Practical Calculation Formulae of Parallel Chord Vierendeel Trusses with Constant Stiffness for Full Loads Derived from "Differenzen-gleichung" Method.

By

Tadaaki SAKAI.

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INTRODUCTION.

With the increase in the actual use in construction of frames with stiff connecting joints, composed of rectangular elements, the proper treatment of indeterminate stresses has been given considerable attention in recent years.

The Vierendeel truss is one of the important examples of such frames. Many authorities have presented various methods of solution for the Vierendeel truss from the standpoint both of accuracy and of rapidity.

For the solution of the parallel chord Vierendeel truss with equal values of stiffness for all the members, comparatively simple methods have already been proposed by Dr. K. Kriso* and T. Nakajima.**

However, as compared with the calculation for statically determinate structures, even their methods still require rather complicated calculation.

The author's thirty six new formulae proposed in this paper give quickly and directly the end moments, direct stresses and shearing stresses in any members and even the truss deflections for parallel chord Vierendeel trusses with equal values of stiffness for all the members and with full joint loads having equal intensity. These formulae were obtained from the application of the "differenzen-gleichung" method.

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* K. Kriso: Statik der Vierendeelträger.

** T. Nakajima: Vierendeel Trusses of Parallel Chords, Civil Engineerings, Japan, Vol. VI, No. 8, 1937.

The grade of accuracy of the results is quite the same as by the slope deflection method and the proposed formulae are also applicable for the preliminary design of Vierendeel trusses of different stiffness in the case of full loads.

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I. PRACTICAL CALCULATION FORMULAE.

In the parallel chord Vierendeel trusses with equal stiffness values of K for all the members and with full joint loads having the equal intensity of P as shown in Fig. 1, one denotes by

h the height of the truss,

λ the panel length of the truss,

n the total number of panels in the truss,

$M_{m\cdot l}$, $M_{\overline{m}\cdot l}$ the moments at the left ends of the m-th upper and lower chord members respectively,

$M_{m\cdot r}$, $M_{\overline{m}\cdot r}$ the moments at the right ends of the m-th upper and lower chord members respectively,

$M_{m\cdot v}$, $M_{\overline{m}\cdot v}$ the moments at the upper and lower ends of the vertical member $m-\overline{m}$ respectively,

N_m, $N_{\overline{m}}$ the direct stresses in the m-th upper and lower chord members respectively,

Q_m, $Q_{\overline{m}}$ the shearing stresses in the m-th upper and lower chord members respectively,

$Q_{m\cdot v}$ the shearing stresses in the vertical member $m-\overline{m}$.

y_m the truss deflection at the panel joint m.

Then, from the results described in the following article the "Practical Calculation Formulae" are proposed for the simple and speedy calculation of the end moments, direct stresses, shearing stresses and truss deflections as follows:

Left End Moments of Chord Members. Coeff.: $-P\cdot\lambda$

$$M_{1\cdot l} = 0.14088\,n-0.1819 \quad \ldots\ldots (1)$$

$$M_{2\cdot l} = 0.12702\,n-0.4551 \quad \ldots\ldots (2)$$

$$M_{3\cdot l} = 0.12525\,n-0.7079 \quad \ldots\ldots (3)$$

$$M_{4\cdot l} = 0.12504\,n-0.9586 \quad \ldots\ldots (4)$$

$$M_{m\cdot l} = 0.125\,n-0.25\,m+0.0416 \quad \text{for} \quad 4<m<n-4 \quad \ldots\ldots (5)$$

$$M_{\overline{m}\cdot l} = M_{m\cdot l} \quad \ldots\ldots (6)$$

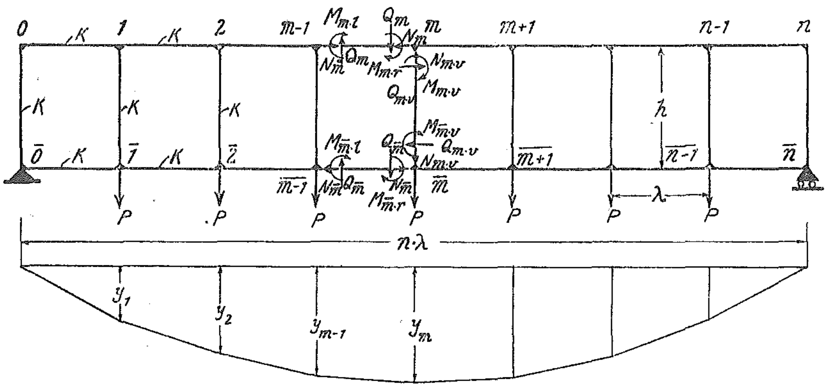

Fig. 1.

Right End Moments of Chord Members. Coeff.: $-P\cdot\lambda$

$M_{1\cdot r} = 0.10912\,n - 0.0681$ .... (7)

$M_{2\cdot r} = 0.12298\,n - 0.2949$ .... (8)

$M_{3\cdot r} = 0.12475\,n - 0.5421$ .... (9)

$M_{4\cdot r} = 0.12495\,n - 0.7913$ .... (10)

$M_{m\cdot r} \doteq 0.125\,n - 0.25\,m + 0.2083 \quad \text{for} \quad 4 < m < n-4$ .... (11)

$M_{\overline{m}\cdot r} = M_{m\cdot r}$ .... (12)

End Moments of Vertical Members. Coeff.: $P\cdot\lambda$

$M_{0\cdot v} = 0.14088\,n - 0.1819$ .... (13)

$M_{1\cdot v} = 0.23614\,n - 0.5231$ .... (14)

$M_{2\cdot v} = 0.24823\,n - 1.0028$ .... (15)

$M_{3\cdot v} = 0.24979\,n - 1.5007$ .... (16)

$M_{4\cdot v} = 0.24996\,n - 1.9997$ .... (17)

$M_{m\cdot v} = 0.25\,n - 0.5\,m \quad \text{for} \quad 4 < m < n-4$ .... (18)

$M_{\overline{m}\cdot v} = M_{m\cdot v}$ .... (19)

Direct Stresses of Chord Members. Coeff.: $\dfrac{P\cdot\lambda}{h}$

$N_1 = 0.28176\,n - 0.3639$ .... (20)

