Temperature Rise of a Conductor due to the Electric Current.

By

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I. Introduction.

Heat generated by electric current is partially dissipated in the surrounding medium through conduction, convection and radiation, and partially produces a temperature rise of the conductor. It is, however, destructive for most electric apparatus or machines to be at too high a temperature. Therefore it is importance to know the relation between the intensity of current and the amount of the temperature rise. Now we are going to treat the phenomena in the wider range of application in order to have an exact and simple form of solution. In this first part we have tried to solve mathematically the problem of temperature rise of a thin wire and a strip with rectangular cross section. Though the results can be applied exactly to the problem of fusion of fuse as shown in the later paper, it will be also applied for the design of the dimension of galvanometer-string and that of electric heater.

For brevity's sake the notations used in this article and their symbols are summarized in the followings:

(Notation) (Unit)

		cm.	gr.	sec.	deg. C°	cal.	amp.	volt.	ohm.
w :	Density of the fuse.	–3	1						
w_1 :	Density of the terminals.	–3	1						
ρ :	Radius of the fuse.	1							
D :	Diameter of the fuse.	1							
l :	Length of the fuse.	1							
t :	Time.			1					
T :	Temperature.				1				
T_m :	Mean temperature.				1				
c :	Specific heat of the fuse.		–1		–1	1			
c_1 :	Specific heat of the terminals.		–1		–1	1			
j :	Heat Stream.	–2		–1		1			
Q :	Heat development.	–3		–1		1			
h :	The Constant of Newton's Law.	–1							
ε :	Thermal Conductivity of the fuse.	–1		–1	–1	1			
ε_1 :	Thermal Conductivity of the terminals.	–1		–1	–1	1			
V :	Voltage of the source.							1	
V_f :	Potential drop between the terminals.							1	
R :	Total circuit resistance, exclusive of the fuse.								1
R_f :	Resistance of the fuse.								1
I :	Total current intensity.						1		
r_f :	Specific resistance of the fuse.	–1							1
r_{fo} :	Specific resistance of the fuse at 0°C.	–1							1
α :	Temperature coefficient of the resistance of the fuse.				–1				

II. The Fundamental Equation and Boundary Conditions.

The fundamental equations of the heat conduction are

(1) $$j = -\epsilon \,\mathrm{Grad}\, T .$$

(2) $$\mathrm{div}\, j + Q = cw \frac{\partial T}{\partial t} .$$

From these equations we have

(3) $$a^2 \Delta T + \frac{1}{cw} Q = \frac{\partial T}{\partial t}$$

where Q is expressed by

Q=Current intensity^2 × Resistance of unit length per unit section.

$$\text{Current intensity} = \frac{\text{Potential difference per unit length.}}{\text{Resistance of unit length per unit section.}}$$

$$= \frac{\left(\frac{V_f}{l}\right)}{r_f \times 4.2} = \frac{V_f}{l\, r_f \times 4.2} .$$

The total electric current is

$$I = \frac{V_f}{R_f}$$

$$= \frac{V_f}{\frac{r_f l}{\pi \rho^2}} = \frac{V}{R_f + R}$$

Suppose that the source is large enough not to be disturbed by the heavy current applied to the wire or strip, then we can consider

V=Constant

Though the resistance of the fuse may be affected by the temperature rise, the total resistance of the circuit is very large against that of the fuse.

Therefore we have

(4) $$Q=\left(\frac{I}{\pi\rho^2}\right)^2 r_f\frac{1}{4.2}\,,$$

(5) $$r_f=r_{f0}(1+\alpha T)\,.$$

Putting

(6) $$b^2=\alpha\frac{I^2}{\pi^2\rho^4}\frac{r_{f0}}{cw}\frac{1}{4.2}\,,$$

(7) $$d^2=\frac{I^2}{\pi^2\rho^4}\frac{r_{f0}}{cw}\frac{1}{4.2}\,,$$

we obtain

(8) $$a^2\varDelta T+d^2+b^2T=\frac{\partial T}{\partial t}\,.$$

The Laplacean $\varDelta$ is represented by the operation

$$\varDelta=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}$$

with respect to the cylindrical wire and

$$\varDelta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\,.$$

with respect to the strip with a rectangular cross section.

Though the large part of the developed heat is consumed in the temperature rise of the material, some part of it must escape from the surface into the surrounding medium in consequence of convection, conduction and radiation.

When the temperature is not extremely high, the heat flow from the surface is calculated by Newton's Law of cooling, that is

(9) $$\frac{\partial T}{\partial n}+hT=0\,.$$

where n is outward normal.

From the terminals, however, some part of the heat must be conducted away. Therefore, at terminals

(10) $$\epsilon\frac{\partial T}{\partial n}=\epsilon_1\frac{\partial T}{\partial n}\,.$$

III. Mathematical Calculations.

From the fundamental equation we have immediately for wire:

(11) $$a^2\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}\right)=\frac{\partial T}{\partial t}-d^2-b^2T\,,$$

for strips

(12) $$a^2\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)=\frac{\partial T}{\partial t}-d^2-b^2T\,,$$

and putting

(13) $$T=u-\frac{d^2}{b^2}\,,$$

we have

(14) $$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}=\frac{1}{a^2}\frac{\partial u}{dt}-\frac{b^2}{a^2}u\,,$$

or

(15) $$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{1}{a^2}\frac{\partial u}{\partial t}-\frac{b^2}{a^2}u\,.$$

First, we are going to solve the equation for wire and putting

(16) $$u=ve^{-p^2t}$$

we have

(17) $$\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{\partial^2 v}{\partial z^2}+\left(\frac{p^2}{a^2}+\frac{b^2}{a^2}\right)v=0\,.$$

and putting

(18) $$\frac{\partial^2 v}{\partial z^2}+q^2v=0$$

we have

(19) $$\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+v\left(\frac{p^2}{a^2}+\frac{b^2}{a^2}-q^2\right)=0\,.$$

(20) $$a^2\lambda_s^2=p^2+b^2-q^2a^2\,.$$

If the boundary conditions are such that

$$(21)\qquad \begin{cases} \dfrac{\partial u}{\partial r}+hu=0 & \text{at} \quad r=\rho\,. \\ \qquad u=0 & \text{at} \quad z=0 \quad \text{and} \quad z=l\,. \\ \qquad u=f(z,\,r) & \text{at} \quad t=0\,, \end{cases}$$

then the solution can be written by the Fourier-Bessel series in the following expression, as is shown in the text books of heat conduction,

or

$$u=T+\frac{d^2}{b^2}$$

$$=\sum_{s=1}^{\infty}\frac{2}{\rho^2}\frac{\lambda_s^2 J_0\!\left(\frac{\lambda_s}{\rho}r\right)e^{-a^2\left\{\left(\frac{n\pi}{l}\right)^2+\lambda_s^2-\frac{b^2}{a^2}\right\}t}}{\{\lambda_s^2+(h\rho)^2\}\{J_0(\lambda_s)\}^2}\int_0^{\rho}xf(x,\,z)J_0\!\left(\frac{\lambda}{\rho}x\right)dx\,.$$

$$(22)\qquad =\sum_{s=1}^{\infty}\sum_{n=1}^{\infty}\frac{4}{\rho^2 l}\frac{\lambda_s^2 J_0\!\left(\frac{\lambda_s}{\rho}r\right)e^{-a^2\left\{\left(\frac{n\pi}{l}\right)^2+\lambda_s^2-\frac{b^2}{a^2}\right\}t}\sin\frac{n\pi}{l}z}{\{\lambda_s^2+(hp)^2\}\{J_0(\lambda_s)\}^2}$$

$$\int_0^{\rho}xJ_0\!\left(\frac{\lambda}{\rho}x\right)dx\int_0^{l}u(x,\,y)\sin\frac{n\pi}{l}y\cdot dy\,.$$

In this case, however, the boundary conditions are different from the above.

$$(23)\qquad T=\psi(r,\,t) \qquad \text{at} \quad z=l,\quad z=0\,.$$

$$(24)\qquad T=0 \qquad \text{at} \quad t=0\,.$$

or

$$(25)\qquad T+\frac{d^2}{b^2}=\psi(r,\,t)+\frac{d^2}{b^2} \qquad \text{at} \quad z=l,\quad z=0\,.$$

$$(26)\qquad \frac{dT}{dr}+hT=0 \qquad \text{at} \quad r=\rho$$

$$(27)\qquad \frac{d\left(T+\frac{d^2}{b^2}\right)}{dr}+h\left(T+\frac{d^2}{b^2}\right)=h\frac{d^2}{b^2} \qquad \text{at} \quad r=\rho$$

$$(28) \qquad \psi+\frac{d^2}{b^2}=\frac{d^2}{b^2} \qquad \text{at} \quad t=0$$

Nevertheless, we can express any function $f\ (r,\ z,\ t)$ by means of the Fourier-Bessel compound series in the following form:

$$(29) \qquad u\,(r,\ z,\ t)=\sum_{s=1}^{\infty}\sum_{n=1}^{\infty}\frac{2}{\rho^2}\frac{2}{l}\frac{\lambda_s^2 J_0\left(\frac{\lambda_s}{\rho}r\right)e^{-a^2\left\{\left(\frac{n\pi}{l}\right)^2+\lambda_s^2-\frac{b^2}{a^2}\right\}t}\sin\frac{n\pi}{l}z}{\{\lambda_s^2+(h\rho)^2\}\,\{J_0(\lambda_s)\}^2}$$

$$\int_0^{\rho}\int_0^{l} xu\,(x,\ y,\ t)\,J_0\left(\frac{\lambda}{\rho}x\right)\sin\frac{n\pi}{l}y\cdot dx\,dy\,.$$

where t is looked upon as a parameter.

Therefore

$$(30) \qquad \frac{\partial^2\left(T+\frac{d^2}{b^2}\right)}{\partial r^2}+\frac{1}{r}\frac{\partial\left(T+\frac{d^2}{b^2}\right)}{\partial r}+\frac{\partial^2\left(T+\frac{d^2}{b^2}\right)}{\partial z^2}$$

$$=\sum_{s=1}^{\infty}\sum_{n=1}^{\infty}\frac{4}{\rho^2 l}\frac{\lambda_s^2 J_0\left(\frac{\lambda_s}{\rho}r\right)\sin\frac{n\pi}{l}z}{\{\lambda_s^2+(h\rho)^2\}\,\{J_0(\lambda_s)\}^2}\int_0^{\rho} xJ_0\left(\frac{\lambda}{\rho}x\right)dx\int_0^{l}\left\{\frac{\partial^2\left(T+\frac{d^2}{b^2}\right)}{\partial x^2}\right.$$

$$\left.+\frac{1}{x}\frac{\partial\left(T+\frac{d^2}{b^2}\right)}{\partial x}+\frac{\partial^2\left(T+\frac{d^2}{b^2}\right)}{dy^2}\right\}\sin\frac{n\pi}{l}y\cdot dy\,.$$

$$=\sum_{s=1}^{\infty}\sum_{n=1}^{\infty}\frac{4}{\rho^2 l}\frac{\lambda_s^2 J_0\left(\frac{\lambda_s}{\rho}r\right)\sin\frac{n\pi}{l}z}{\{\lambda_s^2+(h\rho)^2\}\,\{J_0(\lambda_s)\}^2}\left\{\int_0^{l}\sin\frac{n\pi}{l}y\,dy\left\{\left[J_0\left(\frac{\lambda}{\rho}x\right)x\frac{\partial u}{\partial x}\right]_0^{\rho}\right.\right.$$

$$\left.-\int_0^{\rho}x\frac{du}{\partial x}\frac{d}{dx}J_0\left(\frac{\lambda}{\rho}x\right)dx\right\}+\int_0^{\rho}x\,J_0\left(\frac{\lambda}{\rho}x\right)dx\left\{\left[\frac{\partial u}{\partial y}\sin\frac{n\pi}{l}y\right]_0^{l}\right.$$

$$\left.\left.-\int_0^{l}\frac{n\pi}{l}\cos\frac{n\pi}{l}z\frac{\partial u}{\partial z}dz\right\}\right\}.$$

