On the Thermoelastic Stresses of a Hollow Cylinder

By

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Introduction.

Recently the boundary problem of thermoelasticity has caught the attention of the mathematicians, as there is found in this problem a new field of application of the integral equation and the calculus of variation.[1] It may be truly interesting to treat it from the stand point of mathematical physics. But the practical side has not been yet investigated except in a few special cases.

In the text book of elasticity by Lorenz,[2] we can find this problem fully treated, but the differential equations, from which he has deduced his results are not the so-called " thermoelastic differential equations ".

One of the authors E. Kato has taken interest in this problem from the experiences and observations during his several years of practical work in the Japan Steel Works. Studying in this Faculty, he has completed this essay with Prof. Y. Ikeda.

In the first part the solution of the thermoelastic differential equations about hollow cylinder will be obtained.

In the second part the results will be discussed in detail.

In the third part the method of application will be described.

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1) Roman Krzenziessa. Mathematische Zeitschrift 25 Bd. 2 Heft. Richert Courrant. Acta mathematica 49.

2) H. Lorenz. Technische Elastitätslehre.

Part I. Mathematical Investigation.

I. Thermoelastic differential equations.

ρ: density,
u, v, w: rectangular components of displacement,
X, Y, Z: rectangular components of bodily force,
T: temperature,
λ, μ: Lamé's constants,
θ: $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}$,
$\triangle$: Laplacian operator,
ν: outward normal direction,
τ: tangential direction,
$X^{(\nu)}, Y^{(\nu)}, Z^{(\nu)}$: rectangular components of normal stress,
$X^{(\tau)}, Y^{(\tau)}, Z^{(\tau)}$: rectangular components of tangential stress,
a, b, p: positive physical constants,
t: time.

$$\rho\frac{\partial^2 u}{\partial t^2}-X=(\lambda+\mu)\frac{\partial\theta}{\partial x}+\mu\triangle u-p\frac{\partial T}{\partial x},$$

$$\rho\frac{\partial^2 v}{\partial t^2}-Y=(\lambda+\mu)\frac{\partial\theta}{\partial y}+\mu\triangle v-p\frac{\partial T}{\partial y},$$

$$\rho\frac{\partial^2 w}{\partial t^2}-Z=(\lambda+\mu)\frac{\partial\theta}{\partial z}+\mu\triangle w-p\frac{\partial T}{\partial y},$$

$$\frac{\partial T}{\partial t}=a^2\triangle T-b^2\frac{\partial\theta}{\partial t}.$$

These four equations are called the thermoelastic differential equations. The surface stresses can be expressed by the function of displacement as the following.

$$X^{(\nu)}=(\lambda\theta+2\mu\frac{\partial u}{\partial x}-pT)\cos(\nu x)+\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\cos(\nu y)$$
$$+\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\cos(\nu z),$$

$$Y^{(\nu)}=\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\cos(\nu x)+\left(\lambda\theta+2\mu\frac{\partial v}{\partial y}-pT\right)\cos(\nu y)$$
$$+\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\cos(\nu z),$$

$$Z^{(\nu)}=\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\cos(\nu x)+\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\cos(\nu y)$$
$$+\left(\lambda\theta+2\mu\frac{\partial w}{\partial z}-pT\right)\cos(\nu z),$$

$$X^{(\tau)}=\left(\lambda\theta+2\mu\frac{\partial u}{\partial x}-pT\right)\cos(\tau x)+\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\cos(\tau y)$$
$$+\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\cos(\tau z),$$

$$Y^{(\tau)}=\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\cos(\tau x)+\left(\lambda\theta+2\mu\frac{\partial v}{\partial y}-pT\right)\cos(\tau y)$$
$$+\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\cos(\tau z),$$

$$Z^{(\tau)}=\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\cos(\tau x)+\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\cos(\tau y)$$
$$+\left(\lambda\theta+2\mu\frac{\partial w}{\partial z}-pT\right)\cos(\tau z).$$

We assume that neither acceleration nor external forces exist, and that the cylinder is infinitely long, so that the condition of end can be neglected.

Moreover we assume that the cylinder is cooled or heated symmetrically to its axis and that the initial temperature is also symmetrical to its axis.

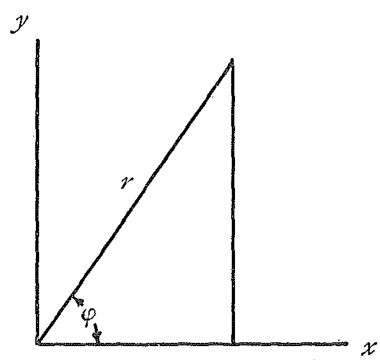

Fig. 1.

In these conditions temperature and displacement are looked upon as the function of radius vector r only. If we take z axis as that of the cylinder, then we have

$$r=\sqrt{x^2+y^2},$$

$$u=R\cos\varphi=\frac{R}{r}x,$$

$$v=R\sin\varphi=\frac{R}{r}y.$$

The differential equations for this problem will be given in the following simple forms:

(1) $$p\frac{\partial T}{\partial r}=(\lambda+2\mu)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(Rr)}{\partial r}\right),$$

(2) $$\frac{\partial T}{\partial t}=\frac{a^2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)-b^2\frac{\partial}{\partial t}\left(\frac{1}{r}\frac{\partial}{\partial r}(Rr)\right).$$

The surface stresses are

(3) $$S^{(\nu)}=(\lambda+2\mu)\frac{\partial R}{\partial r}+\lambda\frac{R}{r}-pT,$$

(4) $$S^{(\tau)}=\lambda\frac{\partial R}{\partial r}+(\lambda+2\mu)\frac{R}{r}-pT.$$

II. Solutions of the equations, when the cylinder it cooled from the outer surface.

Differentiate equation (2) with regard to r and eliminate T by substituting equation (1) in it.

Putting

(5) $$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(Rr)}{\partial r}\right)=U,$$

we have

$$\frac{\partial U}{\partial t}=a^2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rU)\right)-\frac{b^2p}{\lambda+2\mu}\frac{\partial U}{\partial t}.$$

The solution of this equation is well known from the theory of Bessel functions:

(6) $$U=\sum\left\{A_nJ_1(\alpha_nqr)+B_nY_1(\alpha_nqr)\right\}e^{-\alpha_n^2a^2(\lambda+2\mu)t},$$

where A and B are arbitary constants and

$$q^2=\lambda+2\mu+b^2p.$$

From (5), it follows

$$\frac{\partial^2R}{\partial r^2}+\frac{1}{r}\frac{\partial R}{\partial r}-\frac{1}{r^2}R=U.$$

Consequently,

$$(7)\qquad R=\sum\left\{C_n r+D_n\frac{1}{r}-\frac{1}{\alpha_n^2 q^2}\Big(A_n J_1(\alpha_n qr)+B_n Y_1(\alpha_n qr)\Big)\right\}e^{-\alpha_n^2 a^2(\lambda+2\mu)t},$$

where C and D are arbitary constants.
From equation (1),

$$(8)\quad pT=(\lambda+2\mu)\sum\left\{2C_n-\frac{1}{\alpha_n q}\Big(A_n J_0(\alpha_n qr)+B_n Y_0(\alpha_n qr)\Big)+E_n\right\}e^{-\alpha_n^2 a^2(\lambda+2\mu)t},$$

where E is again constant, but it is not independent of the other.
Since equations (7) and (8) must satisfy (2), we have

$$\begin{aligned}\Big(\frac{\lambda+2\mu}{p}+b^2\Big)\sum\Big\{-\alpha_n^2 a^2(\lambda+2\mu)\Big\}\Big\{2C_n-\frac{1}{\alpha_n q}\Big(A_n J_0(\alpha_n qr)\\ +B_n Y_0(\alpha_n qr)\Big)\Big\}e^{-\alpha_n^2 a^2(\lambda+2\mu)t}+\sum E_n\frac{\lambda+2\mu}{p}\Big(-\alpha_n^2 a^2(\lambda+2\mu)\Big)e^{-\alpha_n^2 a^2(\lambda+2\mu)t}\\ =\frac{a^2(\lambda+2\mu)}{p}\sum\Big\{\alpha_n q\Big(A_n J_0(\alpha_n qr)+B_n Y_0(\alpha_n qr)\Big)\Big\}e^{-\alpha_n^2 a^2(\lambda+2\mu)t}.\end{aligned}$$

