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A Study on the Problems of Buckling of the Composite Rectangular Plates

By

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(Received December 27th, 1949)

PREFACE

It seems that the theoretical studies on the elastic stability of long columns with variable cross sections or continuous columns with multiple spans have been extensively investigated by many engineers and elasticians,[1] while the similar problems regarding the rectangular thin plate are left comparatively untouched.[2]

In this paper, the author intends to develope theoretical studies on the buckling of the composite plates composed of any number of elementary rectangular plates each of which has only a common width and is connected with one another in a line with various manners such as rigid, hinged or elastic connections and so on, under the action of compressive forces uniformly distributed along the surrounding edges.

Especially the cases in which the composite plates are simply supported along the two opposite sides parallel to the direction of connection of the elementary plates will be investigated.

In Chaps. I ～ II, the general considerations are developed and in Chap. III, the numerical examples are illustrated for reference to the practical applications on various cases of the edge conditions along the other two opposite end sides, and of the connecting conditions between each elementary plate, and furthermore some additional investigations are shown, and finally in Chap. IV, the author proposes the new method with which we are able to solve conveniently the special cases of composite plates composed of the same kind of elementary plates that—it seems to him—are especially practically important.

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1) We can find many references, for instance, in footnotes of the book by S.Timoshenko, "Theory of Elastic Stability," 1936.

2) We see that there exist some studies on this problems: R. Gran Olssen, "Beiträge zur Statik elastischer Platten veränderlicher Dicke," Z. A. M. M., Band 16, Heft 6, s. 347; S. Timoshenko, "Theory of Plate and Shells," 1940, p.194; Otto Pichler, "Die Biegung Kreissymmetrischer Platten von veränderlicher Dicke," 1928.

Next, let us glimpse some engineering structures in which the composite plates are used. Take, for instance, a case of compression member having the box-shape cross section which is frequently used in bridge constructions, steel towers, steel frame works or airplane constructions.

We can often consider the web plates of that member as uniformly compressed rectangular plates with simply supported edges, neglecting accessary restraints as it is on the safe sides, and if the cross section of the member is square, each of the webs of four sides can be considered, with sufficient precision, as simply supported along the edge lines of the member. And then it is justifiable to take for the ratio of web height and thickness or the slenderness ratio for the strut such a value that the critical stress for the plate is equal to the critical stress for the entire strut. [See, S. Timoshenko "Theory of Elastic Stability," 1936, p.404; Fr. Bleich "Theorie und Berechnung der Eisernen Brücken," 1924, s.226.] When the web plates are locally reinforced with cover plates in order to increase rigidity against bending or twisting forces, these are considered as the plates with discontinuously changed thickness, while if the reinforcing is prepared with internal diaphragms, the webs should have the lateral intermediate supports furnished by those diaphragms.

Now, we can see that those above described things are included in the composite plates studied in this paper, and the web supported especially with equal interval is nothing but one of the continuous plates studied in § 20 or § 22, Chap. IV.

Moreover, we can find many examples of the composite plates or of the continuous plates as structural elements in so-called stressed skin constructions which are often applied in aeronautical structures, in some portions of decks and bulkheads of ships, in walls of a building or in skin plates and diaphragms of various kinds of structures such as cars, girders, machines and so on.

Here, the author takes this opportunity to express his grateful thanks to Late Prof. Dr. S.Iguchi and Prof. Dr. T.Sakai who have given their warmest encouragements and extend thanks to Prof. Dr. T.Kon and Prof. K.Akutsu who read the manuscript. Also, he is grateful for financial support by the Fundamental Scientific Research Grant of the Ministry of Education.

M. Kurata

Sapporo,
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CONTENTS

NOTATIONS

Following notations are frequently used in this paper :

x_r, y_r Rectangular coordinates [suffix r means that the coordinates belong to r-th elementary plate, and that in other notations has the similar meaning]

h_r Thickness of a plate

a_r, b Length and width of a plate

ξ_r $= x_r/a_r$

η_r $= y_r/b$

p, q_r Intensities of distributed loads acting in x_r- and y_r-directions respectively

P_r Q_r Factors concerning critical loads, [see § 1 in this paper]

Q_r' Axial thrust in a supporting beam

D_r Flexural rigidity of a plate

B_r Flexural rigidity of a supporting beam

C_r Torsional rigidity of a supporting beam

μ_r, μ_r', μ_r'' Ratios between some dimensions and elastic constants of the two consecutive elementary plates, [see the footnotes of Table 5]

$\bar{\mu}_r$, $\bar{\mu}_r'$, $\bar{\mu}_r''$ Some coefficients used in the consideration when there exist supporting beams

υ_r, υ_r' Coefficients of restraint along an elastically built Joint

$\varepsilon_r' = D_r\upsilon_{r-1}$ in the case of g, [see § 2-g]

$\varepsilon_r' = D_r\upsilon_r'$ in the case of i, [see § 2-i]

$\varepsilon_{r-1} = D_{r-1}\upsilon_{r-1}$ in the case of i, [see § 2-i]

$\kappa_r = \varepsilon_r \dfrac{\pi}{a_r}$ Another expression of restraint along an elastically built Joint.

$\kappa_r' = \varepsilon_r' \dfrac{\pi}{a_r}$ The same as the above

w_r Deflection of the middle plane of a plate

$\underset{r}{K_m}$, $\underset{r}{L_m}$, $\underset{r}{M_m}$, $\underset{r}{N_m}$, Integration constants

$\underset{r}{A_m}$, $\underset{r}{B_m}$ The same as the above

$f_r(\xi_r)$, $\theta_r(\xi_r)$, $\phi_r(\xi_r)$, $\psi_r(\xi_r)$ Expressions of functions

$\underset{r}{\lambda_1}$, $\underset{r}{\lambda_2}$, $\underset{r}{\omega_1}$, $\underset{r}{\omega_2}$ Notations used for brevity, [see Table-1]

$\underset{r}{\beta_1}$, $\underset{r}{\beta_2}$, $\underset{r}{\gamma_1}$, $\underset{r}{\gamma_2}$ The same as the above, [see Table-3]

φ_r $= \underset{r}{\beta_1} - \underset{r}{\beta_2}$

$\bar{\varphi}_r$ $= \dfrac{1}{\underset{r}{\beta_1}} - \dfrac{1}{\underset{r}{\beta_2}}$

$\underset{r}{\tau_1}$, $\underset{r}{\tau_2}$, $\underset{r}{\tau_3}$, $\underset{r}{\tau_4}$ Notations used for brevity

$\underset{r}{\chi_1}$, $\underset{r}{\chi_2}$, $\underset{r}{\chi_3}$, $\underset{r}{\chi_4}$ The same as the above

CHAPTER I

GENERAL CONSIDERATIONS

§ 1. The Fundamental Differential Equation and its General Solution.

