Mathematical Investigation of the Practical Formula on the Fusion of Fuse.

By

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I. Introduction.

Although it seems as if there is nothing left to be discussed on the phenomena of fusion of fuse, it will not be insignificant to manifest the mathematical meaning of the practical formula for studying it in the wider range of application. For this reason, the authors have intended to solve the problem from the physico-mathematical standpoint. By fuse, we mean the fusible metal used to protect the conductors against excessive heating in consequence of too high currents. The fuses consist of wires or strips connected in the circuit which melt at a given predetermined current in consequence of the Joule's heat developed and thus break the circuit which is there by protected.

The dimensions of the fuse are dependent principally upon the strength of current, and the method of the mounting (openly or closed) besides which we can imagine many factors to be considered for determining the length and cross section of the fuses. Therefore it seems hardly possible to fix all these quantities by formula and they are usually estimated by the aid of the experimental results for normal conditions. In spite of these circumstances, we don't think that the phenomena which lead to the melting of a fuse are so complicated as to be incapable of a mathematical solution. For a theoretically complete calculation, it is necessary to take into account the absolute dimensions, the specific electric conductivity of the material, its temperature coefficient, the specific heat, the latent heat of fusion, specific gravity and specific heat-conductivity of the material, above all the constant of Newton's law of cooling, and the position of the material (horizontal or vertical). All these conditions except the last can affect the melting of a fuse, while the last condition can produce thermo-elastic and mechanical effects. Firstly we have tried to estimate the influences of these conditions and have ascertained that the phenomena which lead to melting can be looked upon as a mathematical problem of heat conduction of metal, in which heat develops, although at the moment of explosion or breaking the circuit, the mechanical condition can affect the explosion or breaking of the fuses very strongly.

If the temperature is very high and near the melting point, the fuse become plastic and breaks on account of its own weight. For this reason it is evident that the breaking of the fuse may be greatly influenced by its position. If we enclose the fuse in a silica tube and lay it horizontal, then the breaking can not occur even if the fuse is in a liquid state. On the other hand even if the temperature of the fuse reaches the melting point, it can not melt without the latent heat of fusion. But this correction can be easily calculated.

II. Practical Formula.

The time necessary for reaching the melting point is given from the formula (91) in the preceding article.

$$(1)\qquad t=\frac{cw\pi^2\rho^4}{\alpha r_{f0}I^2}\,\frac{1}{1-\dfrac{\epsilon\left(\dfrac{\pi^2}{l^2}+\dfrac{2h}{\rho}\right)\pi^2\rho^4}{\alpha r_{f0}I^2}}\log\left[1+\frac{\pi}{4}\alpha T_m\left(1-\frac{\epsilon\left(\dfrac{\pi^2}{l^2}+\dfrac{2h}{\rho}\right)\pi^2\rho^4}{\alpha r_{f0}I^2}\right)\right].$$

Putting

$$(2)\qquad I_m^2=\frac{\epsilon\left(\dfrac{\pi^2}{l^2}+\dfrac{2h}{\rho}\right)\pi^2\rho^4\pi T_m}{\gamma_{f0}(4+\alpha\pi T_m)}\quad ,$$

we have

$$t=\frac{cw\pi^2\rho^4}{\alpha r_{f0}I^2}\,\frac{1}{1-\left(\dfrac{I_m}{I}\right)^2\dfrac{4+\alpha\pi T_m}{\alpha\pi T_m}}\log\left[1+\frac{\pi}{4}\alpha T_m\left\{1-\left(\frac{I_m}{I}\right)^2\frac{4+\alpha\pi T_m}{\alpha\pi T_m}\right\}\right],$$

$$=\left(\frac{\rho^4}{I_m^2}\right)\frac{\left(\frac{I_m}{I}\right)^2 cw\pi^2}{\alpha r_{f0}\left\{1-\left(\frac{I_m}{I}\right)^2\frac{4+\alpha\pi T_m}{\alpha\pi T_m}\right\}}\log\left[\left(1+\frac{\pi}{4}\alpha T_m\right)\left\{1-\left(\frac{I_m}{I}\right)^2\right\}\right],$$

