}{dt} = Cm\varepsilon(t) + W(t) \tag{3.7}$$

$$V(t) = Y\varepsilon(t) \tag{3.8}$$

$$W(t) = \left(Cm - C(t)\right) X(t) + \left(Dm - D(t)\right) H(t) \tag{3.9}$$

where, $\varepsilon(t) = Xm(t) - X(t)$ (3. 10)

$$Cm = Bm^{-1}(I - Am) \tag{3. 11}$$

$$Dm = -Bm^{-1} \tag{3. 12}$$

$$C(t) = B^{-1}(t)\left(I - A(t) - \frac{dB(t)}{dt}\right) \tag{3. 13}$$

$$D(t) = -B^{-1}(t) \tag{3. 14}$$

Let Y be a solution of $Cm^T Y + YCm = -I$ and adaptation principles $C(t)$ and $D(t)$ be

$$C(t) = Kc \otimes \int_0^t V(\tau)X^T(\tau)d\tau + Lc \otimes \left(V(t)X^T(t)\right) + C(0) \tag{3. 15}$$

$$D(t) = Kd \otimes \int_0^t V(\tau)\, H^T(\tau)d\tau + Ld \otimes \left(V(t)H^T(t)\right) + D(0) \tag{3. 16}$$

then the equivalent feedback system will be globally asymptotically hyperstable.

i. e. $\lim_{t\to\infty} ||Xm(t) - X(t)|| = 0$ (3. 17)

$$\lim_{t\to\infty} ||Cm - C(t)|| = 0 \tag{3. 18}$$

$$\lim_{t\to\infty} ||Dm - D(t)|| = 0 \tag{3. 19}$$

Where Kc, Lc, Kd, Ld are the positive matrices and $\otimes$ stands for the operation that is

$$\begin{pmatrix} a_{11} \cdots a_{1n} \\ \vdots \quad\; \vdots \\ a_{n1} \cdots a_{nn} \end{pmatrix} \otimes \begin{pmatrix} b_{11} \cdots b_{1n} \\ \vdots \quad\; \vdots \\ b_{n1} \cdots b_{nn} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} \cdots a_{1n}b_{1n} \\ \vdots \quad\quad\; \vdots \\ a_{n1}b_{n1} \cdots a_{nn}b_{nn} \end{pmatrix} \tag{3. 20}$$

This result shows that the adaptive outputs asymptotically converge to the reference by those adaptation principles.

Furthermore $A(t)$ and $B(t)$ are solved by

$$A(t) = I + D^{-1}(t)\, C(t) - \frac{dD^{-1}(t)}{dt} \tag{3. 21}$$

$$B(t) = -D^{-1}(t) \tag{3. 22}$$

and $\lim_{t\to\infty} ||Am - A(t)|| = 0$ (3. 23)

$$\lim_{t\to\infty} ||Bm - B(t)|| = 0 \tag{3. 24}$$

Theorem 5. (Fundamental Theorem of Model Reference Adaptive I-O System in Discrete Format)

reference model

$$Xm(t+1) = Bm^{-1}(I - Am + Bm)\, Xm(t) - Bm^{-1}H(t) \tag{3. 25}$$

adaptive model

$$X(t+1)=B^{-1}(t+1)\Big(I-A(t)+B(t)\Big)\,X(t)-B^{-1}(t+1)H(t) \tag{3. 26}$$

Define the equivalent feedback system as

$$\varepsilon(t+1)=Cm\varepsilon(t)+W(t+1) \tag{3. 27}$$

$$V(t+1)=Y\varepsilon(t)+LW(t+1) \tag{3. 28}$$

$$W(t+1)=\Big(Cm-C(t+1)\Big)X(t)+\Big(Dm-D(t+1)\Big)H(t) \tag{3. 29}$$

where, $\varepsilon(t)=Xm(t)-X(t)$ (3. 30)