$$=\sum_{s=1}^{\infty}\sum_{n=1}^{\infty}\frac{4}{\rho^2 l}\frac{\lambda_s^2 J_0\left(\frac{\lambda_s}{\rho}r\right)}{\{\lambda_s^2+(h\rho)^2\}\{J_0(\lambda_s)\}^2}\left\{\int_0^l \sin\frac{n\pi}{l}y\,dy\left\{J_0(\lambda_s)\frac{\partial u}{\partial\rho}-\frac{\partial u}{\partial x}\underset{x\to 0}{}\right.\right.$$

$$\left.\left.-\left[ux\frac{dJ_0\left(\frac{\lambda_s}{\rho}x\right)}{dx}\right]_0^\rho\right\}+\int_0^\rho u\frac{d}{dx}\left(x\frac{dJ_0\left(\frac{\lambda}{\rho}x\right)}{dx}\right)dx$$

$$+\int_0^\rho xJ_0\left(\frac{\lambda}{\rho}x\right)dx\left\{\left[-\frac{n\pi}{l}u\cos\frac{n\pi}{l}z\right]_0^l-\left(\frac{n\pi}{l}\right)^2\int_0^l u\sin\frac{n\pi}{l}y\cdot dy\right\}$$

$$=-\sum_{s=1}^{\infty}\sum_{n=1}^{\infty}\frac{4}{\rho^2 l}\frac{\lambda_s^2 J_0\left(\frac{\lambda}{\rho}r\right)\sin\frac{n\pi}{l}z}{\{\lambda_s^2+(h\rho)^2\}\{J_0(\lambda_s)\}^2}\left\{\int_0^\rho\int_0^l u\sin\frac{n\pi}{l}y\cdot x\,J_0\left(\frac{\lambda}{\rho}x\right)dxdy.\right.$$

$$\left\{\left(\frac{n\pi}{l}\right)^2+\frac{\lambda^2}{\rho^2}\right\}-\int_0^l\sin\frac{n\pi}{l}y\cdot dy\left\{J_0(\lambda_s)\frac{\partial u}{\partial\rho}+u\,J_0(\lambda_s)\,\lambda\right\}$$

$$\left.+\int_0^\rho xJ_0\left(\frac{\lambda}{\rho}x\right)dx\left(-\frac{n\pi}{l}u\left\{(-1)^n-1\right\}\right)\right\}$$

The last two integrals remain when n is odd and otherwise reduce to zero

By putting the boundary condition (21) and (25), we obtain

$$(31)\qquad =-\sum_{s=1}^{\infty}\sum_{n=1}^{\infty}\frac{4}{\rho^2 l}\frac{\lambda_s^2 J_0\left(\frac{\lambda}{\rho}r\right)\sin\frac{n\pi}{l}z}{\{\lambda_s^2+(h\rho)^2\}\{J_0(\lambda_s)\}^2}\left\{\int_0^\rho\int_0^l u\sin\frac{n\pi}{l}y\cdot x\cdot J_0\left(\frac{\lambda}{l}x\right)dx\,dy\,.\right.$$

$$\left\{\left(\frac{n\pi}{l}\right)^2+\frac{\lambda^2}{\rho^2}\right\}-h\rho\frac{d^2}{b^2}\left[-\cos\frac{n\pi}{l}y\cdot\frac{1}{\frac{n\pi}{l}}\right]_0^l J_0(\lambda_s)$$

$$\left.+\int_0^\rho xJ_0\left(\frac{\lambda}{\rho}x\right)dx\frac{2n\pi}{l}\left\{\psi(x,t)+\frac{d^2}{b^2}\right\}\right\}.$$

Also we have

$$(32)\qquad \frac{1}{a^2}\frac{\partial u}{\partial t}-\frac{b^2}{a^2}u=-\sum_{s=1}^{\infty}\sum_{n=1}^{\infty}\frac{4}{\rho^2 l}\frac{\lambda_s^2 J_0\left(\frac{\lambda}{\rho}r\right)\sin\frac{n\pi}{l}z}{\{\lambda_s^2+(h\rho)^2\}\{J_0(\lambda_s)\}^2}\int_0^{\rho}\int_0^{l}\left(\frac{\partial u}{\partial t}\frac{1}{a^2}\right.$$

$$\left.-\frac{b^2}{a^2}u\right)\sin\frac{n\pi}{l}y\cdot x\,J_0\left(\frac{\lambda}{\rho}x\right)dx\,dy\,.$$

Therefore, we have from equations (31) and (32)

$$(33)\qquad -\sum_{s=1}^{\infty}\sum_{n=1}^{\infty}\frac{4}{\rho^2 l}\frac{\lambda_s^2 J_0\left(\frac{\lambda}{\rho}r\right)\sin\frac{n\pi}{l}z}{\{\lambda_s^2+(h\rho)^2\}\{J_0(\lambda_s)\}^2}\left\{\int_0^{l}\int_0^{\rho}u\sin\frac{n\pi}{l}\,y\,x\,J_0\left(\frac{\lambda}{\rho}x\right)dx\,dy\right.$$

$$\left\{\left(\frac{n\pi}{l}\right)^2+\frac{\lambda^2}{\rho^2}-\frac{b^2}{a^2}\right\}+\frac{1}{a^2}\frac{d}{dt}\int_0^{l}\int_0^{\rho}u\sin\frac{n\pi}{l}y\cdot x\,J_0\left(\frac{\lambda}{\rho}x\right)dx\,dy$$

$$-h\rho\frac{d^2}{b^2}\frac{l}{n\pi}2J_0(\lambda_s)+\frac{2n\pi}{l}\int_0^{\rho}xJ_0\left(\frac{\lambda}{\rho}x\right)\psi(x,t)\,dx$$

$$\left.+\frac{d^2}{b^2}\frac{\rho^2}{\lambda_s^2}\frac{2n\pi}{l}h\rho\,J_0(\lambda_s)\right\}=0$$

Putting

$$(34)\qquad A=\int_0^{l}\int_0^{\rho}u\sin\frac{n\pi}{l}y\cdot x\,J_0\left(\frac{\lambda}{\rho}x\right)dx\,dy\,.$$

we have the linear differential equation:

$$(35)\qquad \frac{d}{dt}A+\left\{a^2\left(\frac{n\pi}{l}\right)^2+\frac{\lambda^2}{\rho^2}-\frac{b^2}{a^2}\right\}A=a^2h\rho\frac{d^2}{b^2}\frac{l}{n\pi}2\,J_0(\lambda_s)$$

$$+a^2\frac{d^2}{b^2}\frac{2n\pi}{l}\frac{\rho^2}{\lambda_s^2}h\rho J_0(\lambda_s)+a^2\frac{2n\pi}{l}\int_0^{\rho}x\,J_0\left(\frac{\lambda}{\rho}x\right)\psi(x,t)\,dx\,.$$

Solution of the linear differential equation of the first order is well known

$$(36) \quad A=\int_0^t e^{-a^2\left\{\left(\frac{n\pi}{l}\right)^2+\frac{\lambda^2}{\rho^2}-\frac{b^2}{a^2}\right\}(t-\tau)}\left\{2\,a^2 h\rho^3\frac{d^2}{b^2}\frac{J_0(\lambda_s)}{\lambda_s^2}\frac{l}{n\pi}\left\{\frac{\lambda_s^2}{\rho^2}+\frac{n^2\pi^2}{l^2}\right\}\right.$$

$$\left.+a^2\frac{2n\pi}{l}\int_0^\rho x\,J_0\!\left(\frac{\lambda}{\rho}x\right)\psi(x,t)\,dx\right\}d\tau+C\,e^{-a^2\left(\frac{n^2\pi^2}{l^2}+\frac{\lambda^2}{\rho^2}-\frac{b^2}{a^2}\right)t}.$$

$$=2\,a^2h\rho\frac{J_0(\lambda_s)}{\lambda_s^2}\left\{\rho^2\frac{d^2}{b^2}\frac{l}{n\pi}\left(\frac{\lambda_s^2}{\rho^2}+\frac{n^2\pi^2}{l^2}\right)\frac{1-e^{-a^2\left\{\left(\frac{n\pi}{l}\right)^2+\frac{\lambda^2}{\rho^2}-\frac{b^2}{a^2}\right\}t}}{\left\{\frac{\lambda_s^2}{\rho^2}+\left(\frac{n\pi}{l}\right)^2-\frac{b^2}{a^2}\right\}a^2}\right\}$$

$$+\,a^2\frac{2n\pi}{l}\int_0^t\int_0^\rho e^{-a^2\left\{\left(\frac{n\pi}{l}\right)^2+\frac{\lambda^2}{\rho^2}-\frac{b^2}{a^2}\right\}(t-\tau)}x\,J_0\!\left(\frac{\lambda}{\rho}x\right)\psi(z,t)\,dx\,d\tau$$

$$+\,C\,e^{-a^2\left\{\left(\frac{n\pi}{l}\right)^2+\frac{\lambda^2}{\rho^2}-\frac{b^2}{a^2}\right\}t}.$$

By taking the mean value of $\psi\ (x,\ t)$ in the plane of the cross section we can obtain

$$(37) \qquad \psi(t)=\frac{\int_0^\rho x\,J_0\!\left(\frac{\lambda}{\rho}x\right)\psi(x,t)dx}{\int_0^\rho x\,J_0\!\left(\frac{\lambda}{\rho}x\right)dx}.$$

By putting this value of A in the series (29) we have the function u and consequently T as the following

$$(38) \quad T=-\frac{d^2}{b^2}+\sum_{s=1}^{\infty}\sum_{n=1}^{\infty}\frac{4}{\rho^2 l}\frac{\lambda_s^2 J_0\!\left(\frac{\lambda_s}{\rho}r\right)\sin\frac{n\pi}{l}z}{\{\lambda_s^2+(h\rho)^2\}J_0(\lambda_s)}\left\{2\,a^2h\rho\frac{J_0(\lambda_s)}{\lambda_s^2}\right.$$

$$\left\{\rho^2\frac{d^2}{b^2}\frac{l}{n\pi}\left(\frac{\lambda_s^2}{\rho^2}+\frac{n^2\pi^2}{l^2}\right)\frac{1-e^{-a^2\left\{\left(\frac{n\pi}{l}\right)^2+\frac{\lambda^2}{\rho^2}-\frac{b^2}{a^2}\right\}t}}{\left\{\frac{\lambda_s^2}{\rho^2}+\left(\frac{n\pi}{l}\right)^2-\frac{b^2}{a^2}\right\}a^2}\right\}$$

$$+2a^2h\rho\frac{J_0(\lambda_s)}{\lambda_s^2}\rho^2\frac{n\pi}{l}\int_0^t e^{-a^2\left\{\left(\frac{n\pi}{l}\right)^2+\frac{\lambda_s^2}{\rho^2}-\frac{b^2}{a^2}\right\}(t-\tau)}\psi(\tau)d\tau$$

$$+C\frac{2a^2h\rho J_0(\lambda_s)}{\lambda_s^2}e^{-a^2\left\{\left(\frac{n\pi}{l}\right)^2+\frac{\lambda_s^2}{\rho^2}-\frac{b^2}{a^2}\right\}t}\Bigg\}.$$

The constant C is to be chosen so as to satisfy the initial condition at $t=O$, $T=O$.