$$=\frac{r_{f0}(4+\alpha\pi T_m)}{\pi^3\epsilon\left(\frac{\pi^2}{l^2}+\frac{2h}{\rho}\right)T_m\alpha\gamma_{f0}}\frac{\left(\frac{I_m}{I}\right)^2 cw\pi^2}{\left\{1-\left(\frac{I_m}{I}\right)^2\frac{4+\alpha\pi T_m}{\alpha\pi T_m}\right\}}\log\left[\left(1+\frac{\pi}{4}\alpha T_m\right)\left\{1-\left(\frac{I_m}{I}\right)^2\right\}\right],$$

$$(3)\qquad t\left(\frac{\pi^2}{l^2}+\frac{2h}{\rho}\right)=\frac{\frac{cw}{\epsilon}\left(\frac{I_m}{I}\right)^2}{\frac{T_m\pi\alpha}{4+\alpha\pi T_m}-\left(\frac{I_m}{I}\right)^2}\log\left[\left(1+\frac{\pi}{4}\alpha T_m\right)\left\{1-\left(\frac{I_m}{I}\right)^2\right\}\right].$$

Putting the following numerical values for lead wire into these two formulae,

$$(4)\qquad \begin{cases} z=\dfrac{I}{I_m} \\ T_m=300 \text{ (melting point of lead 327°C—room temperature 27°C)} \\ \alpha=43\times10^{-4} \\ w=11.3 \\ c=0.031 \\ \epsilon=0.083 \end{cases}$$

we obtain the simple desirable formulae.

$$(5)\qquad t\left(\frac{\pi^2}{l^2}+\frac{4h}{D}\right)=\frac{8.27}{z^2-1.98}\left[0.7+\log(z^2-1)-\log z^2\right].$$

$$(6)\qquad I_m^2=1.17\times10^5\left(\frac{\pi^2}{l^2}+\frac{4h}{D}\right)D^4.$$

If I/I_m is great, the latent heat will be soon supplied, and the amount of the heat escape from the boundary during the interval will be negligibly small. Thus we might calculate the time necessary for supplying the latent heat under the assumption of adiabatic state.

$$(7)\qquad \frac{r_f I^2}{\pi^2\rho^4}\frac{\pi\rho^2}{4.2}t_{la}=\pi\rho^2 5w\ ,$$

$$(1+\alpha T_m)\frac{\epsilon\left(\frac{\pi^2}{l^2}+\frac{4h}{D}\right)\pi T_m}{4+\alpha T_m\pi}\left(\frac{I}{I_m}\right)^2 t_{la}=5w\ ,$$

$$(8)\qquad \left(\frac{\pi^2}{l^2}+\frac{4h}{D}\right)t_{la}=\frac{w5}{\frac{\pi T_m\epsilon(1+\alpha T_m)}{4+\alpha\pi T_m}z^2}\ .$$

For lead wire

$$(9)\qquad \left(\frac{\pi^2}{l^2}+\frac{4h}{D}\right)t_{la}=\frac{2.52}{z^2}\ ,$$

Therefore,

$$(9)'\qquad \left(\frac{\pi^2}{l^2}+\frac{4h}{D}\right)t=\frac{8.27}{z^2-1.98}\left[0.7+\log(z^2-1)-\log z^2\right]+\frac{2.52}{z^2}\ .$$

But if I/I_m is not great enough, the heat flow into the surrounding medium from the boundary is considerable. In the limiting case of the minimum current, the temperature in the mid-point of the conductor is at the melting point, and the temperatures at both terminals are nearly equal to zero. Therefore, the heat flow from boundary will be greatest in this case.

$$(10)\qquad \frac{r_f I_m^2}{\pi^2\rho^4}\frac{\pi\rho^2}{4.2}t'-\epsilon\frac{\partial T_m}{\partial z}\pi\rho^2 t'-h\epsilon T_m 2\pi\rho t'=0\ .$$

The time t_{la} necessary for obtaining the latent heat will be in the general case:

$$(11)\qquad \frac{r_f I^2}{\pi^2\rho^4}\frac{\pi\rho^2}{4.2}t_{la}-\epsilon\frac{\partial T_m}{\partial z}\pi\rho^2 t_{la}-h\epsilon T_m 2\pi\rho t_{la}=\pi\rho^2 w5\ .$$