This relation must hold in any time and in any position. Therefore

$$\Big(\frac{\lambda+2\mu}{p}+b^2\Big)2C_n+E_n\frac{\lambda+2\mu}{p}=0,$$

or

$$E_n=-\frac{2q^2}{\lambda+2\mu}C_n.$$

Therefore

$$\begin{aligned}(9)\quad pT=(\lambda+2\mu)\sum\Big\{2C_n\Big(1-\frac{q^2}{\lambda+2\mu}\Big)-\frac{1}{\alpha_n q}\Big(A_n J_0(\alpha_n qr)\\ +B_n Y_0(\alpha_n qr)\Big)\Big\}e^{-\alpha_n^2 a^2(\lambda+2\mu)t},\end{aligned}$$

$$\begin{aligned}(10)\quad S^{(\nu)}=\sum e^{-\alpha_n^2 a^2(\lambda+2\mu)t}\Big\{C_n(2q^2-2\mu)-2\mu\frac{D_n}{r^2}+\frac{2\mu}{\alpha_n^2 q^2 r}\Big(A_n J_1(\alpha_n qr)\\ +B_n Y_1(\alpha_n qr)\Big)\Big\}.\end{aligned}$$

In order to determine the constants A, B, C and D, we will take the boundary conditions as the following:

$$
\begin{array}{lll}
\text{I.} & r=r_1, & \dfrac{\partial T}{\partial r}=o \quad , \text{ always;} \\
\text{II.} & r=r_2, & \dfrac{\partial T}{\partial r}+hT=o, \text{ always,} \\
\text{III.} & r=r_1, & S^{(v)}=o \quad , \text{ always,} \\
\text{IV.} & r=r_2, & S^{(v)}=o \quad , \text{ always,} \\
\text{V.} & t=o\,, & T=f(r).
\end{array}
$$

From condition I, it follows

$$
(11) \qquad A_n=K_n Y_1(a_n q r_1), \qquad B_n=-K_n J_1(a_n q r_1),
$$

where K is again a new arbitary constant.
From condition II,

$$
(12) \quad C_n 2h\left(1-\frac{q^2}{\lambda+2\mu}\right)=\frac{h}{a_n q}\left\{A_n J_0(a_n q r_2)+B_n Y_0(a_n q r_2)\right\}-\left\{A_n J_1(a_n q r_2)\right.
$$
$$
\left.+B_n Y_1(a_n q r_2)\right\}.
$$

From conditions III and IV,

$$
(13) \quad \begin{cases} C_n(2q^2-2\mu)-2\mu\dfrac{D_n}{r_1^2}=o, \\ C_n(2q^2-2\mu)-2\mu\dfrac{D_n}{r_2^2}+\dfrac{2\mu}{a_n^2 q^2 r_2}K_n\left\{J_1(a_n q r_2)Y_1(a_n q r_1)-J_1(a_n q r_1)Y_1(a_n q r_2)\right\}. \end{cases}
$$

From (11), (12) and (13), we have the relation

$$
(14) \qquad \left\{Y_1(a_n q r_1)J_0(a_n q r_2)-J_1(a_n q r_1)Y_0(a_n q r_2)\right\}=
$$
$$
\left\{\frac{a_n q r_2}{h r_2}+\frac{2\mu p b^2}{(q^2-\mu)\left(1-\dfrac{r_1^2}{r_2^2}\right)(\lambda+2\mu)a_n q r_2}\right\}\left\{Y_1(a_n q r_1)J_1(a_n q r_2)\right.
$$
$$
\left.-J_1(a_n q r_1)Y_1(a_n q r_2)\right\}.
$$

$a_n q r_2$ must have a certain value to satisfy this relation, or $a_n q r_2$ must be the n-th root of this equation.

Now we have

$$
A_n=K_n Y_1(a_n q r_1), \qquad B_n=-K_n J_1(a_n q r_1),
$$

$$C_n = K_n \frac{\mu}{\alpha_n^2 q^2 r_2 (\mu - q^2)\left(1 - \frac{r_1^2}{r_2^2}\right)} \left\{ J_1(\alpha_n q r_2) Y_1(\alpha_n q r_1) - J_1(\alpha_n q r_1) Y_1(\alpha_n q r_2) \right\},$$

$$D_n = K_n \frac{r_1^2}{\alpha_n^2 q^2 r_2 \left(1 - \frac{r_1^2}{r_2^2}\right)} \left\{ J_1(\alpha_n q r_2) Y_1(\alpha_n q r_1) - J_1(\alpha_n q r_1) Y_1(\alpha_n q r_2) \right\},$$

where K is left alone undetermined.

From (9)

$$(15) \quad T = \frac{\lambda + 2\mu}{p} \sum_{n=1}^{\infty} e^{-\alpha_n^2(\lambda+2\mu)a^2 t} K_n \left[\frac{-pb^2}{\lambda + 2\mu} \frac{2\mu \left\{ J_1(\alpha_n q r_2) Y_1(\alpha_n q r_1) - J_1(\alpha_n q r_1) Y_1(\alpha_n q r_2) \right\}}{\alpha_n^2 q^2 r_2 \left(1 - \frac{r_1^2}{r_2^2}\right)(\mu - q^2)} - \frac{1}{\alpha_n q} \left\{ J_0(\alpha_n q r) Y_1(\alpha_n q r_1) - J_1(\alpha_n q r_1) Y_0(\alpha_n q r) \right\} \right].$$

From (10)

$$(16) \quad S^{(\nu)} = \sum_{n=1}^{\infty} e^{-\alpha_n^2(\lambda+2\mu)a^2 t} K_n \left[-\frac{2\mu}{\alpha_n^2 q^2 r_2} \left\{ J_1(\alpha_n q r_2) Y_1(\alpha_n q r_1) - J_1(\alpha_n q r_1) Y_1(\alpha_n q r_2) \right\} + \frac{2\mu}{\alpha_n^2 q^2 r} \left\{ J_1(\alpha_n q r) Y_1(\alpha_n q r_1) - J_1(\alpha_n q r_1) Y_1(\alpha_n q r) \right\} \right].$$

From (4)

$$(17) \quad S^{(\tau)} = \sum_{n=1}^{\infty} e^{-\alpha_n^2(\lambda+2\mu)a^2 t} K_n \left[-\frac{2\mu \left(1 + \frac{r_1^2}{r_2^2}\right)}{\alpha_n^2 q^2 r_2 \left(1 - \frac{r_1^2}{r_2^2}\right)} \left\{ J_1(\alpha_n q r_2) Y_1(\alpha_n q r_1) - J_1(\alpha_n q r_1) Y_1(\alpha_n q r_2) \right\} - \frac{2\mu}{\alpha_n^2 q^2 r} \left\{ J_1(\alpha_n q r) Y_1(\alpha_n q r_1) - J_1(\alpha_n q r_1) Y_1(\alpha_n q r) \right\} + \frac{2\mu}{\alpha_n q} \left\{ J_0(\alpha_n q r) Y_1(\alpha_n q r_1) - J_1(\alpha_n q r_1) Y_0(\alpha_n q r) \right\} \right].$$

By putting

$$K_n = -\frac{K_n}{\alpha_n q},$$

$$G_n(r)=J_1(\alpha_n qr)Y_1(\alpha_n qr_1)-J_1(\alpha_n qr_1)Y_1(\alpha_n qr),$$
$$W_n(r)=J_0(\alpha_n qr)Y_1(\alpha_n qr_1)-J_1(\alpha_n qr_1)Y_0(\alpha_n qr),$$

we have

$$(15^{\times}\quad T=\sum_{n=1}^{\infty}e^{-\alpha_n^2(\lambda+2\mu)a^2t}K_n\left\{\frac{2\mu b^2G_n(r_2)}{\alpha_n qr_2\left(1-\frac{r_1^2}{r_2^2}\right)(\mu-q^2)}+\frac{\lambda+2\mu}{p}W_n(r)\right\},$$

$$(16^{\times})\qquad S^{(\nu)}=\sum_{n=1}^{\infty}e^{-\alpha_n^2(\lambda+2\mu)a^2t}K_n\left\{\frac{2\mu}{\alpha_n qr_2}G_n(r_2)-\frac{2\mu}{\alpha_n qr}G_n(r)\right\},$$

$$(17^{\times})\qquad S^{(\tau)}=\sum_{n=1}^{\infty}e^{-\alpha_n^2(\lambda+2\mu)a^2t}K_n\left\{\frac{2\mu}{\alpha_n qr_2}\,\frac{r_2^2+r_1^2}{r_2^2-r_1^2}G_n(r_2)\right.$$
$$\left.+\frac{2\mu}{\alpha_n qr}G_n(r)-2\mu W_n(r)\right\}.$$

It is evident from (1), (2) and (6), that

$$\frac{\partial(S^{(\nu)}r)}{\partial r}=S^{(\tau)},$$

which will be also verified from (16) and (17).

