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A Study on Model Reference Adaptive Control In Economic Development (VI)

—Model Reference Adaptive Turnpike Theorem (II)—

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Abstract

The principal purpose of this paper is to improve upon the Model Reference Adaptive Turnpike Theorems presented in our previous paper 5) so that the theorems in 5) can be realized in the case of the relatively unstable system. As a follow up, the usefulness of the theorems obtained in this study is proved by some numerical simulations.

Key Words: Model reference adaptive turnpike theorem, Model reference adaptive system, Turnpike theorem, Dynamic input-output model, Relatively unstable system.

1. Introduction :

We have applied the theory of model reference adaptive system to the dynamic I-O system and developed some fundamental theories about model reference adaptive I-O system in the reference 3). However, as we pointed out in the reference 5), there were two major problems in our paper 3).

(1) How do we choose a reference model or a reference economic growth path ?

(2) What adaptation laws can we apply when the reference model is unstable ?

Our previous paper 5) have given some solutions to those problems, however, these solutions seem to be weak. Namely, as a reference path we adopted the economic growth path which converged to the turnpike. But the mathematical condition which realized that reference path gives some constrained conditions to the eigen values of the reference model, and it is not clear whether those conditions are always satisfied to actual dynamic I-O systems.

This article aims at extending the model reference adaptive turnpike theorems which have been obtained in 5) to be applied to a more general reference model, and also proving the usefulness of the theorems, which are obtained in this study, by some numerical simulations.

2. Reflection on the Previous Paper :

Let us simply review the main results which were obtained in our previous

paper 5). Now a dynamic I-O system under a fixed technology is represented as follows ;

$$Xm(t+1) = Bm^{-1}(I-Am+Bm)\ Xm(t)-Bm^{-1}\ H \tag{1}$$

where, $Xm(t)$: output vector $(n \times 1)$

$\quad\quad H$: fixed final demand vector (exogenously given but excluding private investment) $(n \times 1)$

$\quad Am$: input-output coefficient matrix $(n \times n)$

$\quad Bm$: capital coefficient matrix $(n \times n)$

Let λ be a Frobenius root of $(I-Am)^{-1}Bm$, then the turnpike of (1) is expressed as,

$$Xm(t) = \left(1+\frac{1}{\lambda}\right)^{t}\left(Xm(0)-(I-Am)^{-1}\ H\right)+(I-Am)^{-1}\ H \tag{2}$$

Furthermore let us consider the economic growth path represented by (1) whose initial value is not on the turnpike, and indicate the distance between the path and the turnpike as $d(Xm(t))$. (see Figure 1.) We assume here that $\lim\limits_{t\to\infty} d(Xm(t))=0$ for arbitrary economic growth path represented by (1). Then we can observe that,

$$d\left(Xm(t)\right) = ||Z(t)-Xm(t)|| = ||\Phi\ Xm(t)-\Phi(I-Am)^{-1}\ H|| \tag{3}$$

where, $||\cdot||$: Euclidian norm
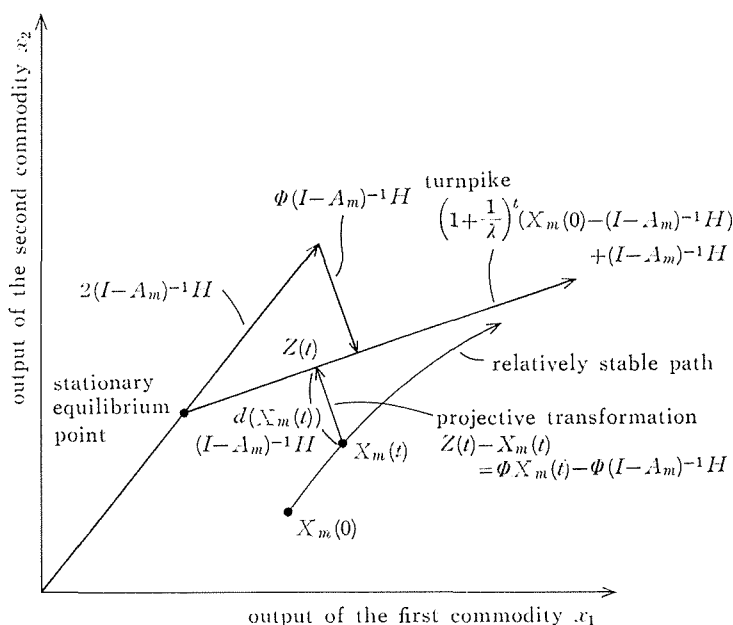
$$\Phi = \eta\eta^{T}||\eta||^{-2}-I$$

Figure 1.   Concept of the Projective Transformation.

η: an eigen vector corresponding to the eigen value $1+\dfrac{1}{\lambda}$ of $Bm^{-1}(I-$

$Am+Bm)$. Where λ stands for a Frobenius root of $(I-Am)^{-1}Bm$.

In our previous paper 5) we called $n \times n$ matrix Φ the projective transformation. That is, Φ defines the linear transformation which draws a perpendicular line from $Xm(t)$ to the turnpike. Now let us define $\hat{X}m(t)$ as follows;

$$\hat{X}m(t) = Z(t) - \hat{X}m(t) + \Phi(I - Am)^{-1} H \tag{4}$$

It is easy to see that $\hat{X}m(t) = \Phi Xm(t)$. By the assumption mentioned above, $\lim_{t \to \infty}(Z(t) - Xm(t)) = 0$ holds, then $\lim_{t \to \infty} \hat{X}m(t) = \Phi(I - Am)^{-1} H$ is realized. Conversely when $\hat{X}m(t)$ is given $Xm(t)$ would be uniquely determined by the relationship $\Phi Xm(t) = \hat{X}m(t)$ if an inverse transformation of Φ could be defined. However, we can observe that the determinant of Φ is zero and also the rank of Φ is one. Therefore we can not define the ordinary inverse transformation of Φ, that is to say, the inverse matrix of Φ. Accordingly we had to introduce another transformation, i. e. the generalized inverse matrix Φ^+ of Φ, then solved $Xm(t)$ by $Xm(t) = \Phi^+ \hat{X}m(t)$. Where the generalized inverse matrix Φ^+ is a $n \times n$ matrix defined by $\Phi\Phi^+\Phi = \Phi$. And Φ^+ is not uniquely determined in general, however, Φ^+ can be defined uniquely with the initial condition $Xm(0) = \Phi^+ \hat{X}m(0)$.

