Parallel Stream Disturbed by Barriers and Gates.

(Continued)

By

Y. IKEDA, S. KOBAYASI and T. UEKI.

In this article it is proposed to draw the streamlines in the domains, which have the following boundaries

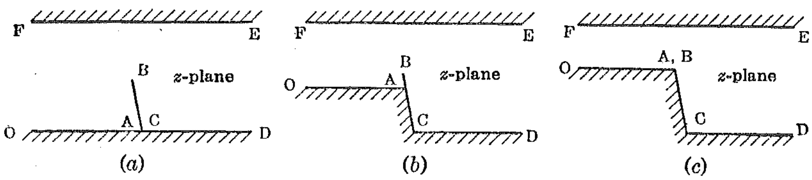


Fig. 1.

where the angle $\angle OAB = 80^\circ$

Each domain can be transformed into a half-infinite plane shown in Fig. 2 by Schwarz-transformation

$$(1) \qquad Z = \int_0^z \frac{(a-z)}{z}(1-z)^{-\frac{5}{9}}(b-z)^{-\frac{4}{9}}dz\ .$$

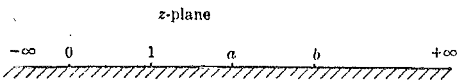


Fig. 2.

If the variable z is changed in t, so that

$$(2) \qquad t^9 = \frac{b-z}{1-z} \qquad \text{or} \qquad z = \frac{b-t^9}{1-t^9}\ ,$$

the domain in z-plane is transformed conformally into the following domain

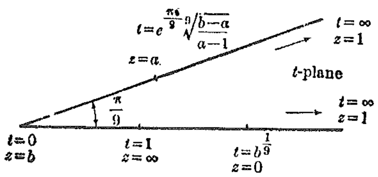


Fig. 3.

Replacing z by t in the integral (1), one obtains

$$Z=9\int_0^t \frac{t^4}{1-t^9}\,dt-9a\int_0^t \frac{t^4}{b-t^9}\,dt\,,$$

or

$$(3)\qquad Z=9\int_0^t \frac{t^4}{1-t^9}\,dt-9\frac{a}{b^{\frac{4}{9}}}\int_0^{\frac{t}{b^{\frac{1}{9}}}} \frac{t^4}{1-t^9}\,dt\,.$$

Now put

$$(4)\qquad f(t)=9\int_0^t \frac{t^4}{1-t^9}\,dt\,,$$

then

$$(5)\qquad Z=f(t)-\frac{a}{b^{\frac{4}{9}}}f\left(\frac{t}{b^{\frac{1}{9}}}\right).$$

By the partial fraction

$$(6)\qquad \frac{9t^4}{1-t^9}=\frac{\alpha_0^{-4}}{\alpha_0-t}+\frac{\alpha_1^{-4}}{\alpha_1-t}+\frac{\alpha_2^{-4}}{\alpha_2-t}+\cdots\cdots+\frac{\alpha_8^{-4}}{\alpha_8-t}\,,$$

where $\alpha_0=1,\quad \alpha_1=e^{i\frac{2\pi}{9}\times 1},\quad \cdots\cdots,\quad$ and $\quad \alpha_8=e^{i\frac{2\pi}{9}\times 8}$,

the integration (3) can be easily performed.

By putting

$$(7)\qquad \begin{aligned}\varphi(t)=-\log(1-t)&+\cos\frac{\pi}{9}\log(e^{\frac{2\pi i}{9}}-t)(e^{-\frac{2\pi i}{9}}-t)\\ &-\cos\frac{2\pi}{9}\log(e^{\frac{4\pi i}{9}}-t)(e^{-\frac{4\pi i}{9}}-t)\\ &+\cos\frac{3\pi}{9}\log(e^{\frac{6\pi i}{9}}-t)(e^{-\frac{6\pi i}{9}}-t)\\ &-\cos\frac{4\pi}{9}\log(e^{\frac{8\pi i}{9}}-t)(e^{-\frac{8\pi i}{9}}-t)\\ &+i\sin\frac{\pi}{9}\log\left(\frac{e^{\frac{2\pi i}{9}}-t}{e^{-\frac{2\pi i}{9}}-t}\right)-i\sin\frac{2\pi}{9}\log\left(\frac{e^{\frac{4\pi i}{9}}-t}{e^{-\frac{4\pi i}{9}}-t}\right)\\ &+i\sin\frac{3\pi}{9}\log\left(\frac{e^{\frac{6\pi i}{9}}-t}{e^{-\frac{6\pi i}{9}}-t}\right)-i\sin\frac{4\pi}{9}\log\left(\frac{e^{\frac{8\pi i}{9}}-t}{e^{-\frac{8\pi i}{9}}-t}\right),\end{aligned}$$

one obtains

$$(8)\qquad f(t)=\varphi(t)-\varphi(0)\ ,$$

where

$$(9)\qquad -\varphi(0)=\frac{4\pi}{9}\sin\frac{\pi}{9}-\frac{8\pi}{9}\sin\frac{2\pi}{9}+\frac{12\pi}{9}\sin\frac{3\pi}{9}-\frac{16\pi}{9}\sin\frac{4\pi}{9}\ .$$

If t lies on the boundary line $t=e^{\frac{\pi i}{9}}\alpha$ ($\alpha=$ real), the integral (4) becomes

$$(10)\qquad f(\alpha e^{\frac{\pi i}{9}})=e^{\frac{5\pi i}{9}}\int_0^\alpha \frac{9\alpha^4}{1+\alpha^9}\,d\alpha\ .$$

Though this integral can be evaluated directly from (7) and (8), it is not easy to determine the arguments of the complex variables.

Therefore it is preferable to put $\alpha e^{\frac{\pi i}{9}}$ instead of t in (6) and to integrate them respectively.

Thus from (6)

$$\frac{9\alpha^4 e^{\frac{4\pi i}{9}}}{1+\alpha^9}$$

$$=e^{\frac{4\pi i}{9}}\left\{\frac{-1}{(-1)-\alpha}-\frac{e^{-\frac{4\pi i}{9}}}{e^{\frac{\pi i}{9}}-\alpha}-\frac{e^{\frac{4\pi i}{9}}}{e^{-\frac{\pi i}{9}}-\alpha}+\frac{e^{-\frac{3\pi i}{9}}}{e^{\frac{3\pi i}{9}}-\alpha}+\frac{e^{\frac{3\pi i}{9}}}{e^{-\frac{3\pi i}{9}}-\alpha}\right.$$

$$\left.-\frac{e^{-\frac{2\pi i}{9}}}{e^{\frac{5\pi i}{9}}-\alpha}-\frac{e^{\frac{2\pi i}{9}}}{e^{-\frac{5\pi i}{9}}-\alpha}+\frac{e^{-\frac{\pi i}{9}}}{e^{\frac{7\pi i}{9}}-\alpha}+\frac{e^{\frac{\pi i}{9}}}{e^{-\frac{7\pi i}{9}}-\alpha}\right\}$$

and

$$(11)\qquad f(\alpha e^{\frac{\pi i}{9}})=e^{\frac{5\pi i}{9}}\left\{\log(-1-\alpha)+\cos\frac{4\pi}{9}\log(e^{\frac{\pi i}{9}}-\alpha)(e^{-\frac{\pi i}{9}}-\alpha)\right.$$

$$-\cos\frac{3\pi}{9}\log(e^{\frac{3\pi i}{9}}-\alpha)(e^{-\frac{3\pi i}{9}}-\alpha)$$

$$+\cos\frac{2\pi}{9}\log(e^{\frac{5\pi i}{9}}-\alpha)(e^{-\frac{5\pi i}{9}}-\alpha)$$

$$-\cos\frac{\pi}{9}\log(e^{\frac{7\pi i}{9}}-\alpha)(e^{-\frac{7\pi i}{9}}-\alpha)$$

$$-i\sin\frac{4\pi}{9}\log\left(\frac{e^{\frac{\pi i}{9}}-\alpha}{e^{-\frac{\pi i}{9}}-\alpha}\right)+i\sin\frac{3\pi}{9}\log\left(\frac{e^{\frac{3\pi i}{9}}-\alpha}{e^{-\frac{3\pi i}{9}}-\alpha}\right)$$

$$\left.-i\sin\frac{2\pi}{9}\log\left(\frac{e^{\frac{5\pi i}{9}}-\alpha}{e^{-\frac{5\pi i}{9}}-\alpha}\right)+i\sin\frac{\pi}{9}\log\left(\frac{e^{\frac{7\pi i}{9}}-\alpha}{e^{-\frac{7\pi i}{9}}-\alpha}\right)\right\}+C$$

The poles of the function

$$\frac{t^4}{1-t^9}$$

are distributed on a circle with radius 1 as shown in Fig. 4.