The deflection surface of the r-th elementary plate of a composite rectangular plate submitted to the compressive forces as shown in Fig. 1 can be characterized by the established equation for the buckled plate as follows[3]:

$$\frac{\partial^4 w_r}{\partial x_r^4} + 2\,\frac{\partial^4 w_r}{\partial x_r^2\,\partial y_r^2} + \frac{\partial^4 w_r}{\partial y_r^4} + \frac{p}{D_r}\frac{\partial^2 w_r}{\partial x_r^2} + \frac{q_r}{D_r}\frac{\partial^2 w_r}{\partial y_r^2} = 0, \qquad (1)$$

where $D_r = \dfrac{E_r\, h_r^3}{12\,(1-\nu_r^2)}$ = Flexural rigidity of the r-th elementary plate;

E_r = Young's modulus;

ν_r = Poisson's ratio;

h_r = Thickness of the plate.

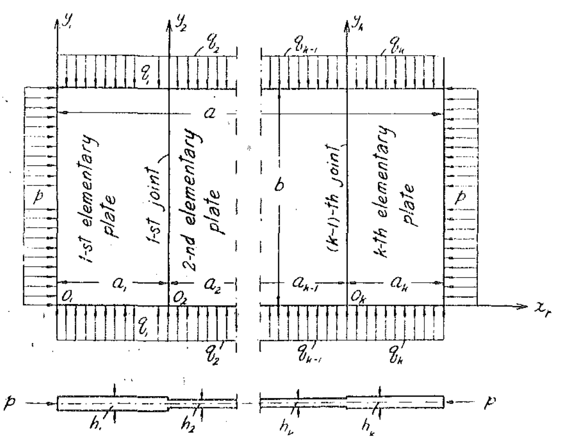

Fig. 1.—Composite plate and system of coodinates.

Now, putting

$$w_r = X_m(\xi_r)\begin{cases}\sin m\pi\eta_r \\ \cos m\pi\eta_r,\end{cases} \qquad \begin{pmatrix}\xi_r = x_r/a_r \\ \eta_r = y_r/b\end{pmatrix}$$

and substituting these in Eq.(1), we obtain linearly independent solutions by separation of variables, that is

$$X_m(\xi_r) = e^{\pi\lambda_{1r}\xi_r}, \qquad e^{-\pi\lambda_{1r}\xi_r}, \qquad e^{\pi\lambda_{2r}\xi_r}, \qquad e^{-\pi\lambda_{2r}\xi_r},$$

or

$$X_m(\xi_r) = f_r^1(\xi_r), \qquad f_r^2(\xi_r), \qquad f_r^3(\xi_r), \qquad f_r^4(\xi_r),$$

where the functions f_r^1, f_r^2, f_r^3, f_r^4 are given as shown in Table 1, corresponding to the magnitude of the following expressions:

3) S. Timoshenko, "Theory of Elastic Stability," 1936, p.324

$$P_r = \frac{p\, a_r^2}{D_r \pi^2}, \qquad Q_r = \frac{q_r\, b^2}{D_r \pi^2}, \qquad \text{(Dimensionless guantities)}$$

thus we can always write the solution of the foregoing differential equation in the form of real quantity.[4)]

Next, considering that Eq. (1) is a symmetrical expression with respect to (x_r, y_r), (p, q_r) and (a_r, b), it is seen that what are obtained by interchanging these in the foregoing solutions are also applicable as solutions. That is

$$w_r = Y_n(\eta_r) \begin{cases} \sin n\pi\xi_r \\ \cos n\pi\xi_r \end{cases},$$

$$Y_n(\eta_r) = e^{\pi \underset{r}{\lambda_1'} \eta_r}, \qquad e^{-\pi \underset{r}{\lambda_1'} \eta_r}, \qquad e^{\pi \underset{r}{\lambda_2'} \eta_r}, \qquad e^{-\pi \underset{r}{\lambda_2'} \eta_r},$$

or

$$Y_n(\eta_r) = g_r^1(\eta_r), \qquad g_r^2(\eta_r), \qquad g_r^3(\eta_r), \qquad g_r^4(\eta_r),$$

where $\underset{r}{\lambda_1'}$, $\underset{r}{\lambda_2'}$ mean what are obtained by interchanging (m, n), (a_r, b) and (P_r, Q_r) in $\underset{r}{\lambda_1}$, $\underset{r}{\lambda_2}$ respectively ; and moreover, the functions g_r^1, g_r^2, g_r^3, g_r^4 mean what are obtained by replacing a_r, b, m, P_r, Q_r and ξ_r in the functions f_r^1, f_r^2, f_r^3, f_r^4 by b, a_r, n, Q_r, P_r and η_r respectively.

Accordingly, the general solution of Eq. (1) is represented as follows:

$$\begin{aligned} w_r = &\sum_m \left\{ \underset{r}{K_m} f_r^1(\xi_r) + \underset{r}{L_m} f_r^2(\xi_r) + \underset{r}{M_m} f_r^3(\xi_r) + \underset{r}{N_m} f_r^4(\xi_r) \right\} \begin{cases} \sin m\pi\eta_r \\ \cos m\pi\eta_r \end{cases} \\ &+ \sum_n \left\{ \underset{r}{K_n} g_r^1(\eta_r) + \underset{r}{L_n} g_r^2(\eta_r) + \underset{r}{M_n} g_r^3(\eta_r) + \underset{r}{N_n} g_r^4(\eta_r) \right\} \begin{cases} \sin n\pi\xi_r \\ \cos n\pi\xi_r \end{cases}, \qquad (2) \\ &(m, n, = 0, 1, 2, 3, \cdots\cdots\cdots\cdots), \end{aligned}$$

where $\underset{r}{K_m}$, $\underset{r}{L_m}$, $\underset{r}{M_m}$, $\underset{r}{N_m}$, $\underset{r}{K_n}$, $\underset{r}{L_n}$, $\underset{r}{M_n}$, $\underset{r}{N_n}$ are integration constants. The above expression has suggested by M. Levy and is the most general form of the solution composed of the two single series. When, especially, the two opposite end sides of a plate are simply supported, these conditions are fulfilled by either one of the series. This form of solution was often used by H. Reissner,[5)] S. Timoshenko and the others.

§ 2. End Side Conditions and Intermediate Joint Conditions.

Hereafter, the two opposite outside edges of a composite plate perpendicular

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4) Dr. Erich Schneider, "Mathematische Schwingungslehre," 1924, ss. 26 ~ 28.
5) A. Nadai, "Elastische Platten," 1925, s. 239.

Table 1.