Now we multiply Φ to both sides of (1), then

$$\Phi \, Xm(t+1) = \Phi \, Bm^{-1}(I - Am + Bm) \, Xm(t) - \Phi \, Bm^{-1} H$$
$$\therefore \quad \hat{X}m(t+1) = \Phi \, Bm^{-1}(I - Am + Bm) \, Xm(t) - \Phi \, Bm^{-1} H \tag{5}$$

Substitute $\hat{X}m(t) = \Phi^+ Xm(t)$ to (5), then

$$\hat{X}m(t+1) = \Phi \, Bm^{-1}(I - Am + Bm) \, \Phi^+ \, \hat{X}m(t) - \Phi \, Bm^{-1} H \tag{6}$$

Because of $\lim_{t \to \infty} \hat{X}m(t) = \Phi(I - Am)^{-1} H$, $\Phi \, Bm^{-1}(I - Am + Bm) \, \Phi^+$ is a stable matrix. Accordingly when we set $Cm = \Phi \, Bm^{-1}(I - Am + Bm) \, \Phi^+$, the Lyapunov's equation mentioned below has a positive definite solution P. i. e. $\exists P = P^T > 0$ s. t.

$$Cm^T P Cm - Cm = -I \tag{7}$$

Then we can derive adaptation laws of $A(t)$ and $B(t)$ which make the gap between reference and adaptive models approach zero asymptotically in the below-stated model reference adaptive I-O system.

reference model

$$\hat{X}m(t+1) = \Phi \, Bm^{-1}(I - Am + Bm) \, \Phi^+ \, \hat{X}m(t) - \Phi \, Bm^{-1} H$$
$$Xm(t) = \Phi^+ \, \hat{X}m(t) \tag{8}$$

adaptive model

$$\hat{X}(t+1) = \Phi B^{-1}(t+1) \, (I - A(t) + B(t)) \, \Phi^+ \, \hat{X}(t) - \Phi B^{-1}(t+1) \, H$$
$$X(t) = \Phi^+ \, \hat{X}(t) \tag{9}$$

However, a problem arising from this context is the assumption that $\lim_{t \to \infty} d(Xm(t)) = 0$. When that assumption is represented in terms of eigen value, the following condition should be realized.

$$\exists i \text{ such that } \mu_i > 1 > |\mu_j| \qquad (j \neq i) \tag{10}$$

where, μ_i: eigen value of $Bm^{-1}(I - Am + Bm)$ $(i = 1, 2, \cdots, n)$

As written in the reference 14), the technological system in an actual dynamic I-O system which satisfies (10) has been hardly observed. And this fact gives us a strog constraint from an actually applicable point of view. Therefore it is very necessary to generalize the main results in 5) so that the results can be applied in the relatively unstable system.

3. Model Reference Adaptive Turnpike Theorem:

In this chapter we shall briefly explain model reference adaptive I-O system which is an application of MRAS to dynamic I-O system, and then show some theorems which were obtained related to the turnpike theorem.

Model reference adaptive I-O system is composed of two models, that is

reference model

$$Xm(t+1) = Bm^{-1}(I - Am + Bm) Xm(t) - Bm^{-1} H(t) \tag{11}$$

adaptive model

$$X(t+1) = B^{-1}(t+1) \left(I - A(t) + B(t) \right) X(t) - B^{-1}(t+1) H(t) \tag{12}$$

where, Xm: n-dim. reference output vector
  Am: $n \times n$ reference input-output coefficient matrix
  Bm: $n \times n$ reference capital coefficient matrix
  $H(t)$: n-dim. final demand vector (exogenously given but excluding private investment)
  $X(t)$: n-dim. adaptive output vector
  $A(t)$: $n \times n$ adaptive input-output coefficient matrix
  $B(t)$: $n \times n$ adaptive capital coefficient matrix

The reference model represents an autonomous growth model under a fixed technology, and the adaptive model is a model which converges asymptotically and stably to the reference output through technological changes whose initial condition is given. (11) and (12) are transformed as

reference model

$$Xm(t+1) = Cm Xm(t) + Dm(t+1) H(t) \tag{13}$$

adaptive model

$$X(t+1) = C(t+1) X(t) + D(t+1) H(t) \tag{14}$$

where, $Cm = Bm^{-1}(I - Am + Bm)$

$$Dm = -Bm^{-1}$$

$$C(t+1) = B^{-1}(t+1)\left(I - A(t) + B(t)\right)$$

$$D(t+1) = -B^{-1}(t+1)$$

A problem arising from this context is how $C(t)$ and $D(t)$ are determined to make $\lim\limits_{t\to\infty} ||Xm(t) - X(t)|| = 0$. This problem is equivalent to proving the asymptotical stability of the below-mentioned error equation obtained by (13)-(14).

$$\varepsilon(t+1) = Cm\,\varepsilon(t) + \left(Cm - C(t+1)\right)X(t) + \left(Dm - D(t+1)\right)H(t) \qquad (15)$$

On the stability of (15) the next theorem holds.

Theorem 1.

reference model

$$Xm(t+1) = CmXm(t) + DmH(t) \qquad (16)$$

adaptive model

$$X(t+1) = C(t+1)X(t) + D(t+1)H(t) \qquad (17)$$

error equation

$$\varepsilon(t+1) = Cm\,\varepsilon(t) + \left(Cm - C(t+1)\right)X(t) + \left(Dm - D(t+1)\right)H(t) \qquad (16)$$

When $k(t)$ is defined as $k(t) = 1 + ||X(t)||^2 + ||H(t)||^2$ and assumed $\exists\tau > 0$ such that $\lambda_{\max} < \sqrt{k(t)}$ for all $t > \tau$, the below-stated adaptation laws of $C(t)$ and $D(t)$ make the error equation asymptotically stable. i. e. $\lim\limits_{t\to\infty} ||Xm(t) - X(t)|| = 0$

$$C(t+1) = C(t) + \frac{1}{k(t)}\,\hat{\varepsilon}(t+1)\,X^T(t) \qquad (19)$$

$$D(t+1) = D(t) + \frac{1}{k(t)}\,\hat{\varepsilon}(t+1)\,H^T(t) \qquad (20)$$

Where $\lambda_{\max}$ stands for the eigen value of Cm which has a maximum absolute value among the eigen values of Cm, and $\hat{\varepsilon}(t+1)$ is a priori error defined by

$$\hat{\varepsilon}(t+1) = Xm(t+1) - C(t)\,X(t) - D(t)\,H(t) \qquad (21)$$

And $A(t)$ and $B(t)$ are solved as follows ;

$$A(t) = I + D^{-1}(t+1)\,C(t+1) - D^{-1}(t) \qquad (22)$$

$$B(t+1) = -D^{-1}(t+1) \qquad (23)$$

Furthermore if there are sufficient linearly independent vectors in the sequence $\{H(t)\}_{t=0}^{\infty}$, then $\lim\limits_{t\to\infty} ||Am - A(t)|| = 0$, and $\lim\limits_{t\to\infty} ||Bm - B(t)|| = 0$ are realized.

(sketch of the proof)

Because the exact proof of this theorem is a complicated one, only a sketch

of the proof is presented here. Now the adaptation laws (19) and (20) are intro-
duced from the next formulae.

$$C(t+1) = C(t) + \varepsilon(t+1) \; X^T(t) \tag{24}$$

$$D(t+1) = D(t) + \varepsilon(t+1) \; H^T(t) \tag{25}$$

These equations may be written as

$$C(t+1) = C(0) + \sum_{i=0}^{t} \varepsilon(i+1) \; X^T(i) \tag{26}$$

$$D(t+1) = D(0) + \sum_{i=0}^{t} \varepsilon(i+1) \; H^T(i) \tag{27}$$

Namely, the adaptation laws of $C(t)$ and $D(t)$ are determined by the past trend
of $\varepsilon(t)$.