$N_2 = 0.75404\,n - 1.4102$ .... (21)

$N_3 = 1.25051\,n - 3.4158$ .... (22)

$N_4 = 1.75009\,n - 6.4173$ .... (23)

$$N_m = \left(\frac{m}{2}-0.25\right)n-\frac{1}{2}m(m-1)-0.4167$$

$$\text{for } 4<m<n-4 \quad \ldots\ldots \quad (24)$$

$$N_{\bar{m}} = -N_m \quad \ldots\ldots \quad (25)$$

Direct Stresses of Vertical Members.

$$N_{0\cdot v} = \frac{n-1}{4}P \quad \text{for lower joint loads} \quad \ldots\ldots \quad (26)$$

$$N_{0\cdot v} = \frac{n+1}{4}P \quad \text{for upper joint loads} \quad \ldots\ldots \quad (27)$$

$$N_{m\cdot v} = -\frac{P}{2} \quad \text{for lower joint loads, for } 0<m<n \quad \ldots \quad (28)$$

$$N_{m\cdot v} = \frac{P}{2} \quad \text{for upper joint loads, for } 0<m<n \quad \ldots \quad (29)$$

Shearing Stresses of Chord Members.

$$Q_m = Q_{\bar{m}} = (0.25\,n+0.25-0.5\,\mathrm{m})P \quad \ldots\ldots \quad (30)$$

Shearing Stresses of Vertical Members.

$$Q_{m\cdot v} = -\frac{2}{h}M_{m\cdot v} \quad \ldots\ldots \quad (31)$$

Truss Deflections. Coeff. : $\frac{P\lambda^2}{EK}$

$$y_1 = 0.05225\,n-0.0796 \quad \ldots\ldots \quad (32)$$

$$y_2 = 0.11345\,n-0.2692 \quad \ldots\ldots \quad (33)$$

$$y_3 = 0.17578\,n-0.5820 \quad \ldots\ldots \quad (34)$$

$$y_4 = 0.23826\,n-1.0194 \quad \ldots\ldots \quad (35)$$

$$y_m = 0.23826\,n+0.0625\left\{(m-4)(n+1)-m(m+1)\right\}+0.2305$$

$$\text{for } 4<m<n-4 \quad \ldots\ldots \quad (36)$$

In regard to the conventional signs of the quantities used in the equations, the sign of the direct stress indicates the properties of the stress, so a plus sign will then signify a compressive stress and a minus sign a tensile stress, while the sign of moments and shearing stresses is considered positive when those stresses tend to cause a clockwise rotation of the member.

Since a truss and its loading are symmetrical about the span centre, it is sufficient to make the calculation for the half of the truss, i.e, on the left side of the span centre.

These "Practical Calculation Formulae" remarkably simplify the calculation of the Vierendeel trusses with constant stiffness.

For example, the direct stresses in the upper chord members of the parallel chord Vierendeel truss with eight panels and with full lower joint loads having the equal intensity of 1000 kg are calculated as follows:

This truss is assumed to have equal values of K for all the members and $\frac{\lambda}{h}=1$.

In this example,

$$n = 8,\quad P = 1000\,\text{kg},\quad \frac{\lambda}{h}=1$$

Substituting these values into the Practical Calculation Formulae (20) to (23), the direct stresses in upper chord members can be instantly obtained as follows:

$$N_1 = (0.28176\times 8-0.3639)\times 1000\times 1 = 1890\,\text{kg}\ \ (1890)$$
$$N_2 = (0.75404\times 8-1.4102)\times 1000\times 1 = 4622\,\text{kg}\ \ (4620)$$
$$N_3 = (1.25051\times 8-3.4158)\times 1000\times 1 = 6588\,\text{kg}\ \ (6590)$$
$$N_4 = (1.75009\times 8-6.4173)\times 1000\times 1 = 7583\,\text{kg}\ \ (7580)$$

The values in brackets are those calculated by Dr. K. Kriso.

II. DERIVATION OF PRACTICAL CALCULATION FORMULAE.

1. Assumptions.

The analysis in this paper is based upon the following assumptions:

(1). The stiffness values of members, i.e., the values of moment of inertia of section divided by length are equally K for the upper and lower chords and kK for the vertical members.

(2). The truss is loaded with P at every joint, whether lower or upper joint.

(3). The connections between the vertical members and chord members are perfectly rigid.

(4). The length of a member is not changed by direct stress and the deformation of a member due to the internal shearing stress is zero.

2. Fundamental Equations of Slope-Deflection.

According to assumptions (3) and (4), the moments at the ends of the member are expressed by the well known slope-deflection equation :

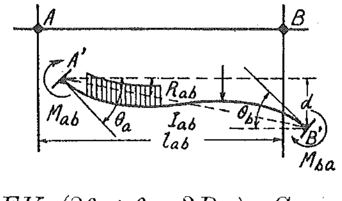

Fig. 2.

$$M_{ab} = 2EK_{ab}(2\theta_a+\theta_b-3R_{ab})-C_{ab} \quad \ldots\ldots\ldots\ldots\ldots \quad (1)$$

In this equation, one denotes by

M_{ab} the end moment at A,
E the modulus of elasticity of the materials,
θ_a, θ_b the joint-rotation angles at the ends A and B respectively,
K_{ab} the stiffness of the member i.e., the moment of inertia of the section divided by the length of the member AB,
R_{ab} the member-revolution angle,
$C_{ab} = \frac{2A}{l_{ab}^2}(3\xi-l_{ab})$, in which A = area of the moment of diagram of a simple beam AB due to intermediate loads; l_{ab} = length of the member AB; ξ = distance of the centroid of the area A from the end B.

The conventional sign of the quantities used in the equation require further explanation as follows:

The end moment of a member is considered positive when it tends to cause a clockwise rotation. The joint-rotation angle is considered positive when the angle has turned clockwise, measured from its initial position. The member-revolution angle is also positive in case of revolution in clockwise direction from the initial position of the member.

In ordinary construction of trusses, panel points carry all the loads and the chord members carry no intermediate loads. In consequence of this loading condition, the load term C_{ab} vanishes in equation (1).