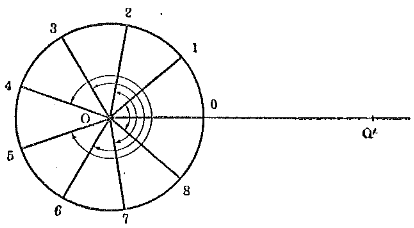


Fig. 4.

If the arguments of the poles are measured in the direction of arrows, the limiting value of the integral (7), when $t = 0$, reduces to $\varphi(0)$ as shown in (9). If t varies from O to O' along the real axis, the arguments of the vectors between the pole and the point t will vary continuously with t. If t tends to infinity, the following terms must have the limiting value $2\pi i$;

$$(12)\quad \begin{cases} \lim\limits_{t\to\infty} \log \dfrac{e^{\frac{2\pi i}{9}}-t}{e^{-\frac{2\pi i}{9}}-t} \to 2\pi i\,, \\ \lim\limits_{t\to\infty} \log \dfrac{e^{\frac{4\pi i}{9}}-t}{e^{-\frac{4\pi i}{9}}-t} \to 2\pi i\,, \\ \lim\limits_{t\to\infty} \log \dfrac{e^{\frac{6\pi i}{9}}-t}{e^{-\frac{6\pi i}{9}}-t} \to 2\pi i\,, \\ \lim\limits_{t\to\infty} \log \dfrac{e^{\frac{8\pi i}{9}}-t}{e^{-\frac{8\pi i}{9}}-t} \to 2\pi i\,. \end{cases}$$

The poles of the function

$$\frac{\alpha^4}{1+\alpha^9}$$

are distributed on a circle with radius 1 as shown in Fig. 5.

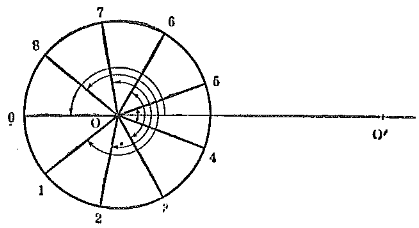


Fig. 5.

Now measuring the arguments of the poles in the direction of arrows, one obtains the limiting values of the integral (11) when $t\to 0$ as follows;

$$0=e^{\frac{5\pi i}{9}}\left\{\pi i+\frac{2\pi}{9}\sin\frac{4\pi}{9}-\frac{6\pi}{9}\sin\frac{3\pi}{9}+\frac{10\pi}{9}\sin\frac{2\pi}{9}-\frac{14\pi}{9}\sin\frac{\pi}{9}\right\}+C.$$

Hence

(13)
$$C=e^{\frac{5\pi i}{9}}\left\{-\pi i-\frac{2\pi}{9}\sin\frac{4\pi}{9}+\frac{6\pi}{9}\sin\frac{3\pi}{9}-\frac{10\pi}{9}\sin\frac{2\pi}{9}+\frac{14\pi}{9}\sin\frac{\pi}{9}\right\}.$$

If α varies from O to O' along the real axis, the arguments of the vectors between the poles and the point t vary continuously with α. Therefore if t tends to infinity,

(14)
$$\begin{cases}\lim\limits_{\alpha\to\infty}\ \log\dfrac{e^{\frac{\pi i}{9}}-\alpha}{e^{-\frac{\pi i}{9}}-\alpha}\ \longrightarrow\ 2\pi i\,,\\[2ex] \lim\limits_{\alpha\to\infty}\ \log\dfrac{e^{\frac{3\pi i}{9}}-\alpha}{e^{-\frac{3\pi i}{9}}-\alpha}\ \longrightarrow\ 2\pi i\,,\\[2ex] \lim\limits_{\alpha\to\infty}\ \log\dfrac{e^{\frac{5\pi i}{9}}-\alpha}{e^{-\frac{5\pi i}{9}}-\alpha}\ \longrightarrow\ 2\pi i\,,\\[2ex] \lim\limits_{\alpha\to\infty}\ \log\dfrac{e^{\frac{7\pi i}{9}}-\alpha}{e^{-\frac{7\pi i}{9}}-\alpha}\ \longrightarrow\ 2\pi i\,.\end{cases}$$

Now from (11), (13) and (14)

$$\begin{aligned}f(\infty e^{\frac{\pi i}{9}})&=e^{\frac{5\pi i}{9}}\Bigg\{\lim_{\alpha\to\infty}\Bigg[\log\left(1+\frac{1}{\alpha}\right)+\log\alpha+\pi i+\cos\frac{4\pi}{9}\left\{\log\left(1-\frac{2}{\alpha}\cos\frac{\pi}{9}\right.\right.\\&\left.\left.+\frac{1}{\alpha^2}\right)+2\log\alpha\right\}-\cos\frac{3\pi}{9}\left\{\log\left(1-\frac{2}{\alpha}\cos\frac{3\pi}{9}+\frac{1}{\alpha^2}\right)+2\log\alpha\right\}+\cos\frac{2\pi}{9}\\&\left\{\log\left(1-\frac{2}{\alpha}\cos\frac{5\pi}{9}+\frac{1}{\alpha^2}\right)+2\log\alpha\right\}-\cos\frac{\pi}{9}\left\{\log\left(1-\frac{2}{\alpha}\cos\frac{7\pi}{9}+\frac{1}{\alpha^2}\right)\right.\\&\left.+2\log\alpha\right\}\Bigg]+2\pi\sin\frac{4\pi}{9}-2\pi\sin\frac{3\pi}{9}+2\pi\sin\frac{2\pi}{9}-2\pi\sin\frac{\pi}{9}-\pi i\\&-\frac{2\pi}{9}\sin\frac{4\pi}{9}+\frac{6\pi}{9}\sin\frac{3\pi}{9}-\frac{10\pi}{9}\sin\frac{2\pi}{9}+\frac{14\pi}{9}\sin\frac{\pi}{9}\Bigg\}\\&=e^{\frac{5\pi i}{9}}\Bigg\{\lim_{\alpha\to\infty}\ \log\alpha\left\{1+2\cos\frac{4\pi}{9}-2\cos\frac{3\pi}{9}+2\cos\frac{2\pi}{9}-2\cos\frac{\pi}{9}\right\}\\&+\frac{16\pi}{9}\sin\frac{4\pi}{9}-\frac{12\pi}{9}\sin\frac{3\pi}{9}+\frac{8\pi}{9}\sin\frac{2\pi}{9}-\frac{4\pi}{9}\sin\frac{\pi}{9}\Bigg\}.\end{aligned}$$

Now

$$(15) \qquad 1+2\cos\frac{4\pi}{9}-2\cos\frac{3\pi}{9}+2\cos\frac{2\pi}{9}-2\cos\frac{\pi}{9}$$

$$=1+e^{\frac{4\pi i}{9}}+e^{-\frac{4\pi i}{9}}-e^{\frac{3\pi i}{9}}-e^{-\frac{3\pi i}{9}}+e^{\frac{2\pi i}{9}}+e^{-\frac{2\pi i}{9}}-e^{\frac{\pi i}{9}}-e^{-\frac{\pi i}{9}}$$

$$=e^{\frac{4\pi i}{9}}\frac{1+e^{-\frac{9\pi i}{9}}}{1+e^{\frac{\pi i}{9}}}=0\,.$$

Therefore it follows

$$(16) \qquad f(\infty e^{\frac{\pi i}{9}})=e^{\frac{5\pi i}{9}}\left(\frac{16\pi}{9}\sin\frac{4\pi}{9}-\frac{12\pi}{9}\sin\frac{3\pi}{9}+\frac{8\pi}{9}\sin\frac{2\pi}{9}-\frac{4\pi}{9}\sin\frac{\pi}{9}\right)$$

$$\doteq\frac{2\pi}{9}\sin\frac{4\pi}{9}-\frac{6\pi}{9}\sin\frac{3\pi}{9}+\frac{10\pi}{9}\sin\frac{2\pi}{9}-\frac{14\pi}{9}\sin\frac{\pi}{9}+i\left(-\frac{2\pi}{9}\cos\frac{4\pi}{9}\right.$$

$$\left.+\frac{2\pi}{9}\cos\frac{3\pi}{9}-\frac{2\pi}{9}\cos\frac{2\pi}{9}+\frac{2\pi}{9}\cos\frac{\pi}{9}-\frac{\pi}{9}+\frac{9\pi}{9}\right).$$

