Cooling Problem and Conformal Representation.

By

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The differential equation of heat conduction is well known and written in the form

$$(1) \qquad \frac{\partial T}{\partial t} = a^2\left\{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right\}$$

where T is temperature, t time and a^2 heat conductivity / specific heat × density. If T is in a stationary state, it follows

$$\frac{\partial T}{\partial t} = 0 .$$

Besides, if T is two dimensional, the differential equation reduces to the form

$$(2) \qquad \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 .$$

Any regular analytic function is satisfied by this differential equation. Now if an analytic function

$$(3) \qquad Z = X + iY = f(x+iy)$$

is given, the loci of $X =$ const. and $Y =$ const. can be drawn easily by the conformal representation by means of the function.

Therefore one can consider the locus of $X =$ const. as the isothermal line which satisfies the differential equation (2). Therefore if the isothermal lines are required to be drawn in a domain, one must find a function (3) $Z = X + iY = f(x+iy)$.

I) If the boundary condition is $T =$ const. or $T = 0$ at every point of the boundary, the isothermal lines can be easily drawn by a function $f(x+iy)$.

II) Next if the boundary condition is $T =$ const. or $T = 0$ in a part of the boundary and $\frac{\partial T}{\partial n} = 0$ in the other part of the boundary, the isothermal lines can be obtained by a function $f(x+iy)$ which satisfies the condition. The above two conditions can be satisfied without much labour, so long as the boundary is simple.

III) But if the third condition

$$\frac{\partial T}{\partial n}+hT=0\,, \tag{4}$$

where n is normal of the boundary and h is constant, must be satisfied; the function $f(x+iy)$ which satisfies this condition can not be obtained easily. However, this is just the case required most frequently in practice.

Now denote the difference of the temperature between two consecutive isothermal lines by ΔT, then from (4)

$$\frac{\Delta T}{\Delta n}=-hT\,. \tag{5}$$

Take $-\Delta T$ as unit temperature difference, it follows from (5) that the length of the normal Δn which produces the temperature difference ΔT at the two ends of the normal Δn, is inversely proportional to T.

Therefore if a boundary curve is given, the length of the normal at the boundary must be

$$\Delta n=-\frac{\Delta T}{hT}\,. \tag{6}$$

Although the original problem is such that the boundary is given, and the distribution of temperature T which satisfies the condition (6) is required, it is too hard to be solved directly, and it may be preferable to solve the inverse problem; the isothermal lines being given by some conformal representation in a domain, the boundary line is sought so that condition (6) is satisfied. In this case it is not so hard to be solved in such an exactness as obtained by the graphical method; Δn has the length of the straight line between the two consecutive isothermal lines and it is perpendicular to the boundary.

Therefore if T_0 and Δn_0 are given at a point of the boundary, one obtains from (6)

$$\frac{\Delta n}{\Delta n_0}=\frac{T_0}{T}\,. \tag{7}$$

Consequently the boundary line can be drawn so that the normal Δn at the points where the boundary intersects the isothermal lines, has the length given by (6) or (7), the boundary thus obtained satisfies desired condition (4) and the isothermal lines are the ones sought.

Therefore if many conformal representations are known by means of such functions that satisfy condition (7), one may obtain the boundary similar to the one given at first, in such an exactness as obtained by the graphical method.

For example, take the conformal representation by means of the functions

$$Z = \cosh^{-1} \frac{1+\sqrt{\frac{z}{z-1}}}{1-\sqrt{\frac{z}{z-1}}} + \cos^{-1} \frac{1-\sqrt{\frac{z}{z-1}}}{1+\sqrt{\frac{z}{z-1}}}$$

and

$$\Psi = T + i\Phi = \log z \, .$$

As this conformal representation has been already published in this Memoir Vol. 4, No. 1.

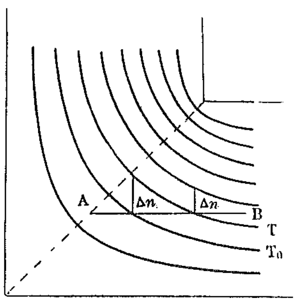


Fig. 1.

The locus of $T=$ const. is shown by the black line in Fig. 1.

Let the straight line AB be the new boundary where the boundary condition

$$\frac{\partial T}{\partial n} + hT = 0$$

must be satisfied.