The heat flow into surrounding medium is smaller than that of the above case, but it may be considered to be nearly equal when I/I_m is very great. Therefore, we may write as an approximation.

$$\frac{r_f(I^2-I_m^2)}{\pi^2\rho^4.4.2}=\frac{w5}{t_{la}}\ .$$

(12) $$\frac{r_{f0}}{4.2\pi^2\rho^4}=\frac{\epsilon\left(\frac{\pi^2}{l^2}+\frac{2h}{\rho}\right)\pi T_m}{4+\alpha\pi T_m}\frac{1}{I_m^2},$$

(13) $$\left(\frac{\pi^2}{l^2}+\frac{4h}{D}\right)t_{la}=\frac{w5}{\frac{\pi T_m\epsilon(1+\alpha T_m)}{4+\alpha\pi T_m}(z^2-1)}.$$

For lead wire:

(14) $$\left(\frac{\pi^2}{l^2}+\frac{4h}{D}\right)t_{la}=\frac{2.52}{z^2-1}.$$

Therefore,

(14)′ $$\left(\frac{\pi^2}{l^2}+\frac{4h}{D}\right)t=\frac{8.27}{z^2-1.98}\left[0.7+\log(z^2-1)-\log z^2\right]+\frac{2.52}{z^2-1}.$$

If the form and material of terminals are given, the terminal temperature will be calculated by the integro-differential equation. On the other hand, if the temperature at the terminals is given, say ψ, then the temperature will be given from (48) in the preceding paper.

(15) $$T_m=\frac{4}{\pi}\frac{d^2}{a^2\left(\frac{\pi^2}{l^2}+\frac{2h}{\rho}\right)-b^2}+\frac{a^2\frac{4\pi}{l^2}\psi}{a^2\left(\frac{\pi^2}{l^2}+\frac{2h}{\rho}\right)-b^2},$$

$$a^2\left(\frac{\pi^2}{l^2}+\frac{2h}{\rho}\right)T_m-\frac{4\pi}{l^2}\psi a^2=d^2\left(1+\frac{\pi}{4}\alpha T_m\right)\frac{4}{\pi}.$$

Putting

(16) $$d^2=\left(\frac{I_{mc}}{I_m}\right)^2\frac{T_m a^2\left(\frac{\pi^2}{l^2}+\frac{2h}{\rho}\right)}{\left(1+\frac{\pi}{4}\alpha T_m\right)}\frac{\pi}{4},$$

we get

$$\left\{1-\left(\frac{I_{mc}}{I_m}\right)^2\right\}\left(\frac{\pi^2}{l^2}+\frac{2h}{\rho}\right)=\frac{4\pi^2}{\pi l^2}\frac{\psi}{T_m},$$

$$1-\left(\frac{I_{mc}}{I_m}\right)^2=\frac{4}{\pi}\frac{1}{1+\frac{2h}{\rho}\frac{l^2}{\pi^2}}\frac{\psi}{T_m},$$

$$\left(\frac{I_{mc}}{I_m}\right)^2=1-\frac{4}{\pi}\frac{1}{1+\frac{2h}{\rho}\frac{l^2}{\pi^2}}\frac{\psi}{T_m},$$

$$I_{mc}=I_m\left(1-\frac{2}{\pi}\frac{1}{1+\frac{2h}{\rho}\frac{l^2}{\pi^2}}\frac{\psi}{T_m}\right). \tag{17}$$

Thus, we may use the formula (9) in the case where t is small and the formula (14) in the case where t is great. In the intermediate case, the time curve will be plotted between the curves traced according to the formulae (9)′ and (14)′. Fortunately as the difference of the two curves is very small, we need not trouble about the inexactitude of the intermediate case. As an example, the curve for lead wire is shown in Fig. 2.

The constant h of the Newton's law is considered to be due to the radiation, convection and conduction. When the wire or strip is thin, the heat flow caused by convection predominates against conduction and radiation. In fact, from the law of Similarity (Ähnlichkeitsgesetz) it follows that the convection is inversely proportional to the dimension (here radius principally).