Inserting (15) one obtains

$$(17) \qquad f(\infty e^{\frac{\pi i}{9}})=\frac{2\pi}{9}\sin\frac{4\pi}{9}-\frac{6\pi}{9}\sin\frac{3\pi}{9}+\frac{10\pi}{9}\sin\frac{2\pi}{9}-\frac{14\pi}{9}\sin\frac{\pi}{9}+\pi i\,.$$

As

$$\frac{\pi}{9}=0.34906585\,,$$

$$\sin\frac{\pi}{9}=0.3420200\,,$$

$$\sin\frac{2\pi}{9}=0.6427724\,,$$

$$\sin\frac{3\pi}{9}=0.8660254\,,$$

$$\sin\frac{4\pi}{9}=0.9848016\,,$$

$$(18) \qquad f(\infty e^{\frac{\pi i}{9}})=-0.553991+\pi i\,.$$

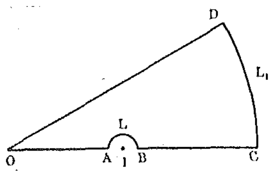


Fig. 6.

Now from the theorem of complex integration the integrals on the contour as shown in Fig. 6 have the relation

$$(19) \qquad \int_O^A + \int_L + \int_B^C + \int_{L_1} + \int_D^O \left(\frac{9t^4}{1-t^9} dt \right) = 0 \, ,$$

since there is no pole of the function

$$\frac{t^4}{1-t^9}$$

in the above domain.

If t lies between O and A, the integral is given by formulae (7), (8) and (9). If t lies on the contour line OD, the integral is given by formulae (11) and (13).

If t lies on the real axis and between B and C, the integral

$$f(t) = \int_0^t \frac{9t^4 dt}{1-t^9}$$

is composed of three integrals;

$$(20) \qquad f(t) = \int_O^A \frac{9t^4 dt}{1-t^9} + \int_L \frac{9t^4 dt}{1-t^9} + \int_B^t \frac{9t^4 dt}{1-t^9} \, .$$

By putting $A = 1-\varepsilon$, $B = 1+\varepsilon$ so that ε tends to 0, one obtains

$$\int_L \frac{9t^4}{1-t^9} dt \to \lim_{\varepsilon \to 0} \int_\pi^0 \frac{9(1+\varepsilon e^{i\theta})^4 \varepsilon i e^{i\theta} d\theta}{1-(1-\varepsilon e^{i\theta})^9} = \pi i \, .$$

The integral (20) becomes

$$\begin{aligned}(21) \quad f(t) = & -\log(t-1) + \cos\frac{\pi}{9}\log(e^{\frac{2\pi i}{9}}-t)(e^{-\frac{2\pi i}{9}}-t) \\ & -\cos\frac{2\pi}{9}\log(e^{\frac{4\pi i}{9}}-t)(e^{-\frac{4\pi i}{9}}-t) \\ & +\cos\frac{3\pi}{9}\log(e^{\frac{6\pi i}{9}}-t)(e^{-\frac{6\pi i}{9}}-t) \\ & -\cos\frac{4\pi}{9}\log(e^{\frac{8\pi i}{9}}-t)(e^{-\frac{8\pi i}{9}}-t) \\ & + i\sin\frac{\pi}{9}\log\left(\frac{e^{\frac{2\pi i}{9}}-t}{e^{-\frac{2\pi i}{9}}-t}\right) - i\sin\frac{2\pi}{9}\log\left(\frac{e^{\frac{4\pi i}{9}}-t}{e^{-\frac{4\pi i}{9}}-t}\right) \\ & + i\sin\frac{3\pi}{9}\log\left(\frac{e^{\frac{6\pi i}{9}}-t}{e^{-\frac{6\pi i}{9}}-t}\right) - i\sin\frac{4\pi}{9}\log\left(\frac{e^{\frac{8\pi i}{9}}-t}{e^{-\frac{8\pi i}{9}}-t}\right) \\ & + \frac{4\pi}{9}\sin\frac{\pi}{9} - \frac{8\pi}{9}\sin\frac{2\pi}{9} + \frac{12\pi}{9}\sin\frac{3\pi}{9} - \frac{16\pi}{9}\sin\frac{4\pi}{9} + \pi i\end{aligned}$$

If t tends to infinity, $f(\infty)$ becomes from (12) and (15)

$$(22)\quad f(\infty) = -2\pi\sin\frac{\pi}{9}+2\pi\sin\frac{2\pi}{9}-2\pi\sin\frac{3\pi}{9}+2\pi\sin\frac{4\pi}{9}$$
$$+\frac{4\pi}{9}\sin\frac{\pi}{9}-\frac{8\pi}{9}\sin\frac{2\pi}{9}-\frac{12\pi}{9}\sin\frac{3\pi}{9}-\frac{16\pi}{9}\sin\frac{4\pi}{9}+\pi i$$
$$= -\frac{14\pi}{9}\sin\frac{\pi}{9}+\frac{10\pi}{9}\sin\frac{2\pi}{9}-\frac{6\pi}{9}\sin\frac{3\pi}{9}+\frac{2\pi}{9}\sin\frac{4\pi}{9}+\pi i\,.$$

From (17) it follows

$$(23)\qquad f(\infty) = f(\infty e^{\frac{\pi i}{9}})\,.$$

It is evident from (19) that

$$\lim_{\substack{D\to\infty\\C\to\infty}}\int_{L_1}\frac{9t^4dt}{1-t^9} = \lim_{r\to\infty}\int_0^{\frac{\pi}{9}}\frac{9r^5e^{4i\theta}ie^{i\theta}d\theta}{1-r^9e^{9i\theta}} \to 0\,.$$

Hence $\qquad t>1$

$$f(t) = \int_\infty^t\frac{9t^4dt}{1-t^9}+f(\infty e^{\frac{\pi i}{9}})\,.$$

The value of the integral on the boundary has been obtained by formulae (7), (8) and (9) when t lies between 0 and 1, and by formula (21) when t lies between 1 and ∞, and lastly by formulae (11) and (13) when t lies on the boundary line $t = e^{\frac{\pi i}{9}}\alpha\,(\alpha = \text{real})$.

When t is a complex variable, the integral can be obtained directly from (7), (8) and (9).

Although the value of $f(t)$ can be calculated by the above formula exactly, it is very complicated to determine the argument of the logarithmus appearing in the formula, and it can hardly be obtained by ordinary effort.

Therefore it is preferable to calculate the integration graphically

$$(24)\qquad (A)\qquad f(t) = \int_0^t\frac{9t^4}{1-t^9}dt \qquad t<1\,,$$

$$(25)\qquad (B)\qquad f(t) = \int_\infty^t\frac{9t^4}{1-t^9}dt+f(\infty) \qquad t>1\,,$$

where $\qquad f(\infty) = f(\infty e^{\frac{\pi i}{9}}) = -0.553991+\pi i\,,$

$$(26)\qquad (C)\qquad f(t) = e^{\frac{5\pi i}{9}}\int_0^\alpha\frac{9\alpha^4d\alpha}{1+\alpha^9}\,.$$

The values of the integral are tabulated by Table I, the second column contains the value of integrals A and B, and the third and the fourth columns contain the real and imaginary parts of the value of the integral C respectively.

If t is complex and denoted by $r\,e^{i\theta}$, the integral

$$f(t)=f(r\,e^{i\theta})=f(r)+9ir^5\int_0^\theta \frac{e^{5i\theta}d\theta}{1-r^9\,e^{9i\theta}} \tag{27}$$

can be calculated by expanding the integrand by an infinite series.