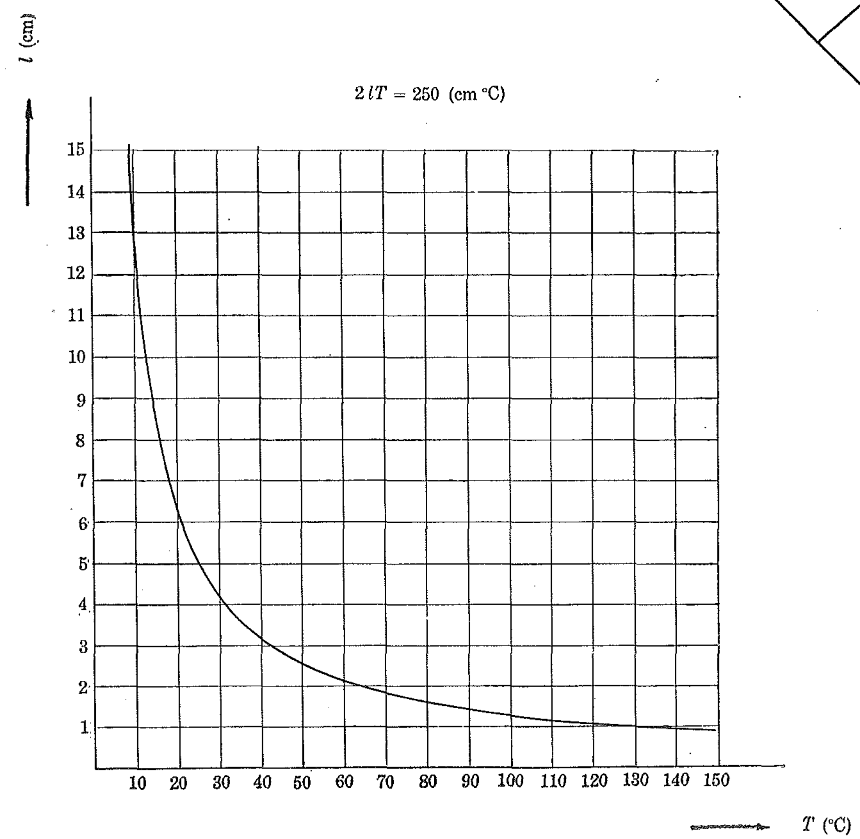
If one draws the normals to this boundary from the points where the boundary intersects the curves of $T=$ const. one may find such a curve that the length of normal Δn between the two consecutive lines of $T=$ const. is inversely proportional to T.

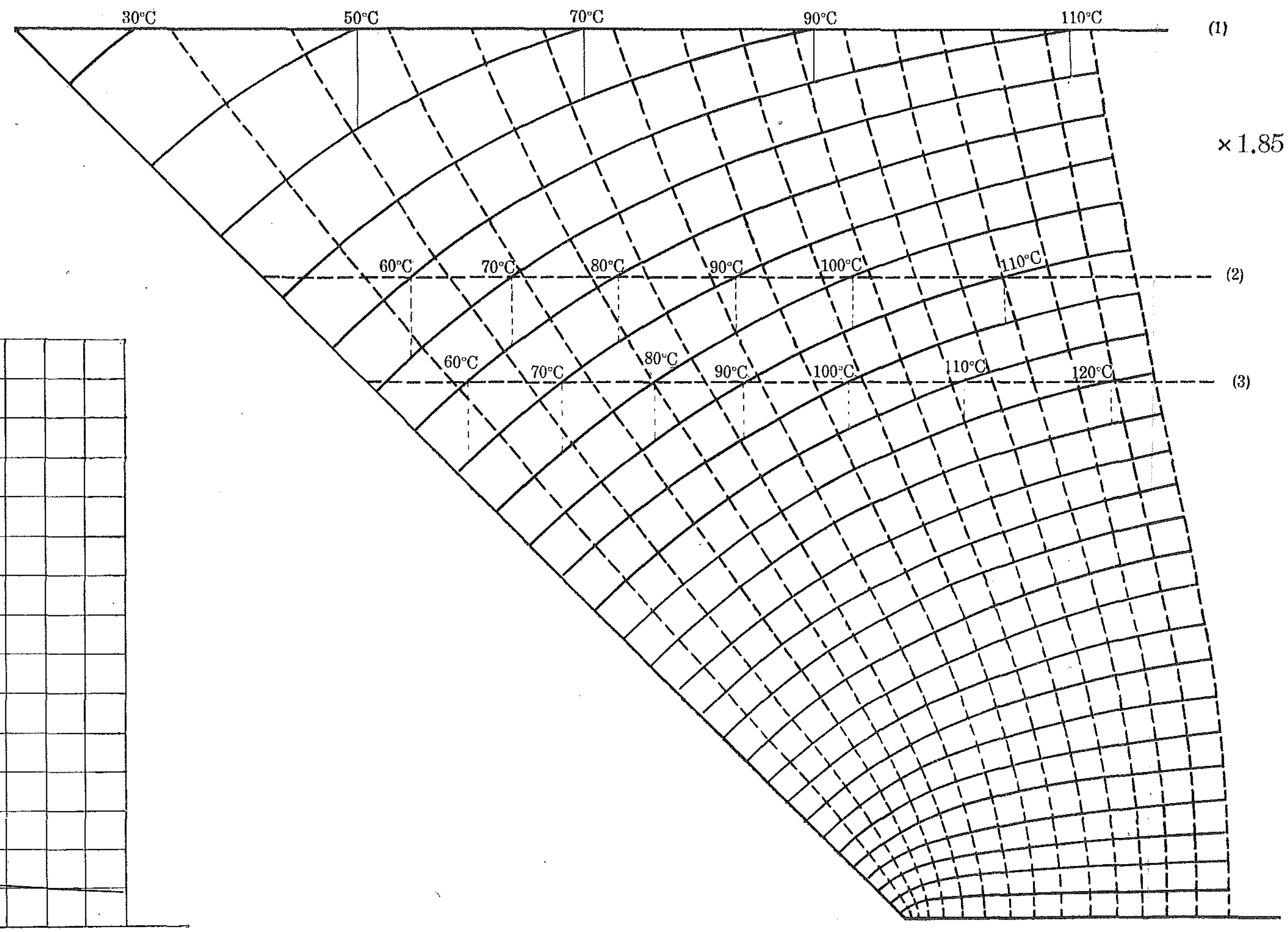
Indeed if $h = \frac{4}{25}$ [1/cm] and $\Delta T = 20°C$, the straight line (1) shown in Pl. I is the new boundary and if $h = \frac{2}{25}$ [1/cm] and $\Delta T = 10°C$, the straight lines (2) and (3) are the other boundaries and the whole lines show the isothermal lines which satisfy the boundary condition (4) or (6) in such an exactness as obtained by the graphical method.

