Streamlines near the Rectangular Edge.

By

Y. IKEDA, S. KOBAYASI, T. SAKURAI and F. TATIBANA.

In this article it is shown how to draw the stream and equi-potential lines in the domain having the following boundaries.

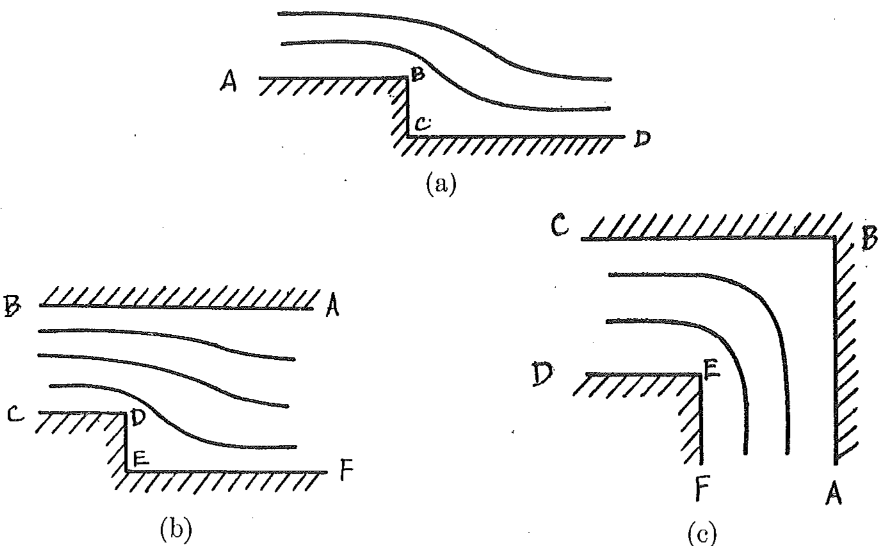
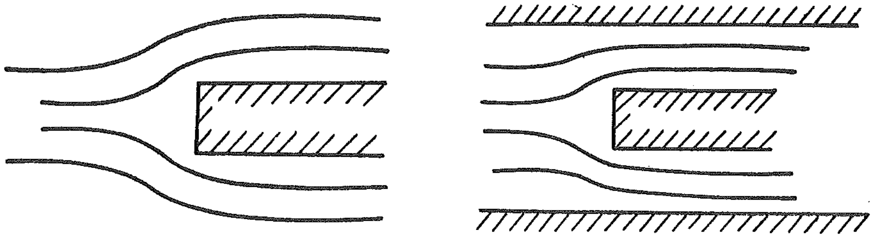
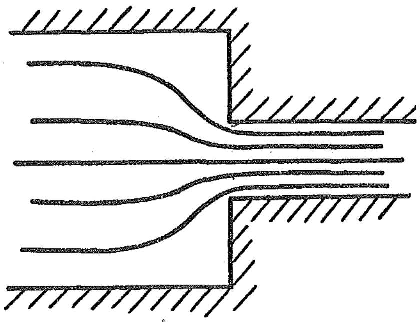


Fig. 1.

If the curves be drawn symmetrically to the boundary line, the following four conformal representations can be obtained directly.



(a′) (b″)


(b′)

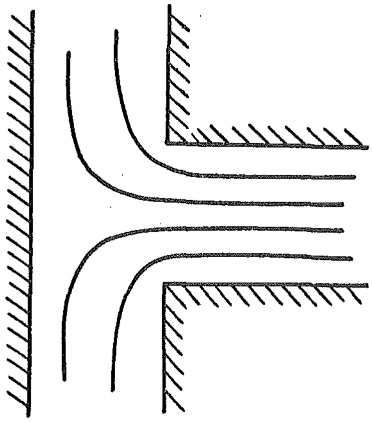

(c′)

Fig. 2.

All these conformal representations are given by Schwarz-transformation. The curves in the domains given in Fig. 1. can be transformed conformally into the curves in the half infinite plane by the following functions respectively ;

$$(1) \qquad Z = k_1\int_0^z \sqrt{\frac{z}{1-z}}\,dz + k_2$$

$$(2) \qquad Z = k_1\int_0^z \frac{1}{z}\sqrt{\frac{a-z}{1-z}}\,dz + k_2$$

$$(3) \qquad Z = k_1\int_0^z \sqrt[4]{\frac{1}{z^3(1-z)}}\,dz + k_2$$

The constant k_1 determines the absolute direction and magnitude, and constant k_2 determines the position of the origin. As only the relative position of the curves is needed, we put $k_1 = 1$ and $k_2 = 0$.
In order to obtain the conformal representation in the domain shown in Fig. 1, we integrate (1)

$$Z = \int\sqrt{\frac{z}{1-z}}\,dz = \sin^{-1}\sqrt{z} - \frac{1}{2}\sin\left\{2\sin^{-1}\sqrt{z}\right\}$$

Now put $\sqrt{z} = u$, then

$$Z = \sin^{-1}u - \frac{1}{2}\sin\left\{2\sin^{-1}u\right\} .$$

From the conformal representation obtained by the function $Z_1=\sin^{-1} u$, it is easily proved that the families of confocal ellipse and hyperbola correspond to the lines parallel to the coordinate axes drawn in the domain shown in Fig. 3 and that the first quadrant of Fig. 3 (a) corresponds to the domain bounded by the dotted line.

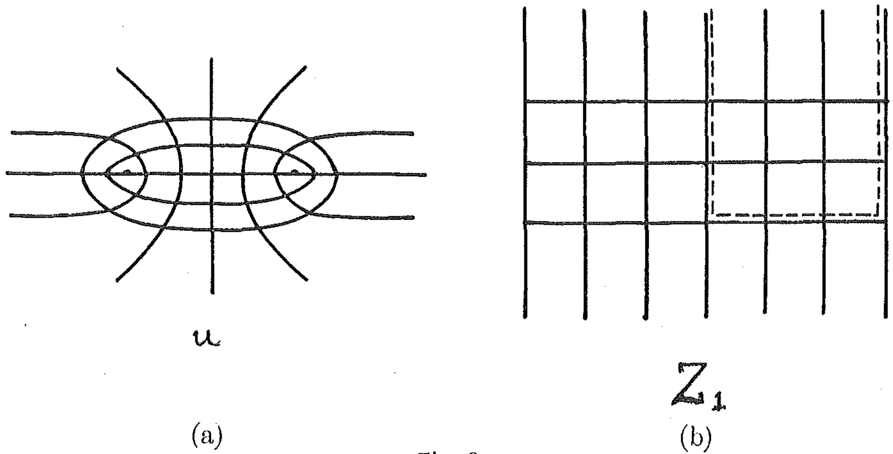


Fig. 3.

Now put $\sin(2Z_1) = Z_2$, then the domain given by Fig. 4 (a) corresponds to the domain given by Fig. 4 (b)

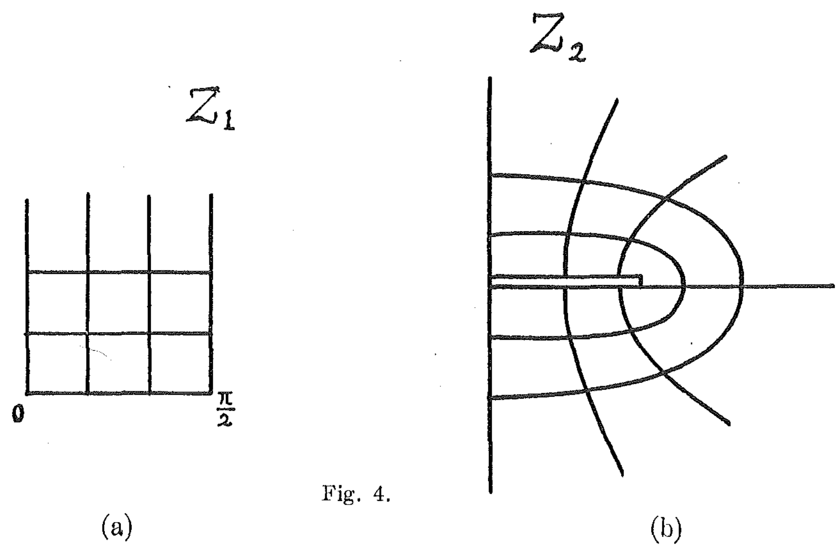


Fig. 4.