Thus if $|t|<1$

$$\begin{aligned}f(r\,e^{i\theta})&=f(\infty)+\int_\infty^r\frac{9t^4}{1-t^9}\,dt\\ &+9r^5\left\{\frac{\cos 5\theta-1}{5}+r^9\frac{\cos 14\theta-1}{14}+\cdots\cdot+r^{9n}\frac{\cos(9n+5)\theta-1}{9n+5}+\cdots\cdot\right\}\\ &+i9r^5\left\{\frac{\sin 5\theta}{5}+r^9\frac{\sin 14\theta}{14}+\cdots\cdot+r^{9n}\frac{\sin(9n+5)\theta}{9n+5}+\cdots\cdot\right\},\end{aligned} \tag{28}$$

if $|t|>1$

$$\begin{aligned}f(r\,e^{i\theta})&=f(r)+\frac{9}{r^4}\left\{\frac{\cos 4\theta-1}{4}+\frac{1}{r^9}\frac{\cos 13\theta-1}{13}+\cdots\cdot\right.\\ &\left.+\frac{1}{r^{9n}}\frac{\cos(9n+4)\theta-1}{9n+4}+\cdots\cdot\right\}-i\frac{9}{r^4}\left\{\frac{\sin 4\theta}{4}+\frac{1}{r^9}\frac{\sin 13\theta}{13}+\cdots\cdot\right.\\ &\left.+\frac{1}{r^{9n}}\frac{\sin(9n+4)}{9n+4}+\cdots\cdot\right\}.\end{aligned} \tag{29}$$

Some values of integral obtained by the graphical integration are checked by the ones calculated from formulae (7), (8) and (9) through tedious calculation, and found to have an error of a negligibly small magnitude.

The loci of real and imaginary parts of the function $f(t)=$ const. are drawn in Pl. I.

Thus the function $f(t)$ has been obtained. From $f(t)$ the stream-lines in the domain shown in Fig. 1 can be easily drawn.

Now

$$Z=f(t)-\frac{a}{b^{\frac{4}{9}}}f\left(\frac{t}{b^{\frac{1}{9}}}\right) \tag{5}$$

and

$$t^9=\frac{b-z}{1-z}. \tag{2}$$

At $z=+0$, or at such t as

$$t^9=\lim_{\varepsilon\to 0}\frac{b-\varepsilon}{1-\varepsilon}=\lim_{\varepsilon\to 0} b\left(1+\varepsilon\frac{b-1}{b}\right),$$

one obtains from (5)

$$Z=f(b^{\frac{1}{9}})-\frac{a}{b^{\frac{4}{9}}}f(1+0)\to -\infty+\pi i-\frac{a}{b^{\frac{4}{9}}}\pi i\,. \tag{30}$$

At $z=-0$, or at such t as

$$t^9=\lim_{\varepsilon\to 0}\frac{b+\varepsilon}{1+\varepsilon}=\lim_{\varepsilon\to 0} b\left(1-\varepsilon\frac{b-1}{b}\right),$$

$$Z=f(b^{\frac{1}{9}})-\frac{a}{b^{\frac{4}{9}}}f(1-0)\to -\infty+\pi i\,. \tag{31}$$

At $z=+\infty$ or at such t as

$$t^9=\lim_{z\to\infty}\frac{\frac{b}{z}-1}{\frac{1}{z}-1}=\lim_{z\to\infty}\left(1-\frac{b}{z}+\frac{1}{z}\right)=\lim_{\varepsilon\to 0}(1-\varepsilon)\,,$$

$$Z=f(1-0)-\frac{a}{b^{\frac{4}{9}}}f\left(\frac{1}{b^{\frac{1}{9}}}\right)\to\infty\,. \tag{32}$$

At $z=-\infty$ or at such t as

$$t^9=\lim_{\varepsilon\to 0}(1+\varepsilon)$$

$$Z=f(1+0)-\frac{a}{b^{\frac{4}{9}}}f\left(\frac{1}{b^{\frac{1}{9}}}\right)\to\infty+\pi i\,. \tag{33}$$

At $z=1$ or $t=\infty$

$$Z=f(\infty)-\frac{a}{b^{\frac{4}{9}}}f(\infty)=\left(1-\frac{a}{b^{\frac{4}{9}}}\right)f(\infty) \tag{34}$$

At $z=b$ or $t^9=0$

$$Z=f(0)-\frac{a}{b^{\frac{4}{9}}}f(0)=\left(1-\frac{a}{b^{\frac{4}{9}}}\right)f(0)\to 0\,. \tag{35}$$

Therefore the boundary of the domain can be drawn as Fig. 7.

If $a = b^{\frac{4}{9}}$, the boundary coincides with the one shown in Fig. 1 (a).

If $a = 1$, the boundary coincides with the one shown in Fig. 1 (c).

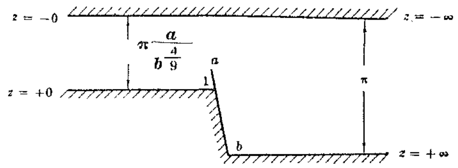


Fig. 7.

In order to draw the streamlines begin from $Z = -\infty$ and end in $Z = +\infty$, the streamlines in z-plane must be drawn which begin from $z = 0$ and end in $z = \infty$, or the streamlines which begin from $t = b^{\frac{1}{9}}$ and end in $t = 1$.

For the purpose, observe the function

(36) $$\Psi = \log z$$

The conformal representation by means of this function will give the streamlines which begin from 0 and end in ∞ and the streamlines are shown in Fig. 8.

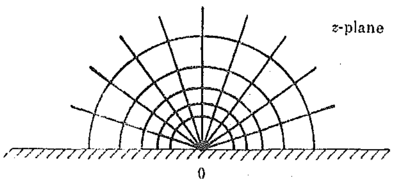


Fig. 8.

Again the loci of the function $\Psi = \text{const.}$ are transformed conformally into the plane ζ by means of the function

(37) $$\zeta = \frac{b-z}{1-z}, \qquad \Psi = \log\frac{b-\zeta}{1-\zeta}.$$

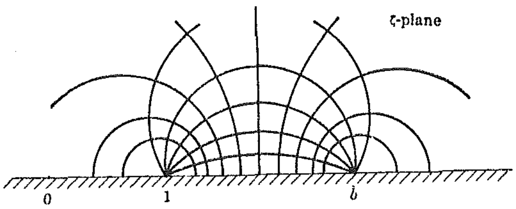


Fig. 9.

Again the loci of $\Psi = \text{const.}$ are transformed conformally into the plane t by means of the function

$$\zeta = t^9$$

Thus the loci of $\Psi = \text{const.}$ are drawn in Fig. 10.

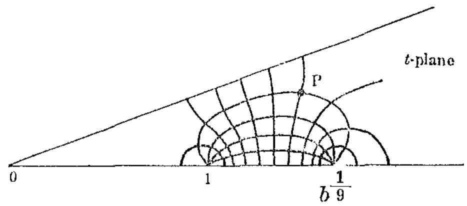


Fig. 10.

Now at first, the constants a and b must be determined. The width of the parallel boundaries is determined by

$$\frac{a}{b^{\frac{4}{9}}}. \tag{39}$$

If the ratio of the width of the parallel boundary at $Z=-\infty$ against the one at $Z=\infty$ is given by say c,

$$b^{\frac{4}{9}}=\frac{a}{c} \qquad \text{or} \qquad b=\left(\frac{a}{c}\right)^{\frac{9}{4}}. \tag{40}$$

The distance between 1 and a is

$$|Z|=f(a)-cf\left(\frac{a}{\left(\frac{a}{c}\right)^{\frac{4}{9}}}\right). \tag{41}$$

If the ratio of this length against the width of the stream is given, a can be obtained, and hence b is determined.

Put

$$\frac{b-1}{2}=\beta \tag{42}$$

and

$$u=\zeta-1-\frac{b-1}{2}, \tag{43}$$

then it follows

$$\Psi=\Phi+i\chi=\log\frac{\zeta-b}{\zeta-1}=\log\frac{u-\beta}{u+\beta}.$$

The loci of $\Phi=$ const. and $\chi=$ const. are both series of circles. Put

$$e^{2\Phi}=k,$$

then the locus of $\Phi = \text{const.}$ is a circle with radius $\frac{2\beta}{|k-1|}\sqrt{k}$, the center being at the distance $\beta\frac{k+1}{k-1}$ from the point $u=0$ or $\zeta = 1+\frac{b-1}{2}$.

The locus of $\chi = \text{const.}$ is also a circle with radius $\frac{\beta}{\sin\chi}$ and center at $\zeta = 1+\frac{b-1}{2}+i\left(\frac{\beta}{\sin\chi}-2\beta\cos^2\frac{\chi}{2}\right)$.