Therefore, h may be written

$$h=\frac{A}{D}+B .$$

From the preliminary experiments of minimum current (the fuse is laid horizontally), we have determined the constants A and B for lead wire by making use of the formula (2).

$$A=0.001 \quad ,$$

$$B=0.0059 .$$

III. Numerical Example.

For the first, we have tried to investigate only the case where the time necessary for fusion is very small, by taking an oscillogram of the current and potential at that moment, as shown in Fig. 1. From (88) of the preceding paper and (7),

$$t=\frac{\pi^2 D^4 wc}{a r_{f0} I^2 16}\left\{\log\left(1+aT_m\right)+\frac{5a}{c\left(1+aT_m\right)}\right\}.$$

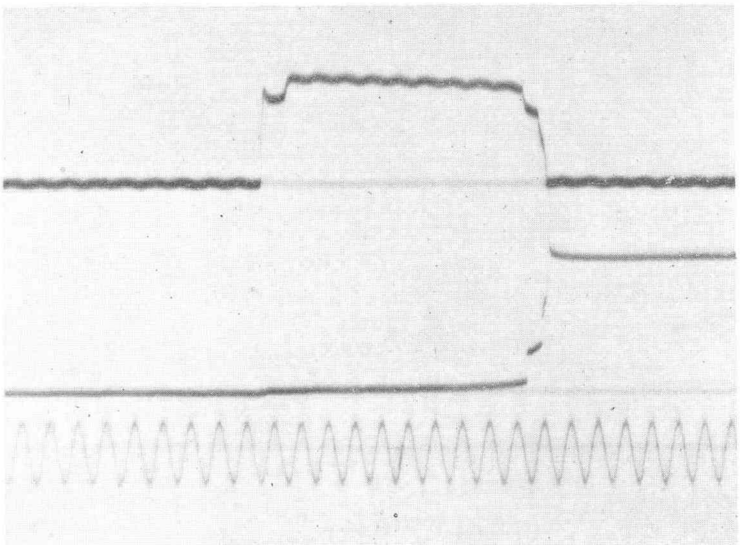

Fig. 1.

For lead wire used in this experiment.

$$\begin{cases} D=0.11\ , \\ w=11.3\ , \\ r_{f0}=20.6\times10^{-6}\ , \\ c=0.031\ , \\ a=43\times10^{-4}\ , \\ T_m=327\text{-room temperature } 17^\circ\text{C.}\ , \\ \quad =310^\circ\text{C.}\ . \end{cases}$$

$$D^4=0.000146\ ,$$

$$1+\alpha T_m=2.34\ ,$$

$$\log(1+\alpha T_m)=.850\ ,$$

$$\frac{5\alpha}{c(1+\alpha T_m)}=0.296\ ,$$

$$\log(1+\alpha T_m)+\frac{5\alpha}{c(1+\alpha T_m)}=1.146\ ,$$

$$\frac{\pi^2 D^4 wc}{\alpha r_{f0}16}=1.5\times10^3\ ,$$

$$t=1.72\times10^3\frac{1}{I^2}\ ,$$

$$I=100\ \text{Amp.}$$

$$t_{\text{cal.}}=0.172\ (\text{sec.})$$

$$t_{\text{obs.}}=\frac{\text{number of waves}}{60}\ ,$$

$$=\frac{10.5}{60}=0.175\ (\text{sec.})$$

Several years ago, Mr. Kudo published a paper (Researches of the Electrotechnical Laboratory of Department of Communications of Japan No. 67) on this problem and explained the phenomena in detail. By taking advantage of his data, we have confirmed that our simple formula can be applied in the general case of fusion of fuse. The numerical calculation for lead wire is shown in the following.

In order to compare with our formula, we have summarized the experimental results of the various lengths and diameters of the fuses in Fig. 3 with $t\left(\frac{\pi^2}{l^2}+\frac{4h}{D}\right)$ as abscissa, percentage of current as ordinate. The curve shows the calculated one, which is the same as shown in Fig. 2. It will be easily seen that it coincides with the observed values.