From these figures one can have the values Z_1 and Z_2 which correspond to the same value of u. Consequently if we plot the loci of the real and imaginary parts of $u=$ const. in Z-plane, we have Pl. I. From the definition $u=\sqrt{z}$ we have the following correspondence.

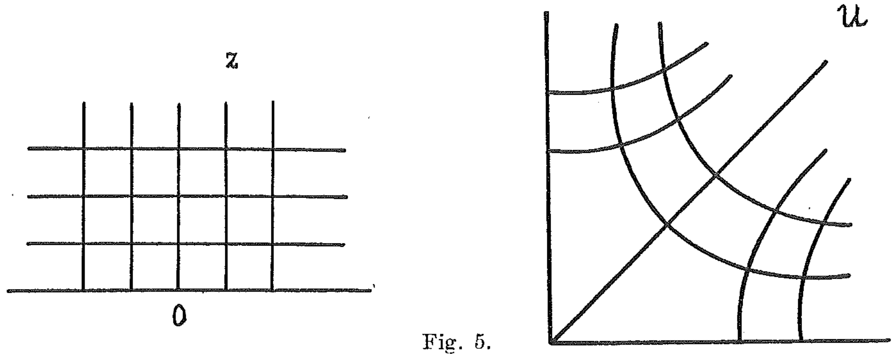


Fig. 5.

(a) (b)

Put Fig. 5 (b) on Fig. 3 (a) and transform the curves of Fig. 5 (b) according to method II conformally in the domain given in Pl. I. Then Pl. II is obtained.

If again the curves in Pl. II be drawn symmetrically to the boundary CD, then Fig. 2 (a′) is obtained which is more accurately drawn in Pl. III.

Now consider the integral (2)

$$Z=\int\frac{1}{z}\sqrt{\frac{a-z}{1-z}}\,dz=\sqrt{a}\,\log\frac{2a-(1+a)z-2\sqrt{a}\sqrt{(1-z)(a-z)}}{z(1-a)}$$

$$-\log\frac{(1+a)-2z-2\sqrt{(1-z)(a-z)}}{1-a}$$

By putting

$$Z_1=\sqrt{a}\,\log\frac{2a-(1+a)z-2\sqrt{a}\sqrt{(1-z)(a-z)}}{z(1-a)}\,,$$

one has directly

$$e^{\frac{Z_1}{\sqrt{a}}}=\frac{2a-(1+a)z-2\sqrt{a}\sqrt{(1-z)(a-z)}}{z(1-a)}$$

$$e^{-\frac{Z_1}{\sqrt{a}}}=\frac{2a-(1+a)z+2\sqrt{a}\sqrt{(1-z)(a-z)}}{z(1-a)}$$

Therefore

$$\frac{e^{\frac{Z_1}{\sqrt{a}}}+e^{-\frac{Z_1}{\sqrt{a}}}}{2}=\frac{2a-(1+a)z}{z(1-a)}=\cosh\frac{Z_1}{\sqrt{a}} .$$

Thus it follows

$$Z_1=\sqrt{a}\cosh^{-1}\left\{\frac{2a-(1+a)z}{z(1-a)}\right\} .$$

In a similar way by putting

$$Z_2=\log\frac{(1+a)-2z-2\sqrt{(1-z)(a-z)}}{1-a} ,$$

it follows

$$e^{Z_2}=\frac{1+a-2z-2\sqrt{(1-z)(a-z)}}{1-a} ,$$

$$e^{-Z_2}=\frac{1+a-2z+2\sqrt{(1-z)(a-z)}}{1-a} .$$

Consequently

$$\frac{e^{Z_2}+e^{-Z_2}}{2}=\frac{1+a-2z}{1-a}=\cosh Z_2 ,$$

$$Z_2=\cosh^{-1}\left\{\frac{1+a-2z}{1-a}\right\} .$$

Put

$$\chi=\frac{2a-(1+a)z}{z(1-a)}=\frac{2a}{1-a}\ \frac{1}{z}-\frac{1+a}{1-a} ,$$

$$b=\frac{2a}{1-a} .$$

Then one obtains

$$z=\frac{b}{\chi+1+b} .$$

Assuming $1>a>0$, one obtains the loci of $x=$ const. and $y=$ const. as shown in Fig. 6.

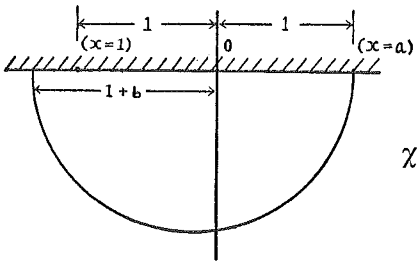
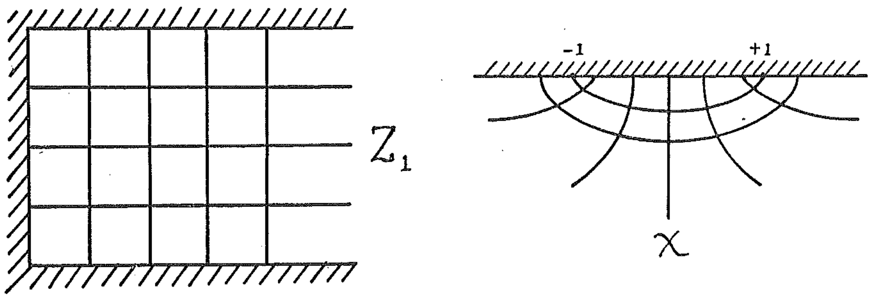


Fig. 6.

The correspondence given by the function $Z=\sin^{-1}z$ is shown in Fig. 3 (a) and (b). Now put $Z=iZ_1-\frac{\pi}{2}$ and $z=e^{i\pi}\chi$ in the formula $Z=\sin^{-1}z$, then one obtains $-\chi=\sin\left(iZ_1-\frac{\pi}{2}\right)=-\cos iZ_1=-\cosh Z_1$.



(a) (b)

Fig. 7.

Thus such a correspondence is obtained as shown in Fig. 7, (a) and (b). Put Fig. 6 on Fig. 7 (b) so that the points $\chi=\pm1$ coincide with each other. According to method II, all the curves in Fig. 6 can be conformally transformed into the domain shown in Fig. 7 (a). Thus Fig. 9 (a) is obtained.

Again by putting

$$\frac{1+a-2z}{1-a}=\chi\ ,$$

One obtains Fig. 8.