Again Pl. II is the conformal representation obtained by the function $Z = f\,(E)$, where $f(E)$ is obtained by the two substitutions $Z = \sin^{-1} z$ and the elliptic integral of the 2nd kind ;

$$E(z) = \int_0^z \sqrt{\frac{1-k^2t}{1-t^2}}\, dt \, .$$

If $h = \frac{1}{500}$ [1/cm], $T = 200°C$ and $\varDelta T = 1.14°C$, then the temperature distribution which satisfies the boundary condition (4) or (6) is drawn as shown in Pl. III, the datum line being at the bottom.





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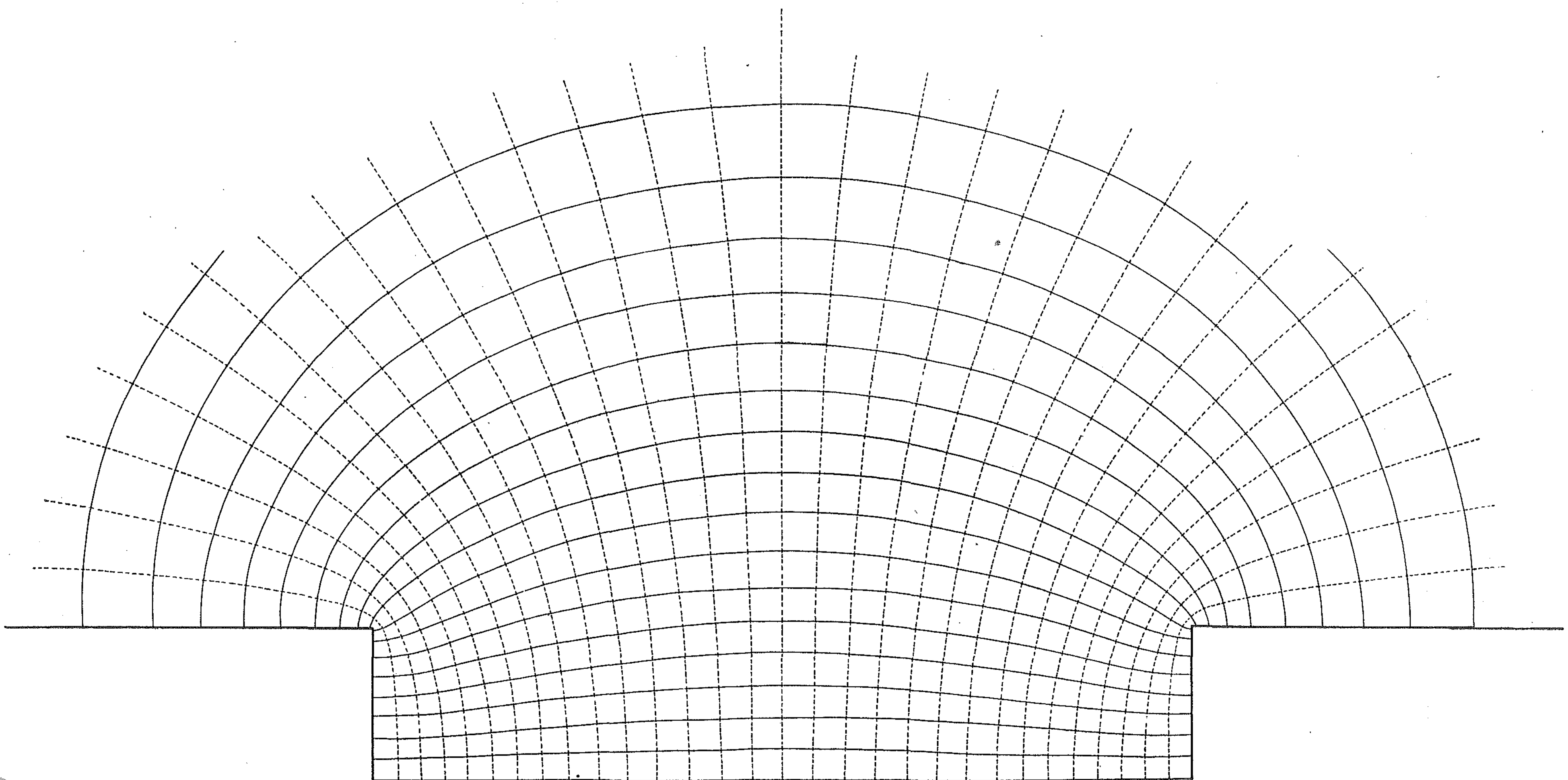