However, because (26) and (27) include $\varepsilon(t+1)$, that is, $X(t+1)$, these have no
practical significance. Therefore let $\varepsilon(t+1)$ be transformed as follows ;

$$\begin{aligned}
\varepsilon(t+1) &= Xm(t+1) - X(t+1) \\
&= Xm(t+1) - \Big(C(t+1) \; X(t) + D(t+1) \; H(t)\Big) \\
&= Xm(t+1) - \Big(C(t) + \varepsilon(t+1) \; X^T(t)\Big) X(t) - \Big(D(t) + \varepsilon(t+1) \; H^T(t)\Big) H(t) \\
&= Xm(t+1) - C(t) \; X(t) - D(t) \; H(t) - \Big(||X(t)||^2 + ||H(t)||^2\Big) \varepsilon(t+1)
\end{aligned}$$

$$\therefore \quad \varepsilon(t+1) = \frac{1}{k(t)} \; \hat{\varepsilon}(t+1) \tag{28}$$

Substitute (28) into (24) and (25), then (19) and (20) are obtained. (19) and (20)
can be practically implemented only using the information of before the t th period.

Next we transform (19) and (20) as follows ;

$$\begin{aligned}
C(t+1) &= C(t) + \frac{1}{k(t)} \Big(Xm(t+1) - C(t) \; X(t) - D(t) \; H(t)\Big) X^T(t) \\
&= C(t) + \frac{1}{k(t)} \Big(Cm Xm(t) + Dm H(t) - C(t) \; X(t) - D(t) \; H(t)\Big) X^T(t) \\
&= C(t) + \frac{1}{k(t)} \Big(Cm \; \varepsilon(t) + \phi(t) \; X(t) + \psi(t) \; H(t)\Big) X^T(t) \tag{29}
\end{aligned}$$

where, $\phi(t) \equiv Cm - C(t)$, $\psi(t) \equiv Dm - D(t)$. Similarly

$$D(t+1) = D(t) + \frac{1}{k(t)} \Big(Cm \; \varepsilon(t) + \phi(t) \; X(t) + \psi(t) \; H(t)\Big) H^T(t) \tag{30}$$

$$\begin{aligned}
\therefore \quad \varepsilon(t+1) &= Xm(t+1) - X(t+1) \\
&= Cm Xm(t) + Dm H(t) - C(t+1) \; X(t) - D(t+1) \; H(t) \\
&= \frac{1}{k(t)} \Big(Cm \; \varepsilon(t) + \phi(t) X(t) + \psi(t) \; H(t)\Big) \tag{31}
\end{aligned}$$

Therefore a proof of the asymptotical stability of (18) results in (31) being asymptoti-

cally stable.   Setting a Lyapunov's function of $y(t+1) = \dfrac{1}{k(t)} Cm y(t)$ as $y^T P y$ $(P = P^T > 0$: positive definite $n \times n$ matrix), we define a Lyapunov's function candidate for (31) by using P as

$$V(t) = \varepsilon^T(t) P \varepsilon(t) + tr \phi^T(t) P \phi(t) + tr \psi^T(t) P \psi(t) \tag{32}$$

Where 'tr' stands for a trace operation of a matrix. $V(t)$ represents some kind of distance of outputs and parameters between the reference and the adaptive models.   With a little complicated calculation, the increment of $V(t)$ is reduced as

$$V(t+1) - V(t) = \varepsilon^T(t) \left(\frac{Cm^T}{\sqrt{k(t)}} P \frac{Cm}{\sqrt{k(t)}} - P \right) \varepsilon(t)$$
$$- \frac{1}{k(t)} \Big(\phi(t) X(t) + \psi(t) H(t) \Big)^T P \Big(\phi(t) X(t) + \psi(t) H(t) \Big) \leqq 0 \tag{33}$$

Consequently $V(t)$ is proved as a Lyapunov's function and $\lim\limits_{t \to \infty} V(t)$ is bounded.

$$\therefore \left| \sum_{t=0}^{\infty} \Big(V(t+1) - V(t) \Big) \right| = \Big| V(\infty) - V(0) \Big| < \infty$$

$$\therefore \left| \sum_{t=0}^{\infty} \Big(V(t+1) - V(t) \Big) \right| = \left| \sum_{t=0}^{\infty} \left\{ \varepsilon^T(t) \left(\frac{Cm^T}{\sqrt{k(t)}} P \frac{Cm}{\sqrt{k(t)}} - P \right) \varepsilon(t) \right. \right.$$
$$\left. \left. - \frac{1}{k(t)} \Big(\phi(t) X(t) + \psi(t) H(t) \Big)^T P \Big(\phi(t) X(t) + \psi(t) H(t) \Big) \right\} \right| < \infty$$

$$\therefore \lim_{t \to \infty} \left\{ \varepsilon^T(t) \left(\frac{Cm^T}{\sqrt{k(t)}} P \frac{Cm}{\sqrt{k(t)}} - P \right) \varepsilon(t) \right.$$
$$\left. - \frac{1}{k(t)} \Big(\phi(t) X(t) + \psi(t) H(t) \Big)^T P \Big(\phi(t) X(t) + \psi(t) H(t) \Big) \right\} = 0$$

$$\therefore \lim_{t \to \infty} \varepsilon(t) = 0 \ \text{ and } \ \lim_{t \to \infty} \Big(\phi(t) X(t) + \psi(t) H(t) \Big) = 0$$

$$\therefore \lim_{t \to \infty} \| Xm(t) - X(t) \| = 0$$

If the sequence $\{ H(t) \}_{t=0}^{\infty}$ contains sufficient linearly independent vectors, then $\lim\limits_{t \to \infty} (\phi(t) X(t) + \psi(t) H(t)) = 0 \Leftrightarrow$

$$\lim_{t \to \infty} \phi(t) = 0 \ \text{ and } \ \psi(t) = 0$$

$$\therefore \lim_{t \to \infty} \| Am - A(t) \| = 0 \ \text{ and } \ \lim_{t \to \infty} \| Bm - B(t) \| = 0$$

(The exact proof of this theorem is shown in 7).)   Q. E. D.