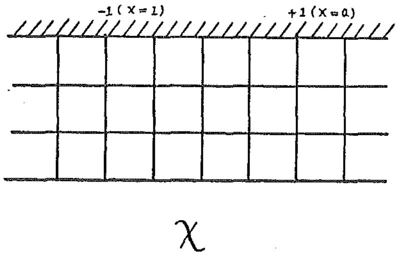
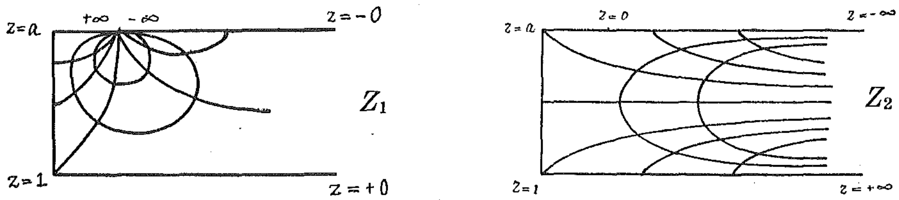


Fig. 8.

Now let Fig. 8 be put on Fig. 7 (b) and in the same way as above, the lines in Fig. 8 be transformed conformally in the domain shown in Fig. 7 (a). Thus Fig. 9 (b) is obtained.



(a) (b)

Fig. 9.

From Fig. 9 (a) and (b) one gets the values of Z_1 and Z_2 which correspond to the same value of z. Therefore one can obtain the conformal representation in Z-plane by method IV, which is shown in Pl. IV.

In order to have the streamlines which begin from $z = 0$ and tend to $z = \infty$, $\zeta = \log z$ is used instead of z itself. The loci of $\xi = \text{const.}$ and $\eta = \text{const.}$ of the function $\zeta = \xi + i\,\eta = \log z$ are the concentric circles and radial lines in z-plane. Put the curves on Fig. 5 (a) and by method II transform all the curves conformally in the same domain with Pl. IV. Thus Pl. V is obtained.

Again if one draws the curves symmetrically to the boundary line AB of Fig. 1 (b) he obtains Pl. VI. If the curves in Pl. V be drawn symmetrically to the boundary line EF, then Pl. VII is obtained.

Lastly we consider the integral

$$Z = k\int \sqrt[4]{\frac{1}{z^3(1-z)}}\,dz = 4e^{\frac{-\pi i}{4}}k\int\frac{dt}{1-t^4}$$

where

$$t^4 = \frac{z}{z-1}$$

This integral can be expressed by the sum of two arc-cosines as shown in the following

$$Z = ke^{\frac{\pi i}{4}}\left\{\cosh^{-1}\frac{1+\sqrt{\frac{z}{z-1}}}{1-\sqrt{\frac{z}{z-1}}} + \cos^{-1}\frac{1-\sqrt{\frac{z}{z-1}}}{1+\sqrt{\frac{z}{z-1}}}\right\}$$

It is possible to put $ke^{\frac{\pi i}{4}} = 1$, as has been remarked at the beginning of this article.

Now put

$$\frac{\sqrt{z-1}+\sqrt{z}}{\sqrt{z-1}-\sqrt{z}} = \zeta\ ,$$

then

$$Z = \cosh^{-1}\zeta + \cos^{-1}\zeta$$

If we put

$$\zeta + \frac{1}{\zeta} = \Psi_1\ ,$$

then

$$\Psi_1 = 2-4z$$

The correspondences between Ψ, z and ζ can be easily seen in Fig. 10, (a), (b) and (c).

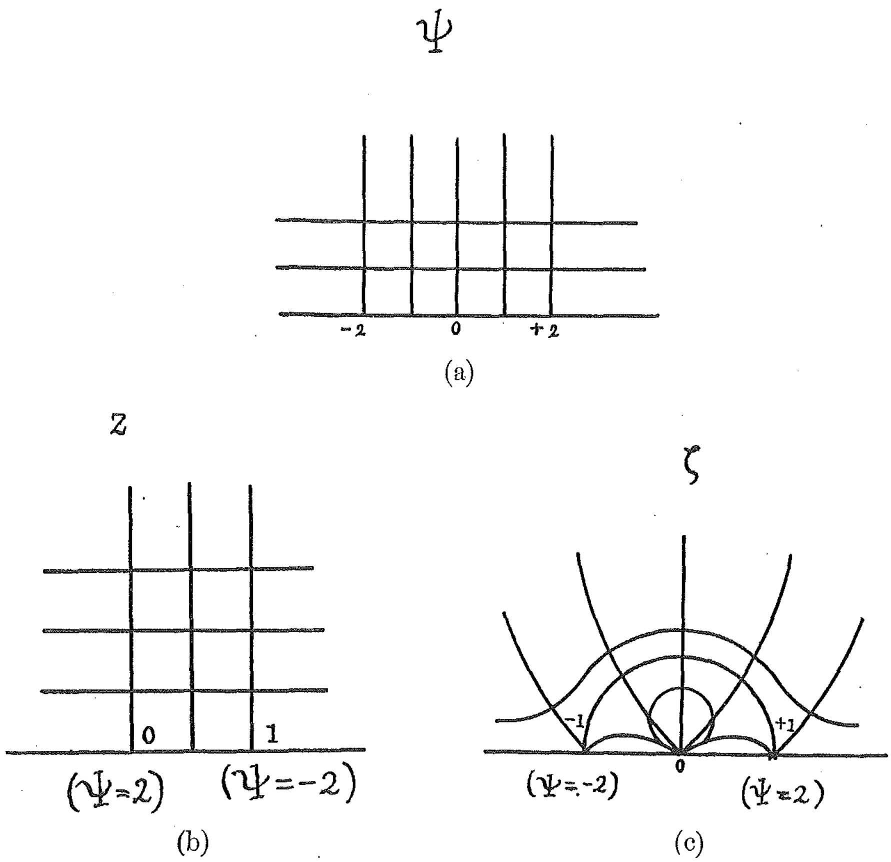


Fig. 10.

If the curves of Fig. 10 (c) be put on Fig. 8, so that the points (1, 0) (—1, 0) of Fig. 10 (c) coincide with the points (—1, 0) (1, 0) of Fig. 8 respectively, and if the curves be transformed conformally in the same domain with Fig. 9 (b), then Fig. 11 (a) is obtained. As easily seen it is very fortunate that the two parts separated by the dotted line in Fig. 11 (a) correspond to the conformal representation by means of the functions $\cosh^{-1}\zeta$ and $\cosh^{-1}\frac{1}{\zeta} = -i\cos^{-1}\frac{1}{\zeta}$ respectively. Therefore the figures are placed in such a position as shown in Fig. 11 (b) and the coordinates of the points which correspond to the same point z are added. Then by method IV Fig. 11 (c) is obtained, which is exactly drawn in Pl. VIII.