In order to determine the constant K, we observe the integrals.

$$I_1=\int_{r_1}^{r_2}rW_n(r)dr,$$

$$I_2=\int_{r_1}^{r_2}\left\{\frac{2\mu b^2G_m(r_2)}{\alpha_m qr_2\left(1-\frac{r_1^2}{r_2^2}\right)(\mu-q^2)}+\frac{\lambda+2\mu}{p}W_m(r)\right\}W_n(r)rdr,$$

$$I_3=\int_{r_1}^{r_2}\left\{\frac{2\mu b^2G_n(r_2)}{\alpha_n qr_2\left(1-\frac{r_1^2}{r_2^2}\right)(\mu-q^2)}+\frac{\lambda+2\mu}{p}W_n(r)\right\}W_n(r)rdr.$$

From the integral formula of products of Bessel functions, it follows

$$I_1=\frac{r_2}{\alpha_n q}G_n(r_2),$$

$$I_2=\frac{2\mu b^2 G_m(r_2)G_n(r_2)}{a_m a_n q^2\left(1-\frac{r_1^2}{r_2^2}\right)(\mu-q^2)}+\frac{\lambda+2\mu}{p}\,\frac{r_2}{a_n^2q^2-a_m^2q^2}\Big\{G_n(r_2)W_m(r_2)a_nq$$

$$-G_m(r_2)W_n(r_2)a_mq\Big\}.$$

By applying the relation (14), we can write

$$I_2=G_mG_n\left[\frac{2\mu b^2}{a_m a_n q^2\left(1-\frac{r_1^2}{r_2^2}\right)(\mu-q^2)}+\frac{\lambda+2\mu}{p}\,\frac{r_2}{a_nq^2-a_mq^2}\left\{\frac{a_nqa_mq-a_nqa_mq}{h}\right.\right.$$

$$\left.\left.+\frac{a_nq2\mu pb^2}{(q^2-\mu)\left(1-\frac{r_1^2}{r_2^2}\right)(\lambda+2\mu)a_mqr_2}-\frac{a_mq2\mu pb^2}{(q^2-\mu)\left(1-\frac{r_1^2}{r_2^2}\right)(\lambda+2\mu)a_nqr_2}\right\}\right],$$

or

$$I_2=0.$$

But in the case where $m=n$, we have

$$I_3=G_n^2(r_2)\frac{2\mu b^2}{(\mu-q^2)\left(1-\frac{r_1^2}{r_2^2}\right)a_n^2q^2}+\frac{\lambda+2\mu}{2p}\Big[r^2G_n^2(r)+r^2W_n^2(r)\Big]_{r=r_1}^{r=r_2}$$

$$=-2\frac{\lambda+2\mu}{\pi^2pa_n^2q^2}+G_n^2(r_2)\left[\frac{2\mu b^2}{(\mu-q^2)\left(1-\frac{r_1^2}{r_2^2}\right)a_n^2q^2}+\frac{\lambda+2\mu}{2p}r_2^2\left\{1\right.\right.$$

$$\left.\left.+\left(\frac{(q^2-\mu)\left(1-\frac{r_1^2}{r_2^2}\right)(\lambda+2\mu)a_n^2q^2r_2+2hpb^2\mu}{h(q^2-\mu)\left(1-\frac{r_1^2}{r_2^2}\right)(\lambda+2\mu)a_nqr_2}\right)^2\right\}\right].$$

Now condition V will be written as follows

$$f(r)=\sum_{n=1}^{\infty}K_n\left\{\frac{2\mu b^2G_n(r_2)}{(\mu-q^2)\left(1-\frac{r_1^2}{r_2^2}\right)a_nqr_2}+\frac{\lambda+2\mu}{p}W_n(r)\right\}. \tag{18}$$

Now multiply (18) by $rW_n(r)$ and integrate from r_1 to r_2, then from the integrals I_1, I_2 and I_3, we can obtain the following relation

$$K_n=\int_{r_1}^{r_2}f(r)\frac{rW_n(r)}{I_3}dr. \tag{19}$$

If the initial temperature is constant in any position of the cylinder, that is ;

$$f(r)=T_o.$$

In this case,

$$K_n=T_o\frac{I_1}{I_3},$$

or

$$(20)\quad K_n=T_oG_n(r_2)\alpha_nqr_2\div\left[G_n^2(r_2)\left[\frac{2\mu b^2}{(\mu-q^2)\left(1-\frac{r_1^2}{r_2^2}\right)}+\frac{\lambda+2\mu}{p}\alpha_n^2q^2r_2^2\left\{1\right.\right.\right.$$
$$\left.\left.\left.+\left(\frac{\alpha_nq}{h}+\frac{2\mu pb^2}{(q^2-\mu)\left(1-\frac{r_1^2}{r_2^2}\right)(\lambda+2\mu)\alpha_nqr_2}\right)^2\right\}\right]-\frac{2(\lambda+2\mu)}{p\pi^2}\right].$$

If b can be neglected, then

$$(21)\qquad K_n=\frac{2pT_o}{\lambda+2\mu}\frac{G_n(r_2)\alpha_nqr_2}{G_n^2(r_2)\alpha_n^2q^2r_2^2\left(1+\frac{\alpha_n^2q^2}{h^2}\right)-\left(\frac{2}{\pi}\right)^2}.$$

If h is assumed to be infinitely great,

$$(22)\qquad K_n=\frac{2pT_o}{\lambda+2\mu}\frac{G_n(r_2)\alpha_nqr_2}{G_n^2(r_2)\alpha_n^2q^2r_2^2-\left(\frac{2}{\pi}\right)^2}.$$

III. Application of the results to the case of heating.

In the preceding, we have investigated the case of cooling, but the mathematical treatment may be applied similarly in the case, where the cylinder is heated from the outer surface.

The boundary conditions I, II and IV are the same in both cases. Newton's law of cooling is, of course, the law of cooling and it can not be applied literally to the case of heating, but the heat, which the cold body absorbs from the hotter atmosphere or furnace can be considered to be proportional to the temperature difference between the cylinder and the atmosphere. If we can keep the furnace at a constant temperature, even after the cold body had been inserted, the heat absorbed will be proportional to the temperature difference from the surface. Of course it is not an exact law in strict sense, but since the law of linear proportionality is a naive law in nature, we are going to apply this law. Now we assume that the cylinder is initially at $-T°C$ and that the furnace is constantly at $O°C$. It is clear that we do not lose the generality by fixing the temperature in such a way.

If we take such a condition, we will have the same results as we have had in the case of cooling. We can obtain equations (15), (16) and (17).

The initial condition is alone different in the sign, but the absolute amount will be equal to the case of heating.

From these considerations, we can conclude that the stresses $S^{(\nu)}$ and $S^{(\tau)}$ are equal in the absolute amount to that of the case of cooling, and opposite in the sense, if the total temperature difference is the same.