By a similar proof mentioned above, we obtain

Theorem 2

reference model

$$Xm(t+1) = Cm Xm(t) + Dm H(t) \tag{34}$$

adaptive model

$$X(t+1) = C(t+1) X(t) + D(t+1) H(t) \tag{35}$$

error equation

$$\varepsilon(t+1) = Cm\varepsilon(t) + \left(Cm - C(t+1)\right) X(t) + \left(Dm - D(t+1)\right) H(t) \tag{36}$$

If $k(t)$ is getting sufficiently large as $t\rightarrow\infty$, then the below-stated adaptation laws of $C(t)$, $D(t)$ make the error equation asymptotically stable.

$$C(t+1) = C(t) + \left(I + \Gamma(t)\right)^{-1} Kc \otimes \hat{\varepsilon}(t+1) X^T(t) \tag{37}$$

$$D(t+1) = D(t) + \left(I + \Gamma(t)\right)^{-1} Kd \otimes \hat{\varepsilon}(t+1) H^T(t) \tag{38}$$

Where Kc and Kd are the matrices which have positive elements ı. e. $Kc = (kc_{ij})$, $Kd = (kd_{ij})$, kc_{ij} and $kd_{ij} > 0$. $\otimes$ stands for a matrix operation as follows ;

$$\begin{pmatrix} a_{11} \cdots a_{1n} \\ \vdots \quad \vdots \\ a_{n1} \cdots a_{nn} \end{pmatrix} \otimes \begin{pmatrix} b_{11} \cdots b_{1n} \\ \vdots \quad \vdots \\ b_{n1} \cdots b_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} & b_{11} \cdots a_{1n} & b_{1n} \\ & \vdots \\ a_{n1} & b_{n1} \cdots a_{nn} & b_{nn} \end{pmatrix} \tag{39}$$

And $\Gamma(t)$ represents the next $n \times n$ diagonal matrix.

$$\Gamma(t) = \begin{pmatrix} \sum_{j=1}^{n} \left\{ kc_{1j} x_j^2(t) + kd_{1j} h_j^2(t) \right\} & & \\ & \ddots & \mathbf{0} \\ \mathbf{0} & \ddots & \\ & & \sum_{j=1}^{n} \left\{ kc_{nj} x_j^2(t) + kd_{nj} h_j^2(t) \right\} \end{pmatrix} \tag{40}$$

Furthermore if the sequence $\{H(t)\}_{t=0}^{\infty}$ include sufficient linearly independent vectors, then

$$\lim_{t\rightarrow\infty} ||Am - A(t)|| = 0 \text{ and } \lim_{t\rightarrow\infty} ||Bm - B(t)|| = 0$$

When a tunpike is chosen as a reference model in Theorem 2, the following theorem is obtained.

Theorem 3.

Let a dynamic I-O system with constant technology be set as

$$Xm(t+1) = Bm^{-1}(I - Am + Bm) Xm(t) - Bm^{-1} H \tag{41}$$

The turnpike from the stationary equilibrium point is represented as

$$Xm(t) = \left(1 + \frac{1}{\lambda}\right)^t \left(Xm(0) - (I - Am)^{-1} H\right) + (I - Am)^{-1} H \tag{42}$$

Where λ stands for the Frobenius root of $(I - Am)^{-1} Bm$, and $Xm(0)$ is the initial value of $Xm(t)$ which is also an eigen vector corresponding to λ. Next let a dynamic I-O system with variable technology be represented as,

$$X(t+1) = B^{-1}(t+1)\left(I - A(t) + B(t)\right) X(t) - B^{-1}(t+1) H \tag{43}$$

When the adaptation laws of $A(t)$ and $B(t)$ are adopted as

$$C(t+1) = C(t) + \left(I + \Gamma(t)\right)^{-1} Kc \otimes \left(\hat{\varepsilon}(t+1) X^T(t)\right) \tag{44}$$
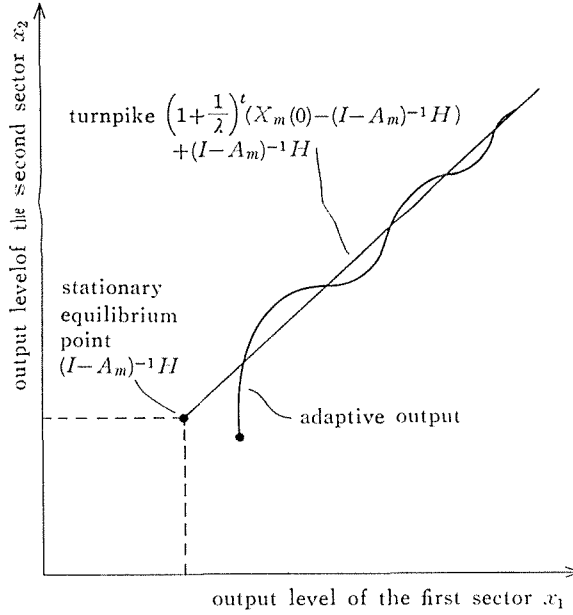
Figure 2. Adaptation process of the adaptive output to the turnpike.

$$D(t+1) = D(t) + \Big(I+\Gamma(t)\Big)^{-1} Kd\otimes\Big(\hat{\varepsilon}(t+1)\,H^T\Big) \tag{45}$$

then $X(t)$ will converge to the turnpike.   Where the notation is the same as that one of Theorem 2.

We shall explain Theorems 1, 2, and 3 in the following context briefly.   Fist of all, the three theorems are realized even in the case that the reference model is relatively unstable.   This fact shows that Theorems 1, 2, and 3 are extensions of the theorem which was proved in our previous paper 5).   And also this result denotes that Theorems 1, 2, and 3 can be practically implemented to any actual dynamic I-O systems.

Secondly, the adaptation laws of $C(t)$ and $D(t)$ only use the reference outputs and exogeneously derived final demand.   Namely, these do not use the information relating to the reference technology, despite the fact that the reference technology is used in the proof.   From the fact that the adaptation laws can identify unknown technologies of a dynamic I-O system, the adaptation process in the theorems are called "adaptive observer" or "adaptive identifier" in the system theory.   Furthermore, the reference technology is not limited to be fixed.   If the absolute values of the eigen values of Cm are bounded when Cm is variable, the adaptation laws will be effective.

Thirdly, a convergence speed of the adaptation process depends on the absolute values of the eigen values of $\dfrac{1}{k(t)}\,Cm$ or $(I+\Gamma(t))^{-1}\,Cm$.   When absolute values of the eigen values are smaller, the convergence speed is faster.   In Theorem 2 and 3, the convergence speed can be controled by Kc and/or Kd, that is, the larger

Kc and/or Kd, the slower the speed. However, if Kc and/or Kd are set to be too big, the adaptive technology seems to change sharply in its initial stages. Therefore, it is difficult but important to select Kc and Kd so that the adaptation process does not lose economic sense. In the dynamic I-O system there are some constraints that are well known, e. g., non-negativity of $A(t)$ and $B(t)$, Solow's condition for the summuation of columns of $A(t)$ and so on.

In our experience, some numerical simulations have shown that some Kc and/or Kd have made the above-mentioned constrained conditions of $A(t)$ or $B(t)$ not to be satisfied. At present, Kc and Kd may be chosen by trial and error.