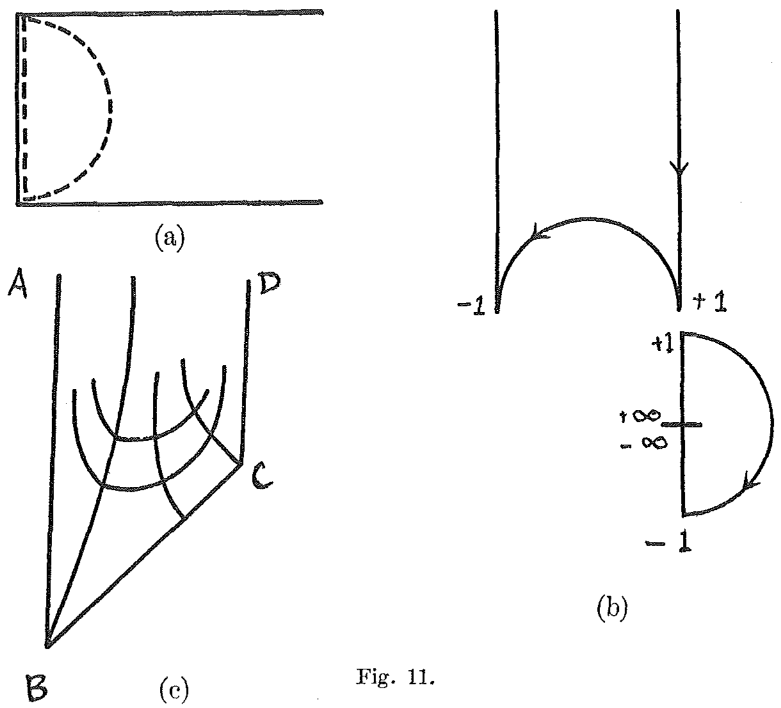
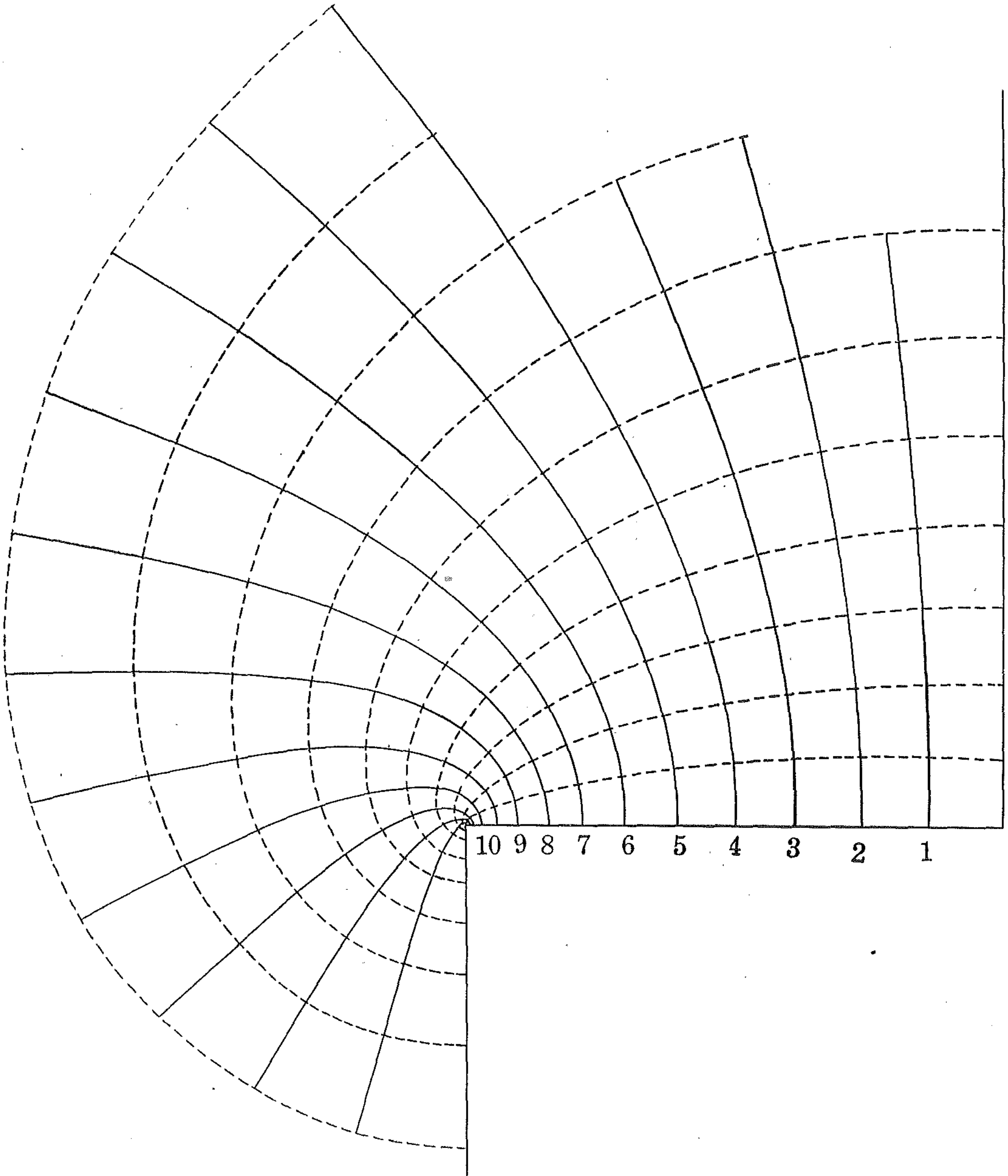


Fig. 11.

Again let $u = \log z$ be used instead of z. The families of concentric circle and radial line can be transformed conformally into two families of curves in the same domain as shown in Fig. 11 (c) according to method II. If the curves thus obtained be drawn symmetrically to the line BC of Fig. 11 (c), Pl. IX is obtained and if again the curves be drawn symmetrically to the boundary line of Pl. IX, Pl. X is obtained.



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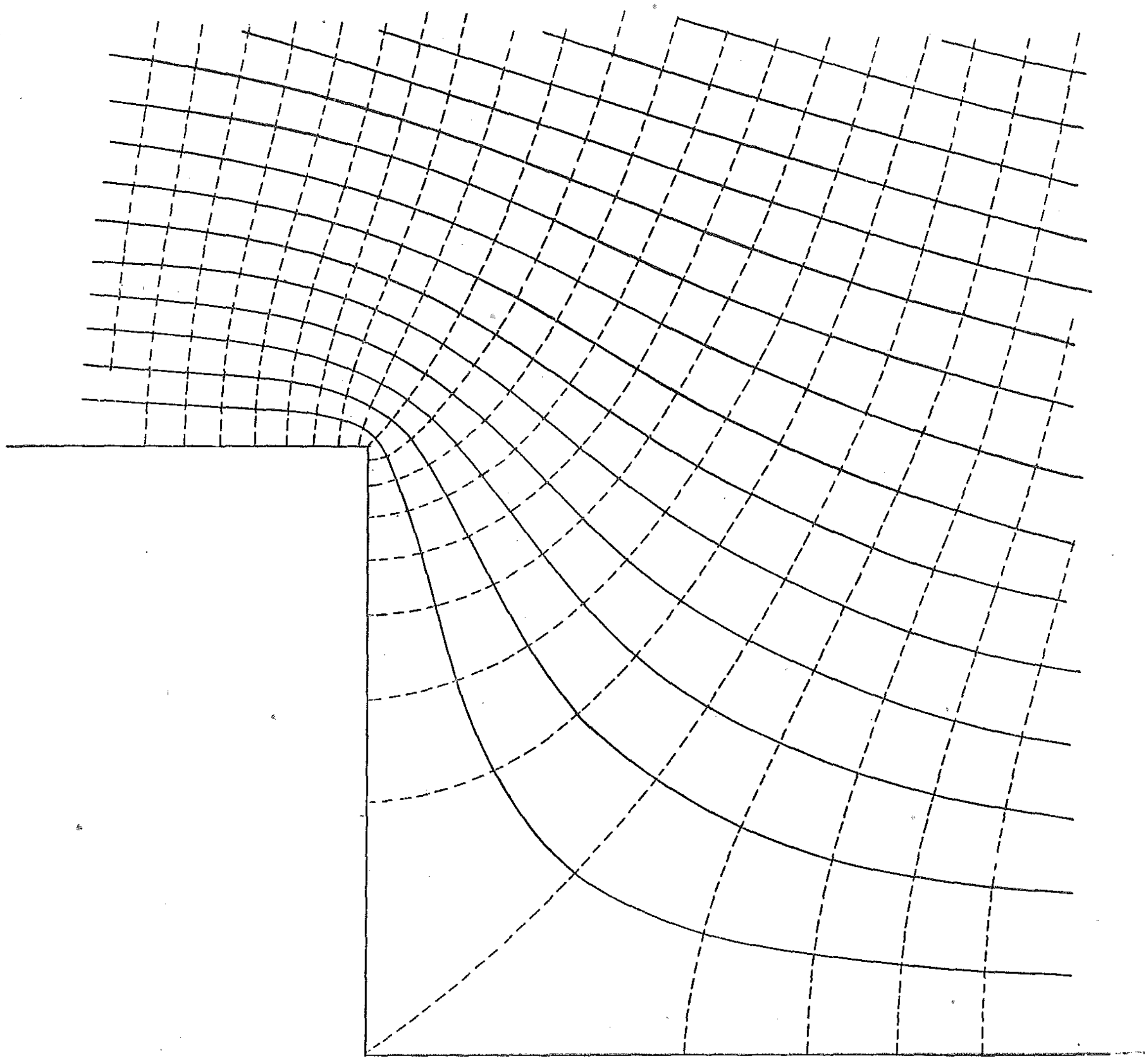