IV. Conclusions obtained from the solutions in the case of cooling.

If we put

$$T_1=\sum_{n=1}^{\infty}K_n e^{-\alpha_n^2 a^2(\lambda+2\mu)t}\frac{\lambda+2\mu}{p}W_n(r),$$

then we can obtain the relation;

$$\int_{r_1}^{r}T_1 r dr=\frac{\lambda+2\mu}{p}r^2\sum_{n=1}^{\infty}K_n e^{-\alpha_n^2 a^2(\lambda+2\mu)t}\frac{G_n(r)}{a_n q r}.$$

It follows from this

$$T=\frac{2\mu b^2}{(r_2^2-r_1^2)(\mu-q^2)}\int_{r_1}^{r_2}Trdr+T_1, \tag{23}$$

$$S^{(\nu)}=\frac{2\mu p}{\lambda+2\mu}\ \frac{1}{r^2}\left\{\frac{r^2-r_1^2}{r_2^2-r_1^2}\int_{r_1}^{r_2}T_1 rdr-\int_{r_1}^{r}T_1 rdr\right\}, \tag{24}$$

(25) $$S^{(\tau)}=\frac{2\mu p}{\lambda+2\mu}\left[\frac{2}{r_2^2-r_1^2}\int_{r_1}^{r_2}T_1rdr-T_1-\frac{1}{r^2}\left\{\frac{r^2-r_1^2}{r_2^2-r_1^2}\int_{r_1}^{r_2}T_1rdr-\int_{r_1}^{r}T_1rdr\right\}\right].$$

If the initial tempetature is constant, then T_1 is also constant at $t=o$. It follows from (24) and (25),

(24×) $$S^{(v)}_{t=o}=o,\ S^{(v)}=\frac{2\mu p}{\lambda+2\mu}\ \frac{1}{r^2}\left\{\frac{r^2-r_1^2}{r_2^2-r_1^2}\int_{r_1}^{r}Trdr-\int_{r_1}^{r}Trdr\right\},$$

(25×) $$S^{(\tau)}_{t=o}=o,\ S^{(\tau)}=\frac{2\mu p}{\lambda+2\mu}\left[\frac{2}{r_2^2-r_1^2}\int_{r_1}^{r}Trdr-T\right]-S^{(v)}.$$

Since the cylinder is supposed to be cooled from the outer surface, if the temperature is decreasing with $r-r_1$, we can conclude from the thermodynamical consideration, that the temperature at any time decreases with $r-r_1$, provided that $b^2=o$. It may be the same for the case where b^2 has a small finite value, though it is not $b^2=o$.

From mean value theorem, we have

$$\int_{r_1}^{r_2}Trdr=T(\theta r_2)\frac{r_2^2-r_1^2}{2},\qquad \int_{r_1}^{r}Trdr=T(\theta' r)\frac{r^2-r_1^2}{2}$$

Now it is clear that

$$T(\theta r_2)<T(\theta' r).$$

Therefore

$$S^{v}<o\qquad r\neq r_2,\ \ r\neq r_1,$$
$$S^{v}=o\qquad r=r_2,\ \ r=r_1.$$

(26) $$S^{(\tau)}=\frac{2\mu p}{\lambda+2\mu}\left\{T(\theta r_2)-T+\left(1-\frac{r_1^2}{r^2}\right)\frac{T(\theta' r)-T(\theta r_2)}{2}\right\}.$$

Since $$T(\theta r_2)>T(r_2),$$

we have $$S^{(\tau)}_{r=r_2}>o.$$

Since $$T(\theta r_2)<T(r_1),$$

we have $$S^{(\tau)}_{r=r_1}<o.$$

If we write

$$S^{(\tau)}=\frac{2\mu p}{\lambda+2\mu}\left\{T(\theta' r)-T\right\}-\left\{T(\theta' r)-T(\theta r_2)\right\}\left\{\frac{1}{2}+\frac{1}{2}\,\frac{r_1^2}{r_2^2}\right\},$$

then $\left\{T(\theta' r)-T\right\}$ increases continuously from 0 with $r-r_1$, and $-\left\{T(\theta' r)-T(\theta r_2)\right\}$ continuously decreases to 0.

Therefore the greatest tension must occur at the outer surface with the amount

$$\frac{aE}{1-\sigma}\left\{T(r_1,t)-T(\theta r_2,t)\right\}, \tag{27}$$

where E is Young's modulus, a linear expansion coefficient and σ Poinson's ratio.

The greatest compression must occur at the inner surface with the amount

$$\frac{aE}{1-\sigma}\left\{T(\theta r_2,t)-T(r_2,t)\right\}.$$

These results are the same as H. Lorenz has deduced.

Therefore if we want to investigate the case where temperature does not vary with time, it is sufficient to apply Lorenz's formula. Though his formula is also valid in the case where temperature varies with time as above shown, the stress at any time can not be calculated directly. Consequently our formula will be the better, when we want to investigate the phenomena which varies with time.

Part II. Physical Investigation.

I. Numerical examples.

Although the constant b^2 is defined by

$$\frac{c_p-c_v}{ac_v}$$

where c_p and c_v are the specific heat for constant pressure and constant

Shozo Saito. Science report of Tohoku University 1910; G. Glünberg. Zeitschrift für Physik. 1926–1 S. 548.

volume respectively, we can not measure c_v directly. Therefore we are obliged to neglect this constant. It could be, however, considered that the calculation does not change so much so far as b^2 is small.

Now we will express the elastic and physical constants by the practical constants as shown in the following;

$$\lambda=\frac{E\sigma}{(1+\sigma)(1-2\sigma)},$$

$$\mu=\frac{E}{2(1+\sigma)},$$

$$p=\alpha\frac{E}{1-2\sigma},$$

$$a^2=\frac{k}{c\rho},$$

where c: specific heat,
k: heat conductivity,
ρ: density.

	Steel	Glass	Unit
Young's modulus ...	2.0×10^{12}	0.7×10^{12}	dyne/cm^2.
Poison's ratio	0.28	0.25	
Expansions coefficient	0.0000115	0.0000080	1/°C
Specific heat	0.114	0.190	cal./gr.°C
Density	7.70	2.50	gr./cm^3.
Conductivity	0.09	0.0025	cal./cm. sec.°C

For example we take glass and assume that the emissive power is 0.007 cal/cm^2. °C sec.

Or $$h=\frac{\text{Emissive power}}{\text{Conductivity}}=3.$$

At first we must calculate the roots of equation (14); the equation is reduced by putting

$$r_2=1, \quad e=\frac{r_2}{r_1}.$$

We write

$$f(aq)=\frac{aq}{3}\left\{Y_1\left(\frac{aq}{e}\right)J_1(aq)-J_1\left(\frac{aq}{e}\right)Y_1(aq)\right\}-\left\{Y_1\left(\frac{aq}{e}\right)J_0(aq)\right.$$
$$\left.-J_1\left(\frac{aq}{e}\right)Y_0(aq)\right\}.$$

The roots are calculated in the case where

$$e=1,\ 1.333,\ 2,\ 4 \text{ and } \infty.$$

	a_1q	a_2q	a_3q	a_4q	a_5q
$e=1.33$	2.48	13.48			
$e=2.00$	2.28	7.20	13.16		
$e=4.00$	1.90	5.16	8.96	13.00	
$e=\infty$	1.79	4.46	7.42	10.46	13.54

It can be easily proved that there is no root in the case where $e=1$, since we have the relation

$$\lim_{e=1} f(aq)=-\frac{2}{\pi aq}$$

by Lommel's formula.

From (21) we can calculate the value K_n. By taking this value we can calculate the temperature at any time and at any position from the formula (15), which is shown in Fig. 2.

Moreover we can calculate directly from equations (16) and (17), the interal stresses due to the temperature gradient.

The curves given in Fig. 3, Fig. 4 and Fig. 5 will be instructive for a study of the relations between stresses and time, and between stresses and the thickness of the cylinder.

It is said that a thin cylindrical vessel can hold stronger than the thick one in temperature treatment. By this calculation, we can recognize the tendency distinctly, although the ratio of the stresses is no longer linearly proportional to the thickness.

The practical meaning of these numerical calculations will be discussed later.