Moreover, as we have from (22)

$$\text{(39)}\quad \frac{d^2}{b^2}=\sum_{s=1}^{\infty}\sum_{n=1}^{\infty}\frac{4}{\rho^2 l}\frac{\lambda_s^2 J_0\left(\frac{\lambda_s}{\rho}r\right)e^{-a^2\left\{\left(\frac{n\pi}{l}\right)^2+\lambda_s^2-\frac{b^2}{a^2}\right\}t}\sin\frac{n\pi}{l}z}{\{\lambda_s^2+(h\rho)^2\}\{J_0(\lambda_s)\}^2}$$

$$\int_0^\rho xJ_0\left(\frac{\lambda}{\rho}x\right)dx\int_0^l\frac{d^2}{b^2}\sin\frac{n\pi}{l}y\cdot dy\ ,$$

we have

$$\text{(40)}\quad T=\sum_{s=1}^{\infty}\sum_{n=1}^{\infty}\frac{4}{l}\frac{J_0\left(\frac{\lambda_s}{\rho}r\right)\sin\frac{n\pi}{l}z}{\{\lambda_s^2+(h\rho)^2\}J_0(\lambda_s)}\frac{2h\rho d^2l(1-e)^{-a^2\left\{\left(\frac{n\pi}{l}\right)^2+\frac{\lambda_s^2}{\rho^2}-\frac{b^2}{a^2}\right\}t}}{n\pi a^2\left(\frac{\lambda_s^2}{\rho^2}+\frac{n^2\pi^2}{l^2}-\frac{b^2}{a^2}\right)}.$$

$$+\sum_{s=1}^{\infty}\sum_{n=1}^{\infty}\frac{4}{l}\frac{J_0\left(\frac{\lambda_s}{\rho}r\right)\sin\frac{n\pi}{l}z}{\{\lambda_s^2+(h\rho)^2\}J_0(\lambda_s)}2a^2h\rho\frac{n\pi}{l}$$

$$\int_0^t e^{-a^2\left\{\left(\frac{n\pi}{l}\right)^2+\frac{\lambda_s^2}{\rho^2}-\frac{b^2}{a^2}\right\}(t-\tau)}\psi(\tau)\,d\tau\ .$$

We define the mean temperature of the cross section by the formula:

$$\text{(41)}\qquad T_m=\frac{\int_0^\rho Trdr}{\int_0^\rho rdr}\ ,$$

then

$$(42)\qquad T_m=\sum_{s=1}^{\infty}\sum_{n=1}^{\infty}\frac{8h\rho\frac{\rho^2}{\lambda^2}\sin\frac{n\pi}{l}z}{l\{\lambda_s^2+(h\rho)^2\}}\frac{\frac{2h\rho d^2}{n\pi}\left(1-e^{-a^2\left\{\left(\frac{n\pi}{l}\right)^2+\frac{\lambda^2}{\rho^2}\right\}}\right)}{a^2\left\{\left(\frac{n\pi}{l}\right)^2+\frac{\lambda_s^2}{\rho^2}-\frac{b^2}{a^2}\right\}}$$

$$+\sum_{s=1}^{\infty}\sum_{n=1}^{\infty}\frac{8}{l}\frac{\sin\frac{n\pi}{l}z\cdot h\rho\frac{\rho^2}{\lambda^2}}{\{\lambda_s^2+(h\rho)^2\}}\frac{2a^2h\rho n\pi}{l}$$

$$\int_0^t e^{-a^2\left\{\left(\frac{n\pi}{l}\right)^2+\frac{\lambda^2}{\rho^2}-\frac{b^2}{a^2}\right\}(t-\tau)}\psi(\tau)\,d\tau\ .$$

Since the radius of the wire is not so large and h is very small, the smallest root of the equation $\lambda J_1(\lambda)=h\rho J_0(\lambda)$ is nearly equal to $\lambda^2=2h\rho$ and $h\rho$ is negligibly small in comparison with λ_s, we can write

$$(43)\qquad T_m=\sum_{s=1}^{\infty}\sum_{n=1}^{\infty}\frac{8h\rho\sin\frac{n\pi}{l}z}{\lambda_s^4a^2\left\{\left(\frac{n\pi}{l}\right)^2+\frac{\lambda^2}{\rho^2}-\frac{b^2}{a^2}\right\}}\frac{2h\rho d^2}{n\pi}\left(1-e^{-a^2\left\{\left(\frac{n\pi}{l}\right)^2+\frac{\lambda^2}{\rho^2}-\frac{b^2}{a^2}\right\}}\right)$$

$$+\sum_{s=1}^{\infty}\sum_{n=1}^{\infty}\frac{8h\rho\sin\frac{n\pi}{l}z}{\lambda_s^4}\frac{a^2 2h\rho n\pi}{l^2}$$

$$\int_0^t e^{-a^2\left\{\left(\frac{n\pi}{l}\right)^2+\frac{\lambda^2}{\rho^2}-\frac{b^2}{a^2}\right\}(t-\tau)}\psi(\tau)\,d\tau\ .$$

But we have the relation between the roots of the first order of the Bessel function.

$$(44)\qquad \begin{cases}\lambda_2 \fallingdotseq \lambda_1+3\\[4pt] \dfrac{1}{\lambda_1^4}\gg\dfrac{1}{\lambda_2^4}\ .\end{cases}$$

Therefore in the case of thin cylinder as we are going to study

$$(45)\qquad T_m=\sum_{n=1}^{\infty}\frac{8h\rho\sin\frac{n\pi}{l}z}{\lambda_1^4a^2\left\{\left(\frac{n\pi}{l}\right)^2+\frac{\lambda_1^2}{\rho^2}-\frac{b^2}{a^2}\right\}}\frac{2h\rho d^2}{n\pi}\left\{1-e^{-a^2\left\{\left(\frac{n\pi}{l}\right)^2+\frac{\lambda^2}{\rho^2}-\frac{b^2}{a^2}\right\}t}\right\}$$

$$+\sum_{n=1}^{\infty}\frac{8h\rho\sin\frac{n\pi}{l}z}{\lambda_1^4}\frac{a^2 2h\rho n\pi}{l^2}\int_0^t e^{-a^2\left\{\left(\frac{n\pi}{l}\right)^2+\frac{\lambda^2}{\rho^2}-\frac{b^2}{a^2}\right\}(t-\tau)}\psi(\tau)\,d\tau\ .$$

$$(46)\qquad \lambda^2=2h\rho$$

or

$$(47)\qquad T_m=\sum_{n=1}^{\infty}\frac{2\sin\frac{n\pi}{l}z}{a^2\left\{\left(\frac{n\pi}{l}\right)^2+\frac{\lambda^2}{\rho^2}-\frac{b^2}{a^2}\right\}}\frac{2d^2}{n\pi}\left(1-e^{-a^2\left\{\left(\frac{n\pi}{l}\right)^2+\frac{\lambda^2}{\rho^2}-\frac{b^2}{a^2}\right\}t}\right)$$

$$+\sum_{n=1}^{\infty}2\sin\frac{n\pi}{l}z\cdot\frac{2a^2n\pi}{l^2}\int_0^t e^{-a^2\left\{\left(\frac{n\pi}{l}\right)^2+\frac{\lambda^2}{\rho^2}-\frac{b^2}{a^2}\right\}(t-\tau)}\psi(\tau)\,d\tau\ .$$

$$(48)\qquad =\sum_{n=1}^{\infty}\frac{4d^2}{l}\frac{\sin\frac{n\pi}{l}z.}{\frac{n\pi}{l}}\int_0^t e^{-a^2\left\{\left(\frac{n\pi}{l}\right)^2+\frac{\lambda^2}{\rho^2}-\frac{b^2}{a^2}\right\}(t-\tau)}d\tau$$

$$+\sum_{n=1}^{\infty}4\sin\frac{n\pi}{l}z\cdot\frac{n\pi}{l^2}a^2\int_0^t e^{-a^2\left\{\left(\frac{n\pi}{l}\right)^2+\frac{\lambda^2}{\rho^2}-\frac{b^2}{a^2}\right\}(t-\tau)}\psi(\tau)\,d\tau\ .$$

$$(49)\qquad \frac{\partial T_m}{\partial z}=\frac{4d^2}{l}\int_0^t\sum_{n=1}^{\infty}\cos\frac{n\pi}{l}z\cdot e^{-a^2\left\{\left(\frac{n\pi}{l}\right)^2+\frac{2h}{\rho}-\frac{b^2}{a^2}\right\}t-\tau)}d\tau$$

$$+4a^2\int_0^t \psi(\tau)\sum_{n=1}^{\infty}\frac{d}{dz}\frac{n\pi}{l^2}\sin\frac{n\pi}{l}ze^{-a^2\left\{\left(\frac{n\pi}{l}\right)^2+\frac{2h}{\rho}-\frac{b^2}{a^2}\right\}(t-\tau)}d\tau\,.$$

$$=\frac{4d^2}{l}\int_0^t e^{-a^2\left(\frac{2h}{\rho}-\frac{b^2}{a^2}\right)(t-\tau)}\sum_{n=1}^{\infty}\cos\frac{n\pi}{l}z\cdot e^{-a^2\left(\frac{n\pi}{l}\right)(t-\tau)}d\tau$$

$$+4a^2\int_0^t \psi(\tau)e^{-a^2\left(\frac{2h}{\rho}-\frac{b^2}{a^2}\right)(t-\tau)}$$

$$\frac{d}{dz}\left(\sum_{n=0}^{\infty}\frac{n\pi}{l^2}\sin\frac{n\pi}{l}z\cdot e^{-a^2\left(\frac{n\pi}{l}\right)^2(t-\tau)}\right)d\tau\,.$$

It may be convenient that we express the series by a definite integral, if possible.

For a small value of z, we may approximate the series by putting

$$(50)\qquad \alpha=\frac{n\pi}{l}z\,,\qquad d\alpha=\frac{2\pi}{l}z\,.$$

and the process will be legitimate when z tends to zero.

Thus

$$(51)\qquad \frac{\partial T_m}{\partial z}=\frac{4d^2}{2\pi z}\int_0^t e^{-a^2\left(\frac{2h}{\rho}-\frac{b^2}{a^2}\right)(t-\tau)}d\tau\int_0^{\infty}\cos\alpha\, e^{-\frac{a^2}{z^2}\alpha^2(t-\tau)}d\alpha$$

$$+4a^2\frac{d}{dz}\int_0^t \psi(\tau)\frac{1}{2\pi z^2}\int_0^{\infty}\alpha\sin\alpha\, e^{-\frac{a^2}{z^2}\alpha^2(t-\tau)}d\alpha\, d\tau\,.$$

$$=\frac{2d^2}{\pi z}\int_0^t e^{-a^2\left(\frac{2h}{\rho}-\frac{b^2}{a^2}\right)(t-\tau)}\frac{1}{2}\sqrt{\frac{z^2\pi}{a^2(t-\tau)}}\,e^{-\frac{z^2}{4a^2(t-\tau)}}d\tau$$

$$+\frac{a^2}{\pi}\frac{d}{dz}\int_0^t \psi(\tau)e^{-a^2\left(\frac{2h}{\rho}-\frac{b^2}{a^2}\right)(t-\tau)}\frac{z}{2}\sqrt{\frac{\pi}{a^2(t-\tau)}}$$

$$\frac{e^{-\frac{z^2}{4a^2(t-\tau)}}}{(t-\tau)}\, d\tau\ .$$

$$\left.\frac{\partial T_m}{dz}\right|_{z=0}=\left[\frac{2d^2}{\pi z}\int_0^t e^{-a^2\left(\frac{2h}{\rho}-\frac{b^2}{a^2}\right)(t-\tau)}\frac{1}{2}\sqrt{\frac{z^2\pi}{a^2(t-\tau)}}\,e^{-\frac{z^2}{4a^2(t-\tau)}}d\tau\right]_{z\to 0}$$

(52) $$=\frac{d^2}{\sqrt{\pi}\,a}\int_0^t e^{-a^2\left(\frac{2h}{b}-\frac{b^2}{a^2}\right)(t-\tau)}\frac{1}{\sqrt{t-\tau}}\,d\tau\ .$$

For the sake brevity we put

(53) $$\varphi=\frac{\epsilon\, d^2}{\sqrt{\pi}\,a}\int_0^t e^{-a^2\left(\frac{2h}{\rho}-\frac{b^2}{a^2}\right)(t-\tau)}\frac{1}{\sqrt{t-\tau}}\,d\tau\ .$$

Now

$$\int_0^t\frac{\varphi(\tau)}{\sqrt{t-\tau}}d\tau=\frac{d^2\epsilon}{\sqrt{\pi}\,a}\int_0^t\frac{d\tau}{\sqrt{t-\tau}}\int_0^\tau\frac{e^{-a^2\left(\frac{2h}{\rho}-\frac{b^2}{a^2}\right)(\tau-\tau_1)}}{\sqrt{t-\tau}}\,d\tau_1\ .$$

(54) $$=\frac{d^2\epsilon}{\sqrt{\pi}a}\int_0^t e^{a^2\left(\frac{2h}{\rho}-\frac{b^2}{a^2}\right)\tau_1}d\tau_1\int_{\tau_1}^t\frac{e^{-a^2\left(\frac{2h}{\rho}-\frac{b^2}{a^2}\right)\tau}}{\sqrt{t-\tau}\sqrt{\tau-\tau_1}}\,d\tau\ .$$