Fourthly, it is possible that Kc and Kd are variables. Now let variable Kc and Kd be noted as $Kc(t)$ and $Kd(t)$. These can be substituted into (37) and (38), then

$$C(t+1) = \sum_{k=0}^{t} \left(I+\Gamma(k)\right)^{-1} \left(Kc(k+1)\otimes\hat{\varepsilon}(k+1)\,X^{T}(k)\right) \tag{46}$$

$$D(t+1) = \sum_{k=0}^{t} \left(I+\Gamma(k)\right)^{-1} \left(Kd(k+1)\otimes\hat{\varepsilon}(k+1)\,H^{T}(k)\right) \tag{47}$$

If $Kc(k)$ and $Kd(k)$ are denoted as small values in the initial stages and varied to larger ones for the present period, the adaptation algorithms may mean an adjustment process which neglects the past and stresses the present. Finally, it may be impossible to convert Theorems 1, 2, and 3 into a continuous format.

4. Numerical Simulations:

In this chapter we present some numerical simulations of the model reference adaptive I-O system.

4.1. Model reference adaptive I-O system with respect to the ordinary turnpike

Let the first numerical simulation defined as follows;

$$Am = \begin{pmatrix} 0.3 & 0.3 \\ 0.2 & 0.2 \end{pmatrix}, \quad Bm = \begin{pmatrix} 1.9 & 2.1 \\ 2.6 & 3.4 \end{pmatrix}, \quad H(t) = \begin{pmatrix} 10 \\ 10 \end{pmatrix}$$

Then $(I-Am)^{-1}Bm = \begin{pmatrix} 4.6 & 5.4 \\ 4.4 & 5.6 \end{pmatrix}$ and its eigen values are $\lambda_1 = 10$ and $\lambda_2 = 0.2$, and $\lambda_1 = 10$ is the Frobenius root. Associated eigen vectors of λ_1 and λ_2 are $(1, 1)^T$ and $(27, -22)^T$. Then the eigen values of $Bm^{-1}(I-Am+Bm)$ are $1+\dfrac{1}{\lambda_1} = 1.1$ and $1+\dfrac{1}{\lambda_2} = 6.0$ with the eigen vectors $(1, 1)^T$ and $(27, -22)^T$. We observe that the reference system is relatively unstable because the eigen value $1+\dfrac{1}{\lambda_1}$ corresponding to the Frobenius root does not have maximum radius between two eigen values. Setting an initial condition $Xm(0) = (10, 10)^T + (22, 18)^T$, the turnpike from the stationary equilibrium point $(I-Am)^{-1}H = (22, 18)^T$ is represented as follows;

$$\begin{pmatrix} xm_1(t) \\ xm_2(t) \end{pmatrix} = 1.1^t \begin{pmatrix} 10 \\ 10 \end{pmatrix} + \begin{pmatrix} 22 \\ 18 \end{pmatrix} \qquad (t = 0, 1, \cdots)$$

Let us simulate an adaptation process whose reference model is the above-stated turnpike.

reference model

$$\begin{pmatrix} xm_1(t+1) \\ xm_2(t+1) \end{pmatrix} = 1.1^{t+1} \begin{pmatrix} 10 \\ 10 \end{pmatrix} + \begin{pmatrix} 22 \\ 18 \end{pmatrix}, \quad \begin{pmatrix} xm_1(0) \\ xm_2(0) \end{pmatrix} = \begin{pmatrix} 10 \\ 10 \end{pmatrix} + \begin{pmatrix} 22 \\ 18 \end{pmatrix} \tag{48}$$

adaptive model

$$X(t+1) = B^{-1}(t+1) \left(I - A(t) + B(t) \right) X(t) - B^{-1}(t+1) H(t) \tag{49}$$

where, $A(-1) = \begin{pmatrix} 0.3 & 0.3 \\ 0.2 & 0.2 \end{pmatrix}$ $B(-1) = B(0) = \begin{pmatrix} 1.9 & 2.1 \\ 2.6 & 3.4 \end{pmatrix}$

$$X(0) = (31, 19)^T, \quad H(t) = (10, 10)^T$$

By Theorem 2 or 3 we adopt the next adaptation laws.

$$C(t+1) = C(t) + \left(I + \Gamma(t) \right)^{-1} \left(Kc \otimes \hat{\varepsilon}(t+1) X^T(t) \right) \tag{50}$$

$$D(t+1) = D(t) + \left(I + \Gamma(t) \right)^{-1} \left(Kd \otimes \hat{\varepsilon}(t+1) H^T(t) \right) \tag{51}$$

where, $C(t+1) = B^{-1}(t+1)(I - A(t) + B(t))$

$$D(t+1) = -B^{-1}(t+1)$$

$$Kc = \begin{pmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \end{pmatrix}, \quad Kd = \begin{pmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \end{pmatrix}$$

$$\Gamma(t) = \begin{pmatrix} \sum_{j=1}^{2} \left\{ kc_{1j} x_j^2(t) + kd_{1j} h_j^2(t) \right\} & 0 \\ 0 & \sum_{j=1}^{2} \left\{ kc_{2j} x_j^2(t) + kd_{2j} h_j^2(t) \right\} \end{pmatrix} \tag{52}$$

The results of the simulation is presented in tables 1~4. These simulations show that the adaptive outputs almost converge to the turnpike prior to the tenth period, and that the variation of $A(t)$ is larger in its initial stages. We also observe that $A(t)$ and $B(t)$ do not approach Am and Bm, respectively, because $H(t)$ is a constant vector, and that the adaptive model chooses another turnpike technology which has the same turnpike as that of the reference model.

4.2. *Model reference adaptive I-O system with respect to the turnpike which has a variable growth rate*

In this section we consider the turnpike which has a variable growth rate, however, a growth rate of a conventional turnpike is constant. Let the reference

Table 1. Simulation results of output levels of the reference
and the adaptive models (section 4. 1.)

period	$xm_1(t)$	$x_1(t)$	$xm_2(t)$	$x_2(t)$
0.	32.000	31.000	28.000	19.000
1.	33.000	38.289	29.000	21.905
2.	34.100	35.012	30.100	29.660
3.	35.310	33.869	31.310	32.645
4.	36.641	36.046	32.641	33.126
5.	38.105	38.288	34.105	33.953
6.	39.716	39.878	35.716	35.586
7.	41.487	41.483	37.487	37.493
8.	43.436	43.403	39.436	39.465
9.	45.580	45.571	41.580	41.589
10.	47.937	47.938	43.937	43.940
11.	50.531	50.529	46.531	46.535
12.	53.384	53.381	49.384	49.389
13.	56.523	56.520	52.523	52.527
14.	59.975	59.972	55.975	55.979
15.	63.773	63.770	59.773	59.776
16.	67.950	67.948	63.950	63.953
17.	72.545	72.543	68.545	68.548
18.	77.599	77.597	73.599	73.602
19.	83.159	83.157	79.159	79.162
20.	89.275	89.274	85.275	85.277

Table 2. Simulation results of the adaptive input-output coefficient matrix $A(t)$ (section 4. 1.)