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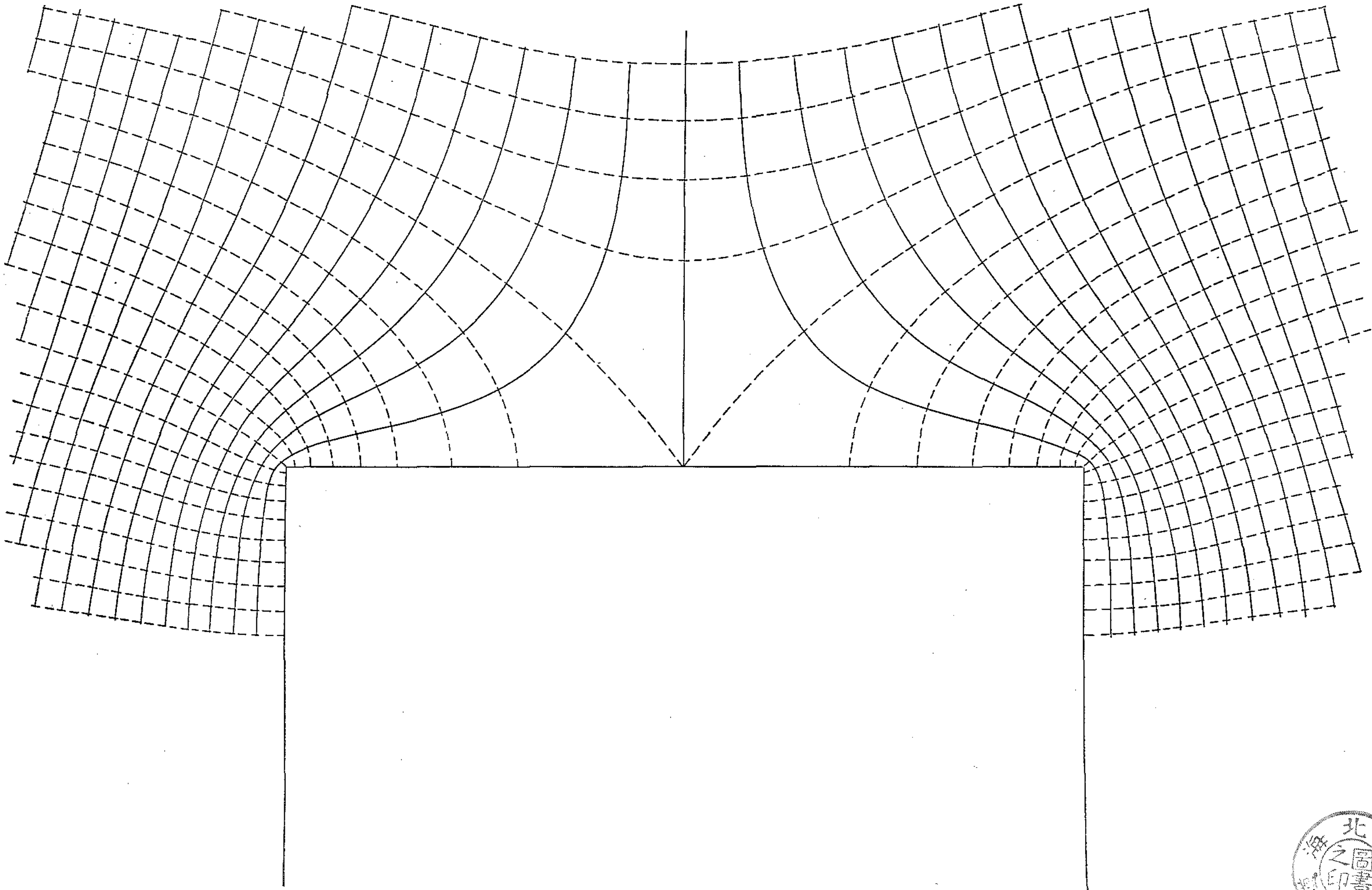

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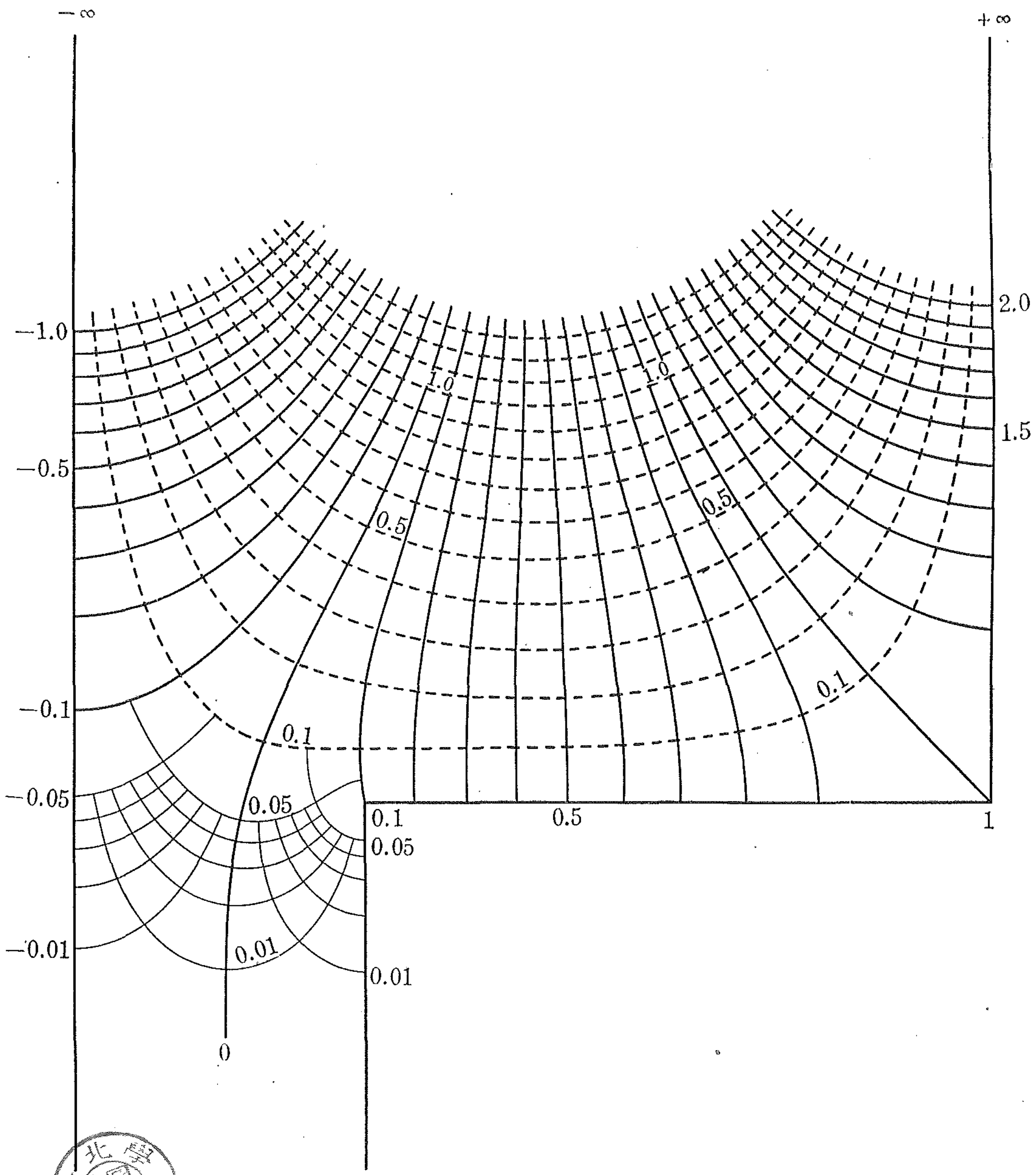