$$Cm=Bm^{-1}(I-Am+Bm) \tag{3. 31}$$

$$Dm=-Bm^{-1} \tag{3. 32}$$

$$C(t+1)=B^{-1}(t+1)\Big(I-A(t)+B(t)\Big) \tag{3. 33}$$

$$D(t+1)=-B^{-1}(t+1) \tag{3. 34}$$

Let Y, L be solutions of

$$Cm^{T}PCm-P=-I \tag{3. 35}$$

$$PCm=Y \tag{3. 36}$$

$$L+L^{T}=P \qquad (P=P^{T}>0) \tag{3. 37}$$

and adaptation principles of $C(t)$ and $D(t)$ be

$$C(t+1)=Kc\otimes\sum_{k=0}^{t}V(k+1)X^{T}(k)+Lc\otimes\Big(V(t+1)\;X^{T}(t)\Big)+C(0) \tag{3. 38}$$

$$D(t+1)=Kd\otimes\sum_{k=0}^{t}V(k+1)H^{T}(k)+Ld\otimes\Big(V(t+1)\;H^{T}(t)\Big)+D(0) \tag{3. 39}$$

then the equivalent feedback system will be globally asymptotically hyperstable.

i. e. $\lim_{t\to\infty}||Xm(t)-X(t)||=0$ (3. 40)

$$\lim_{t\to\infty}||Cm-C(t)||=0 \tag{3. 41}$$

$$\lim_{t\to\infty}||Dm-D(t)||=0 \tag{3. 42}$$

This result shows that the adaptive outputs asymptotically converge to the reference by those adaptation principles. And $A(t)$ and $B(t)$ are solved by

$$A(t)=I+D(t+1)\,C(t+1)+D^{-1}(t) \tag{3. 43}$$

$$B(t+1)=-D^{-1}(t+1) \tag{3. 44}$$

and $\lim_{t\to\infty}||Am-A(t)||=0$ (3. 45)

$$\lim_{t\to\infty}||Bm-B(t)||=0 \tag{3. 46}$$

The prooves of the two theorems are shown in the references 1) and 2).

4. Model Reference Adaptive Turnpike Theorem

There may be many problems in the Theorems 4 and 5, however, the most important is the stability of the reference model. In both theorems, we assume the existences of Y and P which satisfy

$$Cm^T Y + YCm = -I \qquad Y = Y^T > 0 \tag{4. 1}$$

$$Cm^T PCm - P = -I \qquad P = P^T > 0 \tag{4. 2}$$

Of course these equations show the continuous and the discrete Lyapunov's equa tions. So the necessary and sufficient conditions of (4. 1) and (4. 2) are that the real parts of the eigen values of Cm are less than 0 and the absolute values of the eigen values of Cm is less than 1, respectively.

In this way, the behavior of the reference model can be described from a point of view of eigen value as follows. Let (3. 1) and (3. 3) rewrite with respect to the distance $xm(t) = Xm(t) - (I - Am)^{-1} H(t)$ from the stationary equilibrium solution $(I - Am)^{-1} H(t)$, then

$$\frac{dxm(t)}{dt} = Bm^{-1}(I - Am)\, xm(t) \tag{4. 3}$$

$$xm(t+1) = Bm^{-1}(I - Am + Bm)\, xm(t) \tag{4. 4}$$

So the general solutions of (4. 3) and (4. 4) are given by

$$xm(t) = e^{Bm^{-1}(I-Am)t} x_m(0) \tag{4. 5}$$

$$xm(t) = \sum_{i=1}^{n} \alpha_i \mu_i^t h_i \tag{4. 6}$$

where, α_i : constant decided by the initial value $x_m(0)$
μ_i : eigen value of $Bm^{-1}(I - Am + Bm)$
h_i : eigen vector of μ_i and $|h_i| = 1$

From the general solutions (4. 5) and (4. 6) we have three types of behaviors of $xm(t)$, (1) absolutely stable, (2) relatively stable, and (3) relatively unstable, respectively. The meanings of (1), (2) and (3) are explained below again, the reason being that we are not dealing with a continuous case in chapter 2.

(1) *Absolutely Stable*

This case denotes that the real part of the eigen values of $Bm^{-1}(I - Am)$ are less than zero in a continuous format and the absolute values of the eigen values of $Bm^{-1}(I - Am + Bm)$ are less than 1 in discrete format. In this case $\lim_{t\to\infty} x_m(t) = 0$ holds, i. e. $\lim_{t\to\infty} Xm(t) = (I - A)^{-1} H(\infty)$, accordingly $Xm(t)$ asymptotically converges to the stationary equilibrium solution $(I - Am)^{-1} H(\infty)$.