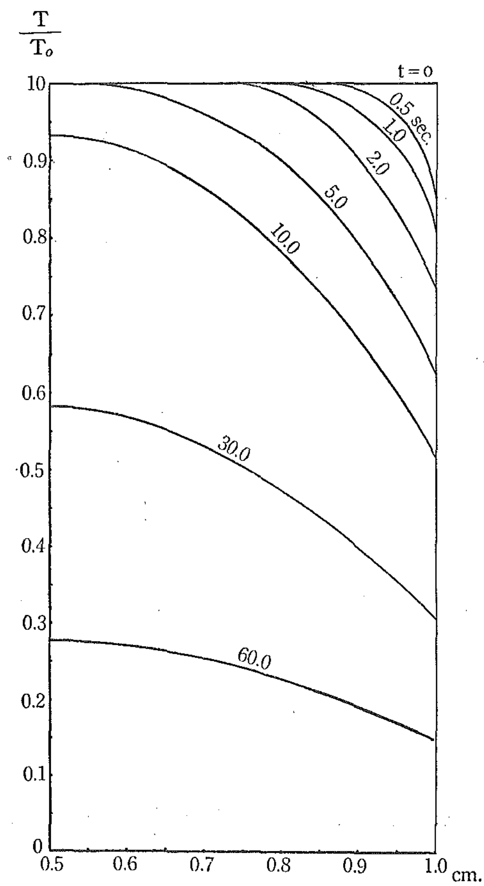

Fig. 2.

Temperature in the case of $e=2$.

II. Physical verifications of the results by means of experiment.

As we have remarked in the preceding, we are obliged to neglect b^2. Now we will verify the results obtained in the preceding, for the results can be mathematically accurate, but one can not conclude that they will be also true to express the natural phenomenon.

The experiment is, however very difficult to accomplish exactly. Therefore we take a simple case where h is great enough to be put $h=\infty$.

Since the elastic constants can not be looked upon as constant in the high temperature, the experiment must be undertaken at as low temperature as possible. Therefore it is preferable to take glass as specimen.

Although the theory is only valid under the condition that the stresses are far from the elastic limit, we assume that the theory will hold in the case where the stresses are equal to the elastic limit.

(I) Expansion coefficient.

We measure the expansion coefficient α by means of the optical lever of the dilatometer and calculate it from the following formula.

$$\alpha=\frac{l'-l}{l(T-T_0)},$$

where

T_0: initial temperature,

T : final temperature,

l : initial length of test piece,

l' : final length of the test piece,

α : expansion coefficient.

$$\alpha=0.000011\pm\triangle\,;\ \triangle=0.000003.$$

where $\triangle$ denotes the maximum deviation from the mean value.

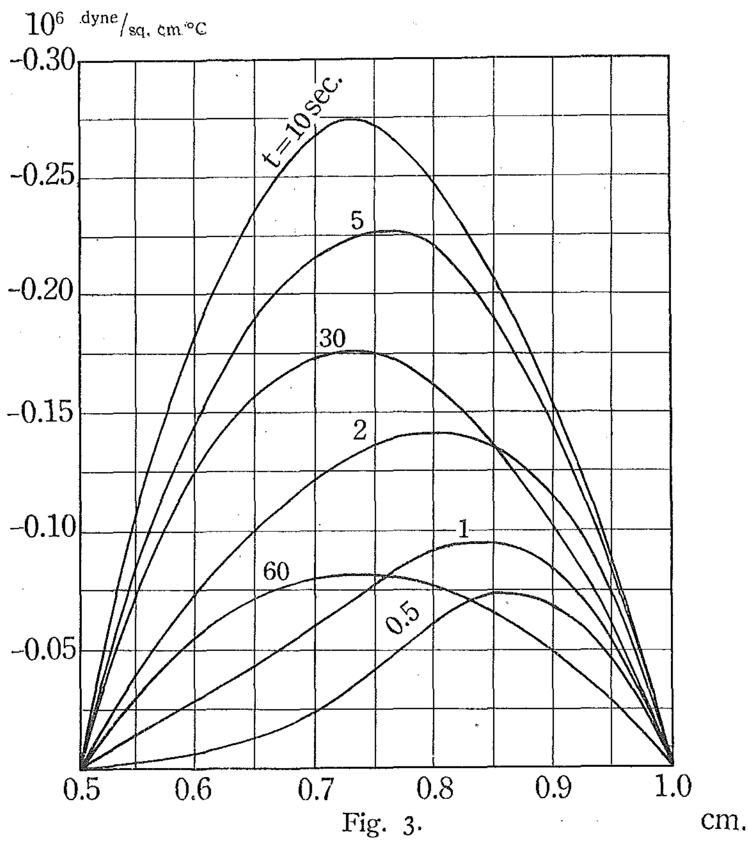

Fig. 3.

$S(\nu)$ in the case of $e=2$.

(II) Simple rigidity.

We measure the twisting angle due to the static twisting moment and we calculate it by the formula;

$$N=\frac{32\ T_m l}{\pi\,(D_1^4-D_2^4)\theta},$$

where T_m : twisting moment,

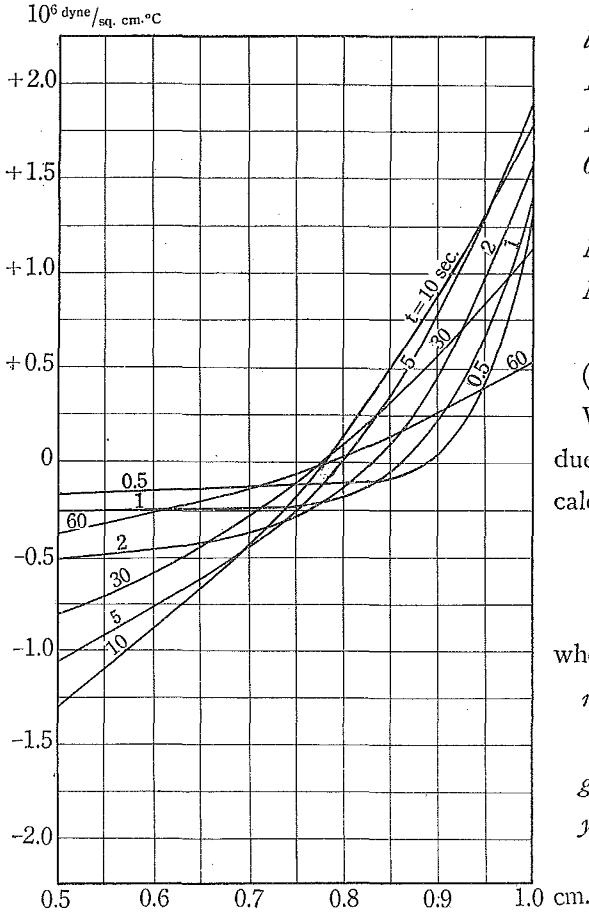

Fig. 4.

$S^{(\tau)}$ in the case of $e=2$.

l : gauge length,

D_1 : outside diameter,

D_2 : inside diameter,

θ : twisting angle measured in radian,

N : simple rigidity.

$N=2.61\times 10^{11}\pm\triangle$;

$\triangle=0.00\times 10^{11}$ dyne/cm^2

(III) Young's modulus.

We measure the deflection due to the bending moment and calculate it by the formula:

$$E=\frac{mgl^3}{48y\frac{\pi}{64}(D_1^4-D_2^4)},$$

where

m : mass of the load at the middle point of the rod,

g : 980 cm/sec^2,

y : deflection due to the load.

$E=6.5\times 10^{11}\pm\triangle$;

$\triangle=1.0\times 10^{11}$ dyne/cm^2

From these data and from the relation

$$E=2N(1+\sigma),$$

we have

$$\sigma=0.25,$$

$$\lambda=\frac{E\sigma}{(1+\sigma)(1-2\sigma)}=2.6\times 10^{11}, \text{ in c.g.s unit,}$$