Changing the variable by

$$\tau=\tau_1+(t-\tau_1)\,\xi\ ,$$

$$d\tau=(t-\tau_1)\,d\xi\ ,$$

$$\begin{cases}\tau=\tau_1\,, & \xi=0\,,\\ \tau=t\,, & \xi=1\,,\end{cases}$$

we obtain

(55) $$\int_0^t\frac{\varphi(t)}{\sqrt{t-\tau}}d\tau=\frac{d^2\epsilon}{\sqrt{\pi}\,a}\int_0^t e^{a^2\left(\frac{2h}{\rho}-\frac{b^2}{a^2}\right)\tau_1}d\tau_1\int_0^1\frac{e^{-a_2\left(\frac{2h}{\rho}-\frac{b^2}{a^2}\right)\left\{\tau_1+(t-\tau)\xi\right\}}}{\sqrt{(1-\xi)\,\xi}}d\xi.$$

$$=\frac{d^2\epsilon}{\sqrt{\pi}a}\int_0^t d\tau_1\int_0^1 \frac{e^{-a^2\left(\frac{2h}{\rho}-\frac{b^2}{a^2}\right)(t-\tau_1)\xi}}{\sqrt{(1-\xi)\xi}}\cdot d\xi\ .$$

(56) $$\frac{1}{\pi}\frac{d}{dz}\int_0^t \psi(\tau)\, e^{-a^2\left(\frac{2h}{\rho}-\frac{b^2}{a^2}\right)(t-\tau)}\frac{z}{2}\sqrt{\frac{\pi}{a^2(t-\tau)}}\frac{e^{-\frac{z^2}{4a^2(t-\tau)}}}{(t-\tau)}d\tau\ .$$

$$=\frac{1}{\sqrt{\pi}}\int_0^t \psi(\tau)\, e^{-a^2\left(\frac{2h}{\rho}-\frac{b^2}{a^2}\right)(t-\tau)}\left\{\frac{1}{2}\sqrt{\frac{1}{a^2(t-\tau)^3}}e^{-\frac{z^2}{4a^2(t-\tau)}}\right.$$

$$\left.-\sqrt{\frac{1}{a^2(t-\tau)^3}}\frac{z^2}{4a^2(t-\tau)}\, e^{-\frac{z^2}{4a^2(t-\tau)}}\right\}d\tau$$

$$=\frac{1}{\sqrt{\pi}}\int_0^t \psi(\tau)\, e^{-a^2\left(\frac{2h}{\rho}-\frac{b^2}{a^2}\right)(t-\tau)}\frac{1}{2}\sqrt{\frac{1}{a^2(t-\tau)^3}}\, e^{-\frac{z^2}{4a^2(t-\tau)}}d\tau$$

$$+\frac{1}{\sqrt{\pi}}\left[\sqrt{\frac{1}{a^2(t-\tau)}}\, e^{-a^2\left(\frac{2h}{\rho}-\frac{b^2}{a^2}\right)(t-\tau)}\psi(\tau)\, e^{-\frac{z^2}{4a^2(t-\tau)}}\right]_0^t$$

$$-\frac{1}{\sqrt{\pi}}\int_0^t e^{-a^2\left(\frac{2h}{\rho}-\frac{b^2}{a^2}\right)(t-\tau)}\frac{\psi(\tau)}{2\sqrt{a^2(t-\tau)^3}}\, e^{-\frac{z^2}{4a^2(t-\tau)}}d\tau$$

$$-\frac{1}{\sqrt{\pi}}\int_0^t \frac{\partial}{\partial\tau}\left\{e^{-a^2\left(\frac{2h}{\rho}-\frac{b^2}{a^2}\right)(t-\tau)}\psi(\tau)\right\}\frac{e^{-\frac{z^2}{4a^2(t-\tau)}}}{\sqrt{a^2(t-\tau)}}d\tau\ .$$

As $\psi\ (0)=0$, we have

(57) $$=-\frac{1}{a\sqrt{\pi}}\int_0^t \frac{e^{-a^2\left(\frac{2h}{\rho}-\frac{b^2}{a^2}\right)(t-\tau)-\frac{z^2}{4a^2(t-\tau)}}}{\sqrt{t-\tau}}\frac{\partial\psi(\tau)}{\partial\tau}d\tau$$

$$-\frac{1}{a\sqrt{\pi}}\int_0^t \frac{a^2\left(\frac{2h}{\rho}-\frac{b^2}{a^2}\right)e^{-a^2\left(\frac{2h}{\rho}-\frac{b^2}{a^2}\right)(t-\tau)-\frac{z^2}{4a^2(t-\tau)}}}{\sqrt{t-\tau}}\psi(\tau)\,d\tau\,.$$

In the limit where $z \to 0$

$$(58)\qquad \epsilon\frac{\partial T}{\partial z}=\varphi(t)-\frac{\epsilon}{a\sqrt{\pi}}\int_0^t \frac{e^{-a^2\left(\frac{2h}{\rho}-\frac{b^2}{a^2}\right)(t-\tau)}}{\sqrt{t-\tau}}\left\{\frac{\partial\psi}{\partial\tau}+a^2\left(\frac{2h}{\rho}-\frac{b^2}{a^2}\right)\psi(\tau)\right\}d\tau\,.$$

In order to find the terminal temperature, we must assume the form of terminal and the method of cooling. Though we can consider many cases, we may assume for mathematical simplicity that the form of the terminal is also a wire with the same cross section, leading straight to infinity and exposed openly in the air. Although it is not the ordinary case, the difference due to this assumption may cancel with some correction the amount heat escaped from the material.

Substituting in the above calculation,

$$(59)\qquad d=0\,,\quad b=0\,,\quad z=-z\,,\quad \epsilon=\epsilon_1\,,\quad h=h_1$$

the solution for the terminals will be given at once

$$(60)\qquad \epsilon_1\frac{\partial T}{\partial z}=\frac{\epsilon_1}{a_1\sqrt{\pi}}\int_0^t \frac{\frac{\partial\psi}{\partial\tau}e^{-a_1^2\frac{2h_1}{\rho}(t-\tau)}}{\sqrt{t-\tau}}d\tau$$

$$+\frac{\epsilon_1}{a_1\sqrt{\pi}}\frac{2h_1}{\rho}a_1^2\int_0^t \frac{\psi(\tau)\,e^{-a_1^2\frac{2h}{\rho}(t-\tau)}}{\sqrt{t-\tau}}d\tau$$

$$(61)\qquad \epsilon\frac{\partial T_m}{\partial z}\Big|_{z\to 0}=\varphi(t)-\frac{\epsilon}{a\sqrt{\pi}}\int_0^t \frac{\frac{\partial\psi}{\partial\tau}}{\sqrt{t-\tau}}e^{\left(b^2-a^2\frac{2h}{\rho}\right)(t-\tau)}d\tau$$

$$+\frac{\epsilon}{a\sqrt{\pi}}\left(b^2-a^2\frac{2h}{\rho}\right)\int_0^t \frac{\psi(\tau)\,e^{\left(b^2-a^2\frac{2h}{\rho}\right)(t-\tau)}}{\sqrt{t-\tau}}d\tau\,.$$

$$(62)\qquad \epsilon_1 \frac{\partial T_m}{\partial z}_{z\to 0} = \frac{\epsilon_1}{a_1\sqrt{\pi}} \int_0^t \frac{\frac{\partial \psi}{\partial \tau} e^{-a_1\frac{2h_1}{\rho}(t-\tau)}}{\sqrt{t-\tau}} d\tau$$

$$+ \frac{\epsilon_1}{a_1\sqrt{\pi}} a_1^2 \frac{2h_1}{\rho} \int_0^t \frac{\psi(\tau)\, e^{-a_1^2\frac{2h_1}{\rho}(t-\tau)}}{\sqrt{t-\tau}} d\tau\ .$$

From the law of the conservation of energy the equation of continuity must be held.

$$(63)\qquad \epsilon \frac{\partial T_m}{\partial z}_{z\to 0} = \epsilon_1 \frac{\partial T_m}{\partial z}_{z\to 0}$$

$$(64)\qquad \sqrt{\pi}\varphi(t) = \int_0^{} \frac{\frac{\epsilon}{a}\frac{\partial \psi}{\partial \tau} e^{\left(b^2-a^2\frac{2h}{\rho}\right)(t-\tau)} - \frac{\epsilon}{a}\left(b^2-a^2\frac{2h}{\rho}\right)\psi\, e^{\left(b^2-a^2\frac{2h}{\rho}\right)(t-\tau)}}{\sqrt{t-\tau}}$$

$$+ \frac{+\frac{\epsilon_1}{a_1}\frac{\partial \psi}{\partial \tau} e^{-a_1^2\frac{2h_1}{\rho}(t-\tau)} + \frac{\epsilon_1}{a_1\sqrt{\pi}} a_1^2 \frac{2h_1}{\rho}\psi e^{-a_1^2\frac{2h_1}{\rho}(t-\tau)}}{\sqrt{t-\tau}} d\tau\ .$$

Putting

$$(65)\qquad \begin{cases} A = \dfrac{\epsilon}{a}\ , \\ B = -\dfrac{\epsilon}{a}\left(b^2-a^2\dfrac{2h}{\rho}\right), \\ C = \dfrac{\epsilon_1}{a_1}\ , \\ E = \dfrac{\epsilon_1}{a_1} a_1^2 \dfrac{2h}{\rho}\ , \\ \alpha = a^2\dfrac{2h}{\rho} - b^2\ , \\ \beta = a_1^2\dfrac{2h_1}{\rho} \end{cases}$$

(66) $\sqrt{\pi}\,\varphi(t)$

$$=\int_0^t \frac{A\frac{\partial\psi}{\partial\tau}e^{-\alpha(t-\tau)}+B\psi e^{-\alpha(t-\tau)}+C\frac{\partial\psi}{\partial\tau}e^{-\beta(t-\tau)}+E\psi e^{-\beta(t-\tau)}}{\sqrt{t-\tau}}d\tau$$

Multiply by $(t-\tau)^{\frac{1}{2}}$ and integrate from 0 to t.

$$\sqrt{\pi}\int_0^t \frac{\varphi(\tau_1)}{\sqrt{t-\tau_1}}d\tau_1=\int_0^t \frac{d\tau_1}{\sqrt{t-\tau_1}}$$

$$\int_0^{\tau_1} \frac{A\frac{\partial\psi}{\partial\tau}e^{-\alpha(\tau_1-\tau)}+B\psi e^{-\alpha(\tau_1-\tau)}+C\frac{\partial\psi}{\partial\tau}e^{-\beta(\tau_1-\tau)}+E\psi e^{-\beta(\tau_1-\tau)}}{\sqrt{\tau_1-\tau}}d\tau$$

$$=\int_0^t \frac{d\tau_1}{\sqrt{t-\tau_1}}\int_0^{\tau_1}\frac{(A+C)\frac{\partial\psi}{\partial\tau}+(B+E)\psi}{\sqrt{\tau_1-\tau}}d\tau$$

$$-\int_0^t \frac{d\tau_1}{\sqrt{t-\tau_1}}\int_0^{\tau_1}\frac{(1-e^{-\alpha(\tau_1-\tau)})\left\{A\frac{\partial\psi}{\partial\tau}+B\psi\right\}}{\sqrt{\tau_1-\tau}}d\tau\ .$$

$$-\int_0^t \frac{d\tau_1}{\sqrt{t-\tau_1}}\int_0^{\tau_1}\frac{(1-e^{-\beta(\tau_1-\tau)})\left\{C\frac{\partial\psi}{\partial\tau}+E\psi\right\}}{\sqrt{\tau_1-\tau}}d\tau$$

$$=\int_0^t\left\{(A+C)\frac{\partial\psi}{\partial\tau}+(B+E)\psi\right\}d\tau\int_\tau^t\frac{d\tau}{\sqrt{t-\tau_1}\,\sqrt{\tau_1-\tau}}\ .$$

$$-\int_0^t\left(A\frac{\partial\psi}{\partial\tau}+B\psi\right)d\tau\int_\tau^t\frac{d\tau_1(1-e^{-\alpha(\tau_1-\tau)})}{\sqrt{\tau_1-\tau}\,\sqrt{t-\tau_1}}$$

$$-\int_0^t\left(C\frac{\partial\psi}{\partial\tau}+E\psi\right)d\tau\int_\tau^t\frac{d\tau_1(1-e^{-\beta(\tau_1-\tau)})}{\sqrt{\tau_1-\tau}\,\sqrt{t-\tau_1}}\ .$$