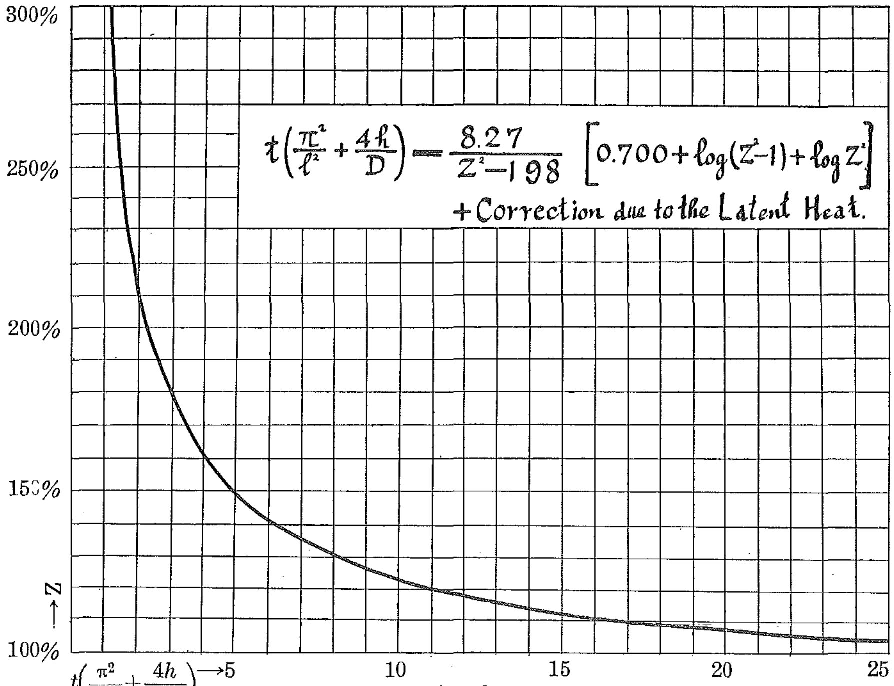
300%
250%
200%
150%
100%
Z
t(π²/l² + 4h/D) = 8.27/(Z²−1 98) [0.700 + log(Z²−1) + log Z²]
+ Correction due to the Latent Heat.
t(π²/l² + 4h/D) → 5
10
15
20
25

Fig. 2.

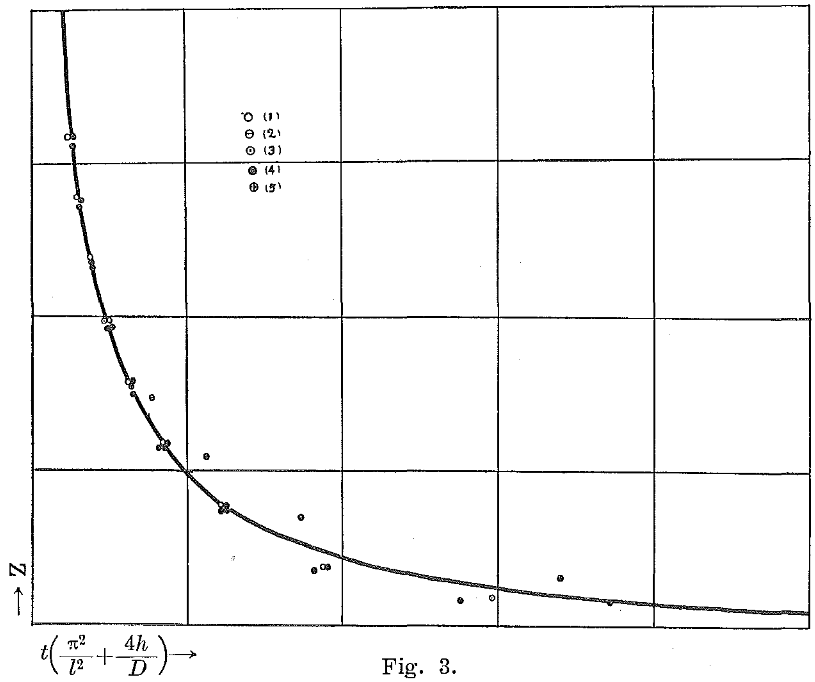
(1)
(2)
(3)
(4)
(5)
Z
t(π²/l² + 4h/D) →

Fig. 3.