period	$a_{11}(t)$	$a_{12}(t)$	$a_{21}(t)$	$a_{22}(t)$
0.	0.30000	0.30000	0.20000	0.20000
1.	0.62715	0.56879	0.23612	0.15931
2.	0.39777	0.28106	0.52595	0.43130
3.	0.32726	0.20064	0.38482	0.24505
4.	0.37502	0.25782	0.31059	0.15719
5.	0.39009	0.27290	0.30446	0.15301
6.	0.38297	0.26358	0.31867	0.16989
7.	0.38042	0.26063	0.32585	0.17732
8.	0.38261	0.26311	0.32749	0.17846
9.	0.38472	0.26527	0.32927	0.18002
10.	0.38604	0.26651	0.33185	0.18264
11.	0.38724	0.26765	0.33469	0.18532
12.	0.38852	0.26889	0.33729	0.18782
13.	0.38978	0.27011	0.33979	0.19024
14.	0.39101	0.27130	0.34224	0.19261
15.	0.39219	0.27244	0.34462	0.19491
16.	0.39334	0.27355	0.34692	0.19714
17.	0.39444	0.27463	0.34913	0.19929
18.	0.39549	0.27565	0.35125	0.20135
19.	0.39651	0.27664	0.35328	0.20333
20.	0.39747	0.27757	0.35521	0.20520

Table 3. Simulation results of the adaptive capital coefficient matrix $B(t)$ (section 4.1.)

period	$b_{11}(t)$	$b_{12}(t)$	$b_{21}(t)$	$b_{22}(t)$
0.	1.90000	2.10000	2.60000	3.40000
1.	1.67496	1.85654	2.11880	2.87940
2.	1.43769	1.56034	1.89652	2.60191
3.	1.43415	1.55592	1.66675	2.31509
4.	1.46470	1.59405	1.62632	2.26462
5.	1.45709	1.58456	1.64126	2.28326
6.	1.44738	1.57243	1.65029	2.29453
7.	1.44629	1.57108	1.64811	2.29181
8.	1.44689	1.57183	1.64427	2.28702
9.	1.44626	1.57103	1.64209	2.28429
10.	1.44519	1.56970	1.64054	.2.28236
11.	1.44429	1.56858	1.63895	2.28037
12.	1.44352	1.56761	1.63736	2.27839
13.	1.44279	1.56671	1.63587	2.27654
14.	1.44211	1.56586	1.63451	2.27484
15.	1.44149	1.56508	1.63326	2.27327
16.	1.44092	1.56436	1.63211	2.27184
17.	1.44039	1.56371	1.63106	2.27053
18.	1.43992	1.56312	1.63012	2.26935
19.	1.43950	1.56260	1.62927	2.26829
20.	1.43912	1.56212	1.62851	2.26735

Table 4. Simulation results of the eigen values of $B^{-1}(t+1)$ $(I-A(t)+B(t))$ (section 4. 1.)

period	$r_1(t)$	$r_2(t)$
0.	6.00000	1.10000
1.	5.73652	1.29381
2.	5.58812	1.18921
3.	5.75979	1.12612
4.	5.80414	1.12497
5.	5.78705	1.12671
6.	5.76966	1.12592
7.	5.77029	1.12479
8.	5.77411	1.12373
9.	5.77527	1.12268
10.	5.77542	1.12162
11.	5.77581	1.12056
12.	5.77637	1.11951
13.	5.77692	1.11846
14.	5.77742	1.11743
15.	5.77790	1.11643
16.	5.77837	1.11544
17.	5.77882	1.11449
18.	5.77924	1.11356
19.	5.77965	1.11267
20.	5.78003	1.11181

model be set as follows;

$$\begin{pmatrix} xm_1(t+1) \\ xm_2(t+1) \end{pmatrix} = \prod_{s=0}^{t} (1.1+0.001 \cdot s) \begin{pmatrix} 100 \\ 100 \end{pmatrix} + \begin{pmatrix} 220 \\ 180 \end{pmatrix}, \begin{pmatrix} xm_1(0) \\ xm_2(0) \end{pmatrix} = \begin{pmatrix} 100 \\ 100 \end{pmatrix} + \begin{pmatrix} 220 \\ 180 \end{pmatrix}$$

$$(53)$$

The growth rate of the reference model from the t th period to the $t+1$ th period is $(10+0.1 \cdot t)\%$. Of course the technology of the reference model is not fixed. Accordingly the technology is assumed to be unidentified because we can apply Theorem 3 even in the case of unidentified technology.

Next we set the adaptive model as follows;

adaptive model

$$X(t+1) = B^{-1}(t+1) \left(I - A(t) + B(t) \right) X(t) - B^{-1}(t+1) \ H(t) \qquad (54)$$

where, $A(-1) = \begin{pmatrix} 0.3 & 0.3 \\ 0.2 & 0.2 \end{pmatrix}$, $\quad B(-1) = B(0) = \begin{pmatrix} 1.9 & 2.1 \\ 2.6 & 3.4 \end{pmatrix}$

$$H(t) = \begin{pmatrix} 100 \\ 100 \end{pmatrix}. \quad \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 100 \\ 100 \end{pmatrix} + \begin{pmatrix} 220 \\ 180 \end{pmatrix}$$

If $A(t)$ and $B(t)$ is fixed as the initial values, the adaptive model has the turnpike with the same direction as the reference, however, the annual growth rate is constant, that is 10%.

When the adaptation laws are chosen as,

$$C(t+1) = C(t) + \left(I + \Gamma(t) \right)^{-1} \left(Kc \otimes \hat{\varepsilon}(t+1) \ X^T(t) \right) \qquad (55)$$

$$D(t+1) = D(t) + \left(I + \Gamma(t) \right)^{-1} \left(Kd \otimes \hat{\varepsilon}(t+1) \ H^T(t) \right) \qquad (56)$$

where, $Kc = \begin{pmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \end{pmatrix}$, $\quad Kd = \begin{pmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \end{pmatrix}$

The adaptation process is depicted in table 5~8. In table 5 it is shown that the adaptive outputs almost coincide with the reference at every period. In table 6 and 7 $A(t)$ and $B(t)$ are uniformly decreasing. These trends depict a technological progress aimed at catching up with the variable growth rate turnpike.

4.3. *Model reference adaptive I-O system associated stock adjustment activities and temporary trade activities with respect to the ordinary turnpike*

In this section we introduce two types of additional activities for the adaptation to the turnpike. The first is the stock adjustment activities which transfer the unused stock of goods in some period to the next period without any cost. The second type is the temporary trade activities which involve the export of unused stock of goods and the import action of urgently needed commodities. These

Table 5.   Simulation results of output levels of the reference and the
adaptive models (section 4. 2.)

period	$xm_1(t)$	$x_1(t)$	$xm_2(t)$	$x_2(t)$
0.	320.000	320.000	280.000	280.000
1.	330.000	330.000	290.000	290.000
2.	341.110	341.110	301.110	301.110
3.	353.463	353.463	313.463	313.463
4.	367.210	367.210	327.210	327.210
5.	382.520	382.520	342.520	342.520
6.	399.584	399.585	359.584	359.584
7.	418.620	418.620	378.620	378.620
8.	439.873	439.873	399.873	399.873
9.	463.619	463.619	423.619	423.619
10.	490.174	490.174	450.174	450.174
11.	519.893	519.893	479.893	479.893
12.	553.181	553.181	513.181	513.181
13.	590.497	590.497	550.497	550.497
14.	632.363	632.363	592.363	592.363
15.	679.373	679.373	639.373	639.373
16.	732.201	732.200	692.201	692.201
17.	791.616	791.616	751.616	751.616
18.	858.495	858.495	818.495	818.495
19.	933.837	933.838	893.837	893.837
20.	1,018.780	1,018.780	978.784	978.784