Case	Limits of P_r or Q_r			$f_r^1(\xi_r)$	$f_r^2(\xi_r)$
1	$\left\{m^2 - \frac{P_r}{2}\left(\frac{b}{a_r}\right)^2\right\}^2 < (m^2 - Q_r)\, m^2$			$\cosh\pi\underset{r}{\omega_1}\xi_r \times \cos\pi\underset{r}{\omega_2}\xi_r$	$\sinh\pi\underset{r}{\omega_1}\xi_r \times \cos\pi\underset{r}{\omega_2}\xi_r$
2	$\left\{m^2 - \frac{P_r}{2}\left(\frac{b}{a_r}\right)^2\right\}^2 > (m^2 - Q_r)\, m^2$	$m^2 < Q_r$		$\cosh\pi\underset{r}{\lambda_1}\xi_r$	$\sinh\pi\underset{r}{\lambda_1}\xi_r$
3		$m^2 > Q_r$	$m^2 - \frac{P_r}{2}\left(\frac{b}{a_r}\right)^2 > 0$	$\cosh\pi\underset{r}{\lambda_1}\xi_r$	$\sinh\pi\underset{r}{\lambda_1}\xi_r$
4			$m^2 - \frac{P_r}{2}\left(\frac{b}{a_r}\right)^2 < 0$	$\cos\pi\underset{r}{\lambda_1}\xi_r$	$\sin\pi\underset{r}{\lambda_1}\xi_r$
5	$\left\{m^2 - \frac{P_r}{2}\left(\frac{b}{a_r}\right)^2\right\}^2 = (m^2 - Q_r)\, m^2$	$m^2 - \frac{P_r}{2}\left(\frac{b}{a_r}\right)^2 > 0$		$\cosh\pi\omega_r\xi_r$	$\pi\xi_r\cosh\pi\omega_r\xi_r$
6		$m^2 - \frac{P_r}{2}\left(\frac{b}{a_r}\right)^2 < 0$		$\cos\pi\omega_r\xi_r$	$\pi\xi_r\cos\pi\omega_r\xi_r$

to x-direction shall be called *end side edges*, and the intermediate edges along which the consecutive elementary plates are jointed *connected edges* or *joints*.

(A) The End Side Edge Conditions.

We have been accustomed to consider the simply supported, clamped, and entirely free edges as the three typical cases of the boundary conditions of a plate.

Now, let us take the above three cases as the conditions of the end side. Then, corresponding to each of them, we can write the expressions as follows:

I) When the end sides $x_1 = 0$, $x_k = a_k$ (or $\xi_1 = 0$, $\xi_k = 1$) are simply supported,

1. $w_1 = 0, \qquad w_k = 0;$

$f_r^3\ (\xi_r)$	$f_r^4\ (\xi_r)$	Remarks
$\cosh\pi\underset{r}{\omega_1}\xi_r \times\sin\pi\underset{r}{\omega_2}\xi_r$	$\sinh\pi\underset{r}{\omega_1}\xi_r \times\sin\pi\underset{r}{\omega_2}\xi_r$	$\underset{r}{\omega_1}=\frac{a_r}{2b}\sqrt{\left\{2m^2-P_r\left(\frac{b}{a_r}\right)^2\right\}+2m\sqrt{m^2-Q_r}}$ $\underset{r}{\omega_2}=\frac{a_r}{2b}\sqrt{-\left\{2m^2-P_r\left(\frac{b}{a_r}\right)^2\right\}+2m\sqrt{m^2-Q_r}}$
$\cos\pi\underset{r}{\lambda_2}\xi_r$	$\sin\pi\underset{r}{\lambda_2}\xi_r$	$\underset{r}{\lambda_1}=\frac{a_r}{b}\sqrt{\left\{m^2-\frac{P_r}{2}\left(\frac{b}{a_r}\right)^2\right\}+\sqrt{\left\{m^2-\frac{P_r}{2}\left(\frac{b}{a_r}\right)^2\right\}^2-(m^2-Q_r)m^2}}$ $\underset{r}{\lambda_2}=\frac{a_r}{b}\sqrt{-\left\{m^2-\frac{P_r}{2}\left(\frac{b}{a_r}\right)^2\right\}+\sqrt{\left\{m^2-\frac{P_r}{2}\left(\frac{b}{a_r}\right)^2\right\}^2-(m^2-Q_r)m^2}}$
$\cosh\pi\underset{r}{\lambda_2}\xi_r$	$\sinh\pi\underset{r}{\lambda_2}\xi_r$	$\underset{r}{\lambda_1}=\frac{a_r}{b}\sqrt{\left\{m^2-\frac{P_r}{2}\left(\frac{b}{a_r}\right)^2\right\}+\sqrt{\left\{m^2-\frac{P_r}{2}\left(\frac{b}{a_r}\right)^2\right\}^2-(m^2-Q_r)m^2}}$ $\underset{r}{\lambda_2}=\frac{a_r}{b}\sqrt{\left\{m^2-\frac{P_r}{2}\left(\frac{b}{a_r}\right)^2\right\}-\sqrt{\left\{m^2-\frac{P_r}{2}\left(\frac{b}{a_r}\right)^2\right\}^2-(m^2-Q_r)m^2}}$
$\cos\pi\underset{r}{\lambda_2}\xi_r$	$\sin\pi\underset{r}{\lambda_2}\xi_r$	$\underset{r}{\lambda_1}=\frac{a_r}{b}\sqrt{-\left\{m^2-\frac{P_r}{2}\left(\frac{b}{a_r}\right)^2\right\}-\sqrt{\left\{m^2-\frac{P_r}{2}\left(\frac{b}{a_r}\right)^2\right\}^2-(m^2-Q_r)m^2}}$ $\underset{r}{\lambda_2}=\frac{a_r}{b}\sqrt{-\left\{m^2-\frac{P_r}{2}\left(\frac{b}{a_r}\right)^2\right\}+\sqrt{\left\{m^2-\frac{P_r}{2}\left(\frac{b}{a_r}\right)^2\right\}^2-(m^2-Q_r)m^2}}$
$\sinh\pi\omega_r\xi_r$	$\pi\xi_r\sinh\pi\omega_r\xi_r$	$\omega_r=\frac{a_r}{b}\sqrt{\left\{m^2-\frac{P_r}{2}\left(\frac{b}{a_r}\right)^2\right\}}$
$\sin\pi\omega_r\xi_r$	$\pi\xi_r\sin\pi\omega_r\xi_r$	$\omega_r=\frac{a_r}{b}\sqrt{-\left\{m^2-\frac{P_r}{2}\left(\frac{b}{a_r}\right)^2\right\}}$

$$2.\quad \frac{\partial^2 w_1}{\partial x_1^2}+\nu_1\frac{\partial^2 w_1}{\partial y_1^2}=0,\qquad \frac{\partial^2 w_k}{\partial x_k^2}+\nu_k\frac{\partial^2 w_k}{\partial y_k^2}=0.$$