By changing the variable

$$\tau_1=\tau+(t-\tau)\xi .$$

$$d\tau_1=(t-\tau)d\xi .$$

$$\begin{cases} \tau_1=\tau , & \xi=0 \\ \tau_1=t , & \xi=1 . \end{cases}$$

$$=\int_0^t \left\{(A+C)\frac{\partial\psi}{\partial\tau}+(B+E)\psi\right\}d\tau \int_0^1 \frac{(t-\tau)d\xi}{\sqrt{(t-\tau)(1-\xi)(t-\tau)\xi}}$$

$$-\int_0^t \left(A\frac{\partial\psi}{\partial\tau}+B\psi\right)d\tau \int_0^1 \frac{(t-\tau)\{1-e^{-\alpha(t-\tau)\xi}\}}{\sqrt{(t-\tau)(1-\xi)(t-\tau)\xi}}d\xi$$

$$-\int_0^t \left(C\frac{\partial\psi}{\partial\tau}+E\psi\right)d\tau \int_0^1 \frac{(t-\tau)\{1-e^{-\beta(t-\tau)\xi}\}}{\sqrt{(t-\tau)(1-\xi)(t-\tau)\xi}}d\xi .$$

Now

$$(67)\quad \sqrt{\pi}\int_0^t \frac{\varphi(\tau_1)}{\sqrt{t-\tau_1}}d\tau_1=\int_0^t \left\{(A+C)\frac{\partial\psi}{\partial\tau}+(B+E)\psi\right\}d\tau \int_0^1 \frac{d\xi}{\sqrt{(1-\xi)\xi}}$$

$$-\int_0^t \left\{A\frac{\partial\psi}{\partial\tau}+B\psi\right\}d\tau \int_0^1 \frac{1-e^{-\alpha(t-\tau)\xi}}{\sqrt{(1-\xi)\xi}}$$

$$-\int_0^t \left\{C\frac{\partial\psi}{\partial\tau}+E\psi\right\}d\tau \int_0^1 \frac{1-e^{-\beta(t-\tau)\xi}}{\sqrt{(1-\xi)\xi}} .$$

The integrals in the right hand will be calculated by aid of Gamma function.

If we substitute the relations (68.1) $\int_0^1 \frac{d\xi}{\sqrt{(1-\xi)\xi}}=\pi$ and (68.2) $\int_0^1 \frac{1-e^{-\alpha(t-\tau)\xi}}{\sqrt{(1-\xi)\xi}}=\pi\Phi_1(t-\tau)$, this equation will be simply written :

$$\int_0^t \frac{\varphi(\tau_1)}{\sqrt{t-\tau_1}}\,d\tau_1 = \pi(A+C)+\pi\int_0^t (B+E)\psi\,d\tau$$

$$-\pi\int_0^t \left\{A\frac{\partial\psi}{\partial\tau}+B\psi\right\}\Phi_1(t-\tau)\,d\tau$$

$$-\pi\int_0^t \left\{C\frac{\partial\psi}{\partial\tau}+E\psi\right\}\Phi_2(t-\tau)\,d\tau\ .$$

$$=\pi(A+C)\,\psi-A\Big[\psi\Phi_1(t-\tau)\Big]_0^t-C\Big[\psi\Phi_2(t-\tau)\Big]_0^t$$

$$+\pi\int_0^t B\psi\Big\{1-\Phi_1(t-\tau)\Big\}d\tau+\pi\int_0^t A\psi\frac{\partial\Phi_1}{\partial\tau}\,d\tau$$

$$+\pi\int_0^t E\psi\Big\{1-\Phi_2(t-\tau)\Big\}d\tau+\pi\int_0^t C\psi\frac{\partial\Phi_2}{\partial\tau}\,d\tau$$

$$(69)\quad \begin{cases} \pi\Phi_1(t-\tau)=\pi-\displaystyle\int_0^1 \frac{e^{-\alpha(t-\tau)\xi}}{\sqrt{(1-\xi)\xi}}\,d\xi \\[2ex] \pi\Phi_2(t-\tau)=\pi-\displaystyle\int_0^1 \frac{e^{-\beta(t-\tau)\xi}}{\sqrt{(1-\xi)\xi}}\,d\xi \end{cases}$$

$$\frac{\sqrt{\pi}}{2}\int_0^t \frac{\varphi(\tau_1)}{\sqrt{t-\tau_1}}\,d\tau_1=\pi(A+C)\psi+\int_0^t B\psi\int_0^1 \frac{e^{-\alpha(t-\tau)\xi}}{\sqrt{(1-\xi)\xi}}\,d\tau\,d\xi$$

$$+\int_0^t E\psi\int_0^1 \frac{e^{-\beta(t-\tau)\xi}}{\sqrt{(1-\xi)\xi}}\,d\tau\,d\xi\ .$$

$$-\int_0^t A\psi\int_0^1 \frac{\alpha\xi e^{-\alpha(t-\tau)\xi}}{\sqrt{(1-\xi)\xi}}d\xi-\int_0^t C\psi\int_0^1 \frac{\beta\xi e^{-\beta(t-\tau)\xi}}{\sqrt{(1-\xi)\xi}}d\xi.$$

(70)
$$=\pi(A+C)\psi(t)+\int_0^t B\psi d\tau\int_0^1\sqrt{\frac{1-\xi}{\xi}}e^{-\alpha(t-\tau)\xi}d\xi$$

$$+\int_0^t E\psi d\tau\int_0^1\sqrt{\frac{1-\xi}{\xi}}e^{-\beta(t-\tau)\xi}d\xi\ .$$

The left side of this equation has already been calculated, therefore

(71)
$$\frac{\epsilon_1 d^2}{a}\int_0^t d\tau\int_0^1 \frac{e^{-a^2\left(\frac{2h}{\rho}-\frac{b^2}{a^2}\right)(t-\tau)\xi}}{\sqrt{(1-\xi)\xi}}d\xi=\left(\frac{\pi\epsilon}{a}+\frac{\pi\epsilon_1}{2a_1}\right)\psi(t)$$

$$+\alpha\frac{\epsilon}{a}\int_0^t\psi\int_0^1\sqrt{\frac{1-\xi}{\xi}}\,e^{-\alpha(t-\tau)\xi}d\xi$$

$$+\beta\frac{\epsilon_1}{a_1}\int_0^t\psi\int_0^1\sqrt{\frac{1-\xi}{\xi}}\,e^{-\beta(t-\tau)\xi}d\xi\ .$$

This is an integral equation of Volterra's type. Since the kern is analytic function of $(t-\tau)$, the solution can be obtained easily. But the function $\psi(t)$ must be a monotonous function, therefore the last integral will be written from the mean value theorem.

$$\alpha\frac{\epsilon}{a}\psi\mu_1\int_0^1\sqrt{\frac{1-\xi}{\xi}}\,e^{-\alpha(t-\tau)\xi}d\xi\ .$$

where $0<\mu_1<1$

and

$$\beta\frac{\epsilon_1}{a_1}\psi\mu_2\int_0^1\sqrt{\frac{1-\xi}{\xi}}\,e^{-\beta(t-\tau)\xi}d\xi$$

where $0<\mu_2<1$.

(72)

$$\psi=\frac{\dfrac{\epsilon d^2}{\sqrt{\pi}\,a\alpha}\displaystyle\int_0^1\frac{1-e^{-\alpha t\xi}}{\sqrt{(1-\xi)\xi}\cdot\xi}d\xi}{\left(\dfrac{\epsilon}{a}+\dfrac{\epsilon_1}{2a_1}\right)\sqrt{\pi}+\dfrac{\epsilon}{\sqrt{\pi}\,a}\mu_1\displaystyle\int_0^1\sqrt{\frac{1-\xi}{\xi}}e^{-\alpha(t-\tau)\xi}d\xi+\dfrac{\epsilon_1}{a_1}\mu_2\displaystyle\int_0^1\sqrt{\frac{1-\xi}{\xi}}e^{-\beta(t-\tau)\xi}d\xi}$$

If αt and βt are very small, we have:

$$\psi=\frac{\dfrac{\epsilon d^2}{\sqrt{\pi}\,a\alpha}\displaystyle\int_0^1\frac{\alpha t}{\sqrt{(1-\xi)\xi}}d\xi}{\left(\dfrac{\epsilon}{a}+\dfrac{\epsilon_1}{a_1}\right)\sqrt{\pi}+\dfrac{\epsilon}{a}\dfrac{\mu_1}{\sqrt{\pi}}\displaystyle\int_0^1\frac{\alpha t\sqrt{1-\xi}}{\sqrt{\xi}}d\xi+\dfrac{\epsilon_1}{a_1}\dfrac{\mu_2}{\sqrt{\pi}}\displaystyle\int_0^1\frac{\beta t\sqrt{1-\xi}}{\sqrt{\xi}}d\xi}$$

$$=\frac{\dfrac{\epsilon d^2t}{\sqrt{\pi}\,a}\dfrac{\left\{\Gamma\left(\frac{1}{2}\right)\right\}^2}{\Gamma(1)}-\dfrac{\epsilon d^2}{\sqrt{\pi}\,a}\dfrac{\alpha t^2}{2}\dfrac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(2)}}{\left(\dfrac{\epsilon}{a}+\dfrac{\epsilon_1}{a_1}\right)\sqrt{\pi}+\dfrac{\epsilon}{a}\dfrac{\alpha t}{\sqrt{\pi}}\mu_1\dfrac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)}+\dfrac{\epsilon_1}{a_1}\dfrac{\beta t}{\sqrt{\pi}}\mu_2\dfrac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)}}$$

$$=\frac{\dfrac{\epsilon d^2t}{a}\sqrt{\pi}+\dfrac{\epsilon d^2}{a}\sqrt{\pi}\dfrac{\alpha t^2}{4}}{\left(\dfrac{\epsilon}{a}+\dfrac{\epsilon_1}{a_1}\right)\sqrt{\pi}+\dfrac{\epsilon}{a}\dfrac{\alpha t}{2}\mu_1+\dfrac{\epsilon_1\beta t}{a_1 2}\mu_2}$$

$$=\frac{d^2t\left(1+\dfrac{\alpha t}{4}\right)}{\left(1+\dfrac{\epsilon_1}{\epsilon}\dfrac{a}{a_1}\right)\left\{1+\dfrac{\beta t}{2}\mu+\dfrac{\dfrac{\alpha t}{2}\mu_1-\dfrac{\beta t}{2}\mu_2}{1+\dfrac{\epsilon_1}{\epsilon}\dfrac{a}{a_1}}\right\}}$$

$$(73) \qquad = \frac{d^2t}{1+\frac{\epsilon_1}{\epsilon}\frac{a}{a_1}}$$

When αt is large, we can evaluate the integral, for

$$\int_0^\varepsilon \frac{1-e^{-\alpha t\xi}}{\sqrt{1-\xi}\,\xi^{\frac{3}{2}}}d\xi=\int_0^\varepsilon \frac{1-e^{-\alpha t\xi}}{\xi^{\frac{3}{2}}}d\xi+\int_0^\varepsilon \frac{1-e^{-\alpha t\xi}}{2\xi^{\frac{1}{2}}}d\xi$$

$$=\left|-2\xi^{-\frac{1}{2}}(1-e^{-\alpha t\xi})\right|_0^\varepsilon-\int_0^\varepsilon -2\xi^{-\frac{1}{2}}\alpha t e^{-\alpha t\xi}d\xi$$

$$+\epsilon^{\frac{1}{2}}-\int_0^\varepsilon \frac{e^{-\alpha t\xi}}{2\xi^{\frac{1}{2}}}d\xi\,.$$

$$=\left|-2\xi^{-\frac{1}{2}}(1-e^{-\alpha t\xi})\right|_0^\varepsilon+\epsilon^{\frac{1}{2}}-\int_0^\varepsilon \frac{e^{-\alpha t\xi}(1-4\alpha t)}{2\xi^{\frac{1}{2}}}d\xi\,.$$

$$\xi^{\frac{1}{2}}=x$$

$$\frac{1}{2}\xi^{-\frac{1}{2}}d\xi=dx$$

$$\begin{cases}\xi=0\,, & x=0\\ \xi=\epsilon\,, & x=\sqrt{\epsilon}\end{cases}$$

$$=-\frac{2}{\sqrt{\epsilon}}(1-e^{-\alpha t\varepsilon})+\epsilon^{\frac{1}{2}}-\int_0^{\sqrt{\varepsilon}} e^{-\alpha tx^2}(1-4\alpha t)dx$$

$$=-\frac{2}{\sqrt{\epsilon}}(1-e^{-\alpha t\varepsilon})+\epsilon^{\frac{1}{2}}-\frac{1-4\alpha t}{\sqrt{\alpha t}}\int_0^{\sqrt{\varepsilon}} e^{-\alpha tx^2}\sqrt{\alpha t}\,dx\,.$$