TABLE I. The Amount of $\frac{\pi^2}{l^2}+\frac{4h}{D}$

Number of groups of the experiment	D	l	$h=\frac{0.001}{D}+0.0059$	$\frac{4h}{D}$	$\frac{\pi^2}{l^2}+\frac{4h}{D}$
(1)	0.08382	10.16	0.0178	0.850	0.948
(2)	0.08382	5.08	0.0178	0.850	1.243
(3)	0.19812	10.16	0.0107	0.215	0.313
(4)	0.19812	5.08	0.0107	0.215	0.6085
(5)	0.3165	10.16	0.0091	0.114	0.212

TABLE II. Temperature rise of terminals

Number of groups of the experiment	Temperature rise of terminals in °C	ψ $\frac{\text{Temperature rise}}{300}\times 100\ \%$
(1)	14.2	4.7
(2)	18.3	6.2
(3)	35	11.8
(4)	14	4.5
(5)	21	7.0

TABLE III. Minimum current for melting the fuse.

Number of groups of the experiment	Cal. Min. Cur. when $\psi=0$ $I_m=\sqrt{1.17\times10^5\left(\frac{\pi^2}{l^2}+\frac{4h}{D}\right)D^4}$	Cal. Min. Cur. with correction due to ψ $I_{mc}=I_m\left(1-\frac{2}{\pi}\times\frac{1}{1+\frac{4h}{D}\frac{l^2}{\pi^2}}\frac{\psi}{T_m}\right)$	I_m obs.	$\frac{I_m \text{ obs.}}{I_m}$
(1)	7.42	7.4	7.3	0.98
(2)	8.45	8.38	8.2	0.97
(3)	23.85	23.4	23.8	0.995
(4)	33.00	32.6	32.5	0.985
(5)	50.00	49.0	49.3	0.985

TABLE IV. Time necessary for fusion of the first group.

(1) group

$\frac{I}{I_{m\,\text{obs.}}}$ in %	$\frac{I}{I_m}$ in %	$t\left(\frac{\pi^2}{l^2}+\frac{4h}{D}\right)=0.948\,t$	t obtained from the curve in Fig. 2.	$t_{\text{obs.}}$
110	108	20.2	21.3	26
120	118	12.0	12.6	14
140	137	6.6	6.9	7
160	157	4.3	4.5	4.5
180	177	3.1	3.2	3.2
200	196	2.4	2.5	

$t_{\text{obs}}\left(\frac{\pi^2}{l^2}+\frac{4h}{D}\right)$ is plotted in Fig. 3. with the notation ○.

TABLE V. Time necessary for fusion of the second group.

(2) group.

$\frac{I}{I_{m\,\text{obs.}}}$ in %	$\frac{I}{I_m}$ in %	$t\left(\frac{\pi^2}{l^2}+\frac{4h}{D}\right)=1.243\,t$	t obtained from the curve in Fig. 2.	$t_{\text{obs.}}$
110	106.5	23.0	18.5	26
120	116.2	13.8	11.1	14
140	136	6.8	5.45	7
160	155	4.5	3.61	4.5
180	174	3.3	2.65	3.1
200	194	2.5	2.01	

$t_{\text{obs.}}\left(\frac{\pi^2}{l^2}+\frac{4h}{D}\right)$ is plotted in Fig. 3. with the notation ⊖.

TABLE VI. Time necessary for fusion of the third group.

(3) group.

$\frac{I}{I_{m\,\text{obs.}}}$ in %	$\frac{I}{I_m}$ in %	$t\left(\frac{\pi^2}{l^2}+\frac{4h}{D}\right)=0.313\,t$	t obtained from the curve in Fig. 2.	$t_{\text{obs.}}$
110	109.5	19	60.5	46
120	119.5	11.1	35.4	30
140	139.4	6.1	19.5	19.5
160	159.3	4.2	13.4	13.5
180	179.1	3.0	9.55	10.0
200	199.0	2.3	7.3	7.6
220	219.0	1.85	5.9	6.0
240	238.8	1.5	4.8	4.6
260	258.6	1.3	4.15	4.0

$t_{\text{obs.}}\left(\frac{\pi^2}{l^2}+\frac{4h}{D}\right)$ is plotted in Fig. 3. with the notation ⊙.