II) When they are clamped,

$$1.\quad w_1=0,\qquad w_k=0;$$

$$2.\quad \frac{\partial w_1}{\partial x_1}=0,\qquad \frac{\partial w_k}{\partial x_k}=0.$$

III) When they are entirely free,

$$1.\quad \frac{\partial^2 w_1}{\partial x_1^2}+\nu_1\frac{\partial^2 w_1}{\partial y_1^2}=0,\qquad \frac{\partial^2 w_k}{\partial x_k^2}+\nu_k\frac{\partial^2 w_k}{\partial y_k^2}=0;$$

$$2.\quad \left[\frac{\partial^3 w_1}{\partial x_1^3}+(2-\nu_1)\frac{\partial^3 w_1}{\partial x_1\partial y_1^2}\right]+\frac{p}{D_1}\frac{\partial w_1}{\partial x_1}=0,^*$$

* See APPENDIX II.

$$\left[\frac{\partial^3 w_k}{\partial x_k^3} + (2-\nu_k)\frac{\partial^3 w_k}{\partial x_k \partial y_k^2}\right] + \frac{p}{D_k}\frac{\partial w_k}{\partial x_k} = 0\,.$$

(B) The Connected Edge Conditions.

Supposing practical examples of constructions, it seems that the following cases of the conditions are sufficient to be investigated. We shall now introduce such expressions for an intermediate connected edge $x_{r-1}=a_{r-1}$, $x_r=0$ (or $\xi_{r-1}=1$, $\xi_r=0$) when $\eta_{r-1}=\eta_r$.

I) The cases of the rigidly connected joint.—Such cases can be realized when composite plates are constructed by welding or riveting, or initially produced as one body, with an abrupt change of thickness along such an edge line. Moreover, in the present case the following under-classification may be taken, assuming various kinds of lateral supportings along the joint.

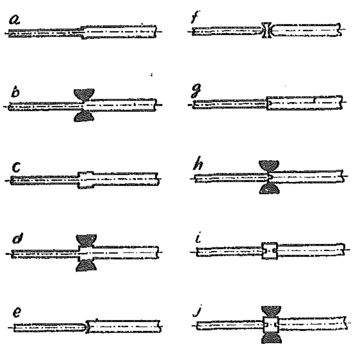


Fig. 2.—Types of intermediate joints.

a. No supporting [Fig. 2, *a*].—The connected edge conditions are now expressed as follows at such a joint:

1) Both deflections of the two adjacent elementary plates are equal to each other at the joint, that is

$$w_{r-1} = w_r.$$

2) Both slope displacements are continuous to each other, that is

$$\frac{\partial w_{r-1}}{\partial x_{r-1}} = \frac{\partial w_r}{\partial x_r}\,.$$

3) Both bending moments transmitted from the adjacent elementary plates must be in equilibrium state, that is

$$D_{r-1}\left(\frac{\partial^2 w_{r-1}}{\partial x_{r-1}^2} + \nu_{r-1}\frac{\partial^2 w_{r-1}}{\partial y_{r-1}^2}\right) = D_r\left(\frac{\partial^2 w_r}{\partial x_r^2} + \nu_r\frac{\partial^2 w_r}{\partial y_r^2}\right).$$

4) Both lateral pressures transmitted from these adjacent plates must be in equilibrium state, that is

$$D_{r-1}\left[\frac{\partial^3 w_{r-1}}{\partial x_{r-1}^3} + (2-\nu_{r-1})\frac{\partial^3 w_{r-1}}{\partial x_{r-1}\partial y_{r-1}^2}\right] = D_r\left[\frac{\partial^3 w_r}{\partial x_r^3} + (2-\nu_r)\frac{\partial^3 w_r}{\partial x_r \partial y_r^2}\right].$$

b. Simple supporing [Fig. 2, *b*]:

1) No deflections exist there; *i. e.*,

$$w_{r-1} = 0; \qquad\qquad w_r = 0.$$

2) Both slope displacements are equal to each other; *i. e.*,

$$\frac{\partial w_{r-1}}{\partial x_{r-1}}=\frac{\partial w_r}{\partial x_r}.$$

3) The bending moments are balanced with each other, then

$$D_{r-1}\left(\frac{\partial^2 w_{r-1}}{\partial x_{r-1}^2}+\nu_{r-1}\frac{\partial^2 w_{r-1}}{\partial y_{r-1}^2}\right)=D_r\left(\frac{\partial^2 w_r}{\partial x_r^2}+\nu_r\frac{\partial^2 w_r}{\partial y_r^2}\right).$$

c. Elastic supporting in 1-st way [Fig. 2, *c*].—As the types of elastic supportings are too various to be investigated, we shall limit these to the two comparatively simple cases. In the first instance, the composite plate is supported by an elastic beam along the joint:

1) Both deflections are equal to each other at the joint; *i. e.*,

$$w_{r-1}=w_r.$$

2) Both slope displacements are equal to each other; *i. e.*,

$$\frac{\partial w_{r-1}}{\partial x_{r-1}}=\frac{\partial w_r}{\partial x_r}.$$

3) The bending moments transmitted from the plates and the twisting moment in the supporting beam must be in equilibrium state; *i. e.*,

$$D_{r-1}\left(\frac{\partial^2 w_{r-1}}{\partial x_{r-1}^2}+\nu_{r-1}\frac{\partial^2 w_{r-1}}{\partial y_{r-1}^2}\right)-D_r\left(\frac{\partial^2 w_r}{\partial x_r^2}+\nu_r\frac{\partial^2 w_r}{\partial y_r^2}\right)=C_{r-1}\frac{\partial}{\partial y_r}\left(\frac{\partial^2 w_r}{\partial x_r\partial y_r}\right)$$

$$=C_{r-1}\frac{\partial^2}{\partial y_r^2}\left(\frac{\partial w_r}{\partial x_r}\right).\text{[6]}$$

where C_{r-1} means the torsional rigidity of a supporting beam.

4) The lateral pressures transmitted from the two adjacent plates and the lateral pressure caused in the distorted beam by the axial thrust must be balanced with the reaction force of the supporting beam. Thus

$$B_{r-1}\frac{\partial^4 w_r}{\partial y_r^4}=D_{r-1}\left[\frac{\partial^3 w_{r-1}}{\partial x_{r-1}^3}+(2-\nu_{r-1})\frac{\partial^3 w_{r-1}}{\partial x_{r-1}\partial y_{r-1}^2}\right]$$

$$-D_r\left[\frac{\partial^3 w_r}{\partial x_r^3}+(2-\nu_r)\frac{\partial^3 w_r}{\partial x_r\partial y_r^2}\right]-Q'_{r-1}\frac{\partial^2 w_r}{\partial y_r^2}.\text{[7]}$$

where B_{r-1} means the flexural rigidity of the supporting beam,

Q'_{r-1} means the axial force in the supporting beam.

d. Elastic supporting in 2-nd way [Fig. 2, *d*].—The connected edge is prevented from deflecting and restrained by the twisting resistance of a supporting beam; this case is considered to occur too when the composite plate is rigidly

6) S. Timoshenko, "Theory of Elastic Stability," p. 302, and "Plates and Shells," 1940, p. 92.