For large value of $\sqrt{\alpha t}\,x$,

$$=-\frac{2}{\sqrt{\epsilon}}(1-e^{-\alpha t\varepsilon})+\epsilon^{\frac{1}{2}}-\frac{1-4\alpha t}{\sqrt{\alpha t}}\left(\frac{1}{2}\sqrt{\pi}-\frac{1}{2}\frac{e^{-\alpha tx^2}}{\sqrt{\alpha t}\,x}\right)$$

$$=-\frac{2}{\sqrt{\epsilon}}(1-e^{-\alpha t\epsilon})+\epsilon^{\frac{1}{2}}-\frac{1-4\alpha t}{\sqrt{\alpha t}}\frac{1}{2}\sqrt{\pi}.$$

(74) $$=-\frac{2}{\sqrt{\epsilon}}+\sqrt{\epsilon}-\frac{1-4\alpha t}{\sqrt{\alpha t}}\frac{1}{2}\sqrt{\pi}.$$

For small value of $\sqrt{\alpha t}\,x$,

$$=-\frac{2}{\sqrt{\epsilon}}(1-e^{-\alpha t\epsilon})+\epsilon^{\frac{1}{2}}-\frac{1-4\alpha t}{\sqrt{\alpha t}}\left(\frac{1}{2}\sqrt{\alpha t}\,x\right)$$

$$=-2\alpha t\sqrt{\epsilon}+\epsilon^{\frac{1}{2}}-\frac{1}{2}(1-4\alpha t)\sqrt{\epsilon}.$$

(75) $$=\frac{1}{2}\sqrt{\epsilon}.$$

and

$$\int_{\epsilon}^{1}\frac{1-e^{-\alpha t\xi}}{\sqrt{1-\xi}\,\xi^{\frac{3}{2}}}d\xi=\int_{\epsilon}^{1}\frac{1}{\sqrt{1-\xi}\,\xi^{\frac{3}{2}}}d\xi$$

$$\xi^{\frac{1}{2}}=x$$

$$\frac{1}{2}\xi^{-\frac{1}{2}}d\xi=dx$$

$$\begin{cases}\xi=1\,, & x=1\\ \xi=\epsilon\,, & x=\sqrt{\epsilon}\end{cases}$$

$$=\int_{\sqrt{\epsilon}}^{1}\frac{2}{\sqrt{1-x^2}\,x^2}dx$$

$$=2\left|\frac{-\sqrt{1-x^2}}{x}\right|_{\sqrt{\epsilon}}^{1}$$

$$=2\frac{\sqrt{1-\epsilon}}{\sqrt{\epsilon}}$$

(76) $$=\frac{2}{\sqrt{\epsilon}}-\sqrt{\epsilon}.$$

(77) $$\int_0^1 \frac{1-e^{-\alpha t\xi}}{\sqrt{1-\xi}\,\xi^{\frac{3}{2}}}d\xi=\frac{4\alpha t}{\sqrt{\alpha t}}\frac{1}{2}\sqrt{\pi}=2\sqrt{\alpha t}\sqrt{\pi}.$$

In the same way,

(78) $$\int_0^\varepsilon \frac{(1-e^{-\alpha t\xi})\sqrt{1-\xi}}{\xi^{\frac{3}{2}}}d\xi=-\frac{2}{\sqrt{\epsilon}}-\sqrt{\epsilon}+\frac{1+4\alpha t}{\sqrt{\alpha t}}\frac{1}{2}\sqrt{\pi}.$$

$$\int_\varepsilon^1 \frac{(1-e^{-\alpha t\xi})\sqrt{1-\xi}}{\xi^{\frac{3}{2}}}d\xi=\int_{\sqrt{\varepsilon}}^1 \frac{\sqrt{1-\xi}}{\xi^{\frac{3}{2}}}d\xi$$

$$=\int_{\sqrt{\varepsilon}}^1 \frac{2\sqrt{1-x^2}}{x^2}d\xi$$

$$=-2\left|\frac{\sqrt{1-x^2}}{x}+\sin^{-1}x\right|_{\sqrt{\varepsilon}}^1$$

$$=-\pi+2\frac{\sqrt{1-\epsilon}}{\sqrt{\epsilon}}+2\sqrt{\epsilon}$$

(79) $$=-\pi+2\frac{1}{\sqrt{\epsilon}}+\sqrt{\epsilon}.$$

(80) $$\int_0^1 \frac{(1-e^{-\alpha t\xi})\sqrt{1-\xi}}{\xi^{\frac{3}{2}}}d\xi=2\sqrt{\alpha t}\sqrt{\pi}-\sqrt{\pi}.$$

Thus

$$\psi=\frac{\dfrac{2d^2\epsilon\sqrt{t}}{a_1\sqrt{\alpha}}}{\left(\dfrac{\epsilon}{a}+\dfrac{\epsilon_1}{a_1}\right)\sqrt{\pi}+\dfrac{\epsilon\mu_1}{\sqrt{\pi}a}(2\sqrt{\alpha t}\sqrt{\pi}-\sqrt{\pi})+\dfrac{\epsilon_1\mu_2}{\sqrt{\pi}a_1}(2\sqrt{\beta t}\sqrt{\pi}-\sqrt{\pi})}$$

$$=\frac{\dfrac{2d^2\sqrt{t}}{a\sqrt{\alpha}}}{\dfrac{\epsilon_1}{a_1}\sqrt{\pi}+2\dfrac{\epsilon\sqrt{\alpha t}}{a}\mu_1+2\dfrac{\epsilon_1\sqrt{\beta t}}{a_1}\mu_2}$$

$$\mu_1 \doteqdot 1\ , \qquad \mu_2 \doteqdot 1\ .$$

$$=\frac{\dfrac{d^2}{a}}{1+\dfrac{1}{2}\dfrac{\epsilon_1}{\epsilon}\dfrac{a}{a_1}\left(2\sqrt{\beta}+\dfrac{\sqrt{\pi}}{\sqrt{t}}\right)\dfrac{1}{\sqrt{\alpha}}}$$

(81) $$=\frac{d^2}{a}\left(\frac{\sqrt{\alpha}\,\epsilon}{a}\Bigg/\frac{\sqrt{\beta}\,\epsilon_1}{a_1}\right)$$

if the αt and βt tend to infinity,

(82) $$\psi=\frac{d^2}{b^2}$$

while ϵ_1 is very large, and αt and βt are not infinity.

(83) $$\psi=\frac{\dfrac{d^2\sqrt{\alpha t}}{a\sqrt{\pi}}}{\dfrac{\epsilon_1}{\epsilon}\dfrac{a}{a_1}}=\frac{d^2}{a\sqrt{\pi}}\frac{\epsilon}{\epsilon_1}\frac{a_1}{a}\sqrt{\alpha t}$$

Thus, we have obtained mathematically the temperature rise of the terminals.

In the case where αt and βt are very small in spite of the signs of α, β.

(73) $$\psi=\frac{d^2 t}{\left(1+\dfrac{\epsilon_1}{a_1}\dfrac{a}{\epsilon}\right)}$$

$$\int_0^t \psi(\tau)\, e^{\left\{b^2-a^2\left(q^2+\frac{2h}{\rho}\right)\right\}(t-\tau)}\, d\tau$$

$$=\frac{d^2}{1+\frac{\epsilon_1}{a_1}\frac{a}{\epsilon}}\int_0^t \tau e^{\left\{b^2-a^2\left(q^2+\frac{2h}{\rho}\right)\right\}(t-\tau)}d\tau$$

$$=\frac{d^2 e^{\left\{b^2-a^2\left(q^2+\frac{2h}{\rho}\right)\right\}t}}{1+\frac{\epsilon_1}{a_1}\frac{a}{\epsilon}}\int_0^t \tau e^{-\left\{b^2-a^2\left(q^2+\frac{2h}{\rho}\right)\right\}(t-\tau)}d\tau$$

$$=\frac{d^2 e^{\left\{b^2-a^2\left(q^2+\frac{2h}{\rho}\right)\right\}t}}{\left(1+\frac{\epsilon_1}{a_1}\frac{\epsilon}{a}\right)}\left|\frac{e^{-\left\{b^2-a^2\left(q^2+\frac{2h}{\rho}\right)\right\}\tau}}{\left\{b^2-a^2\left(q^2+\frac{2h}{\rho}\right)\right\}\tau}\right.$$

$$\left.\left\{-\left\{b^2-a^2\left(q^2+\frac{2h}{\rho}\right)\right\}\tau-1\right\}\right|_0^t$$

$$=\frac{d^2 e^{\left\{b^2-a^2\left(q^2+\frac{2h}{\rho}\right)\right\}t}}{\left(1+\frac{\epsilon_1}{a_1}\frac{a}{\epsilon}\right)\left\{b^2-a^2\left(q^2+\frac{2h}{\rho}\right)\right\}^2}\left\{e^{-\left\{b^2-a^2\left(q^2+\frac{2h}{\rho}\right)\right\}t}\right.$$

$$\left.\left\{-\left\{b^2-a^2\left(q^2+\frac{2h}{\rho}\right)\right\}t-1\right\}+1\right\}$$

(84) $$=\frac{-d^2}{\left(1+\frac{\epsilon_1}{a_1}\frac{a}{\epsilon}\right)\left\{b^2-a^2\left(q^2+\frac{2h}{\rho}\right)\right\}^2}\left\{\left\{1+\left(b^2-a^2\left(q^2+\frac{2h}{\rho}\right)\right)t\right\}\right.$$

$$\left.-e^{\left\{b^2-a^2\left(q^2+\frac{2h}{\rho}\right)\right\}t}\right\}$$

From (48) we obtain finally

$$T_m=\sum\left\{\frac{4d^2\sin\frac{n\pi}{l}z}{n\pi}\,\frac{1}{a^2\left(q^2+\frac{2h}{\rho}\right)-b^2}-\frac{4d^2\sin\frac{n\pi}{l}z}{\pi}\,\frac{e^{-a^2\left(\frac{\pi^2}{l^2}+\frac{2h}{\rho}\right)t+b^2t}}{a^2\left(\frac{\pi^2}{l^2}+\frac{2h}{\rho}\right)-b^2}\right.$$

$$+\sum \frac{4n\pi a^2}{l^2}\sin\frac{n\pi}{l}z\frac{-d^2}{\left(1+\frac{\epsilon_1}{a_1}\frac{a}{\epsilon}\right)\left\{b^2-a^2\left(q^2+\frac{2h}{\rho}\right)\right\}^2}$$

$$\left[\left\{1+\left\{b^2-a^2\left(q^2+\frac{2h}{\rho}\right)\right\}t\right\}-e^{\left\{b^2-a^2\left(q^2+\frac{2h}{\rho}\right)\right\}t}\right]$$

(85)
$$=\sum \frac{4d^2\sin\frac{n\pi}{l}z}{n\pi}\frac{1}{a^2\left(q^2+\frac{2h}{\rho}\right)-b^2}-\frac{4d^2\sin\frac{n\pi}{l}z}{\pi}\frac{e^{-a^2\left(\frac{\pi^2}{l^2}+\frac{2h}{\rho}\right)t+b^2t}}{a^2\left(\frac{\pi^2}{l^2}+\frac{2h}{\rho}\right)-b^2}$$

$$+\frac{4\pi a^2}{2l^2}\sin\frac{\pi}{l}z\frac{d^2t^2}{\left(1+\frac{\epsilon_1}{a_1}\frac{a}{\epsilon}\right)}.$$

The last terms of the right hand disappear, if t is very small.