(2) *Relatively Stable*

This case depicts that in a continuous format there exists an eigen value of $Bm^{-1}(I - Am)$ which is positive and has a maximum absolute value among the other eigen values. Also in discrete format there exists an eigen value which is greater than 1 and has a maximum absolute value among the other eigen values.

In this case the asymptotial behavior of $x_m(t)$ is dominated by the maximum eigen value, say μ_1, and the sectorwise share rate of $x_m(t)$ converges to the one of the half line from the origin $\alpha_1 e^{\mu_1 t} h_1$ or $\alpha_1 \mu_1^t h_1$, respectively.

(3) *Relatively Unstable*

This case shows that in continuous format there exists some complex eigen value of $Bm^{-1}(I-Am)$ whose an absolute value is maximum among the other eigen values. In discrete format there exists some complex or negative eigen value of $Bm^{-1}(I-Am+Bm)$ whose the absolute value is maximum among the other eigen values. This case, same as that in chapter 2, looses an economic sense.

These above-mentioned cases are shown in Figure 5. We can immediately see that the eigen values assumed in the theorem 4 and theorem 5 are different from ones of the case of relative stability which reflects normal economic thinking and that the theorems 4 and 5 can not be realized except in the case of absolute stability. Therefore some transformations are needed so that the theorems 4 and 5 are realized under the condition of relative stability.

To achieve this objective, let us introduce a projective transformation as depicted in Figure 6. Let us assume that (3. 1) and (3. 3) are relatively stable and $H(t)$ is a constant final demand vector only for simplicity purposes. Further let us assume that the output trajectory represented by (3. 1) and/or (3. 3) asymptotically converges to the Turnpike $c\mu^t+(I-A)^{-1}H$ ($t>0$, c is a constant and μ is a Frobenius root of Cm.) from the equilibrium solution $X=(I-A)^{-1}H$. Under these assumptions we

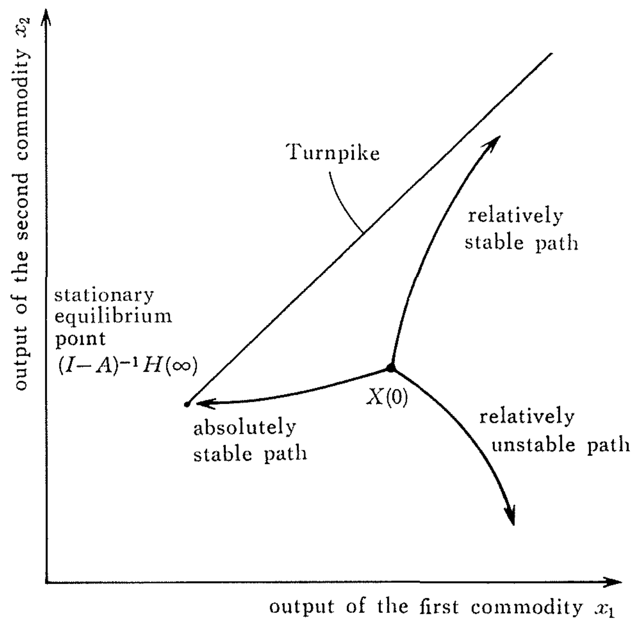

Figure 5. Behaviours of Outputs in Absolutely Stable, Relatively Stable, and Relatively Unstable.