7) S. Timoshenko, "Theory of Elastic Stability," p. 346.

connected along the joint with an elastic thin wall perpendicular to the plane of the composite plate:

1) No deflections exist along the joint, *i. e.*,

$$w_{r-1} = 0; \qquad w_r = 0.$$

2) Both slope displacements are equal to each other, then

$$\frac{\partial w_{r-1}}{\partial x_{r-1}} = \frac{\partial w_r}{\partial x_r}.$$

3) The bending moments transmitted from the plates and the twisting moment along the joint must be in equilibrium state, then

$$D_{r-1}\left(\frac{\partial^2 w_{r-1}}{\partial x_{r-1}^2} + \nu_{r-1}\frac{\partial^2 w_{r-1}}{\partial y_{r-1}^2}\right) - D_r\left(\frac{\partial^2 w_r}{\partial x_r^2} + \nu_r\frac{\partial^2 w_r}{\partial y_r^2}\right) = C_{r-1}\frac{\partial^2}{\partial y_r^2}\left(\frac{\partial w_r}{\partial x_r}\right).$$

II) **The cases of the hinged joint.**—These are the cases where composite plates are considered such as composed of elementary plates by jointing with perfectly smooth hinges: —a skirting-board may be one of the examples in such cases. Assuming the various kinds of lateral supportings as before, the under-classification may be taken in the following manner.

e. No supporting [Fig. 2, *e*]:

1) Both deflections are equal to each other; *i. e.*,

$$w_{r-1} = w_r.$$

2) The bending moments vanish along the joint; *i. e.*,

$$\frac{\partial^2 w_{r-1}}{\partial x_{r-1}^2} + \nu_{r-1}\frac{\partial^2 w_{r-1}}{\partial y_{r-1}^2} = 0; \qquad \frac{\partial^2 w_r}{\partial x_r^2} + \nu_r\frac{\partial^2 w_r}{\partial y_r^2} = 0.$$

3) The lateral pressures must be in equilibrium state; *i. e.*,

$$D_{r-1}\left[\frac{\partial^3 w_{r-1}}{\partial x_{r-1}^3} + (2-\nu_{r-1})\frac{\partial^3 w_{r-1}}{\partial x_{r-1}\partial y_{r-1}^2}\right] + p\frac{\partial w_{r-1}}{\partial x_{r-1}}$$

$$= D_r\left[\frac{\partial^3 w_r}{\partial x_r^3} + (2-\nu_r)\frac{\partial^3 w_r}{\partial x_r\partial y_r^2}\right] + p\frac{\partial w_r}{\partial x_1}.^*$$

Next, supposing the case of the simple supporting along this joint, it becomes that the connecting conditions vanish along the joint. Therefore, such a case should be omitted from that of the composite plate.

f. Elastic supporting in 1-st way [Fig. 2, *f*]. —The joint is supported by an elastic beam as already stated:

1) Both deflections coincide with each other; *i. e.*,

$$w_{r-1} = w_r.$$

2) Both bending moments vanish, then

* See APPENDIX II.

$$\frac{\partial^2 w_{r-1}}{\partial x_{r-1}^2} + \nu_{r-1} \frac{\partial^2 w_{r-1}}{\partial y_{r-1}^2} = 0 ; \qquad \frac{\partial^2 w_r}{\partial x_r^2} + \nu_r \frac{\partial^2 w_r}{\partial y_r^2} = 0 .$$

3) The lateral pressures transmitted from the plates, the lateral pressure caused by the axial thrust in the beam, and the reaction force of the supporting beam must be in equilibrium state. Thus

$$B_{r-1} \frac{\partial^4 w_{r-1}}{\partial y_{r-1}^4} = D_{r-1} \left[\frac{\partial^3 w_{r-1}}{\partial x_{r-1}^3} + (2-\nu_{r-1}) \frac{\partial^3 w_{r-1}}{\partial x_{r-1} \partial y_{r-1}^2} \right] + p \frac{\partial w_{r-1}}{\partial x_{r-1}}$$

$$- D_r \left[\frac{\partial^3 w_r}{\partial x_r^3} + (2-\nu_r) \frac{\partial^3 w_r}{\partial x_r \partial y_r^2} \right] - p \frac{\partial w_r}{\partial x_r} - Q'_{r-1} \frac{\partial^2 w_r}{\partial y_r^2} .$$

III) The cases of the elastically built joint.—These are the cases where the types of joint are situated between both the extreme cases of the rigidly connected joint and the smoothly hinged one. Then, the two concecutive elementary plates have a certain difference between their slope displacements along the joint. It is general that such a difference of slope are assumed to be linearly proportional to the bending moment,[8] and then we can take, as the constant of ratio, the angle displacement which is occured by an unit bending moment distributed along the connected edge. Thus, the constant of ratio described above will be called *coefficient of restraint* and denoted by υ_{r-1} concerning the $(r-1)$-th joint.

Now, the under-classification by types of the lateral supportings may be taken in the following manner.

g. No supporting [Fig. 2, *g*]:

1) Both deflections coincide with each other; *i. e.*,

$$w_{r-1} = w_r .$$

2) A certain difference between both the slope displacements occurs according to the bending moment; *i. e.*,

$$\frac{\partial w_{r-1}}{\partial x_{r-1}} - \frac{\partial w_r}{\partial x_r} = - \varepsilon'_r \left(\frac{\partial^2 w_r}{\partial x_r^2} + \nu_r \frac{\partial^2 w_r}{\partial y_r^2} \right) ,$$

where

$$\varepsilon'_r = D_r \upsilon_{r-1} .$$

3) Both bending moments coincide with each other; *i. e.*,

$$D_{r-1} \left(\frac{\partial^2 w_{r-1}}{\partial x_{r-1}^2} + \nu_{r-1} \frac{\partial^2 w_{r-1}}{\partial y_{r-1}^2} \right) = D_r \left(\frac{\partial^2 w_r}{\partial x_r^2} + \nu_r \frac{\partial^2 w_r}{\partial y_r^2} \right) .$$

4) The lateral pressures must be in equilibrium state, then

8) S. Timoshenko, "Plates and Shells," p. 17; Fr. Bleich, "Theorie und Berechnung der Eisernen Brücken," s. 221; Ferd. Schleicher, "Kreisplatten auf Elastischer Unterlage," s. 64.