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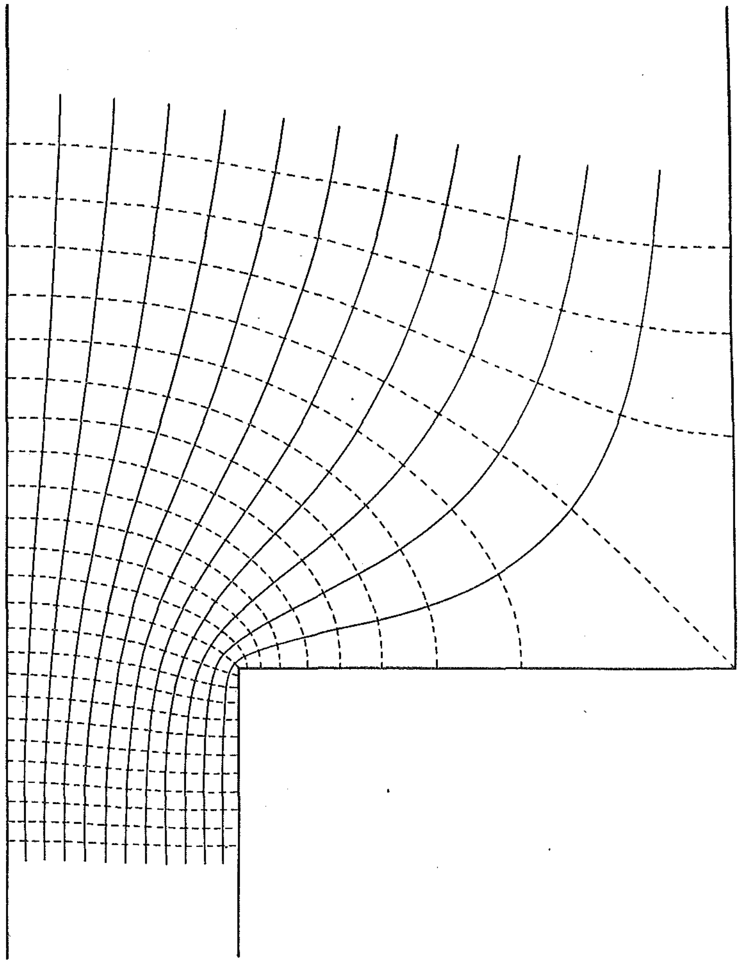

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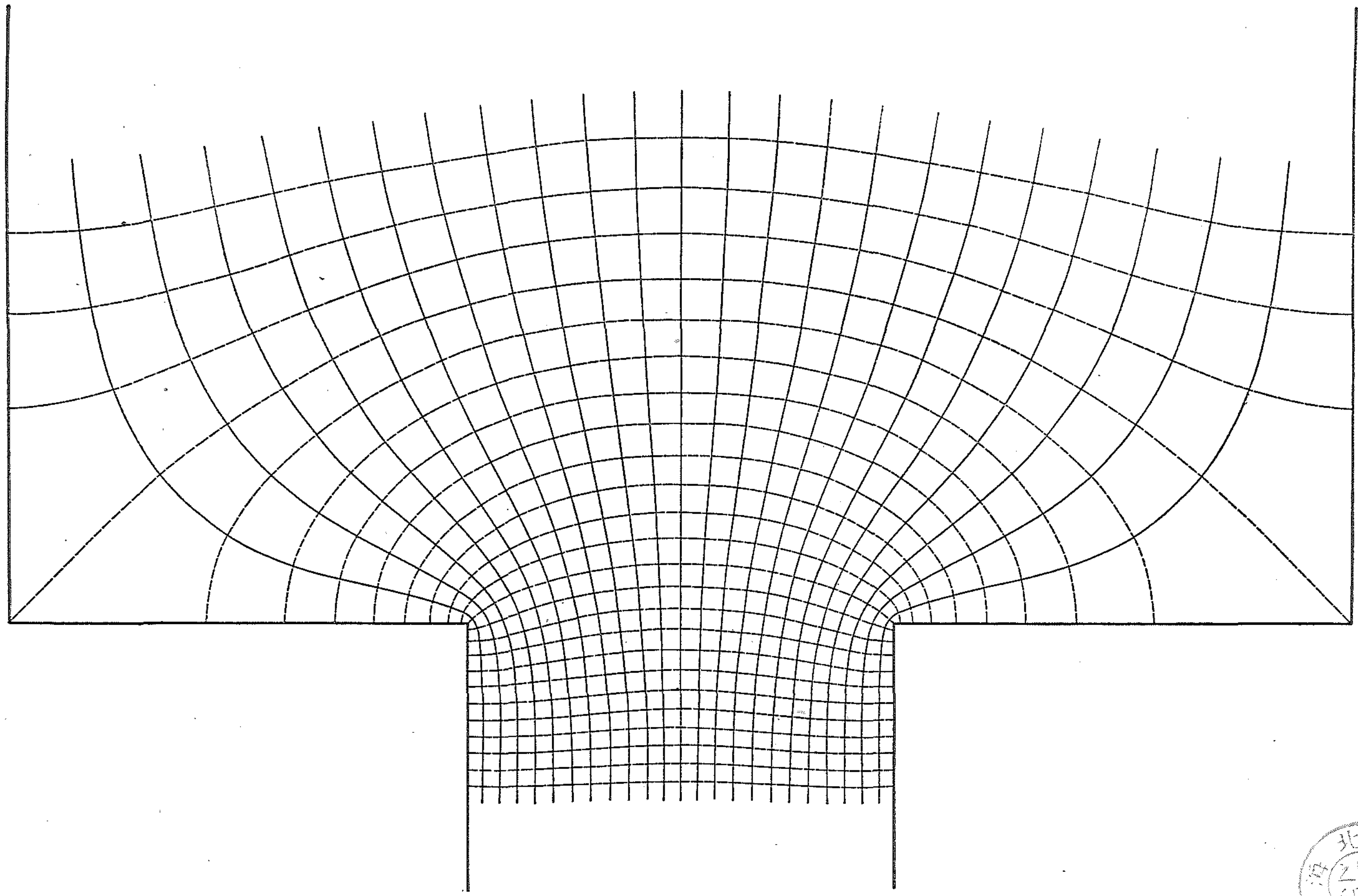