(86)
$$T_m=\sum \frac{4d^2\sin\frac{n\pi}{l}z}{n\pi}\frac{1}{a^2\left(q^2+\frac{2h}{\rho}\right)-b^2}-\frac{4d^2\sin\frac{n\pi}{l}z}{n}\frac{e^{-a^2\left(\frac{\pi^2}{l^2}+\frac{2h}{\rho}\right)t+b^2t}}{a^2\left(\frac{\pi^2}{l^2}+\frac{2h}{\rho}\right)-b^2}.$$

For the first approximation, we have for the ordinary l.

(87)
$$T_m=\frac{4d^2}{\pi}\frac{1-e^{-a^2\left(\frac{\pi^2}{l^2}+\frac{2h}{\rho}\right)t+b^2t}}{a^2\left(\frac{\pi^2}{l^2}+\frac{2h}{\rho}\right)-b^2}$$

For infinitely long l,

$$T_m=\frac{d^2}{a^2\frac{2h}{\rho}-b^2}\sum_{n=1}^{\infty}\frac{4}{\pi}\frac{\sin n\pi}{n}\left(1-e^{-a^2\frac{2h}{\rho}t+b^2t}\right)$$

From Fourier's series it follows $\sum_{n=1}^{\infty}\frac{4}{\pi}\frac{\sin n\pi}{n}=1$

(88) $$T_m=\frac{d^2}{a^2\left(\frac{2h}{\rho}-\frac{b^2}{a^2}\right)}\left(1-e^{-a^2\frac{2h}{\rho}t+b^2t}\right)$$

(89) $$=\frac{r_{f0}I^2}{\epsilon\pi^2\rho^3}\frac{1}{2h-\alpha\frac{r_{f0}I^2}{\epsilon\pi^2\rho^3}}\left(1-e^{-a^2\frac{2h}{\rho}t+b^2t}\right)$$

(90) $$T_m=\sqrt{\frac{hT_m\epsilon}{(1+\alpha T_m)r_{f0}}}\sqrt{2\pi\rho\cdot\pi\rho^2}.$$

(91) $$t=\frac{1}{b^2}\frac{1}{1-\frac{a^2}{b^2}\left(\frac{\pi^2}{l^2}+\frac{2h}{\rho}\right)}\log\left\{1-\frac{\pi}{4}\frac{b^2}{d^2}\left[1+\frac{a^2}{b^2}\left(\frac{\pi^2}{l^2}+\frac{2h}{\rho}\right)\right]T_m\right\}$$

Next, if $\sqrt{\alpha t}$ is large
from equation (91)

$$\psi(t)=\frac{d^2}{\alpha}\left(\frac{\sqrt{\alpha}\,\epsilon}{a}\middle/\frac{\sqrt{\beta}\,\epsilon_1}{a_1}\right)$$

Therefore,

$$\int_0^t\psi(t)\,e^{\left\{b^2-a^2\left(q^2+\frac{2h}{\rho}\right)\right\}(t-\tau)}d\tau$$

(92) $$=\frac{d^2}{\alpha}\left(\frac{\sqrt{\alpha}\,\epsilon}{a}\middle/\frac{\sqrt{\beta}\,\epsilon_1}{a_1}\right)\frac{e^{\left\{b^2-a^2\left(q^2+\frac{2h}{\rho}\right)\right\}t}-1}{b^2-a^2\left(q^2+\frac{2h}{\rho}\right)}$$

From equation (48)

$$T_m=d^2\left[\frac{4\sin\frac{\pi}{l}z}{\pi}+\frac{4\pi a^2}{l^2}\frac{1}{\alpha}\left(\frac{\sqrt{\alpha}\,\epsilon}{a}\middle/\frac{\sqrt{\beta}\,\epsilon_1}{a_1}\right)\right]\frac{1-e^{\left\{b^2-a^2\left(q^2+\frac{2h}{\rho}\right)\right\}t}}{a^2\left(\frac{\pi^2}{l^2}+\frac{2h}{\rho}-\frac{b^2}{a^2}\right)}$$

Finally the time necessary for reaching T_m.

(93)

$$t=\frac{1}{b^2-a^2\left(\frac{\pi^2}{l^2}+\frac{2h}{\rho}\right)}\log\left\{1-\frac{T_m a^2\left(\frac{\pi^2}{l^2}+\frac{2h}{\rho}-\frac{b_2}{a^2}\right)}{d^2\left[\frac{4\sin\frac{\pi}{l}z}{\pi}+\frac{4\pi a^2}{l^2}\frac{1}{\alpha}\left(\frac{\sqrt{\alpha}\,\epsilon}{a}\Big/\frac{\sqrt{\beta}\,\epsilon_1}{a_1}\right)\right]}\right\}$$

(94)

$$T_m=d^2\left[\frac{4\sin\frac{\pi}{l}z}{\pi}+\frac{4\pi a^2}{l^2}\frac{1}{\alpha}\left(\frac{\sqrt{\alpha}\,\epsilon}{a}\Big/\frac{\sqrt{\beta}\,\epsilon_1}{a_1}\right)\right]\frac{1}{a^2\left(\frac{\pi^2}{l^2}+\frac{2h}{\rho}-\frac{b^2}{a^2}\right)}.$$

For the strip, the differential equation is

$$a^2\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+d^2+b^2T=\frac{\partial T}{\partial t} \tag{95}$$

$$\left\{\begin{array}{lll}\frac{\partial T}{\partial x}+hT=0 & \text{at } x=0 & \text{and } x=k\\ \frac{\partial T}{\partial y}+hT=0 & \text{at } y=0 & \text{and } y=\delta\\ T=0 & \text{at } t=0 & \end{array}\right. \tag{96}$$

$$\epsilon\frac{\partial T}{\partial z}=-\epsilon_1\frac{\partial T}{\partial z} \quad \text{at } z=0 \quad \text{and } z=l \tag{97}$$

$$T=\psi(t, x, y) \quad \text{at } z=0 \quad \text{and } z=l \tag{98}$$

In the same way as the above case we can expand $u(x, y, z, t)$ (t as a parameter) by a series of trigonometric functions, in which each term satisfies the condition

$$\left.\begin{array}{l}\frac{\partial u_n}{\partial x}+hu_n=0\\ \frac{\partial u_n}{\partial y}+hu_n=0\\ \frac{\partial u_n}{\partial z}=0\end{array}\right\} \text{at the boundary} \tag{99}$$

(100) $u(x, y, z)$

$$=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\frac{4(\alpha_m\cos\alpha_m x+h\sin\alpha_m x)(\beta_n\cos\beta_n y+h\sin\beta_n y)\frac{2}{l}\sin\frac{p\pi}{l}z}{\{\alpha_m^2+h^2)k+2h\}\{\beta_n^2+h^2)\delta+2h\}}$$

$$\times\int_0^l\int_0^\delta\int_0^k(\alpha_m\cos\alpha_m x+h\sin\alpha_m x)(\beta_n\cos\beta_n y+h\sin\beta_n y)\sin\frac{p\pi}{l}z.$$

$$u(x, y, z)dxdydz\ .$$

where α_m, β_n are the roots of

(101) $$\tan\alpha_m k=\frac{2\alpha_m h}{\alpha_m^2-h^2},\quad \tan\beta_n\delta=\frac{2\beta_n h}{\beta_n^2-h^2}.$$

Again we calculate the following equation

(102)

$$\frac{\partial^2\left(T+\frac{d^2}{b^2}\right)}{\partial x^2}+\frac{\partial^2\left(T+\frac{d^2}{b^2}\right)}{\partial y^2}+\frac{\partial^2\left(T+\frac{d^2}{b^2}\right)}{\partial z^2}$$

$$=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\frac{4(\alpha_m\cos\alpha_m x+h\sin\alpha_m x)(\beta_n\cos\beta_n y+h\sin\beta_n y)\frac{2}{l}\sin\frac{p\pi}{l}z.}{\{(\alpha_m^2+h^2)k+2h\}\{(\beta_n^2+h^2)\delta+2h\}}$$

$$\times\left\{\int_0^k\frac{\partial^2 u}{\partial x^2}(\alpha_m\cos\alpha_m x+h\sin\alpha_m x)dx\int_0^l\int_0^\delta(\beta_n\cos\beta_n y+h\sin\beta_n y)\sin\frac{p\pi}{l}z.dydz.\right.$$

$$+\int_0^\delta\frac{\partial^2 u}{\partial y^2}(\beta_n\cos\beta_n y+h\sin\beta_n y)dy\int_0^l\int_0^k(\alpha_m\cos\alpha_m x+h\sin\alpha_m x)\sin\frac{p\pi}{l}z.dx.dz\ .$$

$$\left.+\int_0^l\frac{\partial^2 u}{\partial z^2}\sin\frac{p\pi}{l}z.\,dz\int_0^\delta\int_0^k(\alpha_m\cos\alpha_m x+h\sin\alpha_m x)(\beta_n\cos\beta_n y+\sin\beta_n y)dxdy\right\}.$$

Consider the integral,

$$\alpha_m\int_0^k\frac{\partial^2 u}{\partial x^2}\cos\alpha_m x dx$$

$$=\alpha_m\left|\frac{\partial u}{\partial x}\cos\alpha_m x\right|_0^k+\alpha_m^2\int_0^k\frac{\partial u}{\partial x}\sin\alpha_m x dx\ .$$

$$=a_m \left| \frac{\partial u}{\partial x} \cos a_m x \right|_0^k + a_m^2 \left| u \sin a_m x \right|_0^k - a_m^3 \int_0^k u \cos a_m x dx \,.$$

and

$$h \int_0^k \frac{\partial^2 u}{\partial x^2} \sin a_m x dx$$

$$= h \left| \frac{\partial u}{\partial x} \sin a_m x \right|_0^k - h a_m \int_0^k \frac{\partial u}{\partial x} \cos a_m x dx \,.$$

$$= h \left| \frac{\partial u}{\partial x} \sin a_m x \right|_0^k - h a_m \left| u \cos a_m x \right|_0^k - h a_m^2 \int_0^k u \sin a_m x dx \,.$$

Thus

$$\int_0^k \frac{\partial^2 u}{\partial x^2} (a_m \cos a_m x + h \sin a_m x) dx \tag{103}$$

$$= \left| a_m h \cos a_m k + a_m^2 u \sin a_m k + h \frac{\partial u}{\partial x} \sin a_m k - h a_m u \cos a_m k \right|_k$$

$$+ \left| -a_m \frac{\partial u}{\partial x} + h a_m u \right|_0 - a_m^2 \int_0^k u (a_m \cos a_m x + h \sin a_m x) dx \,.$$

From the boundary conditions, we have

$$\begin{cases} \left| -\frac{\partial u}{\partial x} + hu \right|_0 = h \frac{d^2}{b^2} \\ \left| \frac{\partial u}{\partial x} + hu \right|_k = h \frac{d^2}{b^2} \\ (a_m^2 - h^2) \sin a_m k = 2 a_m h \cos a_m k \,. \end{cases} \tag{104}$$

Therefore,

$$= a_m h \frac{d^2}{b^2} \left(\cos a_n k + \frac{h \sin a_m k}{a_m} + 1 \right) - a_m^2 \int_0^k u (a_m \cos a_m x + h \sin a_m x) dx$$

$$=\alpha_m h\frac{d^2}{b^2}2\cos\frac{\alpha_m k}{2}\left(\cos\frac{\alpha_m k}{2}+\frac{h}{\alpha_m}\sin\frac{\alpha_m k}{2}\right)-\alpha_m{}^2\int_0^k u(\alpha_m\cos\alpha_m x$$

$$+h\sin\alpha_m x)dx\,.$$

$$=\alpha_m h\frac{d^2}{b^2}2\cos^2\frac{\alpha_m k}{2}\left(1+\frac{h^2}{\alpha_m{}^2}\right)-\alpha_m{}^2\int_0^k u(\alpha_m\cos\alpha_m x+h\sin\alpha_m x)dx\,.$$

$$=\alpha_m h\frac{d^2}{b^2}2\cos^2\frac{\alpha_m k}{2}\left(1+\tan^2\frac{\alpha_m k}{2}\right)-\alpha_m{}^2\int_0^k u(\alpha_m\cos\alpha_m x+h\sin\alpha_m x)dx$$

$$(105)\qquad =2\alpha_m h\frac{d^2}{b^2}-\alpha_m{}^2\int_0^k u(\alpha_m\cos\alpha_m x+h\sin\alpha_m x)dx\,.$$

$$\int_0^k(\alpha\cos\alpha x+h\sin\alpha x)dx$$

$$=\left|\sin\alpha x\right|_0^k-\frac{h}{\alpha}\left|\cos\alpha x\right|_0^k$$

$$=\sin\alpha k-\frac{h}{\alpha}(\cos\alpha k-1)$$

$$=2\sin\frac{\alpha k}{2}\left(\cos\frac{\alpha k}{2}+\frac{h}{\alpha}\sin\frac{\alpha k}{2}\right)$$

$$=2\sin^2\frac{\alpha k}{2}\left(\frac{\alpha}{h}+\frac{h}{\alpha}\right)$$

$$=2\frac{h}{\alpha}\sin^2\frac{\alpha k}{2}\left(1+\frac{\alpha^2}{h^2}\right)$$

$$(106)\qquad =2\frac{h}{\alpha}$$

Similarly it follows for β.