$$D_{r-1}\left[\frac{\partial^3 w_{r-1}}{\partial x_{r-1}^3}+(2-\nu_{r-1})\frac{\partial^3 w_{r-1}}{\partial x_{r-1}\partial y_{r-1}^2}\right]+p\frac{\partial w_{r-1}}{\partial x_{r-1}}$$

$$=D_r\left[\frac{\partial^3 w_r}{\partial x_r^3}+(2-\nu_r)\frac{\partial^3 w_r}{\partial x_r\partial y_r^2}\right]+p\frac{\partial w_r}{\partial x_r}.$$

It is obvious that this case approaches that of a rigidly connected joint, or of a hinged joint respectively, if the value of ε_r' in the above expressions becomes smaller and smaller, or larger and larger, and it will finally coincide with the foregoing case-*a*. or -*e*.

h. *Simple supporting* [Fig. 2, *h*]:

1) No deflections exist there; *i. e.*,

$$w_{r-1}=0; \qquad w_r=0.$$

2) A difference between both slope displacements occurs according to the bending moment; *i. e.*,

$$\frac{\partial w_{r-1}}{\partial x_{r-1}}-\frac{\partial w_r}{\partial x_r}=-\varepsilon_r'\left(\frac{\partial^2 w_r}{\partial x_r^2}+\nu_r\frac{\partial^2 w_r}{\partial y_r^2}\right).$$

3) The bending moments must be in equilibrium state; *i. e.*,

$$D_{r-1}\left(\frac{\partial^2 w_{r-1}}{\partial x_{r-1}^2}+\nu_{r-1}\frac{\partial^2 w_{r-1}}{\partial y_{r-1}^2}\right)=D_r\left(\frac{\partial^2 w_r}{\partial x_r^2}+\nu_r\frac{\partial^2 w_r}{\partial y_r^2}\right).$$

i. *Elastic supporting in 1-st way* [Eig. 2, *i*].—As the most general case, we shall consider about the case in which two consecutive plates connect elastically with both sides of a supporting beam respectively:

1) We can equate the deflection of the plate on one side of the beam to that on the other side, assuming that the beam is sufficiently slender. Then

$$w_{r-1}=w_r.$$

2) We can introduce the condition relating to the slope displacement in the following manner. Now, assuming the shape of section of the deflection surface at the $(r-1)$-th joint as shown in Fig. 3, the angle of rotation of any cross section of the beam is

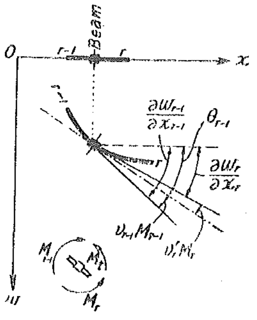


Fig. 3.

$$\theta_{r-1}=-\frac{\partial w_{r-1}}{\partial x_{r-1}}+\upsilon_{r-1}M_{r-1}=-\frac{\partial w_r}{\partial x_r}-\upsilon_r'M_r \qquad (2.\ 1)$$

where, M_{r-1}, M_r mean the bending moments transmitted from the $(r-1)$-th elementary plate and the r-th respectively; υ_{r-1}, υ_r' are the coefficients of restraint with which the $(r-1)$-th elementary plate and the r-th are elastically connected to the supporting beam respectively. Next, substituting the following expressions for M_{r-1} and M_r in Eq. (2. 1):

$$\left.\begin{aligned} M_{r-1} &= -D_{r-1}\left(\frac{\partial^2 w_{r-1}}{\partial x_{r-1}^2} + \nu_{r-1}\frac{\partial^2 w_{r-1}}{\partial y_{r-1}^2}\right); \\ M_r &= -D_r\left(\frac{\partial^2 w_r}{\partial x_r^2} + \nu_r\frac{\partial^2 w_r}{\partial y_r^2}\right), \end{aligned}\right\} \tag{2. 2}$$

we finally get

$$\frac{\partial w_{r-1}}{\partial x_{r-1}} - \frac{\partial w_r}{\partial x_r} = -\varepsilon_{r-1}\left(\frac{\partial^2 w_{r-1}}{\partial x_{r-1}^2} + \nu_{r-1}\frac{\partial^2 w_{r-1}}{\partial y_{r-1}^2}\right) - \varepsilon_r'\left(\frac{\partial^2 w_r}{\partial x_r^2} + \nu_r\frac{\partial^2 w_r}{\partial y_r^2}\right),$$

where

$$\varepsilon_{r-1} = D_{r-1}\,\upsilon_{r-1}, \qquad \varepsilon_r' = D_r\,\upsilon_r'.$$

3) Denoting by $-M_t$ the resisting moment which is caused by the twist of the suppoiting beam, the equilibrium of the moments at the position of the beam is expressed by the following equation:

$$M_{r-1} - M_r - M_t = 0, \tag{2. 3}$$

Now, by the consideration about twist of a beam, it has known that

$$M_t = C_{r-1}\frac{\partial^2\theta_{r-1}}{\partial y_{r-1}^2}.\ ^{9)}$$

Substituting (2. 1) into the above equation, we obtain

$$M_t = -C_{r-1}\frac{\partial^2}{\partial y_r^2}\left[\frac{\partial w_r}{\partial x_r} - \varepsilon_r'\left(\frac{\partial^2 w_r}{\partial x_r^2} + \nu_r\frac{\partial^2 w_r}{\partial y_r^2}\right)\right]. \tag{2. 4}$$

Hence, substituting the expressions (2. 2) and (2. 4) into Eq. (2. 3.), we finally get

$$D_{r-1}\left(\frac{\partial^2 w_{r-1}}{\partial x_{r-1}^2} + \nu_{r-1}\frac{\partial^2 w_{r-1}}{\partial y_{r-1}^2}\right) - D_r\left(\frac{\partial^2 w_r}{\partial x_r^2} + \nu_r\frac{\partial^2 w_r}{\partial y_r^2}\right)$$
$$= C_{r-1}\frac{\partial^2}{\partial y_r^2}\left[\frac{\partial w_r}{\partial x_r} - \varepsilon_r'\left(\frac{\partial^2 w_r}{\partial x_r^2} + \nu_r\frac{\partial^2 w_r}{\partial y_r^2}\right)\right].$$