TABLE VII. Time necessary for fusion of the fourth group.

(4) group.

$\frac{I}{I_{m\,\text{obs.}}}$ in %	$\frac{I}{I_m}$ in %	$t\left(\frac{\pi^2}{l^2}+\frac{4h}{D}\right)$ $=0.6085\,t$	t obtained from the curve in Fig. 2.	$t_{\text{obs.}}$
110	108	19.0	31.2	31
120	118	11.8	19.6	19
140	138	6.3	10.4	10
160	158	4.3	7.0	7
180	175	3.2	5.2	5
200	197	2.6	4.2	4

$t_{\text{obs.}}\left(\frac{\pi^2}{l^2}+\frac{4h}{D}\right)$ is plotted in Fig. 3. with the notation ●.

TABLE VIII. Time necessary for fusion of the fifth group.

(5) group.

$\frac{I}{I_{m\,\text{obs.}}}$ in %	$\frac{I}{I_m}$ in %	$t\left(\frac{\pi^2}{l^2}+\frac{4h}{D}\right)$ $=0.212\,t$	t obtained from the curve in Fig. 2.	$t_{\text{obs.}}$
110	108	20.5	97	65
120	118	11.5	54	43
140	138	6.3	29.7	27
160	158	4.3	20.3	20
180	175	3.2	15.1	15
200	197	2.6	12.3	12
220	217	1.9	8.95	9
240	236	1.5	7.1	7.2
260	256	1.3	6.1	6

$t_{\text{obs.}}\left(\frac{\pi^2}{l^2}+\frac{4h}{D}\right)$ is plotted in Fig. 3. with the notation ⊕.

IV. Summary.

From the theoretical calculation, we have obtained the minimum current for fusion of the fuse:

$$(2)\qquad I_m^2=\frac{\epsilon\left(\frac{\pi^2}{l^2}+\frac{4h}{D}\right)\pi^2D^4\pi T_m}{16\,r_{f0}(4+\alpha\pi T_m)}\,.$$

under the assumption of the terminal temperature being zero. If the temperature of the terminals is ψ, then the correction will be given by

$$I_{mc}=I_m\left(1-\frac{2}{\pi}\frac{1}{1+\frac{4h}{D}\frac{l^2}{\pi^2}}\frac{\psi}{T_m}\right). \tag{17}$$

The time necessary for fusion is the time necessary for reaching the melting point added to the time necessary for supplying the latent heat of fusion, which is given by the formulae (3), (8) and (13).

For use when t is long

$$t\left(\frac{\pi^2}{l^2}+\frac{4h}{D}\right)=\frac{\frac{cw}{\epsilon}\left(\frac{I_m}{I}\right)^2}{\frac{T_m\pi a}{4+a\pi T_m}-\left(\frac{I_m}{I}\right)^2}\log\left[\left(1+\frac{\pi}{4}aT_m\right)\left\{1-\left(\frac{I_m}{I}\right)^2\right\}\right]$$

$$+\frac{w5}{\frac{\pi T_m\epsilon(1+aT_m)}{4+a\pi T_m}\left(\frac{I}{I_m}\right)^2}.$$

For use when t is small

$$t\left(\frac{\pi^2}{l^2}+\frac{4h}{D}\right)=\frac{\frac{cw}{\epsilon}\left(\frac{I_m}{I}\right)^2}{\frac{T_m\pi a}{4+a\pi T_m}-\left(\frac{I_m}{I}\right)^2}\log\left[\left(1+\frac{\pi}{4}aT_m\right)\left\{1-\left(\frac{I_m}{I}\right)^2\right\}\right]$$

$$+\frac{w5}{\frac{\pi T_m\epsilon(1+aT_m)}{4+a\pi T_m}\left(\frac{I}{I_m}\right)^2}.$$

The formula is verified by making use of lead wire. Coincidence with the experimental results has been attained almost perfectly.

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