$$(107)\quad \int_0^l \frac{\partial^2 u}{\partial z^2}\sin\frac{p\pi}{l}z\cdot dz$$

$$=\left[\frac{\partial u}{\partial z}\sin\frac{p\pi z}{l}\right]_0^l - p\pi\int_0^l \frac{\partial u}{\partial z}\cos\frac{p\pi}{l}z dz$$

$$=-p\pi u\left\{(-1)^p-1\right\}u(0)-\int_0^l u\sin\frac{p\pi}{l}z\cdot dz\,.$$

and

$$(108)\qquad u(0)=u(l)$$

$$=-\frac{d^2}{b^2}-\psi(x,\,y,\,t)$$

By putting

$$(109)\quad \psi(t)=\frac{\int_0^\delta\int_0^k(\alpha\cos\alpha x+h\sin\alpha x)(\beta\cos\beta y+h\sin\beta y)\psi(\tau,\,x,\,y)\,dxdy}{\int_0^\delta\int_0^k(\alpha\cos\alpha x+h\sin\alpha x)(\beta\cos\beta y+h\sin\beta y)dxdy},$$

We obtain the relation,

$$(110)\quad \varDelta u=\sum_{m=1}^\infty\sum_{n=1}^\infty\sum_{p=1}^\infty\frac{4(\alpha\cos\alpha x+h\sin\alpha x)(\beta\cos\beta y+h\sin\beta y)\frac{2}{l}\sin\frac{p\pi}{l}z}{\{(\alpha^2+h^2)k+2h\}\{(\beta^2+h^2)\delta+2h\}}$$

$$\times\left\{8h^2\frac{d^2}{b^2}(\alpha^2+\beta^2+q^2)\frac{1}{\alpha_m\beta_n q}+8\frac{h^2 q}{\alpha_m\beta_n}\psi(\tau)\right.$$

$$-(\alpha^2+\beta^2+q^2)\int_0^l\int_0^\delta\int_0^k(\alpha\cos\alpha x+h\sin\alpha x)(\beta\cos\beta y+h\sin\beta y)$$

$$\left.\times\sin qz\cdot u\cdot dx\,dy\,dz\right\}$$

Also we have

$$=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\frac{4(\alpha\cos\alpha x+h\sin\alpha x)(\beta\cos\beta y+h\sin\beta y)\frac{2}{l}\sin qz.}{\{(\alpha^2+h^2)k+2\}\{(\beta^2+h^2)\delta+2h\}}$$

$$\times\int_0^l\int_0^\delta\int_0^k(\alpha\cos\alpha x+h\sin\alpha x)(\beta\cos\beta y+h\sin\beta y)\sin qz.$$

$$\left(\frac{\partial u}{\partial t}\frac{1}{a^2}-\frac{b^2}{a^2}u\right)dxdydz$$

(111) Putting

$$A=\int_0^l\int_0^\delta\int_0^k(\alpha\cos\alpha x+h\sin\alpha x)(\beta\cos\beta y+h\sin\beta y)\sin qz.\,udxdydz\ ,$$

we have the linear differential equation:

$$\text{(112)}\quad \frac{d}{dt}A-Aa^2\left(\frac{b^2}{a^2}-\alpha_m{}^2-\beta_n{}^2-q^2\right)$$

$$=8h^2\frac{a^2d^2}{b^2}(\alpha_m{}^2+\beta_n{}^2+q^2)\frac{1}{\alpha_m\beta_n q}+4\frac{h^2qa^2}{\alpha_m\beta_n}\psi(t)\ .$$

Solving this equation, we have:

$$\text{(113)}\quad A=\int_0^t e^{-a^2\left(\alpha_m{}^2+\beta_n{}^2+q^2-\frac{b^2}{a^2}\right)(t-\tau)}\left\{8\frac{d^2a^2}{b^2}(\alpha^2+\beta^2+q^2)\frac{h^2}{\alpha\beta q}\right.$$

$$\left.+8\frac{h^2qa^2}{\alpha\beta}\psi(\tau)\right\}d\tau$$

$$=\frac{8\frac{d^2}{b^2}a^2\left(\alpha^2+\beta^2+q^2-\frac{b^2}{a^2}+\frac{b^2}{a^2}\right)}{a^2\left(\alpha^2+\beta^2+q^2-\frac{b^2}{a^2}\right)}\frac{h^2}{\alpha\beta q}\left(1-e^{-a^2\left(\alpha^2+\beta^2+q^2-\frac{b^2}{a^2}\right)t}\right)$$

$$+4\frac{h^2qa^2}{\alpha\beta}\int_0^t e^{-a^2\left(\alpha^2+\beta^2+q^2-\frac{b^2}{a^2}\right)(t-\tau)}\psi(\tau)d\tau+Ce^{-a^2\left(\alpha^2+\beta^2+q^2-\frac{b^2}{a^2}\right)t},$$

(114)

$$T=-\frac{d^2}{b^2}+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\frac{4(\alpha_m\cos\alpha_m x+h\sin\alpha_m x)(\beta_n\cos\beta_n y+h\sin\beta_n y)\frac{2}{l}\sin qz}{\{(\alpha_m^2+h^2)k+2h\}\{(\beta_n^2+h^2)\delta+2h\}}$$

$$\times\left\{\frac{8\frac{d^2}{b^2}a^2\left(\alpha^2+\beta^2+q^2-\frac{b^2}{a^2}+\frac{b^2}{a^2}\right)}{a^2\left(\alpha^2+\beta^2+q^2-\frac{b^2}{a^2}\right)}\frac{h^2}{\alpha\beta q}\left(1-e^{-a^2\left(\alpha^2+\beta^2+q^2-\frac{b^2}{a^2}\right)t}\right)\right.$$

$$+8\frac{h^2qa^2}{\alpha\beta}\int_0^t e^{-a^2\left(\alpha^2+\beta^2+q^2-\frac{b^2}{a^2}\right)(t-\tau)}\psi(\tau)d\tau$$

$$\left.+Ce^{-a^2\left(\alpha^2+\beta^2+q^2-\frac{b^2}{a^2}\right)t}\right\}.$$

The arbitary constant C is to be determined so as to vanish at $t=0$, and $\frac{d^2}{b^2}$ can be expanded by the series of (100), to get

(115)

$$=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\frac{4(\alpha\cos\alpha x+h\sin\alpha x)(\beta\cos\beta y+h\sin\beta y)\frac{2}{l}\sin qz}{\{(\alpha^2+h^2)k+2h\}\{(\beta^2+h^2)\delta+2h\}}$$

$$\times\left\{\frac{8\frac{d^2}{a^2}h^2\left(1-e^{-a^2\left(\alpha^2+\beta^2+q^2-\frac{b^2}{a^2}\right)t}\right)}{\left(\alpha^2+\beta^2+q^2-\frac{b^2}{a^2}\right)\alpha\beta q}\right.$$

$$\left.+8\frac{h^2qa^2}{\alpha\beta}\int_0^t e^{-a^2\left(\alpha^2+\beta^2+q^2-\frac{b^2}{a^2}\right)(t-\tau)}\psi(\tau)\,d\tau\right\}.$$

We will calculate the mean temperature by the formula :

(116)

$$T_m=\frac{\int_0^\delta\int_0^k Tdxdy}{\int_0^\delta\int_0^k dxdy}.$$

Again we neglect the terms α_m, β_n $(m>2, n>2)$, as the terms containing them are very small compared with the first term;

$$T_m=\sum_{p=1}^{\infty}\frac{4\times 2\frac{h}{\alpha}\times 2\frac{h}{\beta}\frac{2}{l}\sin qz.\frac{1}{k\delta}\frac{1}{\alpha\beta}}{\{(\alpha^2+h^2)k+2h\}\{(\beta^2+h^2)\delta+2h\}}$$

$$\times\left\{\frac{8\frac{d^2}{a^2}h^2\left(1-e^{-a^2\left(\alpha^2+\beta^2+q^2-\frac{b^2}{a^2}\right)t}\right)}{\left(\alpha^2+\beta^2+q^2-\frac{b^2}{a^2}\right)q}\right.$$

$$\left.+8\,h^2 x a^2\int_0^t e^{-a^2\left(\alpha^2+\beta^2+q^2-\frac{b^2}{a^2}\right)(t-\tau)}\psi(\tau)\,d\tau\right\}.$$

$$=\sum_{p=1}^{\infty}\frac{4\frac{2}{l}\sin qz\times 8h^2}{h^2\{kl+4\}\{h\delta+4\}}\left\{\frac{\frac{d^2}{a^2}\left(1-e^{-a^2\left(\alpha^2+\beta^2+q^2-\frac{b^2}{a^2}\right)t}\right)}{\alpha^2+\beta^2+q^2-\frac{b^2}{a^2}}\right.$$

$$\left.+qa^2\int_0^t e^{-a^2\left(\alpha^2+\beta^2+q^2-\frac{b^2}{a^2}\right)(t-\tau)}\psi(\tau)\,d\tau\right\}$$

$$(117)\quad =\sum_{p=1}^{\infty}\frac{4}{l}\sin qz\left\{\frac{\frac{d^2}{a^2}\left(1-e^{-a^2\left(\alpha^2+\beta^2+q^2-\frac{b^2}{a^2}\right)t}\right)}{\left(\alpha^2+\beta^2+q^2-\frac{b^2}{a^2}\right)q}\right.$$

$$\left.+qa^2\int_0^t e^{-a^2\left(\alpha^2+\beta^2+q^2-\frac{b^2}{a^2}\right)(t-\tau)}\psi(\tau)\,d\tau\right\}.$$

Since k and δ are small,

$$(118)\qquad \alpha^2=\frac{2h}{k}\,,\qquad \beta=\frac{2h}{\delta}$$

(119) $$\alpha^2+\beta^2 = 2h\left(\frac{1}{k}+\frac{1}{\delta}\right)$$

$$= h.\frac{\text{periphery of the cross section}}{\text{cross section}}$$

Compare this result with the case of wire, and substituting

$$\alpha^2+\beta^2 \quad \text{for} \quad \frac{\lambda^2}{\rho^2}\,\frac{(h.\ \text{periphery of the cylindrical wire})}{\text{cross section}},$$

it is easily seen that these two formula reduce to identical forms, therefore, we can obtain the following result in perfectly similar way.

For the infinitely long l,

(120) $$T_m = \frac{d^2}{a^2\left\{2h\left(\frac{1}{k}+\frac{1}{\delta}\right)-\frac{b^2}{a^2}\right\}},$$

(121) $$I_m = \sqrt{\frac{hT_m\epsilon}{(1+\alpha T_m)r_{f0}}}\sqrt{2(k+\delta)k\delta}$$

For the first approximation, the time necessary for reaching the mean temperature T_m.

(122) $$t=\frac{1}{b^2}\,\frac{1}{1-\frac{a^2}{b^2}\left(\frac{\pi^2}{l^2}+2h\left(\frac{1}{k}+\frac{1}{\delta}\right)\right)}\log\left\{1+\frac{\pi}{4}\frac{b^2}{a^2}\left\{1-\frac{a^2}{b^2}\left(\frac{\pi^2}{l^2}+2h\left(\frac{1}{k}+\frac{1}{\delta}\right)\right)\right\}T_m\right\}.$$