4) From the equilibrium of the lateral pressures transmitted from the plates, the lateral pressure caused by the axial thrust in the beam, and the reaction force of the beam, the expression used in the foregoing case-f is also valid in this case; *i. e.*,

$$B_{r-1}\frac{\partial^4 w_{r-1}}{\partial y_{r-1}^4} = D_{r-1}\left[\frac{\partial^3 w_{r-1}}{\partial x_{r-1}^3} + (2-\nu_{r-1})\frac{\partial^3 w_{r-1}}{\partial x_{r-1}\partial y_{r-1}^2}\right] + p\frac{\partial w_{r-1}}{\partial x_{r-1}}$$
$$- D_r\left[\frac{\partial^3 w_r}{\partial x_r^3} + (2-\nu_r)\frac{\partial^3 w_r}{\partial x_r\partial y_r^2}\right] - p\frac{\partial w_r}{\partial x_r} - Q'_{r-1}\frac{\partial^2 w_r}{\partial y_r^2}.$$

j. Elastic supporting in 2-nd way [Fig. 2, *j*].—This is the case in which the connected edge is prevented from deflecting and restrained by the twisting

9) S. Timoshenko, "Theory of Elastic Stability," p. 302; "Theory of Plates and Shells," p .93.

resistance of a supporting beam:

1) No deflections exist there; *i. e.*,

$$w_{r-1}=0; \qquad w_r=0.$$

2) The analogous expression as in the case of i can also be written with respect to the slope displacement; *i. e.*,

$$\frac{\partial w_{r-1}}{\partial x_{r-1}}-\frac{\partial w_r}{\partial x_r}=-\varepsilon_{r-1}\left(\frac{\partial^2 w_{r-1}}{\partial x_{r-1}^2}+\nu_{r-1}\frac{\partial^2 w_{r-1}}{\partial y_{r-1}^2}\right)-\varepsilon_r'\left(\frac{\partial^2 w_r}{\partial x_r^2}+\nu_r\frac{\partial^2 w_r}{\partial y_r^2}\right).$$

3) The bending moments transmitted from the plates and the resisting moment caused by the twist of the beam must be balanced, and then we get

$$D_{r-1}\left(\frac{\partial^2 w_{r-1}}{\partial x_{r-1}^2}+\nu_{r-1}\frac{\partial^2 w_{r-1}}{\partial y_{r-1}^2}\right)-D_r\left(\frac{\partial^2 w_r}{\partial x_r^2}+\nu_r\frac{\partial^2 w_r}{\partial y_r^2}\right)$$
$$=C_{r-1}\frac{\partial^2}{\partial y_r}\left[\frac{\partial w_r}{\partial x_r}-\varepsilon_r'\left(\frac{\partial^2 w_r}{\partial x_r^2}+\nu_r\frac{\partial^2 w_r}{\partial y_r^2}\right)\right].$$

Putting $C_{r-1}=0$ in all of the above expressions, we can obtain such a case that the composite plate is not influenced by the twist of supporting beams. Finally, the various cases of boundary conditions obtained above are tabulated in Table 2.

§ 3. Expression of Critical Load.

If the integration constants in the general solution (2) can be determined to fulfil such given boundary conditions as considered in the previous section, the problems are completely solved.

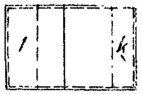
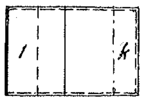
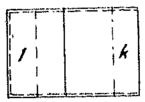
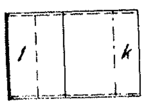
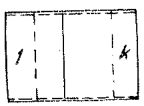
But, owing to the character of homogeneous boundary value problems, it is possible only in the case of particular values—namely characteristic numbers—of p or q that the solutions which are different from zero can be obtained, and then such particular values of p or q are called *critical loads* or *buckling loads* under which buckling occurs.

Moreover, the following considerations are necessary for determination of these values. Now, if either one of p or q is initially given as a constant thrust per unit length, the other comes to act as the critical load, and also if both of them hold a functional relation between themselves, one is to be represented by the other.

Next, concerning $q_1, q_2, \ldots\, q_k$, we can establish the following relations between them. That is to say, the total contractions of the elementary plates in the direction of y-axis $e_1, e_2, \ldots\, e_k$ must be the same ones because, if not so, the boundary conditions are broken. Hence

$$e_1=\frac{q_1}{E_1h_1}=e_2=\frac{q_2}{E_2h_2}=\cdots\cdots=e_r=\frac{q_r}{E_rh_r}=\cdots\cdots=e_k=\frac{q_k}{E_kh_k},$$

Table 2.

Composite plates		End side conditions (2 places)		Rigidly connected joints: No supporting (m_1 places)		Rigidly connected joints: Simple supporting (m_2 places)		Rigidly connected joints: Elastic supporting in 1-st way (m_3 places)		Rigidly connected joints: Elastic supporting in 2-nd way (m_4 places)		Hinged joints: No supporting (n_1 places)		Hinged joints: Elastic supporting in 1-st way (n_2 places)		Elastically built joints: No supporting (o_1 places)		Elastically built joints: Simple supporting (o_2 places)		Elastically built joints: Elastic supporting in 1-st way (o_3 places)		Elastically built joints: Elastic supporting in 2-nd way (o_4 places)	
1-st kind		$\mathrm{I})-1)_1$ $\mathrm{I})-2)_1$ $\mathrm{I})-1)_k$ $\mathrm{I})-2)_k$	4 conditions	$a-1)$ $a-2)$ $a-3)$ $a-4)$	$4m_1$ conditions	$b-1)_{r-1}$ $b-1)_r$ $b-2)$ $b-3)$	$4m_2$ conditions	$c-1)$ $c-2)$ $c-3)$ $c-4)$	$4m_3$ conditions	$d-1)_{r-1}$ $d-1)_r$ $d-2)$ $d-3)$	$4m_4$ conditions	$e-1)$ $e-2)_{r-1}$ $e-2)_r$ $e-3)$	$4n_1$ conditions	$f-1)$ $f-2)_{r-1}$ $f-2)_r$ $f-3)$	$4n_2$ conditions	$g-1)$ $g-2)$ $g-3)$ $g-4)$	$4o_1$ conditions	$h-1)_{r-1}$ $h-1)_r$ $h-2)$ $h-3)$	$4o_2$ conditions	$i-1)$ $i-2)$ $i-3)$ $i-4)$	$4o_3$ conditions	$j-1)_{r-1}$ $j-1)_r$ $j-2)$ $j-3)$	$4o_4$ conditions
2-nd kind		$\mathrm{II})-1)_1$ $\mathrm{II})-2)_1$ $\mathrm{I})-1)_k$ $\mathrm{I})-2)_k$	4 conditions	The same as the above												[Total number of conditions: $4\sum_{s=1}^{4}(m_s+o_s)+4\sum_{s=1}^{2}n_s$]							
3-rd kind		$\mathrm{II})-1)_1$ $\mathrm{II})-2)_1$ $\mathrm{II})-1)_k$ $\mathrm{II})-2)_k$	4 conditions	The same as the above																			
4-th kind		$\mathrm{I})-1)_1$ $\mathrm{I})-2)_1$ $\mathrm{III})-1)_k$ $\mathrm{III})-2)_k$	4 conditions	The same as the above																			
5-th kind		$\mathrm{II})-1)_1$ $\mathrm{II})-2)_1$ $\mathrm{III})-1)_k$ $\mathrm{III})-2)_k$	4 conditions	The same as the above																			
6-th kind		$\mathrm{III})-1)_1$ $\mathrm{III})-2)_1$ $\mathrm{III})-1)_k$ $\mathrm{III})-2)_k$	4 conditions	The same as the above																			

Remarks.—In the above table, for instance, the notation $\mathrm{I})-2)_k$ means the expression concerning the k-th elementary plate in the foregoing item: The End Side Edge Conditions $\mathrm{I})-2$, and also the notation $f-2)_r$ means that it concerns the r-th plate in the foregoing item: The Connected Edge Conditions f-2).