Putting in equation (1)

$$2E\theta_a = \varphi_a, \quad 2E\theta_b = \varphi_b, \quad -6ER_{ab} = \Psi_{ab}, \quad C_{ab} = 0$$

the slope-deflection equation becomes

$$M_{ab} = K_{ab}(2\varphi_a+\varphi_b+\Psi_{ab}) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

For the member which has no member-revolution angle, the term ψ_m vanishes and the slope-deflection equation becomes

$$M_{ab} = K_{ab}(2\varphi_a + \varphi_b) \qquad \text{(3)}$$

In Fig. 3 the intersections of the neutral axes of the chord members with the neutral axes of the vertical members are denoted by $0, 1, 2, \ldots, m-1, m, m+1, \ldots, n$ for the upper chord panel joints and by $\bar{0}, \bar{1}, \bar{2}, \ldots, \overline{m-1}, \bar{m}, \overline{m+1}, \ldots, \bar{n}$ for the lower chord panel joints, beginning at the left and reading toward the centre.

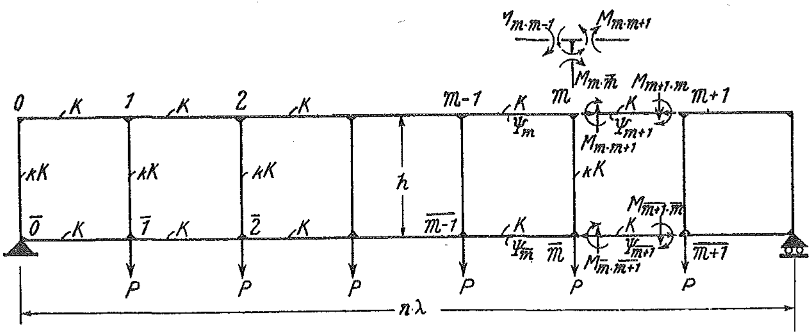

Fig. 3.

Assumption (4) makes the value of ψ equal both for an upper chord member and the lower chord member in the same panel.

Therefore,

$$\psi_m = \psi_{\bar{m}} \qquad \text{(4)}$$

where, ψ_m and $\psi_{\bar{m}}$ denote member-revolution angles of the m-th upper and lower chord members respectively numbered from the left support of the truss, that is, $\psi_m = \psi_{m-1 \cdot m}$ and $\psi_{\bar{m}} = \psi_{\overline{m-1} \cdot \bar{m}}$.

For all the vertical members the value of ψ becomes equal according to the same assumption and moreover this value is zero in the case of symmetrical loadings as the case mentioned in assumption (2).

Therefore,

$$\psi_{m\bar{m}} = 0 \qquad \text{(5)}$$

Next, when the chord sections are designed in order that the value of stiffness for the upper chord member may be equal to that of the lower chord member in the same panel, as the truss mentioned in assumption (1), the values of φ at both extremities of each vertical member become equal.

Therefore,

$$\varphi_m = \varphi_{\bar{m}} \quad \text{......................... (6)}$$

Thus, the following relations are also given:

$$\left.\begin{aligned} M_{m\cdot m-1} &= M_{\bar{m}\cdot\overline{m-1}} \\ M_{m\cdot m+1} &= M_{\bar{m}\cdot\overline{m+1}} \\ M_{m\cdot\bar{m}} &= M_{\bar{m}\cdot m} \end{aligned}\right\} \quad \text{...................... (7)}$$

Therefore, in the special case such as the truss mentioned in assumption (1), the treatment of the problem become very simplified.

3. Joint-Equilibrium Equation.

At any panel joint m excepting the joints at the extremities of the truss one gets (Fig. 3)

$$\left.\begin{aligned} M_{m\cdot m+1} &= K(2\varphi_m+\varphi_{m+1}+\Psi_{m+1}) \\ M_{m\cdot\bar{m}} &= 3kK\varphi_m \\ M_{m\cdot m-1} &= K(2\varphi_m+\varphi_{m-1}+\Psi_m) \end{aligned}\right\} \quad \text{.......... (8)}$$

Substituting these into the joint-equilibrium condition

$$M_{m\cdot m+1}+M_{m\cdot\bar{m}}+M_{m\cdot m-1}=0 \quad \text{................ (9)}$$

one gets the following joint-equilibrium equation:

$$\varphi_{m-1}+(4+3k)\varphi_m+\varphi_{m+1}+\Psi_m+\Psi_{m+1}=0 \quad \text{.......... (10)}$$

At panel joints 0 and n,

$$\left.\begin{aligned} M_{01} &= K(2\varphi_0+\varphi_1+\Psi_1) \\ M_{0\bar{0}} &= 3kK\varphi_0 \end{aligned}\right\} \quad \text{................... (11)}$$

and

$$\left.\begin{aligned} M_{n\cdot n-1} &= K(2\varphi_n+\varphi_{n-1}+\Psi_n) \\ M_{n\cdot\bar{n}} &= 3kK\varphi_n \end{aligned}\right\} \quad \text{............. (12)}$$

Substituting these into the joint-equilibrium conditions

$$M_{01}+M_{0\bar{0}}=0 \quad \text{............................ (13)}$$
$$M_{n\cdot n-1}+M_{n\cdot\bar{n}}=0, \quad \text{........................ (14)}$$

joint-equilibrium equation at the extremities of the truss are obtained as follows:

$$(2+3k)\varphi_0+\varphi_1+\Psi_1 = 0 \quad \dots\dots\dots\dots \quad (15)$$
$$(2+3k)\varphi_n+\varphi_{n-1}+\Psi_n = 0 \quad \dots\dots\dots\dots \quad (16)$$

4. Panel-Equilibrium Equation.

As shown in Fig. 3, imagining two vertical sections very near by panel points m and $m+1$, one gets the equilibrium condition of the upper and lower chord moments:

$$M_{m\cdot m+1}+M_{m+1\cdot m}+M_{\bar{m}\cdot\overline{m+1}}+M_{\overline{m+1}\cdot\bar{m}}+S_{m+1}\cdot\lambda = 0 \quad \dots\dots \quad (17)$$

or using the relation of (7), this condition becomes

$$2(M_{m\cdot m+1}+M_{m+1\cdot m})+S_{m+1}\cdot\lambda = 0 \quad \dots\dots\dots\dots \quad (18)$$

where S_{m+1} denotes the shearing force for the $(m+1)$-th panel numbered from the left support of the truss and λ length of panel.