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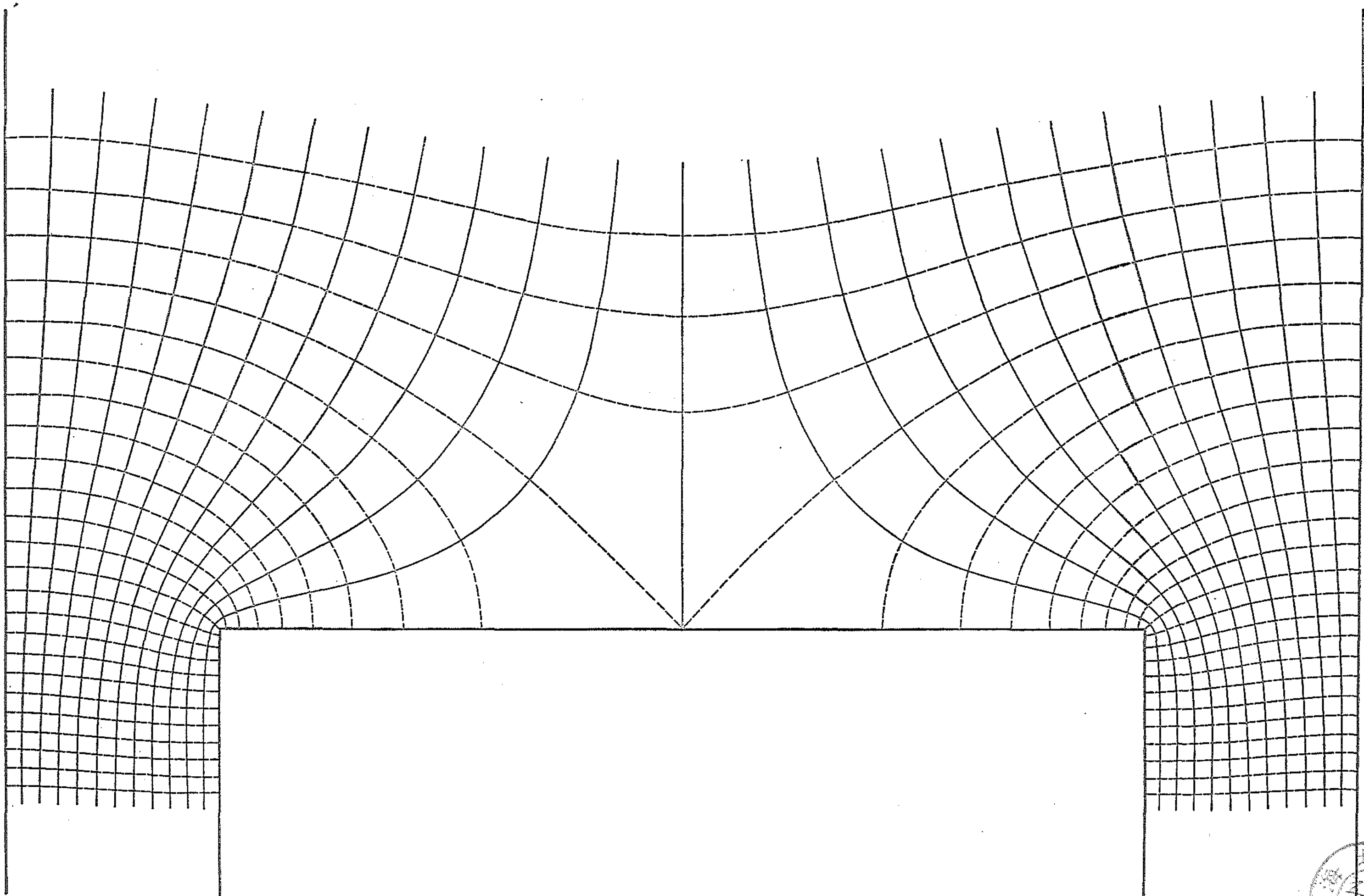

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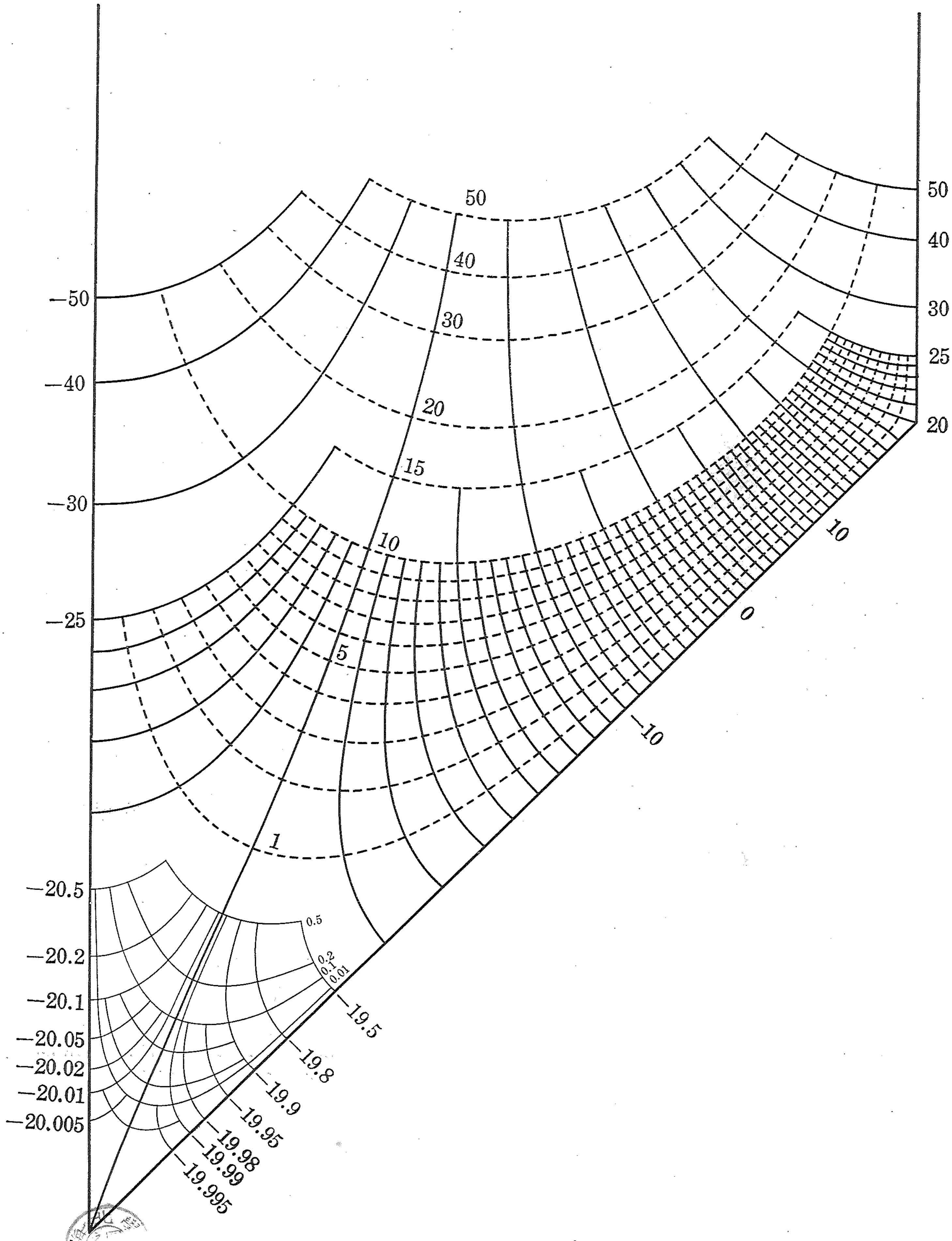