Table 6.   Simulation results of the adaptive input-output coefficient matrix $A(t)$ (section 4. 2.)

period	$a_{11}(t)$	$a_{12}(t)$	$a_{21}(t)$	$a_{22}(t)$
0.	0.30000	0.30000	0.20000	0.20000
1.	0.29829	0.29816	0.19743	0.19724
2.	0.29730	0.29725	0.19596	0.19588
3.	0.29622	0.29625	0.19433	0.19437
4.	0.29502	0.29514	0.19254	0.19270
5.	0.29372	0.29393	0.19058	0.19089
6.	0.29230	0.29262	0.18846	0.18892
7.	0.29077	0.29120	0.18616	0.18680
8.	0.28914	0.28968	0.18371	0.18452
9.	0.28739	0.28806	0.18109	0.18209
10.	0.28555	0.28635	0.17833	0.17952
11.	0.28361	0.28453	0.17541	0.17680
12.	0.28157	0.28264	0.17236	0.17396
13.	0.27946	0.28066	0.16918	0.17099
14.	0.27726	0.27861	0.16590	0.16791
15.	0.27500	0.27648	0.16250	0.16472
16.	0.27268	0.27429	0.15902	0.16144
17.	0.27031	0.27205	0.15546	0.15808
18.	0.26789	0.26976	0.15184	0.15464
19.	0.26544	0.26743	0.14816	0.15115
20.	0.26296	0.26507	0.14445	0.14760

Table 7.   Simulation results of the adaptive capital coefficient matrix $B(t)$ (section 4.2.)

period	$b_{11}(t)$	$b_{12}(t)$	$b_{21}(t)$	$b_{22}(t)$
0.	1.90000	2.10000	2.60000	3.40000
1.	1.90000	2.10000	2.60000	3.40000
2.	1.90093	2.10113	2.60139	3.40170
3.	1.90195	2.10239	2.60293	3.40358
4.	1.90306	2.10374	2.60459	3.40561
5.	1.90426	2.10521	2.60639	3.40781
6.	1.90554	2.10677	2.60830	3.41015
7.	1.90688	2.10841	2.61032	3.41262
8.	1.90829	2.11013	2.61244	3.41520
9.	1.90975	2.11192	2.61462	3.41787
10.	1.91125	2.11374	2.61687	3.42062
11.	1.91276	2.11560	2.61915	3.42340
12.	1.91429	2.11747	2.62144	3.42621
13.	1.91582	2.11933	2.62373	3.42900
14.	1.91732	2.12117	2.62598	3.43176
15.	1.91879	2.12297	2.62819	3.43446
16.	1.92022	2.12472	2.63033	3.43708
17.	1.92160	2.12639	2.63239	3.43959
18.	1.92290	2.12799	2.63435	3.44199
19.	1.92414	2.12951	2.63621	3.44426
20.	1.92530	2.13093	2.63795	3.44639

Table 8.   Simulation results of the eigen values of $B^{-1}(t+1)$ $(I-A(t)+B(t))$ (section 4.2.)

period	$r_1(t)$	$r_2(t)$
0.	6.00000	1.10000
1.	6.00000	1.10000
2.	6.00000	1.10031
3.	6.00000	1.10068
4.	6.00000	1.10109
5.	6.00000	1.10155
6.	6.00000	1.10207
7.	6.00000	1.10263
8.	6.00000	1.10325
9.	6.00000	1.10392
10.	6.00000	1.10465
11.	6.00000	1.10543
12.	6.00000	1.10627
13.	6.00000	1.10716
14.	6.00000	1.10810
15.	6.00000	1.10908
16.	6.00000	1.11010
17.	6.00000	1.11116
18.	6.00000	1.11226
19.	6.00000	1.11338
20.	6.00000	1.11453

activities are formulated as follows;

$$K(t) = S(t) X(t) \tag{57}$$

$$EM(t) = W(t) X(t) \tag{58}$$

where, $K(t)$: stock adjustment activity vector $(n \times 1)$
　　　$S(t)$: stock adjustment coefficient matrix $(n \times n)$
　　$EM(t)$: temporary trade activity vector $(n \times 1)$
　　　$W(t)$: temporary trade coefficient matrix $(n \times n)$

Then the balance equation of the adaptive model is transformed as,

$$X(t) = A(t) X(t) + B(t+1) X(t+1) - B(t) X(t)$$
$$+ S(t+1) X(t+1) - S(t) X(t) + W(t) X(t) + H(t)$$
$$\therefore \quad X(t+1) = \Big(B(t+1) + S(t+1) \Big)^{-1} \Big(I - A(t) + B(t) - W(t) + S(t) \Big) X(t)$$
$$- \Big(B(t+1) + S(t+1) \Big)^{-1} H(t) \tag{59}$$

When we adopt the reference model which is the same as in section 4.1. and assume that $A(t) = Am$, $B(t) = Bm$ then the system can be represented as follows:

reference model

$$\begin{pmatrix} xm_1(t+1) \\ xm_2(t+1) \end{pmatrix} = 1.1^{t+1} \begin{pmatrix} 10 \\ 10 \end{pmatrix} + \begin{pmatrix} 22 \\ 18 \end{pmatrix} \tag{60}$$

adaptive model

$$X(t+1) = \Big(Bm + S(t+1) \Big)^{-1} \Big(I - Am + Bm + S(t) - W(t) \Big) X(t)$$
$$- \Big(Bm + S(t+1) \Big)^{-1} H(t) \tag{61}$$

where, $Am = \begin{pmatrix} 0.3 & 0.3 \\ 0.2 & 0.2 \end{pmatrix}$, $Bm = \begin{pmatrix} 1.9 & 2.1 \\ 2.6 & 3.4 \end{pmatrix}$

$$X(0) = (31, 19)^T, \quad H(t) = (10, 10)^T$$

Setting $C(t+1)$ and $D(t+1)$ as,

$$C(t+1) = \Big(Bm + S(t+1) \Big)^{-1} \Big(I - Am + Bm + S(t) - W(t) \Big) \tag{62}$$

$$D(t+1) = -\Big(Bm + S(t+1) \Big)^{-1} \tag{63}$$

the same adaptation algorithm in Theorem 2 or 3 can be utilized for this problem and $S(t)$ and $W(t)$ are obtained. However, it should be noted here that $K(t)$ must be positive, therefore some amount of caution is needed for setting up Kc and Kd. Here Kc and Kd are applied as,

$$Kc = \begin{pmatrix} 0.0001 & 0.0001 \\ 0.0001 & 0.0001 \end{pmatrix}, \quad Kd = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

and $S(t) \geqq 0$ is taken into account in the constrained condition of the adaptation.