While

$$\left.\begin{aligned} M_{m\cdot m+1} &= K(2\varphi_m+\varphi_{m+1}+\Psi_{m+1}) \\ M_{m+1\cdot m} &= K(2\varphi_{m+1}+\varphi_m+\Psi_{m+1}) \end{aligned}\right\} \quad \dots\dots\dots\dots \quad (19)$$

Therefore, equation (18) becomes

$$3\varphi_m+3\varphi_{m+1}+2\Psi_{m+1} = -\frac{S_{m+1}\cdot\lambda}{2K} \quad \dots\dots\dots\dots \quad (20)$$

In a similar way at the m-th panel,

$$3\varphi_{m-1}+3\varphi_m+2\Psi_m = -\frac{S_m\cdot\lambda}{2K} \quad \dots\dots\dots\dots \quad (21)$$

Equations (20) and (21) are called respectively panel-equilibrium equations at the $(m+1)$-th and m-th panels.

Summing up these two equations gives

$$3\varphi_{m-1}+6\varphi_m+3\varphi_{m+1}+2\Psi_m+2\Psi_{m+1} = -\frac{\lambda}{2K}(S_m+S_{m+1}) \quad \dots\dots \quad (22)$$

At the panels of both extremities of the truss, the panel-equilibrium equation becomes

$$3\varphi_0+3\varphi_1+2\psi_1 = -\frac{S_1\cdot\lambda}{2K} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (23)$$

$$3\varphi_{n-1}+3\varphi_n+2\psi_n = -\frac{S_n\cdot\lambda}{2K} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (24)$$

5. Differenzengleichung of φ.

Eliminating ψ from equations (10) and (22) gives

$$\varphi_{m-1}-(2+6k)\varphi_m+\varphi_{m+1} = -\frac{\lambda}{2K}(S_m+S_{m+1}) \quad \ldots\ldots \quad (25)$$

This equation is a differenzengleichung containing a series of unknown φ only.

For the full joint loads having the equal intensity of P as related in assumption (1), shearing forces in the $(m-1)$-th, m-th and $(m+1)$-th panels are expressed as follows:

$$\left.\begin{aligned} S_{m-1} &= P\left\{\frac{n+1}{2}-(m-1)\right\} \\ S_m &= P\left\{\frac{n+1}{2}-m\right\} \\ S_{m+1} &= P\left\{\frac{n+1}{2}-(m+1)\right\} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad (26)$$

where n denotes the total number of panels in a span.

Therefore

$$S_m+S_{m+1} = P(n-2m)$$

Substituting this value in equation (25) gives

$$\varphi_{m-1}-(2+6k)\varphi_m+\varphi_{m+1} = -\frac{P\lambda}{2K}(n-2m) \quad \ldots\ldots\ldots \quad (25')$$

Next, eliminating ψ_1 from equations (15) and (23) gives

$$-(1+6k)\varphi_0+\varphi_1 = -\frac{S_1\lambda}{2K} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (27)$$

or

$$-(1+6k)\varphi_0+\varphi_1 = -\frac{P\lambda}{2K}(n-1) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (27')$$

In a similar way, eliminating ψ_n from equations (16) and (24) gives

$$-(1+6k)\varphi_n+\varphi_{n-1} = -\frac{S_n\lambda}{2K} \qquad \text{(28)}$$

or

$$-(1+6k)\varphi_n+\varphi_{n-1} = \frac{P\lambda}{4K}(n-1) \qquad \text{(28')}$$

Equations (27) and (28) are the boundary conditions for the differenzengleichung of (25).

6. Particular Solution of Differenzengleichung.

A particular solution of the above obtained differenzengleichung, that is, the value of φ_m in the case when the local effect of the non-uniformity of structure at the ends of the truss depending on the abrupt change of the boundary condition is neglected, is obtained as follows :

A particular solution of the above differenzengleichung is to take the form

$$\varphi_m = am+b \qquad \text{(29)}$$

where a and b are constants to be determined here.

Substituting this equation in (25′) one gets

$$a(m-1)+b-(2+6k)(am+b)+a(m+1)+b = -\frac{P\lambda}{2K}(n-2m)$$

or

$$-6kam6-kb = \frac{P\lambda}{K}m-\frac{P\lambda}{2K}n \qquad \text{(30)}$$

Therefore

$$-6ka = \frac{P\lambda}{K}$$

$$6kb = \frac{P\lambda}{2K}n$$

or

$$\left.\begin{aligned} a &= -\frac{P\lambda}{6kK} \\ b &= \frac{P\lambda}{12kK}n \end{aligned}\right\} \qquad \text{(31)}$$

Substituting these values in equation (29) a particular solution is determined as follow :

$$\varphi_m = \frac{P\lambda}{12kK}(n-2m) \quad \text{.....................} \quad (32)$$

or

$$\varphi_m = \frac{\lambda}{12kK}(S_m+S_{m+1}) \quad \text{.................} \quad (32')$$

In the special case of $k=1$,

$$\varphi_m = \frac{P\lambda}{12K}(n-2m) \quad \text{.....................} \quad (32'')$$

Substituting equation (32′) in (21) one gets a particular solution for the member-revolution angle Ψ_m as follows :

$$\frac{3\lambda}{12kK}(S_m+S_{m-1})+\frac{3\lambda}{12kK}(S_m+S_{m+1})+2\Psi_m = -\frac{S_m\lambda}{2K}$$

or

$$\Psi_m = -\frac{\lambda}{8kK}(S_{m-1}+2S_m+S_{m+1})-\frac{S_m\lambda}{4K}$$

While, from the relation of (26) one gets

$$S_{m-1}+S_{m+1} = 2S_m$$

Therefore

$$\Psi_m = -\frac{S_m\lambda}{K}\left(\frac{1}{2k}+\frac{1}{4}\right) \quad \text{.....................} \quad (33)$$

or

$$\Psi_m = -\frac{P\lambda}{K}\left(\frac{1}{2k}+\frac{1}{4}\right)\left(\frac{n+1}{2}-m\right) \quad \text{.........} \quad (33')$$