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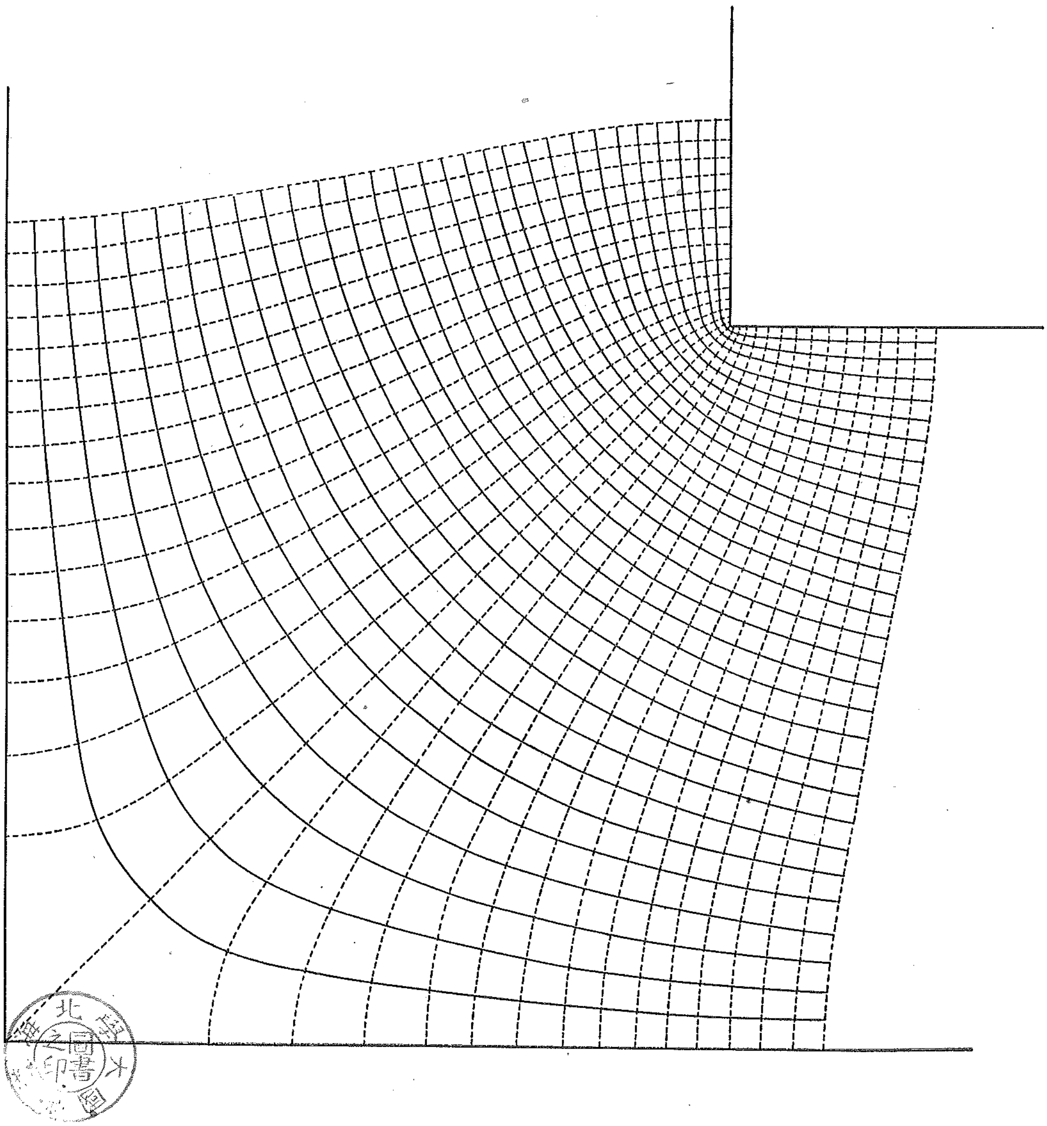

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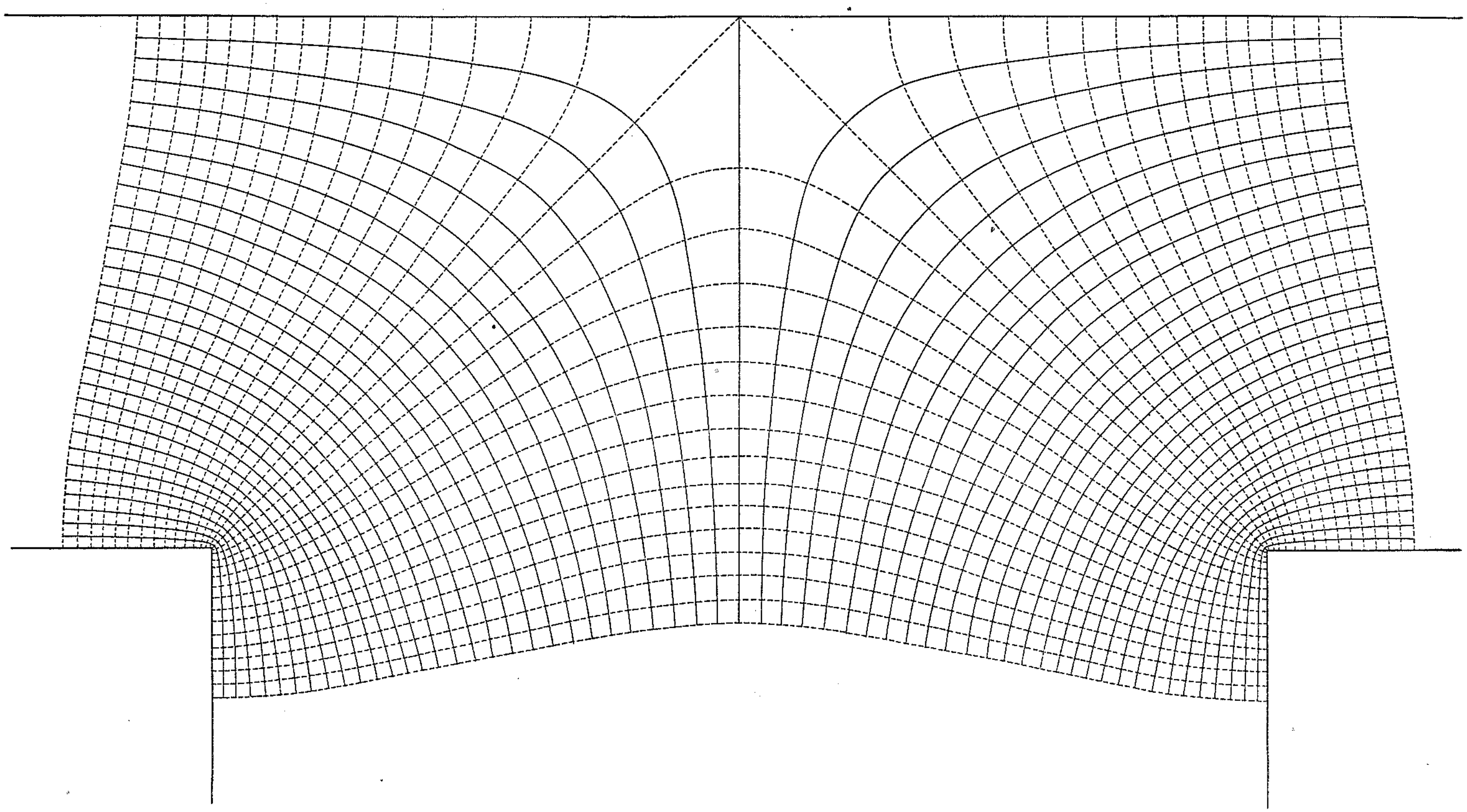

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