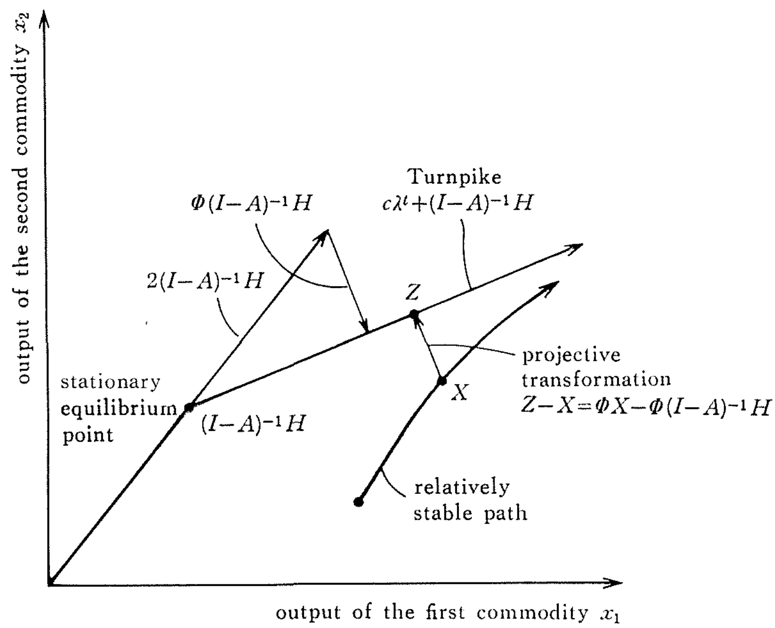

Figure 6. Turnpike and Projective Transformation.

shall define the projective transformation as the foot of a perpendicular line from the relatively stable trajectory to Turnpike. (see Figure 6) Let $Z(t)$ be a foot of a perpendicular line from a point $X(t)$ on the relatively stable trajectory to the Turnpike. From this we obtain the following equations.

$$Z(t)-X(t)=\Phi X(t)-\Phi(I-A)^{-1}H \tag{4.7}$$

$$\Phi=\eta\eta^T\|\eta\|^{-2}-I \tag{4.8}$$

η: an eigen vector of μ

Where Φ is the $n\times n$ matrix which stands for the projective transformation. Let $\hat{X}(t)$ be $\hat{X}(t)\equiv Z(t)-X(t)+\Phi(I-A)^{-1}H$, then $\hat{X}(t)=\Phi X(t)$. We shall introduce a generalized inverse matrix Φ^+ of Φ because of the singularity of Φ i.e. $\det\Phi=0$, then we get $X(t)=\Phi^+\hat{X}(t)$. Here the generalized inverse matrix is defined as the matrix which satisfies the relation $\Phi\Phi^+\Phi=\Phi$. Though the generalized inverse matrix is not decided uniquely in general we are able to decide uniquely by using the initial value of $X(t)$.

By introducing Φ^+, (3.1) and (3.3) can be written in these forms.

$$\frac{d\hat{X}(t)}{dt}=\Phi B^{-1}(I-A)\Phi^+\hat{X}(t)-\Phi B^{-1}H \tag{4.9}$$

$$X(t+1)=\Phi B^{-1}(I-A+B)\Phi^+\hat{X}(t)-\Phi B^{-1}H \tag{4.10}$$

We can see easily $\lim_{t\to\infty}\hat{X}(t)=\Phi(I-A)^{-1}H$, so (4.9) and (4.10) are asymptotically stable. From this point of view the projective transformation is characterized as the coordinate transformation which converts the unstable system into an asymptotically stable one.

Concequently, when we adopt the technological system which asymptotically converges to a Turnpike as a reference model, we derive the following theorems.
Theorem 6. (Model Reference Adaptive Turnpike Theorem in Continuous Format)

reference model

$$\frac{d\hat{X}m(t)}{dt} = \Phi Bm^{-1}(I-Am)\,\Phi^{+}\hat{X}m(t) - \Phi Bm^{-1}H(t) \tag{4.11}$$

$$Xm(t) = \Phi^{+}\hat{X}m(t) \tag{4.12}$$

adaptive model

$$\frac{d\hat{X}(t)}{dt} = \Phi B^{-1}(t)\left(I - A(t) - \frac{dB(t)}{dt}\right)\Phi^{+}\hat{X}(t) - \Phi^{+}\hat{X}(t) - \Phi B^{-1}(t)\,H(t) \tag{4.13}$$