In the special case of $k=1$,

$$\Psi_m = -\frac{3P\lambda}{4K}\left(\frac{n}{2}-m+\frac{1}{2}\right) \quad \text{................} \quad (33'')$$

Substituting equations (32′) and (33) in the equations

$$M_{m-1\cdot m} = K(2\varphi_{m-1}+\varphi_m+\Psi_m),$$
$$M_{m\cdot m-1} = K(2\varphi_m+\varphi_{m-1}+\Psi_m),$$
$$M_{m\cdot \overline{m}} = 3kK\varphi_m$$

the moment at the left and right ends of the m-th upper chord members and that at the upper end of the vertical member $m-\overline{m}$ become as follows:

$$M_{m-1\cdot m}=M_{m\cdot l}=\lambda\left\{\frac{1}{6k}S_{m-1}-\left(\frac{1}{4k}+\frac{1}{4}\right)S_m+\frac{1}{12k}S_{m+1}\right\} \quad \cdots\cdots (34)$$

or

$$M_{m-1\cdot m}=M_{m\cdot l}=-P\lambda\left(\frac{1}{8}n-\frac{1}{4}m-\frac{1}{12k}+\frac{1}{8}\right) \quad \cdots\cdots (34')$$

$$M_{m\cdot m-1}=M_{m\cdot r}=\lambda\left\{\frac{1}{12k}S_{m-1}-\left(\frac{1}{4k}+\frac{1}{4}\right)S_m+\frac{1}{6k}S_{m+1}\right\} \quad \cdots\cdots (35)$$

or

$$M_{m,\,m-1}=M_{m\cdot r}=-P\lambda\left(\frac{1}{8}n-\frac{1}{4}m+\frac{1}{12k}+\frac{1}{8}\right) \quad \cdots\cdots (35')$$

$$M_{m\cdot\overline{m}}=M_{m\cdot v}=\frac{\lambda}{4}(S_m+S_{m+1}) \quad \cdots\cdots (36)$$

or

$$M_{m\cdot\overline{m}}=M_{m\cdot v}=P\lambda\left(\frac{1}{4}n-\frac{1}{2}m\right) \quad \cdots\cdots (36')$$

In the special case of $k=1$,

$$M_{m-1\cdot m}=M_{m\cdot l}=-P\lambda\left(\frac{1}{8}n-\frac{1}{4}m+\frac{1}{24}\right) \quad \cdots\cdots (34'')$$

$$M_{m\cdot m-1}=M_{m\cdot r}=-P\lambda\left(\frac{1}{8}n-\frac{1}{4}m+\frac{5}{24}\right) \quad \cdots\cdots (35'')$$

$$M_{m\cdot\overline{m}}=M_{m\cdot v}=P\lambda\left(\frac{1}{4}n-\frac{1}{2}m\right) \quad \cdots\cdots (36'')$$

The above obtained expressions of $M_{m-1\cdot m}$, $M_{m\cdot m-1}$ and $M_{m\cdot\overline{m}}$ are applicable in the case when the local effect of the non-uniformity of structure at the both ends of the truss depending on the abrupt change of the boundary condition is not taken into consideration. But this effect is considered to be negligibly small for the end moments of members whose situation is over about four panels distant from the ends of the truss and therefore, for such members, the end moments can be easily and directly calculated from the above obtained fromulae. These formulae are all linear about n, namely the end moments of any member vary linearly by the increase of the number of panels in a span.

7. General Solution of Differenzengleichung.

For the local effect of the non-uniformity of structure at both ends of the truss, depending on the abrupt change of the boundary condition, equations (32) to (36) which were obtained as the particular solution of differenzengleichung of (25) are not applicable to the members near the ends of the truss.

For the members near each end of the truss, unknown quantities should be found from the general solution.

The general solution* of the differenzengleichung of (25′), that is, the value of φ in the case when the local effect of the non-uniformity of structure at both ends of the truss is taken into consideration are obtained as follows :

A characteristic equation of the differenzengleichung of (25′) is

$$\gamma^2-2(1+3k)\gamma+1=0 \quad \text{.................. (37)}$$

Let γ_1 and γ_2 be two roots of this characteristic equation, then the general solution is to be expressed as follows:

$$\varphi_m=\bar{\varphi}_m+C_1\gamma_1{}^m+C_2\gamma_2{}^m \quad \text{................ (38)}$$

where $\bar{\varphi}_m$ is the particular solution.

The second and third terms in the right hand side of the above equation represent the effect of the non-uniformity of structure at both ends of the truss and the coefficients C_1 and C_2 are ones to be determined from the boundary conditions of equations (27′) and (28′).

For the Vierendeel truss with equal values of stiffness for all the members and with full joint loads having equal intensity of P, differenzengleichung, particular solution, characteristic equation, general solution and boundary conditions become as follow :

Differenzengleichung : $$\varphi_{m-1}-8\varphi_m+\varphi_{m+1}=-\frac{P\lambda}{2K}(n-2m) \quad (39)$$

Particular solution : $$\bar{\varphi}_m=\frac{P\lambda}{12K}(n-2m) \quad \text{.............. (40)}$$

Characteristic equation : $$\gamma^2-8\gamma+1=0 \quad \text{.............. (41)}$$

General solution : $$\varphi_m=\frac{P\lambda}{12K}(n-2m)+C_1\gamma_1{}^m+C_2\gamma_2{}^m \quad \text{.... (42)}$$

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* Paul Funk : Die Linearen Differenzengleichungen und ihre Anwendung in der Theorie der Baukonstruktionen.