$$\mu=\frac{E}{2(1+\sigma)}=0.6\times 10^{11}, \quad ,,$$

$$p=\frac{\alpha E}{1-2\sigma}=1.43\times 10^{7}, \quad ,,$$

$$\frac{p}{\lambda+2\mu}=0.183\times 10^{-4}\quad , \quad 1/°C$$

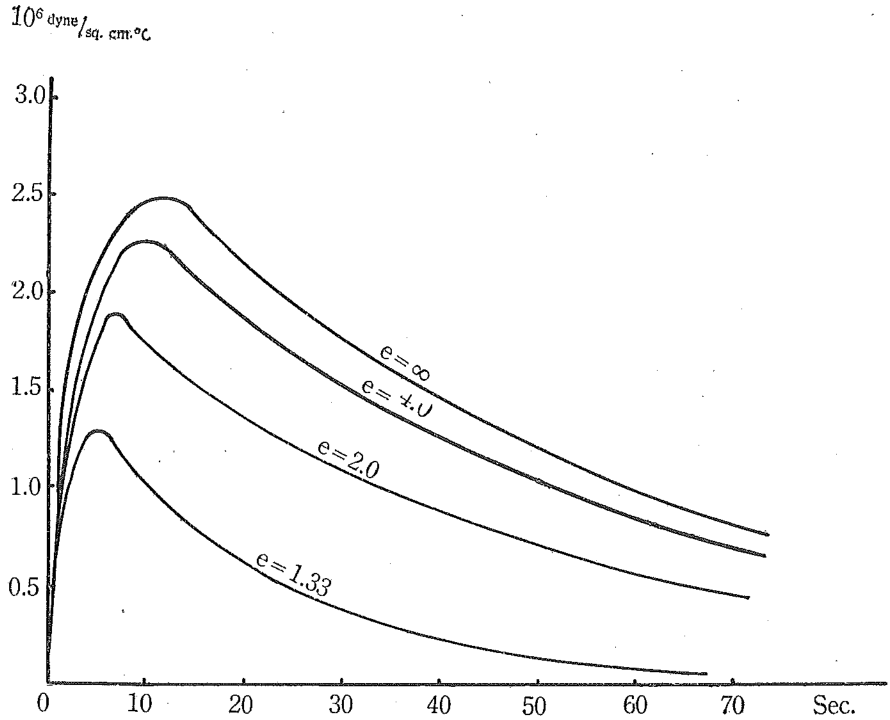

Fig. 5.

The hoop tension at the outer surface.

At the same time when we measure the Young's modulus, we measure also the breaking strength, which will be given in the following formula;

$$f=\frac{MD_1}{\frac{\pi}{32}(D_1^4-D_2^4)}$$

where M: bending moment,

f : breaking strength.

Piece No.	f
1	8.22×10^8 $dyne/_{cm^2}$
2	7.00×10^8 ,,
3	6.93×10^8 ,,
4	7.03×10^8 ,,
5	5.96×10^8 ,,
6	5.28×10^8 ,,
mean	6.74×10^8 ,,

By taking these values and above obtained aqr_2, we can calculate the greatest hoop tension. Since the greatest hoop tension occurs always at the outer surface, we can easily calculate the maximum hoop tension with respect to time, but in the case where $h=\infty$, it follows immediately from equation (27), that the tension becomes instantly the maximum. The amount is

$$\frac{\alpha E}{1-\sigma}T_o,$$

if the cylinder is initially at T°C and it is thrown into the medium at $0°C$.

If the maximum value of the greatest hoop tension is equal to the breaking strength, that is

$$\frac{\alpha E}{1-\sigma}T_o=6.74\times 10^8,$$

a crack will occur.

Now we heat the test piece of glass in the electric furnace and thrown it into the water at $0°C$ and record the temperature, when the crack has taken place in the glass.

In the following, the observed critical temperatures T_c are written

e	T_c
1.33	$85.4\ °C \pm 7°$,
2.	$75.\ °C \pm 3°$,
4.	$73.6\ °C \pm 5°$,
∞	$70.5\ °C \pm 6°$,

while the calculated value from the above formula is

$$T_c=72°C$$

It may be considered that owing to many causes—partial heating, unsymmetry of the form, defects of the material etc.—there must be a tendency to crack at the lower temperature than that calculated. However the experiment shows that it may be only the case where the cylinder is solid; from the experiment we can conclude that the crack must occur at higher temperature in the other three cases. There must anyhow be some reason to explain this fact. Indeed it is clear that we can not have the case where $h=\infty$ in reality. If h is not mathematically infinitely great, the greatest tensions must be different from one other, according to the value of e.

In order to treat these general cases we must use our formula obtained in part I. Here we insist on the convenience of our formula. According to the curve in Fig. 7, it will be easily seen that this experiment must take place, when h is very great. It can also be calculated by means of asymptotic expansion of the Bessel functions, that h is about 200.

Part III. Practical Contributions of the results.

I. Method of applications to practice.

When the heated material is cooled, there will occur a temperature gradient, which will cause the elastic strains and stresses. If the stress exceeds its elastic limit or its breaking strength, the material should be permanently deformed or subjected to crack.

We encounter these phenomenon, when a large ingot is heated too rapidly or when a forged work is cooled in the cold air of the strict winter day after the forged treatment.

For safety's sake, we must spend a considerable time to reach a required temperature, because the smaller the cooling rate is, the smaller the temperature gradient occurs in the materials. On the other hand, it is preferable to cool or heat as quickly as possible for the saving of fuel and time of treatment. Therefore we seek the minimum time spent to reach the required temperature under such a circumstance that the internal stress is lower than its allowable stress.

The roots $a_n qr$ of equation (14) depend upon many constants λ, μ, p, b^2, a^2, r_1, r_2 and h. Among these constants we have only one constant which is under our control; that is the coefficient h of Newton's law of cooling, which is defined by

$$h=\frac{emmisive\ power}{conductivity}.$$

The emissive power is concidered to be due to the radiation, convection and conduction of the medium, where the material is exposed. We can have the various values of $\alpha_n q r_2$ by changing the condition of cooling.

In the case of cooling, the roots are calculated and represented by the curves in Fig. 6.

Now the greatest hoop tension occurs at the outer surface, and this must be the cause of cracking.

From equation ($17^{\times}$), we have

$$S_{r_2}^{(\tau)}=\sum_{n=1}^{\infty} e^{-\alpha_n^2 q^2 a^2 t} K_n\left\{\frac{2\mu}{\alpha_n q r_2}\,\frac{r_2^2+r_1^2}{r_2^2-r_1^2}G_n(r_2)+\frac{2\mu}{\alpha_n q r_2}G_n(r_2)-2\mu W_n(r_2)\right\},$$

provided that $b^2=o$.

Since we have from (14)

$$W_n(r_2)=\frac{\alpha_n q r_2}{h r_2}G_n(r_2),$$

we can write by taking the value K_n given in equation (21),

$$(28)\qquad S_{r_2}^{(\tau)}=2\mu\frac{2pT_o}{\lambda+2\mu}\sum_{n=1}^{\infty}e^{-\alpha_n^2q^2a^2t}\frac{G_n^2(r_2)\left\{\dfrac{2r_2^2}{r_2^2-r_1^2}-\dfrac{\alpha_n^2q^2r_2^2}{hr_2}\right\}}{G_n^2(r_2)\alpha_n^2q^2r_2^2\left(I+\dfrac{\alpha_n^2q^2r_2^2}{h^2r_2^2}\right)-\left(\dfrac{2}{\pi}\right)^2}$$

From equation

$$\frac{dS_{r_2}^{(\tau)}}{dt}=o,$$

or

$$(29)\qquad \frac{dS_{r_2}^{(\tau)}}{dt}=-2\mu\frac{2pT_o}{\lambda+2\mu}\sum_{n=1}^{\infty}e^{-\alpha_n^2q^2a^2t}\frac{a^2G_n^2(r_2)\,\alpha_n^2q^2\left\{\dfrac{2r_2^2}{r_2^2-r_1^2}-\dfrac{\alpha_n^2q^2r_2^2}{hr_2}\right\}}{G_n^2(r_2)\alpha_n^2q^2r_2^2\left(I+\dfrac{\alpha_n^2q^2r_2^2}{h^2r_2^2}\right)-\left(\dfrac{2}{\pi}\right)^2},$$

we can determine the time where $S_{r_2}^{(\tau)}$ is the maximum.