The results of the simulations are presented in tables $9 \sim 13$. In table 9 it is shown that the adaptive outputs almost converge to the turnpike before the 5 th period, however, the behavior of the adaptive outputs fluctuate a little larger in the initial stages. The reason may be that the stock adjustment activities do not work because of the constrained condition $S(t) \geqq 0$. From the results depicted in table $10 \sim 13$, we see that $K(t)$ and $EM(t)$ work effectively in only three periods after the initial time and take on very little values after the third period. This result shows that the adjustment of the adaptive model to the turnpike only works in the pre-third period and the adaptive model takes on the turnpike technology after the third period.

5. Conclusion

This study investigated Model Reference Adaptive Turnpike Theorem following the results in our previous paper 5). In this paper we have succeeded in removing the assumption of the relative stability which was a major problem in 5), and developed the theorems which could be applied to a more general dynamic I-O system. Furthermore the usefulness of the theorems obtained in this article was proved by some numerical simulations. And it is hoped that those theorems would play an

Table 9. Simulation results of output levels of the reference
and the adaptive models (section 4. 3.)

period	$xm_1(t)$	$x_1(t)$	$xm_2(t)$	$x_2(t)$
0.	32.000	31.000	28.000	19.000
1.	33.000	53.500	29.000	1.500
2.	34.100	34.856	30.100	29.425
3.	35.310	35.360	31.310	31.270
4.	36.641	36.632	32.641	32.649
5.	38.105	38.104	34.105	34.106
6.	39.716	39.715	35.716	35.716
7.	41.487	41.486	37.487	37.488
8.	43.436	43.435	39.436	39.437
9.	45.580	45.579	41.580	41.580
10.	47.937	47.937	43.937	43.938
11.	50.531	50.530	46.531	46.532
12.	53.384	53.383	49.384	49.385
13.	56.523	56.522	52.523	52.524
14.	59.975	59.974	55.975	55.976
15.	63.773	63.771	59.773	59.774
16.	67.950	67.949	63.950	63.951
17.	72.545	72.543	68.545	68.546
18.	77.599	77.598	73.599	73.600
19.	83.159	83.158	79.159	79.160
20.	89.275	89.274	85.275	85.276

Table 10. Simulation results of the stock adjustment coefficient matrix $S(t)$ (section 4.3.)

period	$s_{11}(t)$	$s_{12}(t)$	$s_{21}(t)$	$s_{22}(t)$
0.	0.00000	0.00000	0.00000	0.00000
1.	0.00000	0.00000	0.00000	0.00000
2.	0.40360	0.49329	0.00000	0.00000
3.	0.00000	0.00000	0.21650	0.26461
4.	0.00000	0.00000	0.00000	0.00000
5.	0.00329	0.09403	0.00000	0.00000
6.	0.00174	0.00212	0.00000	0.00000
7.	0.00168	0.00205	0.00000	0.00000
8.	0.00177	0.00217	0.00000	0.00000
9.	0.00186	0.00227	0.00000	0.00000
10.	0.00196	0.00240	0.00000	0.00000
11.	0.00206	0.00252	0.00000	0.00000
12.	0.00218	0.00266	0.00000	0.00000
13.	0.00229	0.00280	0.00000	0.00000
14.	0.00243	0.00297	0.00000	0.00000
15.	0.00257	0.00314	0.00000	0.00000
16.	0.00273	0.00334	0.00000	0.00000
17.	0.00292	0.00357	0.00000	0.00000
18.	0.00311	0.00380	0.00000	0.00000
19.	0.00330	0.00404	0.00000	0.00000
20.	0.00354	0.00433	0.00000	0.00000

Table 11. Simulation results of levels of the stock adjustment activities $K(t)$ (section 4.3.)

period	$K_1(t)$	$K_2(t)$
0.	0.000	0.000
1.	0.000	0.000
2.	28.583	0.000
3.	0.000	15.930
4.	0.000	0.000
5.	0.263	0.000
6.	0.145	0.000
7.	0.147	0.000
8.	0.162	0.000
9.	0.179	0.000
10.	0.199	0.000
11.	0.221	0.000
12.	0.247	0.000
13.	0.277	0.000
14.	0.313	0.000
15.	0.352	0.000
16.	0.399	0.000
17.	0.457	0.000
18.	0.521	0.000
19.	0.594	0.000
20.	0.686	0.000

Table 12. Simulation results of the temporary trade coefficient matrix $W(t)$ (section 4.3.)

period	$w_{11}(t)$	$w_{12}(t)$	$w_{21}(t)$	$w_{22}(t)$
0.	0.00000	0.00000	0.00000	0.00000
1.	-0.44850	-0.53814	-0.00177	-0.00005
2.	0.40374	0.49333	-0.24241	-0.28873
3.	0.00014	0.00004	0.21473	0.26456
4.	-0.00352	-0.00436	$-.000177$	-0.00005
5.	0.00150	0.00175	-0.00176	-0.00005
6.	0.00001	-0.00008	-0.00176	-0.00005
7.	-0.00015	-0.00027	-0.00176	-0.00005
8.	-0.00015	-0.00028	-0.00176	-0.00004
9.	-0.00018	-0.00031	-0.00176	-0.00004
10.	-0.00019	-0.00031	-0.00175	-0.00004
11.	-0.00022	-0.00035	-0.00175	-0.00004
12.	-0.00023	-0.00036	-0.00175	-0.00004
13.	-0.00027	-0.00041	-0.00175	-0.00003
14.	-0.00028	-0.00042	-0.00174	-0.00003
15.	-0.00032	-0.00046	-0.00174	-0.00003
16.	-0.00038	-0.00052	-0.00174	-0.00002
17.	-0.00039	-0.00054	-0.00173	-0.00002
18.	-0.00042	-0.00057	-0.00173	-0.00002
19.	-0.00049	-0.00065	-0.00172	-0.00001
20.	-0.00054	-0.00070	-0.00171	-0.00000

Table 13. Simulation results of levels of the temporary trade activities $EM(t)$ (section 4.3.)

period	$EM_1(t)$	$EM_2(t)$
0.	0.000	0.000
1.	-24.802	-0.095
2.	28.589	-16.945
3.	0.006	15.866
4.	-0.271	-0.066
5.	0.117	-0.069
6.	-0.003	-0.072
7.	-0.016	-0.075
8.	-0.018	-0.078
9.	-0.021	-0.082
10.	-0.022	-0.086
11.	-0.027	-0.090
12.	-0.030	-0.095
13.	-0.037	-0.100
14.	-0.040	-0.106
15.	-0.048	-0.113
16.	-0.059	-0.120
17.	-0.066	-0.127
18.	-0.074	-0.135
19.	-0.093	-0.144
20.	-0.107	-0.153

important role in a stage of a corroborative study.

Areas worth examining for futrher research and analysis include introduction of the adaptation process by a control of the private capital investment and/or R & D investment to make the adaptation law endogeneously possible.

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