Boundary conditions: $-7\varphi_0+\varphi_1=-\frac{P\lambda}{4K}(n-1)$ .... (43)

$$-7\varphi_n+\varphi_{n-1}=\frac{P\lambda}{4K}(n-1) \quad \text{........ (44)}$$

Solving the characteristic equation of (41) gives

$$\left.\begin{aligned}\gamma_1&=7.872983\\ \gamma_2&=0.127017\end{aligned}\right\} \quad \text{........................ (45)}$$

Substituting these values in equation (42), the general solution in the case of $k=1$ becomes

$$\varphi_m=\frac{P\lambda}{12K}(n-2m)+7.872983^m C_1+0.127017^m C_2 \quad \text{....... (46)}$$

or

$$\varphi_m=\frac{P\lambda}{12K}(n-2m)+0.127017^{-m} C_1+0.127017^m C_2 \quad \text{....... (46')}$$

From this equation

$$\left.\begin{aligned}\varphi_0\ &=\frac{P\lambda}{12K}n+C_1+C_2\\ \varphi_1\ &=\frac{P\lambda}{12K}(n-2)+0.127017^{-1}C_1+0.127017C_2\\ \varphi_{n-1}&=\frac{P\lambda}{12K}(2-n)+0.127017^{1-n}C_1+0.127017^{n-1}C_2\\ \varphi_n\ &=-\frac{P\lambda}{12K}n+0.127017^{-n}C_1+0.127017^{n}C_2\end{aligned}\right\} \quad \text{..... (47)}$$

Substituting these equations in the boundary conditions of (43) and (44), the simultaneous equations by which the coefficients C_1 and C_2 are to be determined are obtained as follows:

$$\left.\begin{aligned}0.872983C_1-6.872983C_2&=\frac{P\lambda}{12K}(3n+5)\\ -6.872983\times0.127017^{-n}C_1+0.872983\times0.127017^{n}C_2&=-\frac{P\lambda}{12K}(3n+5)\end{aligned}\right\}$$

.... (48)

Solving the above simultaneous equations gives

$$\left.\begin{aligned} C_1 &= \frac{-6.872983+0.872983\times 0.127017^{n}}{0.762099\times 0.127017^{n}-4.723789\times 0.127017^{-n}}\frac{P\lambda}{12K}(3n+5) \\ C_2 &= \frac{-0.872983+6.872983\times 0.127017^{-n}}{0.762099\times 0.127017^{n}-4.723789\times 0.127017^{-n}}\frac{P\lambda}{12K}(3n+5) \end{aligned}\right\} \quad (49)$$

Substituting the above values in equation (46′), the general solution is obtained.

In the expressions of C_1 and C_2,

$$\left.\begin{aligned} &0.872983\times 0.127017^{n} \ll 6.872983 \\ &0.872983 \ll 6.872983\times 0.127017^{-n} \\ &0.762099\times 0.127017^{n} \ll 4.723789\times 0.127017^{-n} \end{aligned}\right\} \quad \cdots\cdots (50)$$

Therefore the values of C_1 and C_2 may be simplified as follows:

$$\left.\begin{aligned} C_1 &= \frac{0.145497}{0.127017^{-n}}\frac{P\lambda}{12K}(3n+5) \\ C_2 &= -0.145497\frac{P\lambda}{12K}(3n+5) \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots\cdots (51)$$

Using these values, the general solution becomes

$$\varphi_m = \frac{P\lambda}{12K}\left\{(n-2m)+0.145497(0.127017^{n-m}-0.127017^{m})(3n+5)\right\} \quad \cdots\cdots\cdots (52)$$

In the case when the total number of the panels in a span is comparatively great, the above formula becomes approximately as follows:

In the case of $m<\frac{n}{2}$,

$$\varphi_m = \frac{P\lambda}{12K}\left\{(n-2m)-0.145497\times 0.127017^{m}(3n+5)\right\} \quad \cdots\cdots\cdots (53)$$

In the case of $m>\frac{n}{2}$,

$$\varphi_m = \frac{P\lambda}{12K}\left\{(n-2m)+0.145497\times 0.127017^{n-m}(3n+5)\right\} \quad \ldots\ldots (54)$$

Equations (53) and (54) are the formulae which give the values of joint-rotation angle in the case when the local effect of the non-uniformity of structure at the ends of the truss is taken into consideration.

8. Derivation of Practical Calculation Formulae for φ, Ψ and M.

From equation (53), the joint-rotation angles near the left end of the truss become as follows :

$$\varphi_0 = \frac{P\lambda}{K}(0.04696n-0.0606) \quad \ldots\ldots\ldots\ldots\ldots\ldots (55)$$

$$\varphi_1 = \frac{P\lambda}{K}(0.07871n-0.1744) \quad \ldots\ldots\ldots\ldots\ldots\ldots (56)$$

$$\varphi_2 = \frac{P\lambda}{K}(0.08274n-0.3343) \quad \ldots\ldots\ldots\ldots\ldots\ldots (57)$$

$$\varphi_3 = \frac{P\lambda}{K}(0.08326n-0.5001) \quad \ldots\ldots\ldots\ldots\ldots\ldots (58)$$

$$\varphi_4 = \frac{P\lambda}{K}(0.08332n-0.6667) \quad \ldots\ldots\ldots\ldots\ldots\ldots (59)$$

In the case of $m>4$, the value of $-(0.145497\times 0.127017^{m})(3n+5)$ in equation (53), that is, the effect of the non-uniformity of structure at the left end of the truss becomes negligibly small compared with the value of $(n-2m)$ and thereby the equation becomes

$$\varphi_m = \frac{P\lambda}{12K}(n-2m) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (60)$$

This formula coincides with the already obtained equation (32′).

Since a truss and its loading are symmetrical about the span centre the calculation of the joint-rotation angle at the right hand side of the truss can be omitted.

The above formulae (56) to (60) have a very simple form and the value of φ varies linearly by the increase of the total number of panels in a span.

The Author calls these formulae "Practical Calculation Formulae of Joint-Rotation Angle".

Substituting these Practical Calculation Formulae of the joint-rotation angle in equation (21), that is,

$$3\varphi_{m-1}+3\varphi_m+2\psi_m=-\frac{S_m\lambda}{2K}$$

or

$$\psi_m=-\frac{3}{2}(\varphi_{m-1}+\varphi_m)-\frac{P\lambda}{8K}(n-2m-1) \quad \text{.......... (61)}$$

one gets the Practical Calculation Formulae of the member-revolution angle of the members near the left end of the truss as follows:

$$\psi_1=-\frac{P\lambda}{K}(0.31350n-0.4775) \quad \text{.............. (62)}$$

$$\psi_2=-\frac{P\lambda}{K}(0.36717n-1.1380) \quad \text{.............. (63)}$$

$$\psi_3=-\frac{P\lambda}{K}(0.37400n-1.8766) \quad \text{.............. (64)}$$

$$\psi_4=-\frac{P\lambda}{K}(0.37487n-2.6252) \quad \text{.............. (65)}$$

$$\psi_m\doteqdot-\frac{P\lambda}{K}(0.375n-0.75_m+0.375) \text{ for } m>4 \quad \text{.. (66)}$$

Formula (66) coincides with equation (33'').