For the first approximation, we have

$$e^{(\alpha_2^2q^2-\alpha_1^2q^2)a^2t_1}=-\frac{\dfrac{G_2^2\alpha_2^2q^2r_2^2\left\{\dfrac{2r_2^2}{r_2^2-r_1^2}-\dfrac{\alpha_2^2q^2r_2^2}{hr_2}\right\}}{G_2^2\alpha_2^2q^2r_2^2\left(I+\dfrac{\alpha_n^2q^2r_2^2}{h^2r_2^2}\right)-\left(\dfrac{2}{\pi}\right)^2}}{\dfrac{G_1^2\alpha_1^2q^2r_2^2\left\{\dfrac{2r_2^2}{r_2^2-r_1^2}-\dfrac{\alpha_1^2q^2r_2^2}{hr_2}\right\}}{G_1^2\alpha_1^2q^2r_2^2\left(I+\dfrac{\alpha_1^2q^2r_2^2}{h^2r_2^2}\right)-\left(\dfrac{2}{\pi}\right)^2}}.$$

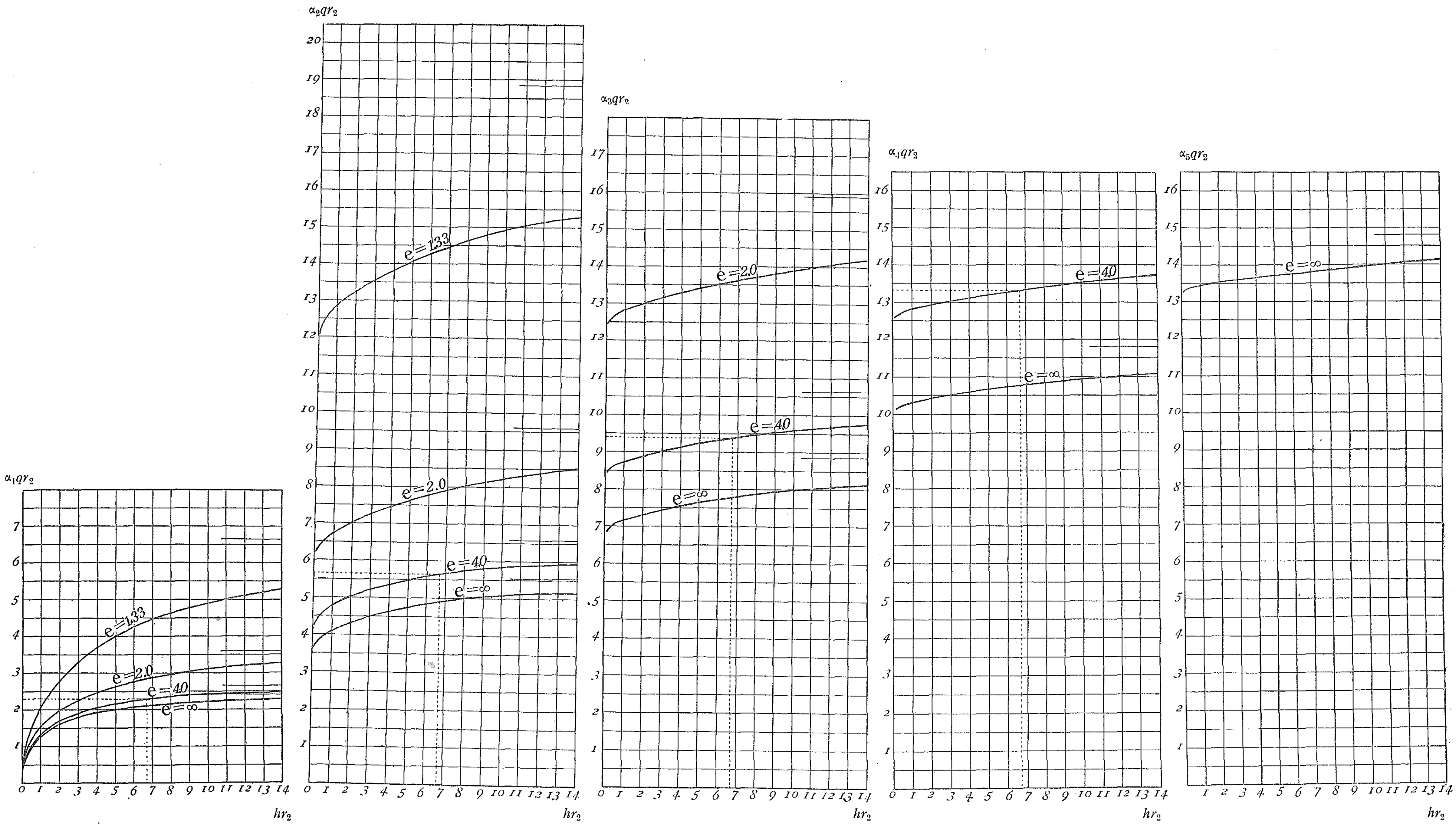

Fig. 6.

$$e=\frac{r_2}{r_1}$$

The horizontal line denotes the asymptotic value of the curve just below it.

For the second approximation, we have

$$(30) \qquad e^{(\alpha_2^2 q^2-\alpha_1^2 q^2)a^2 t^2} = -\frac{G_2^2 a_2^2 q^2 r\left\{\dfrac{2r_2^2}{r_2^2-r_1^2}-\dfrac{a_2^2 q^2 r_2^2}{hr_2}\right\}}{G_2^2 a_2^2 q^2 r_2^2\left(1+\dfrac{a_n^2 q^2 r_2^2}{h^2 r_2^2}\right)-\left(\dfrac{2}{\pi}\right)^2} \div$$

$$\left[\frac{G_1^2 a_1^2 q^2 r_2^2\left\{\dfrac{2r_2^2}{r_2^2-r_1^2}-\dfrac{a_1^2 q^2 r_2^2}{hr_2}\right\}}{G_1^2 a_1^2 q^2 r_2^2\left(1+\dfrac{a_n^2 q^2 r_2^2}{h^2 r^2}\right)-\left(\dfrac{2}{\pi}\right)^2}\right.$$

$$\left.+\frac{G_3^2 a_3^2 q^2 r_2^2\left\{\dfrac{2r_2^2}{r_2^2-r_1^2}-\dfrac{a_3^2 q^2 r_2^2}{hr_2}\right\}e^{(\alpha_1^2 q^2-\alpha_3^2 q^2)a^2 t_1}}{G_3^2 a_3^2 q^2 r_2^2\left(1+\dfrac{a_3^2 q^2 r_2^2}{h^2 r_2^2}\right)-\left(\dfrac{2}{\pi}\right)^2}\right].$$

From the practical calculation, it follows that it is sufficient to take this second approximation as the time which will give the maximum value of $S_{r_2}^{(\tau)}$, for the various hr_2, so far as hr_2 is not so great.

If we put the value t_2 in question (28), we can have the maximum value of $S_{r_2}^{(\tau)}$. If we take the values of constants such as measured in experiment, we obtain the maximum values of $S_{r_2}^{(\tau)}$ for various hr_2 which are shown in Fig. 7.

Now for the definite hr_2, there is a maximum value of $S_{r_2}^{(\tau)}$ and the cooling process will be safe under such a condition that $S_{r_2}^{(\tau)}$ does not exceed the breaking strength of the material.

Again for the definite value of T_o and other material constants, the maximum tension will be given by the curve as shown in Fig. 7 for various hr_2.

Now draw the horizontal line with the amount

$$y=\frac{\textit{breaking strength}}{T_o}$$

in Fig. 7, then the line will intersect with the curves of maximum tension. If hr_2 is smaller than the intersecting point $hr_2^{\times}$, the maximum tension corresponding to this hr_2 will be smaller than the breaking stress. Therefore $h^{\times}$ will indicate the critical value of the cooling rate, and when r_2 is great, this critical value $h^{\times}$ must be the smaller. The cooling process is therefore safe under the condition that the cooling rate is slower than the value corresponding to this critical value $hr^{\times}_2$.

To this critical value of $hr^{\times}_2$ correspond some roots of equation (14), which can be obtained by the curves in Fig. 6.

When $hr^{\times}_2$ and the roots of (14) are given, it is very easy to calculate the cooling curve from equations (15) and (22). If we cool the material in question at such a cooling rate, there must occur no cracking.

For example:

Again take the hollow cylinder of glass. The value of the constants are the same that experimented in part II.

Assume that

$$r_2=7 \text{ cm.}$$
$$r_1=7/4 \text{ cm.}$$
$$T_0=160^{\circ} C$$

From Fig. 7, we have

$$hr_2=6.7.$$

From Fig. 6, we have

$$a_1qr_2=2.26, \quad a_2qr_2=5.64, \quad a_3qr_2=9.37, \quad a_4qr_2=13.32$$

The cooling curve is given in Fig. 8.

If we cool more slowly than the rate which this curve shows, the treatment will be safe from cracking.

These considerations can be applied also in the case of heating, but it is impossible to keep the furnace temperature constant in practice. In the workshop, the material in question is inserted at first into the low temperature furnace and gradually heated to reach high temperature. Therefore the rate of heating is more controllable than that of cooling.