The intersecting points of the two series of circles are obtained by solving the two quadratic equations of circles, or more easily obtained by drawing the two series of circles.

Denote the polar coordinates of the point (ξ, η) in ζ-plane by r, θ then $r=\sqrt{\xi^2+\eta^2}$, $\theta = \tan^{-1}\frac{\eta}{\xi}$ can be determined. By conformal representation by means of the function $\zeta = t^9 = \rho^9 e^{9i\varphi}$, r and θ transform into ρ and φ by the relations respectively;

$$r = e^{\mu}, \quad e^{\frac{\mu}{9}} = \rho \quad \text{and} \quad \frac{\theta}{9} = \varphi .$$

Thus Fig. 10 can be obtained.

Now observe point P in Fig. 10. Put Fig. 10 on Pl. I, so that the boundaries of the two figures coincide with each other and read the position t_P of P by the curvilinear coordinates, shown on Pl. I; say X_1 and Y_1. In similar manner, read the curvilinear coordinates of the point $\frac{t}{b^{\frac{1}{9}}}$; say X_2 and Y_2. If the values of $X_1-\frac{a}{b^{\frac{4}{9}}}X_2$ and $Y_1-\frac{a}{b^{\frac{4}{9}}}Y_2$ are marked on Fig. 1, these values correspond to the rectangular coordinates of Z of the point P transformed in Z-plane.

Thus all the intersecting points can be transformed. If the points thus transformed are connected by a smooth curve, the streamlines which begin from $-\infty$ and end in $+\infty$ can be drawn.

For example put

$$b = 15.053$$

$$b^{\frac{1}{9}} = 1.3517 \quad \text{or} \quad \frac{1}{b^{\frac{1}{9}}} = 0.7398$$

If $a = b^{\frac{4}{9}}$, $a = 3.3383$. In the case, the streamlines are shown in Pl. II.

If $a = \frac{1}{3}b^{\frac{4}{9}}$, $a = 1.1128$. In the case, the streamlines are shown in Pl. III.

If $a = \frac{2}{3}b^{\frac{4}{9}}$, $a = 2.2256$. In the case, the streamlines are shown in Pl. IV.

Table I.

t	$\int_0^t \frac{9t^4}{1-t^9}\,dt$	$\cos\frac{4\pi}{9}\int_0^t \frac{9t^4}{1+t^9}\,dt$	$\sin\frac{4\pi}{9}\int_0^t \frac{9t^4}{1+t^9}\,dt$
0.01	—	—	—
0.02	—	—	—
0.03	—	—	—
0.04	—	—	—
0.05	—	—	—
0.06	—	—	—
0.07	—	—	—
0.08	—	—	0.00001
0.09	—	—	0.00001
0.10	0.00001	—	0.00002
0.11	0.00002	—	0.00003
0.12	0.00003	0.00001	0.00005
0.13	0.00006	0.00001	0.00007
0.14	0.00007	0.00002	0.00010
0.15	0.00013	0.00002	0.00014
0.16	0.00018	0.00003	0.00019
0.17	0.00025	0.00005	0.00026
0.18	0.00033	0.00006	0.00033
0.19	0.00044	0.00008	0.00044
0.20	0.00057	0.00010	0.00057
0.21	0.00073	0.00013	0.00073
0.22	0.00092	0.00016	0.00092
0.23	0.00115	0.00020	0.00114
0.24	0.00142	0.00025	0.00141
0.25	0.00175	0.00031	0.00173
0.26	0.00213	0.00037	0.00211
0.27	0.00257	0.00045	0.00255
0.28	0.00309	0.00054	0.00305
0.29	0.00341	0.00064	0.00364
0.30	0.00409	0.00076	0.00431
0.31	0.00487	0.00090	0.00508
0.32	0.00578	0.00105	0.00598
0.33	0.00676	0.00123	0.00697
0.34	0.00790	0.00142	0.00807
0.35	0.00917	0.00164	0.00932
0.36	0.01060	0.00189	0.01072
0.37	0.01221	0.00217	0.01230
0.38	0.01399	0.00248	0.01405

Table I.—(*Continued*)

t	$\int_0^t \frac{9t^4}{1-t^9}\,dt$	$\cos\frac{4\pi}{9}\int_0^t \frac{9t^4}{1+t^9}\,dt$	$\sin\frac{4\pi}{9}\int_0^t \frac{9t^4}{1+t^9}\,dt$
0.39	0.01598	0.00284	0.01608
0.40	0.01816	0.00320	0.01817
0.41	0.02059	0.00362	0.02055
0.42	0.02326	0.00409	0.02318
0.43	0.02620	0.00460	0.02608
0.44	0.02943	0.00516	0.02925
0.45	0.03296	0.00577	0.03273
0.46	0.03683	0.00644	0.03653
0.47	0.04104	0.00717	0.04066
0.48	0.04563	0.00797	0.04517
0.49	0.05062	0.00883	0.05008
0.50	0.05604	0.00977	0.05540
0.51	0.06191	0.01078	0.06115
0.52	0.06826	0.01188	0.06737
0.53	0.07513	0.01306	0.07405
0.54	0.08252	0.01435	0.08140
0.55	0.09050	0.01571	0.08912
0.56	0.09909	0.01719	0.09749
0.57	0.11012	0.01877	0.10647
0.58	0.11823	0.02047	0.11610
0.59	0.12886	0.02229	0.12640
0.60	0.14025	0.02422	0.13736
0.61	0.15227	0.02629	0.14907
0.62	0.16549	0.02851	0.16168
0.63	0.17943	0.03087	0.17509
0.64	0.19432	0.03337	0.18927
0.65	0.21021	0.03603	0.20432
0.66	0.22716	0.03884	0.22029
0.67	0.24523	0.04182	0.23718
0.68	0.26448	0.04498	0.25508
0.69	0.28499	0.04831	0.27395
0.70	0.30682	0.05182	0.29389
0.71	0.33007	0.05552	0.31489
0.72	0.35481	0.05942	0.33698
0.73	0.38104	0.06354	0.36037
0.74	0.40918	0.06787	0.38490
0.75	0.43903	0.07236	0.41040
0.76	0.47082	0.07707	0.43708

Table I.—(*Continued*)

t	$\int_0^t \frac{9t^4}{1-t^9}\,dt$	$\cos\frac{4\pi}{9}\int_0^t \frac{9t^4}{1+t^9}\,dt$	$\sin\frac{4\pi}{9}\int_0^t \frac{9t^4}{1+t^9}\,dt$
0.77	0.50470	0.08198	0.46493
0.78	0.54084	0.08720	0.49451
0.79	0.55948	0.09262	0.52527
0.80	0.62060	0.09816	0.55668
0.81	0.66469	0.10389	0.58919
0.82	0.71192	0.10986	0.62303
0.83	0.76263	0.11601	0.65790
0.84	0.81683	0.12238	0.69404
0.85	0.87101	0.12894	0.73124
0.86	0.93905	0.13565	0.76930
0.87	1.00828	0.14253	0.80834
0.88	1.08384	0.14957	0.84827
0.89	1.16678	0.15676	0.88904
0.90	1.25844	0.16409	0.93059
0.91	1.36058	0.17154	0.97283
0.92	1.47559	0.17909	1.01568
0.93	1.60686	0.18674	1.05905
0.94	1.75933	0.19446	1.10286
0.95	1.94073	0.20224	1.14699
0.96	2.16417	0.21007	1.19137
0.97	2.45463	0.21792	1.23589
0.98	2.87046	0.22578	1.28048
0.99	3.61998	0.23363	1.32501
1.00	∞	0.24146	1.36941
1.01	* 3.62031	0.24926	1.41360
1.02	2.87101	0.25699	1.45748
1.03	2.45536	0.26466	1.50097
1.04	2.16481	0.27225	1.54401
1.05	1.94156	0.27973	1.58645
1.06	1.76028	0.28711	1.62829
1.07	1.60753	0.29439	1.66955
1.08	1.47589	0.30154	1.71012
1.09	1.36041	0.30856	1.74995
1.10	1.25769	0.31545	1.78900
1.11	1.16532	0.32219	1.82722
1.12	1.08154	0.32878	1.86462