In a similar way, substituting the Practical Calculation Formulae of φ and ψ in the slope-deflection equation the Practical Calculation Formulae of the end moments of members can be obtained as shown in the preceding section.

9. Derivation of Practical Calculation Formulae for N,Q and y.

The relations among the end moments, direct stresses and shearing stresses can be obtained as follows:

Make an imaginary vertical section near to joint $m-1$ as shown in Fig. 4 and cut the upper and lower chord very near to the vertical $(m-1)-(\overline{m-1})$. Then, equilibrium conditions of $\sum H=0$ and $\sum M=0$ give:

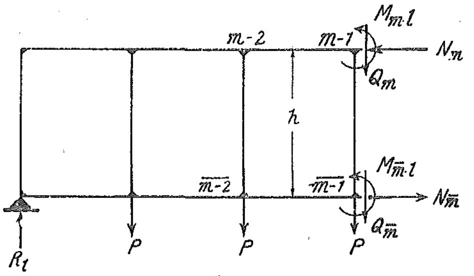

Fig. 4

$$N_m = -N_{\bar m} \quad \text{........} \quad (67)$$

and

$$M_{m\cdot l}+M_{\bar m\cdot l} = (\mathfrak{M}_{m-1}-N_m h) \quad \text{........} \quad (68)$$

where $\mathfrak{M}_{m-1}$ denotes the bending moment dut to the given load system at the panel joint $m-1$ in the case when the truss is assumed as a simple beam.

Equation (67) shows that the direct stress in the upper chord member is one of compression while that in the lower chord member is tensile. Their absolute values are equal in the same panel.

When the chord sections are designed in order that the value of stiffness for the upper chord member may be equal to that of the lower chord member in the same panel, as the truss mentioned in assumption (1), there exists the relation of equation (7). Therefore, equation (68) becomes

$$M_{m\cdot l} = M_{\bar m\cdot l} = \frac{1}{2}(\mathfrak{M}_{m-1}-N_m h) \quad \text{.............} \quad (69)$$

or

$$N_m = \frac{1}{h}(\mathfrak{M}_{m-1}-2M_{m\cdot l}) \quad \text{..................} \quad (69')$$

In a similar way, one gets

$$M_{m\cdot r} = M_{\bar m\cdot r} = \frac{1}{2}(N_m h-\mathfrak{M}_m) \quad \text{..............} \quad (70)$$

or

$$N_m = \frac{1}{h}(2M_{m\cdot r}+\mathfrak{M}_m) \quad \text{..................} \quad (70')$$

Next, considering two imaginary vertical sections near the panel joint m and $\bar m$ at both sides of the vertical member $m-\bar m$, the equilibrium of forces acting on this vertical member and the relation represented by equation (7) give

$$M_{m\cdot v} = M_{\bar m\cdot v} = \frac{h}{2}(N_{m+1}-N_m) \quad \text{.............} \quad (71)$$

Considering two imaginary vertical sections near both the ends of the m-th upper chord member, the equilibrium of forces acting of this member gives

$$Q_m\lambda = -(M_{m\cdot l}+M_{m\cdot r})$$

Substituting equations (67) and (70) in the above equation, the shearing stress induced in the upper chord member becomes

$$Q_m = \frac{1}{2\lambda}(\mathfrak{M}_m-\mathfrak{M}_{m-1}) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (73)$$

Also, there exists the well known following relation between $\mathfrak{M}_m$ and S_m where S_m is the shearing force in the m-th panel in the case when the truss is assumed as a simple beam :

$$S_m\lambda = \mathfrak{M}_m-\mathfrak{M}_{m-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (74)$$

Therefore

$$Q_m = \frac{1}{2}S_m \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (72)$$

In a similar way, considering the equilibrium of forces acting on the lower chord member one gets

$$Q_{\overline{m}} = \frac{1}{2}S_m \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (73)$$

Considering two imaginary horizontal sections near the two ends of the vertical member $m-\overline{m}$, the equilibrium of forces acting on this member gives

$$Q_{m\cdot v}h = -(M_{m\cdot v}+M_{\overline{m}\cdot v})$$

or

$$Q_{m\cdot v} = -\frac{2}{h}M_{m\cdot v} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (74)$$

Substituting the relation of (71) in the above equation, the shearing stress which acts in the vertical member $m-\overline{m}$ can be expressed by the following equation, too :

$$Q_{m\cdot v} = N_m-N_{m+1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (74')$$

At the last, considering the equilibrium of forces acting at panel point $\overline{m}$, the direct stress induced in the vertical member $m-\overline{m}$, excluding the vertical members at the two extremities of the truss, becomes

$$-N_{m\cdot v}+Q_{\bar{m}}-Q_{\overline{m+1}}-P=0$$

or

$$-N_{m\cdot v}=Q_{\overline{m+1}}-Q_{m}+P$$

or

$$-N_{m\cdot v}=\frac{1}{2}(S_{m+1}-S_{m})+P$$

While,

$$S_{m+1}=S_{m}-P$$

Therefore,

$$N_{m\cdot v}=-\frac{P}{2}\quad\text{(tension)}\quad\dots\dots\dots\dots\dots\dots\dots\quad(75)$$

In the case when the upper chord joints are loaded and the lower chord joints are free from loads, the stress in the vertical member $m-\bar{m}$ becomes

$$N_{m\cdot v}=\frac{P}{2}\quad\text{(compression)}\quad\dots\dots\dots\dots\dots\dots\quad(76)$$

For two verticals at the extremities of the truss, the direct stresses can be expressed as follows, considering the equilibrium of forces acting at the joints at the extremities.

$$\left.\begin{aligned}N_{0\cdot v}&=\frac{1}{2}(R_{l}-P_{\bar{0}})\\N_{n\cdot v}&=\frac{1}{2}(R_{r}-P_{\bar{n}})\end{aligned}\right\}\quad\dots\dots\dots\dots\dots\dots\dots\dots\quad(77)$$

where R_l and R_r denote the reactions at the left and right supports of the truss respectively and $P_{\bar{0}}$ and $P_{\bar{n}}$ denote the joint loads at joint $\bar{0}$ and $\bar{n}$ respectively.