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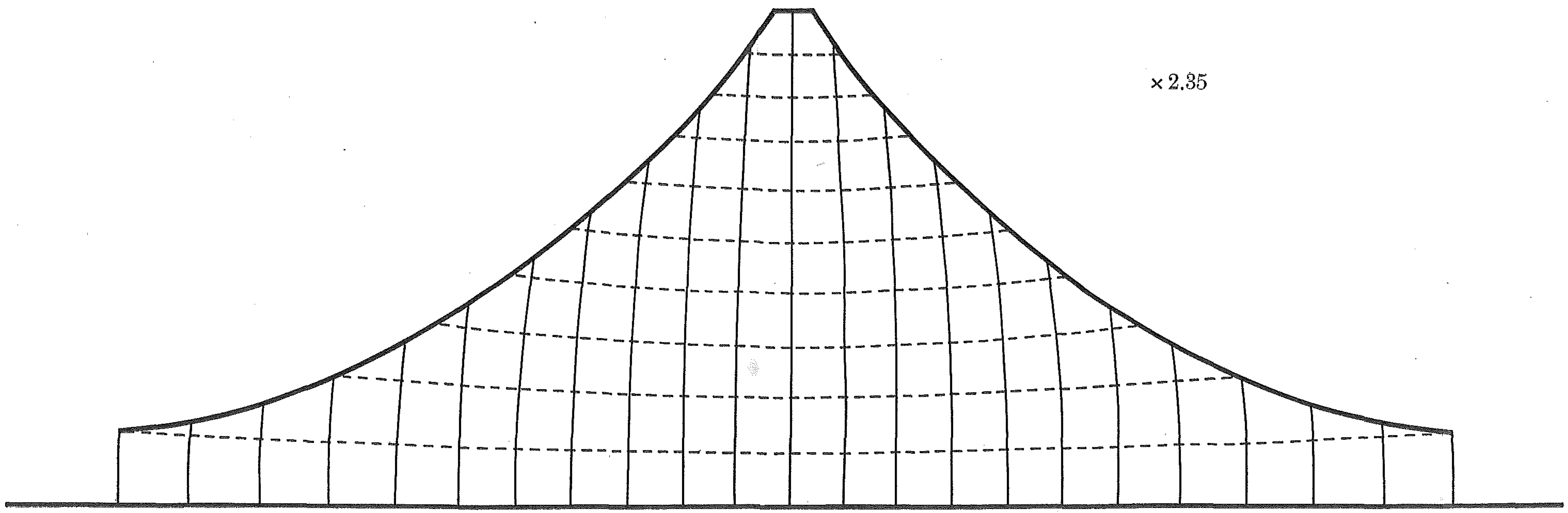

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