$$X(t) = \Phi^{+}\hat{X}(t) \tag{4.14}$$

Define the equivalent feedback system as

$$\frac{d\varepsilon(t)}{dt} = Cm\,\varepsilon(t) + W(t) \tag{4.15}$$

$$V(t) = Y\,\varepsilon(t) \tag{4.16}$$

$$W(t) = \left(Cm - C(t)\right)X(t) + \left(Dm - D(t)\right)H(t) \tag{4.17}$$

where,

$$\varepsilon(t) = Xm(t) - X(t) \tag{4.18}$$

$$Cm = \Phi Bm^{-1}(I-Am)\,\Phi^{+} \tag{4.19}$$

$$Dm = -\Phi Bm^{-1} \tag{4.20}$$

$$C(t) = \Phi B^{-1}(t)\left(I - A(t) - \frac{dB(t)}{dt}\right)\Phi^{+} \tag{4.21}$$

$$D(t) = -\Phi B^{-1}(t) \tag{4.22}$$

Let Y be a solution of $CmY + YCm = -I$ and the adaptation principles of $C(t)$ and $D(t)$ be

$$C(t) = Kc \otimes \int_0^t V(\tau)\,\hat{X}^T(\tau)\,d\tau + Lc \otimes \left(V(t)\hat{X}^T(t)\right) + C(0) \tag{4.23}$$

$$D(t) = Kd \otimes \int_0^t V(\tau)\,H^T(\tau)\,d\tau + Ld \otimes \left(V(t)\,H^T(t)\right) + D(0) \tag{4.24}$$

then the equivalent feedback system will be globally asymptotically hyperstable.

i. e.

$$\lim_{t\to\infty} \|\hat{X}m(t) - \hat{X}(t)\| = 0 \tag{4.25}$$

$$\lim_{t\to\infty} \|Cm - C(t)\| = 0 \tag{4.26}$$

$$\lim_{t\to\infty} \|Dm - D(t)\| = 0 \tag{4.27}$$

This result shows that the output of the adaptive model converges to Turnpike.

And $X(t)$, $A(t)$, $B(t)$ are solved by

$$X(t)=\Phi^{+}\hat{X}(t) \tag{4.28}$$

$$A(t)=I+D^{-1}(t)\,C(t)\,\Phi-\frac{dD^{-1}(t)}{dt}\,\Phi \tag{4.29}$$

$$B(t)=-D^{-1}(t)\,\Phi \tag{4.30}$$

and the next equations realized.

$$\lim_{t\to\infty}||Xm(t)-X(t)||=0 \tag{4.31}$$

$$\lim_{t\to\infty}||Am-A(t)||=0 \tag{4.32}$$

$$\lim_{t\to\infty}||Bm-B(t)||=0 \tag{4.33}$$

Theorem 7. (Model Reference Adaptive Turnpike Theorem in Discrete Format)

reference model

$$\hat{X}m(t+1)=\Phi Bm^{-1}(I-Am+Bm)\,\Phi^{+}\hat{X}m(t)-\Phi Bm^{-1}H(t) \tag{4.34}$$

$$Xm(t)=\Phi^{+}\hat{X}m(t) \tag{4.35}$$

adaptive model

$$\hat{X}(t+1)=\Phi B^{-1}(t+1)\Big(I-A(t)+B(t)\Big)\,\Phi^{+}\hat{X}(t)-\Phi B^{-1}(t+1)\,H(t) \tag{4.36}$$

$$X(t)=\Phi^{+}\hat{X}(t) \tag{4.37}$$

Define the equivalent feedback system as

$$\varepsilon(t+1)=Cm\varepsilon(t)+W(t+1) \tag{4.38}$$

$$V(t+1)=Y\varepsilon(t)+LW(t+1) \tag{4.39}$$

$$W(t+1)=\Big(Cm-C(t+1)\Big)\,X(t)+\Big(Dm-D(t+1)\Big)\,H(t) \tag{4.40}$$

where, $$\varepsilon(t)=\hat{X}m(t)-\hat{X}(t) \tag{4.41}$$

$$Cm=\Phi Bm^{-1}(I-Am+Bm)\,\Phi^{+} \tag{4.42}$$

$$Dm=-\Phi Bm^{-1} \tag{4.43}$$

$$C(t+1)=\Phi B^{-1}(t+1)\Big(I-A(t)+B(t)\Big)\,\Phi^{+} \tag{4.44}$$

$$D(t+1)=-\Phi B^{-1}(t+1) \tag{4.45}$$

Let Y and L be solutions of

$$Cm^{T}PCm-Cm=-I \tag{4.46}$$

$$PCm=Y \tag{4.47}$$

$$L+L^{T}=P \qquad (P=P^{T}>0) \tag{4.48}$$

and the adaptation principles of $C(t)$ and $D(t)$ be

$$C(t+1)=Kc\otimes\sum_{k=0}^{t}V(k+1)\hat{X}^T(k)+Lc\otimes V(t+1)\hat{X}^T(t)+C(0) \qquad (4.49)$$