The phenomena will be different, however, from the case we have investigated. But even when the cylinder is inserted in the cold furnace, the flame must be at a high temperature. Therefore if we assume that it would be the same as when the initial temperature is kept constant, we could apply the theory obtained in the preceding. Even if we want to heat in such a rate as the straight line in Fig. 9, the surface temperature of the cylinder will take the dotted line curve in real process of heating, while this curve shows an inverse relation to the curve in Fig. 8. Conse-

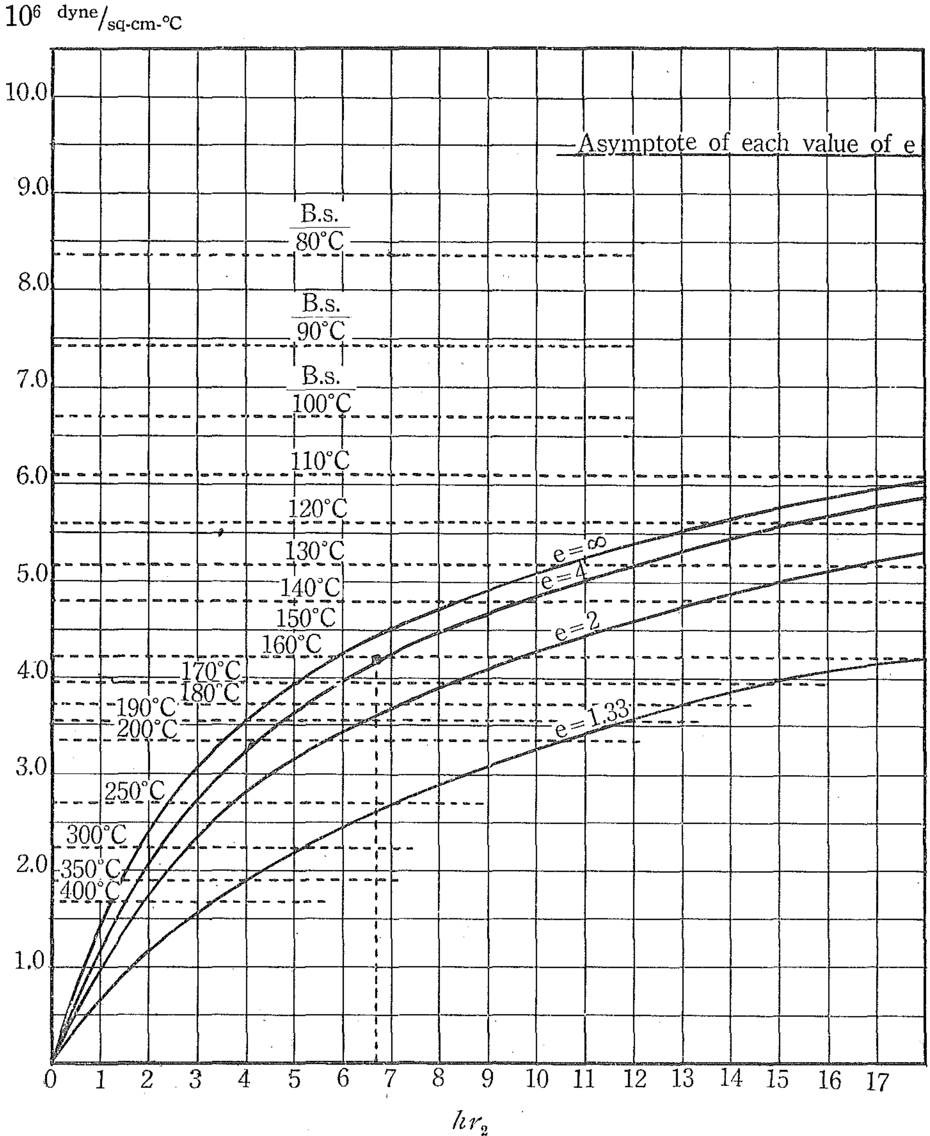

Fig. 7.

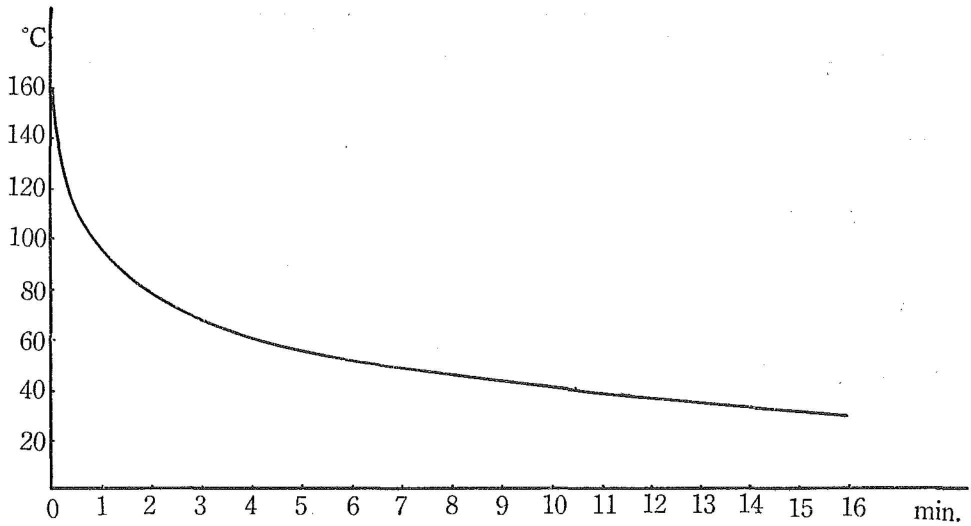

Fig. 8.

Critical cooling curve.

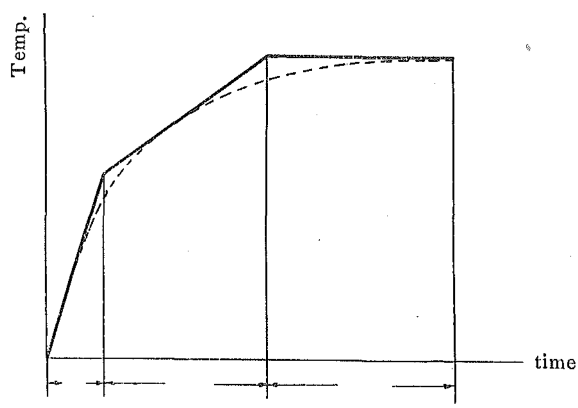

Fig. 9.

quently the maximum tension would occur at the inner surface and it would be the principal cause of cracking. If we heat cylinder more slowly than the heating rate, whose absolute amount is equal to the cooling rate shown in Fig. 8, the heating process is safe from cracking. Lastly we want to remark that in practice we used to heat the inner side of the cylinder preliminary to inserting it into the furnace. It is clear that this process will diminish the tension of the inner surface greatly, since the process will diminish the temperature difference between the mean temperature and that of the inner surface.

II. Summary.

The thermoelastic differential equations are solved in the condition that the inner surface is kept adiabatic and the outer surface is cooled. The stresses are given as the function of time and position. Therefore we can discuss directly the surface stresses as the function of time. The application of these results to the case of heating is also discussed and it is concluded that the surface stresses are equal to that in the case of heating in the absolute amount and opposite in the sign.

In Part II, some numerical examples are calculated, from which we can see how much influence on the maximum tension the thickness of the cylinder has. Under the supposition $h=\infty$, the experiment is undertaken with glass. The results are different, however, slightly from those calculated. The origine of this discrepancy is supposed to be in the assumption that $h=\infty$. Indeed it is proved from that calculation in Part III.

In Part III, the maximum hoop tension of the cylinder with various thicknesses is calculated.

The roots of equation (14) are shown in Fig. 6 for the various hr_2. The method, by which we can find the critical cooling rate is described. As a example, the critical cooling rate of the glass cylinder is calculated; it is concluded that the cooling process is safe from cracking, if we cool the cylinder more slowly than the rate shown in Fig. 8.

The case of heating is also discussed similarly.

We express our hearty thanks to Mr. S. KATTO engineer of Japan Steel Works, who have given us some data on the glass.

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