* $\int_\infty^t \frac{9t^4}{1-t^9}\,dt$ —0.55399

Table I.—(*Continued*)

t	$\int_{\infty}^{t} \frac{9t^4}{1-t^9}\,dt - 0.55399$	$\cos\frac{4\pi}{9}\int_0^t \frac{9t^4}{1+t^9}\,dt$	$\sin\frac{4\pi}{9}\int_0^t \frac{9t^4}{1+t^9}\,dt$
1.13	1.00499	0.33522	1.90114
1.14	0.93463	0.34151	1.93679
1.15	0.86957	0.34764	1.97157
1.16	0.80915	0.35361	2.00544
1.17	0.75291	0.35943	2.03840
1.18	0.70039	0.36508	2.07045
1.19	0.65116	0.37057	2.10162
1.20	0.60489	0.37591	2.13189
1.21	0.56128	0.38110	2.16130
1.22	0.52009	0.38613	2.18986
1.23	0.48115	0.39101	2.21754
1.24	0.44428	0.39574	2.24438
1.25	0.40931	0.40033	2.27039
1.26	0.37607	0.40478	2.29560
1.27	0.34445	0.40908	2.32002
1.28	0.31435	0.41325	2.34366
1.29	0.28564	0.41732	2.36666
1.30	0.25823	0.42120	2.38876
1.31	0.23206	0.42499	2.41022
1.32	0.20702	0.42866	2.43104
1.33	0.18306	0.43213	2.45075
1.34	0.16013	0.43563	2.47057
1.35	0.13816	0.43895	2.48939
1.36	0.11708	0.44216	2.50762
1.37	0.09686	0.44527	2.52524
1.38	0.07745	0.44826	2.54222
1.39	0.05878	0.45116	2.55867
1.40	0.04084	0.45398	2.57465
1.41	0.02359	0.45671	2.59011
1.42	0.00699	0.45935	2.60508
1.43	−0.00897	0.46190	2.61957
1.44	−0.02436	0.46437	2.63358
1.45	−0.03920	0.46677	2.64716
1.46	−0.05349	0.46908	2.66031
1.47	−0.06727	0.47133	2.67303
1.48	−0.08056	0 47350	2.68535
1.49	−0.09341	0.47561	2.69730
1.50	−0.10579	0.47765	2.70886

Table I.—(*Continued*)

t	$\int_{\infty}^{t}\frac{9t^4}{1-t^9}dt-0.55399$	$\cos\frac{4\pi}{9}\int_0^t\frac{9t^4}{1+t^9}dt$	$\sin\frac{4\pi}{9}\int_0^t\frac{9t^4}{1+t^9}dt$
1.51	−0.11776	0.47962	2.72006
1.52	−0.12932	0.48153	2.73091
1.53	−0.14048	0.48339	2.74143
1.54	−0.16027	0.48518	2.75161
1.55	−0.16170	0.48692	2.76148
1.56	−0.17179	0.48853	2.77061
1.57	−0.18155	0.49025	2.78033
1.58	−0.19100	0.49183	2.78932
1.59	−0.20014	0.49338	2.79811
1.60	−0.20900	0.49489	2.80668
1.61	−0.21757	0.49641	2.81529
1.62	−0.22179	0.49775	2.82286
1.63	−0.23393	0.49912	2.83063
1.64	−0.24175	0.50044	2.83814
1.65	−0.24929	0.50172	2.84540
1.66	−0.25662	0.50297	2.85247
1.67	−0.26372	0.50418	2.85933
1.68	−0.27061	0.50535	2.86599
1.69	−0.27730	0.50649	2.87246
1.70	−0.28379	0.50760	2.87873
1.71	−0.29009	0.50867	2.88484
1.72	−0.29621	0.50972	2.89076
1.73	−0.30214	0.51074	2.89652
1.74	−0.30791	0.51172	2.90212
1.75	−0.31351	0.51268	2.90756
1.76	−0.31895	0.51361	2.91284
1.77	−0.32423	0.51452	2.91799
1.78	−0.32937	0.51540	2.92300
1.79	−0.33437	0.51626	2.92786
1.80	−0.33922	0.51710	2.93260
1.81	−0.34394	0.51791	2.93722
1.82	−0.34854	0.51870	2.94170
1.83	−0.35300	0.51947	2.94606
1.84	−0.35735	0.52022	2.95030
1.85	−0.36157	0.52095	2.95443
1.86	−0.36569	0.52165	2.95845
1.87	−0.36969	0.52234	2.96236
1.88	−0.37359	0.52302	2.96617

Table I.—(*Continued*)

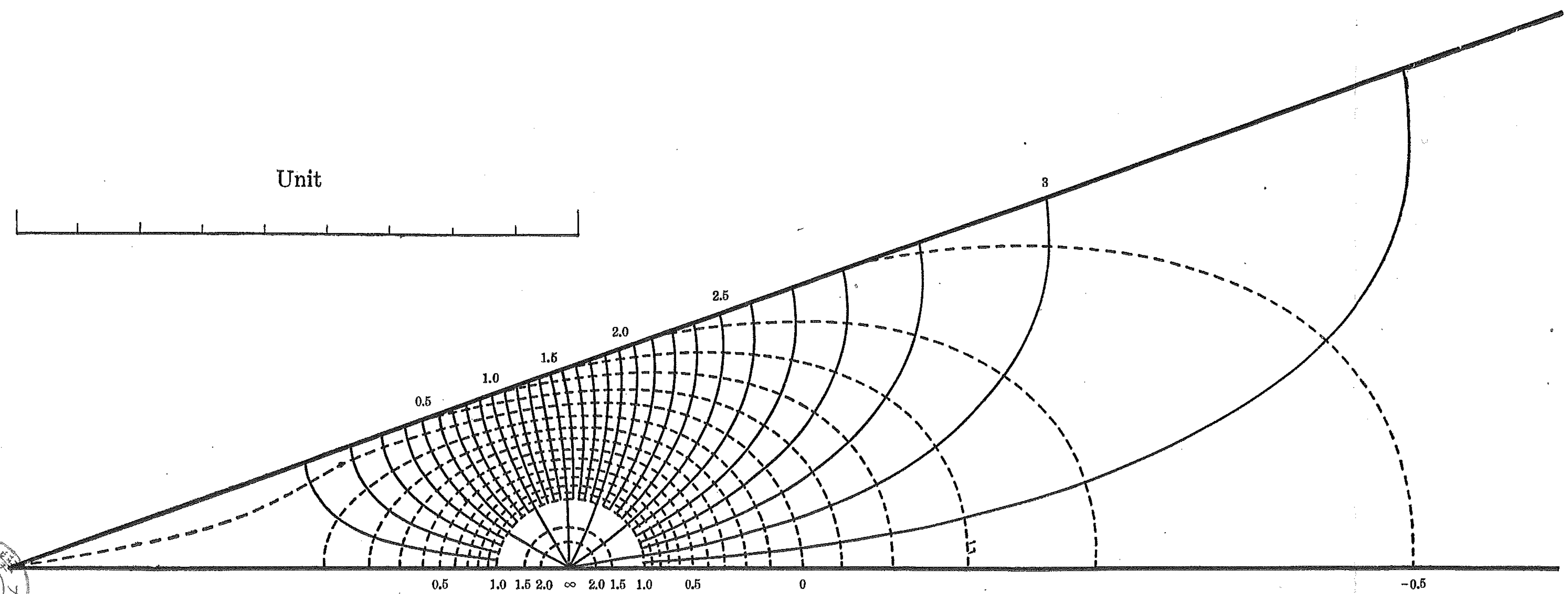
t	$\int_{\infty}^{t} \frac{9t^4}{1-t^9}\,dt-0.55399$	$\cos\frac{4\pi}{9}\int_0^t \frac{9t^4}{1+t^9}\,dt$	$\sin\frac{4\pi}{9}\int_0^t \frac{9t^4}{1+t^9}\,dt$
1.89	−0.37738	0.52367	2.96988
1.90	−0.38108	0.52431	2.97350
1.91	−0.38468	0.52493	2.97702
1.92	−0.38818	0.52553	2.98045
1.93	−0.39160	0.52612	2.98380
1.94	−0.39492	0.52670	2.98704
1.95	−0.39816	0.52726	2.99022
1.96	−0.40132	0.52780	2.99330
1.97	−0.40440	0.52833	2.99632
1.98	−0.40740	0.52885	2.99926
1.99	−0.41033	0.52936	3.00213
2.00	−0.41318	0.52985	3.00492
2.01	−0.41597	0.53033	3.00765
2.02	−0.41868	0.53080	3.01032
2.03	−0.42133	0.53126	3.01291
2.04	−0.42391	0.53171	3.01545
2.05	−0.42643	0.53214	3.01792
2.06	−0.42889	0.53257	3.02034
2.07	−0.43129	0.53298	3.02269
2.08	−0.43364	0.53339	3.02499
2.09	−0.43601	0.53378	3.02724
2.10	−0.43816	0.53417	3.02944
2.11	−0.44034	0.53455	3.03157
2.12	−0.44247	0.53492	3.03367
2.13	−0.44455	0.53528	3.03571
2.14	−0.44658	0.53563	3.03771
2.15	−0.44856	0.53597	3.03965
2.16	−0.45050	0.53631	3.04156
2.17	−0.45240	0.53663	3.04343
2.18	−0.45425	0.53696	3.04524
2.19	−0.45606	0.53727	3.04702
2.20	−0.45783	0.53758	3.04877
2.21	−0.45956	0.53788	3.05047
2.22	−0.46042	0.53817	3.05213
2.23	−0.46291	0.53846	3.05375
2.24	−0.46452	0.53874	3.05535
2.25	−0.46610	0.53901	3.05690
2.26	0.46766	0.53928	3.05842