$$D(t+1)=Kd\otimes\sum_{k=0}^{t}V(k+1)H^T(k)+Ld\otimes V(t+1)H^T(t)+D(0) \qquad (4.50)$$

then the equivalent feedback system will be globally asymptotically hyperstable.

i. e. $$\lim_{t\to\infty}\|\hat{X}m(t)-\hat{X}(t)\|=0 \qquad (4.51)$$

$$\lim_{t\to\infty}\|Cm-C(t)\|=0 \qquad (4.52)$$

$$\lim_{t\to\infty}\|Dm-D(t)\|=0 \qquad (4.53)$$

This result shows that the output of the adaptive model converges to Turnpike.

And $X(t)$, $A(t)$ and $B(t)$ are solved by

$$X(t)=\Phi^{+}\hat{X}(t) \qquad (4.54)$$

$$A(t)=I+D^{-1}(t+1)\,C(t+1)\,\Phi-D^{-1}(t)\,\Phi \qquad (4.55)$$

$$B(t)=-D^{-1}(t+1) \qquad (4.56)$$

and the following equations are realized.

$$\lim_{t\to\infty}\|Xm(t)-X(t)\|=0 \qquad (4.57)$$

$$\lim_{t\to\infty}\|Am-A(t)\|=0 \qquad (4.58)$$

$$\lim_{t\to\infty}\|Bm-B(t)\|=0 \qquad (4.59)$$

The proofs of the above two theorems are similar to ones of Theorem 4 and Theorem 5.

5. Conclusion

This study developed the fundamental theory of the model reference adaptive input-output system applying the theory of model reference adaptive system to the dynamic input-output model.

By the main results of this study we can see that there exist some adaptation processes which converge to the Turnpike even if the initial technological system is unstable. And also the results may lead us to the solution of the problems between the stability of the actual exonomic growth and the unstability of the dynamic input-output table.

Because this study is in its initial stages, a number of problems were encountered in the development of this framework. There is therefore the need for continuous and constant appraisal of its efficiency.

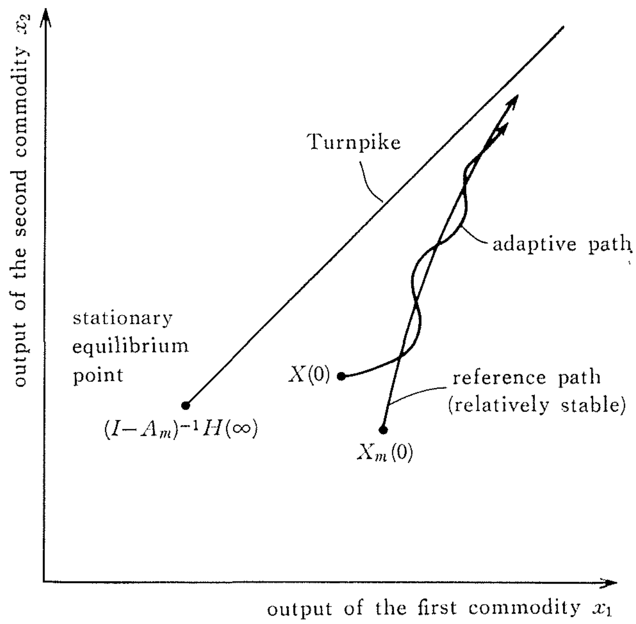

Figure 7. Adaptation Process in Model Reference Adaptive Turnpike Theorem.

Issues worth noting include

(1) Introducing the adaptation process by the control of the private capital or R & D investment.

(2) Simplification of the adaptation principle.

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