In the case when the upper chord joints are loaded and the lower chord joints are free from loads, the direct stresses in the vertical members at the extremities become

$$\left.\begin{aligned}N_{0\cdot v}&=\frac{1}{2}(R_{l}+P_{0})\\N_{n\cdot v}&=\frac{1}{2}(R_{r}+P_{n})\end{aligned}\right\}\quad\dots\dots\dots\dots\dots\dots\dots\dots\quad(78)$$

where P_0 and P_n denote the joint loads at joint $\bar{0}$ and $\bar{n}$ respectively.

In the above obtained relations, $\mathfrak{M}_m$ and S_m are to be given as the known values for the given loading system. In the case when the lower or upper joints are fully loaded with loads of equal intensity of P,

$$\mathfrak{M}_m = \frac{1}{2}P(n-1)m\lambda - P\left\{1+2+\ldots\ldots+(m-1)\right\}\lambda$$

$$= P\lambda\left\{\frac{1}{2}m(n-1)-\frac{1}{2}m(m-1)\right\}$$

or

$$\mathfrak{M}_m = \frac{1}{2}P\lambda m(n-m) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (79)$$

Regarding the value of S_m, it is expressed by equation (26) as already mentioned.

Therefore, if the end moments are solved, all the other quantities may also be determined and the shearing stresses in chord members and direct stresses in the vertical member can be directly determined by panel shear and joint load.

Substituting the Practical Calculation Formulae of the end moments into equation (69′) and (70′), the Practical Calculation Formulae of the direct stresses can be obtained. For the shearing stresses the Practical Calculation Formulae are also determined using the above obtained relations.

The Practical Calculation Formulae of the truss deflection can be derived as follows:

Between the vertical deflection at joint m with respect to joint $m-1$, d_m, and the member revolution angle Ψ_m, there exists the following relation:

$$d_m = R_m\lambda \quad \text{or} \quad -\frac{\Psi_m\lambda}{6E}$$

Therefore, the truss deflection at joint m, y_m, is expressed as follows:

$$y_m = \sum_{m=1}^{m} d_m \quad \text{or} \quad -\frac{\lambda}{6E}\sum_{m=1}^{m}\Psi_m \qquad \ldots\ldots\ldots\ldots\ldots \quad (80)$$

Substituting equations (62) to (66) into the above equation, the Practical Calculation Formulae of truss deflection can be obtained.

Every one of these Practical Calculation Formulae in the case when the stiffness of all members is equally K is given in the preceding section. For the case when the stiffness is K for all the chord members and kK for all the vertical members, the Practical Calculation Formulae may be derived in a similar way if necessary.

III. ACCURACY OF THE RESULTS BY PRACTICAL CALCULATION FORMULAE.

In the actual calculation, the Practical Calculation Formulae give very good results for even a truss with as few panels as three or four. For example, in Table 1 there are shown the values of the direct stresses by the actual exact calculation and the Practical Calculation Formulae proposed in this paper. The values in a brackets are those calculated by the Practical Calculation Formulae.

These results show that the result by the proposed Practical Calculation Formulae have reliability to the fourth or fifth figures of the number.

Table 1.

Coeff.: $\frac{P\lambda}{h}$

No. of Panels in Span: n	Direct Stress in Upper Chord Members					
	N_1	N_2	N_3	N_4	N_5	N_6
2	0.2143 (0.1996)					
3	0.4839 (0.4814)	0.8710 (0.8519)				
4	0.7636 (0.7631)	1.6091 (1.6060)				
5	1.0451 (1.0449)	2.3607 (2.3600)	2.8402 (2.8367)			
6	1.3268 (1.3267)	3.1143 (3.1140)	4.0878 (4.0873)			
7	1.6087 (1.6084)	3.8683 (3.8681)	5.3371 (5.3378)	5.8344 (5.8333)		
8	1.8903 (1.8902)	4.6223 (4.6221)	6.5883 (6.5883)	7.5840 (7.5834)		
9	2.1720 (2.1719)	5.3759 (5.3762)	7.8387 (7.8388)	9.3340 (9.3335)	9.8335 (9.8334)	
10	2.4537 (2.4537)	6.1304 (6.1302)	9.0894 (9.0893)	11.0841 (11.0836)	12.0834 (12.0834)	
20	5.2713 (5.2713)	13.6706 (13.6706)	21.5944 (21.5944)	28.5845 (28.5845)	34.5835 (34.5835)	39.5833 (39.5833)
30	8.0889 (8.0889)	21.2110 (21.2110)	34.0995 (34.0995	46.0854 (46.0854)	57.0836 (57.0836)	67.0833 (67.0833)

SUMMARY AND CONCLUSION.

The general conclusions to be drawn from the investigations described in this paper are included in this resumé below:

The proposed Practical Calculation Formulae are applicable for the computation of the direct stress, end moment, shearing stress and truss deflection for the paralled chord Vierendeel truss with equal values of stiffness for all the members and with full joint loads having equal intensity.

In the design of the Vierendeel trurss with different stiffness, it is customarily assumed for the preliminary design that all the members have the same stiffness value and thereby the Practical Calculation Formulae must conduce to this purpose, too.

The special feature of the new method consists in the remarkable rapidity of the calculation. Calculation time estimated is given in minutes, because the required quantities are to be directly calculated from the very simple linear formulae.

By the proposed Practical Calculation Formulae, the selective calculation of requircd quantities is possible independently of the other quantities, selecting any desired ones and there is no labour to solve simultaneous equations.

The results by the proposed Practical Calculation Formulae have reliability to the fourth or fifth figures of the numbers of results.

With regard to the Practical Calculation Formulae for the maximum stress due to a live load, they will be reserved for some future paper.