Table I.—(*Continued*)

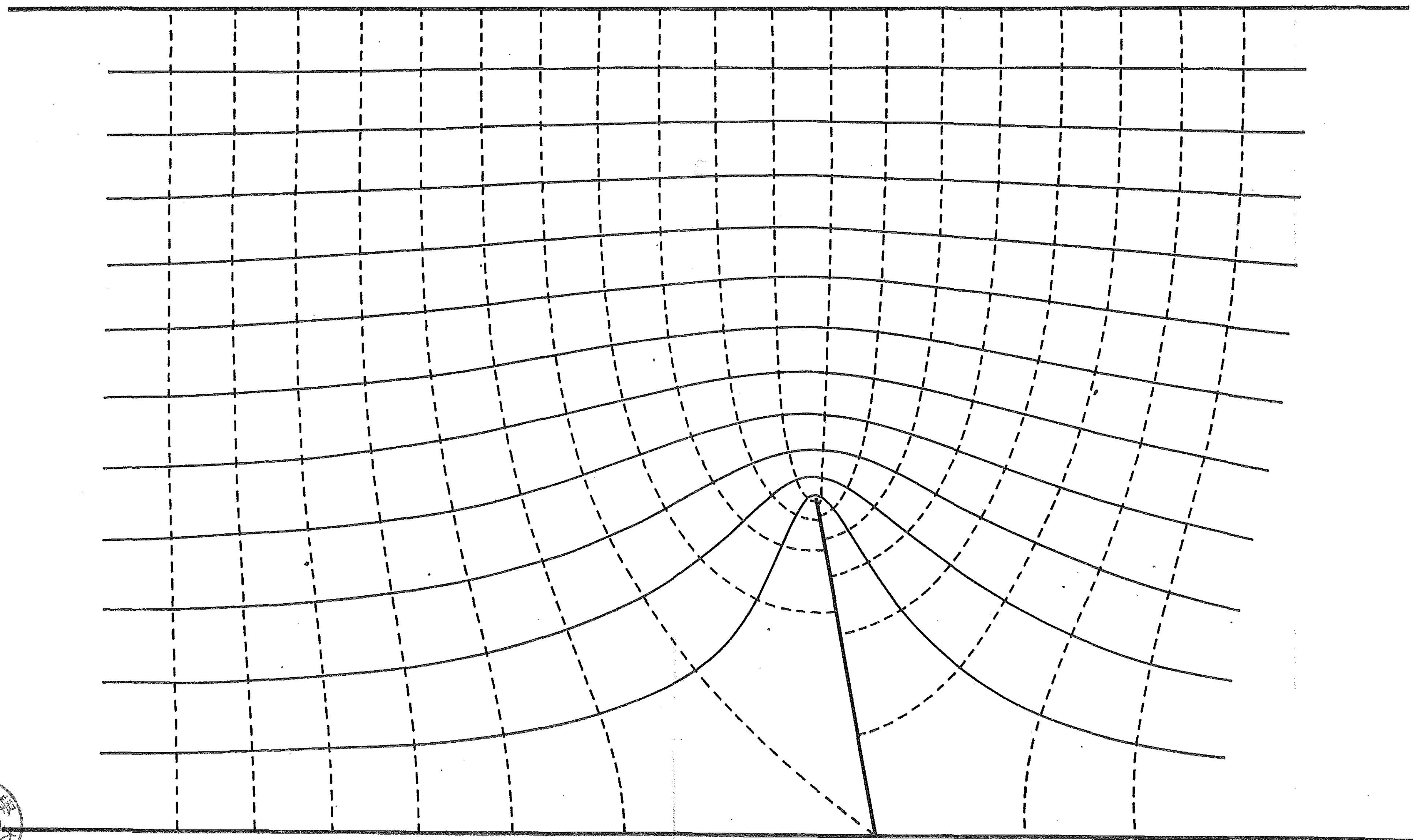
t	$\int_{\infty}^{t} \frac{9t^4}{1-t^9} dt - 0.55399$	$\cos \frac{4\pi}{9} \int_0^t \frac{9t^4}{1+t^9} dt$	$\sin \frac{4\pi}{9} \int_0^t \frac{9t^4}{1+t^9} dt$
2.27	−0.46916	0.53954	3.05991
2.28	−0.47064	0.53980	3.06136
2.29	−0.47209	0.54005	3.06279
2.30	−0.47351	0.54030	3.06418
2.31	−0.47490	0.54054	3.06554
2.32	−0.47625	0.54077	3.06688
2.33	−0.47758	0.54100	3.06818
2.34	−0.47888	0.54123	3.06946
2.35	−0.48015	0.54145	3.07071
2.36	−0.48230	0.54166	3.07193
2.37	−0.48261	0.54188	3.07313
2.38	−0.48381	0.54208	3.07430
2.39	−0.48497	0.54229	3.07546
2.40	−0.48612	0.54248	3.07658
2.41	−0.48723	0.54268	3.07767
2.42	−0.48833	0.54287	3.07875
2.43	−0.48941	0.54305	3.07981
2.44	−0.49046	0.54324	3.08085
2.45	−0.49149	0.54342	3.08187
2.46	−0.49250	0.54359	3.08286
2.47	−0.49349	0.54376	3.08384
2.48	−0.49446	0.54393	3.08479
2.49	−0.49546	0.54410	3.08573
2.50	−0.49635	0.54426	3.08664
2.51	−0.49726	0.54442	3.08754
2.52	−0.49816	0.54457	3.08843
2.53	−0.49904	0.54473	3.08929
2.54	−0.49990	0.54488	3.09014
2.55	−0.50074	0.54502	3.09097
2.56	−0.50157	0.54517	3.09178
2.57	−0.50238	0.54531	3.09258
2.58	−0.50318	0.54544	3.09337
2.59	−0.50396	0.54558	3.09413
2.60	−0.50472	0.54571	3.09489
2.61	−0.50547	0.54584	3.09562
2.62	−0.50621	0.54597	3.09635
2.63	−0.50693	0.54610	3.09706
2.64	−0.50764	0.54622	3.09776

Table I.—(*Continued*)

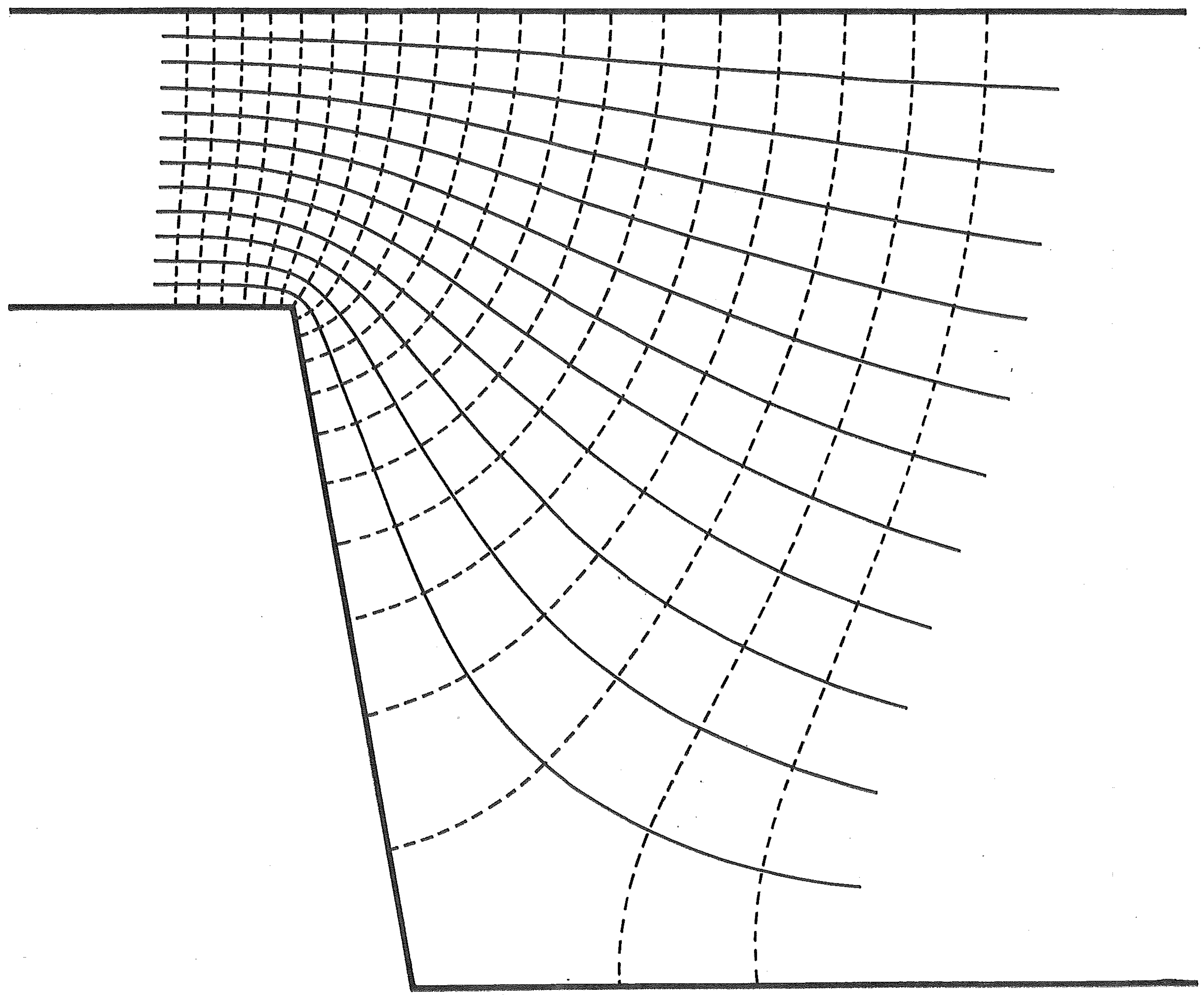
t	$\int_{\infty}^{t} \frac{9t^4}{1-t^9}\,dt - 0.55399$	$\cos\frac{4\pi}{9}\int_0^t \frac{9t^4}{1+t^9}\,dt$	$\sin\frac{4\pi}{9}\int_0^t \frac{9t^4}{1+t^9}\,dt$
2.65	−0.50834	0.54634	3.09844
2.66	−0.50902	0.54646	3.09912
2.67	−0.50969	0.54658	3.09978
2.68	−0.51035	0.54669	3.10042
2.69	−0.51100	0.54680	3.10106
2.70	−0.51163	0.54691	3.10168
2.71	−0.51225	0.54702	3.10229
2.72	−0.51286	0.54712	3.10289
2.73	−0.51345	0.54723	3.10348
2.74	−0.51405	0.54733	3.10406
2.75	−0.51463	0.54743	3.10464
2.76	−0.51520	0.54753	3.10519
2.77	−0.51575	0.54763	3.10574
2.78	−0.51630	0.54772	3.10628
2.79	−0.51684	0.54781	3.10680
2.80	−0.51746	0.54791	3.10732
2.81	−0.51788	0.54800	3.10784
2.82	−0.51839	0.54808	3.10834
2.83	−0.51889	0.54817	3.10883



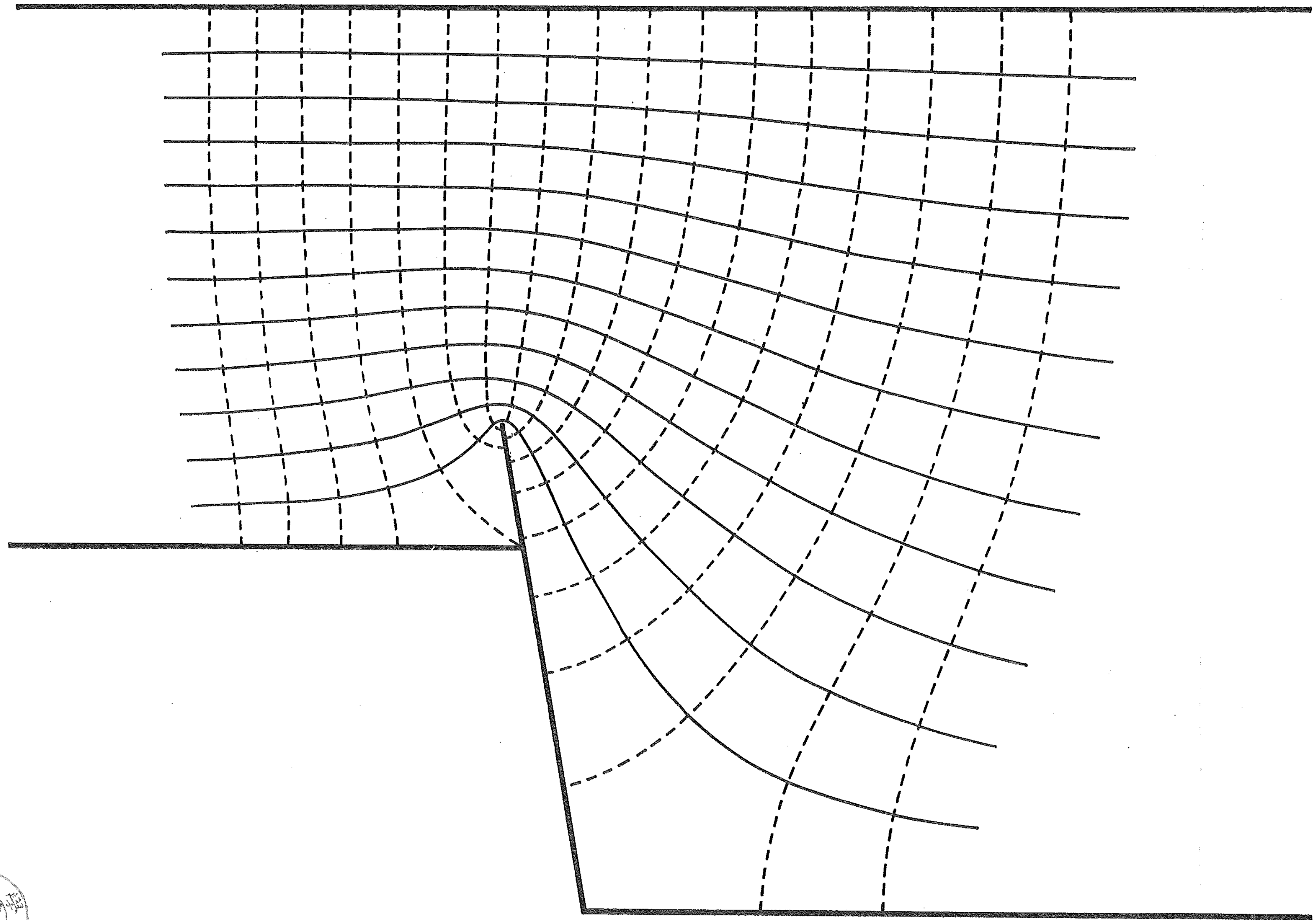
Y. Ikeda, S. Kobayasi and T. Ueki: Parallel Stream Disturbed by Barriers and Gates.



Y. Ikeda, S. Kobayasi and T. Ueki: Parallel Stream Disturbed by Barriers and Gates.



Y. Ikeda, S. Kobayasi and T. Ueki: Parallel Stream Disturbed by Barriers and Gates.



Y. Ikeda, S. Kobayasi and T. Ueki: Parallel Stream Disturbed by Barriers and Gates.