Simply supported end edge
Clamped end edge
Free end edge
Rigidly connected joint
Hinged joint
Elastically built joint

or

$$q_1 : q_2 : \cdots\cdots : q_r : \cdots\cdots : q_k = E_1 h_1 : E_2 h_2 : \cdots\cdots : E_r h_r : \cdots\cdots : E_k h_k .$$

From this, with any q_r, we are able to represent all the others, and then it will be possible to represent Q_1, Q_2, Q_3,$\cdots\cdots Q_r$,$\cdots\cdots Q_k$ with any one of them.

On the other hand, using the expressions

$$P_1 = \frac{p a_1^2}{D_1 \pi^2}, \quad P_2 = \frac{p a_2^2}{D_2 \pi^2}, \cdots\cdots, \quad P_r = \frac{p a_r^2}{D_r \pi^2}, \cdots\cdots, \quad P_k = \frac{p a_k^2}{D_k \pi^2},$$

we can write

$$P_1 : P_2 : \cdots\cdots : P_r : \cdots\cdots : P_k = \frac{a_1^2}{D_1} : \frac{a_2^2}{D_2} : \cdots\cdots : \frac{a_r^2}{D_r} : \cdots\cdots : \frac{a_k^2}{D_k} .$$

Therefore, in the same way as before, we are able to represent with any P_r all the others.

Considering that the factors P_r and Q_r are dimensionless guantities, we shall, hereafter, use them as the expressions of critical loads for convenience of computations.

CHAPTER II

GENERAL FORMULA

(A) GENERAL TREATMENTS.

§ 4. Derivation of the Equations for the calculation of Critical Loads.

Taking only the single sine-series concerning m in the expression (**2**) as a solution, we have

$$w_r = \sum \left\{ K_{m_r} f_r^1(\xi_r) + L_{m_r} f_r^2(\xi_r) + M_{m_r} f_r^3(\xi_r) + N_{m_r} f_r^4(\xi_r) \right\} \sin m\pi\eta_r, \qquad (\mathbf{3})$$

$$(m = 1,\ 2,\ 3, \ldots\ldots\ldots\ldots\ldots\ldots).$$

This satisfies the end side conditions of simple supporting at $y_r = 0$ or $y_r = b$ from the beginning, that is

$$w_r = 0, \qquad \frac{\partial^2 w_r}{\partial y_r^2} + \nu_r \frac{\partial^2 w_r}{\partial x_r^2} = 0, \qquad \text{when} \quad \eta_r = 0 \quad \text{and} \quad \eta_r = 1.$$

Therefore, the solution can completely be settled, if it is possible to determine the constants K_{m_r}, L_{m_r}, M_{m_r}, N_{m_r} in such a way that the remaining end side conditions and the intermediate joint conditions are satisfied.

To do this, we shall start with introducing the necessary functions as follows:

$$\frac{\partial w_r}{\partial x_r} = \sum \frac{\pi}{a_r} \left\{ K_{m_r} \theta_r^1(\xi_r) + L_{m_r} \theta_r^2(\xi_r) + M_{m_r} \theta_r^3(\xi_r) + N_{m_r} \theta_r^4(\xi_r) \right\} \sin m\pi\eta_r; \quad (4.1)$$

$$\frac{\partial^2 w_r}{\partial x_r^2} + \nu_r \frac{\partial^2 w_r}{\partial y_r^2} = \sum \left(\frac{\pi}{a_r} \right)^2 \left\{ K_{m_r} \phi_r^1(\xi_r) + L_{m_r} \phi_r^2(\xi_r) + M_{m_r} \phi_r^3(\xi_r) + N_{m_r} \phi_r^4(\xi_r) \right\} \sin m\pi\eta_r\ ; \quad (4.2)$$

$$\frac{\partial^3 w_r}{\partial x_r^3} + (2-\nu_r) \frac{\partial^3 w_r}{\partial x_r \partial y_r^2} = \sum \left(\frac{\pi}{a_r} \right)^3 \left\{ K_{m_r} \psi_r^1(\xi_r) + L_{m_r} \psi_r^2(\xi_r) + M_{m_r} \psi_r^3(\xi_r) + N_{m_r} \psi_r^4(\xi_r) \right\} \sin m\pi\eta_r\ ; \quad (4.3)$$

$$\frac{\partial^2}{\partial y_r^2} \left(\frac{\partial w_r}{\partial x_r} \right) = -\sum \left(\frac{\pi}{a_r} \right) \left(\frac{m\pi}{b} \right)^2 \left\{ K_{m_r} \theta_r^1(\xi_r) + L_{m_r} \theta_r^2(\xi_r) + M_{m_r} \theta_r^3(\xi_r) + N_{m_r} \theta_r^4(\xi_r) \right\} \sin m\pi\eta_r\ ; \quad (4.4)$$

$$\frac{\partial^2 w_r}{\partial y_r^2} = -\sum\left(\frac{m\pi}{b}\right)^2\left\{K_{m\atop r} f_r^1(\xi_r)+L_{m\atop r} f_r^2(\xi_r)+M_{m\atop r} f_r^3(\xi_r)+N_{m\atop r} f_r^4(\xi_r)\right\}\sin m\pi\eta_r\,; \quad (4.5)$$

$$\frac{\partial^4 w_r}{\partial y_r^4} = \sum\left(\frac{m\pi}{b}\right)^4\left\{K_{m\atop r} f_r^1(\xi_r)+L_{m\atop r} f_r^2(\xi_r)+M_{m\atop r} f_r^3(\xi_r)+N_{m\atop r} f_r^4(\xi_r)\right\}\sin m\pi\eta_r\,; \quad (4.6)$$

$$\frac{\partial^2}{\partial y_r^2}\left[\frac{\partial w_r}{\partial x_r} - \varepsilon_r'\left(\frac{\partial^2 w_r}{\partial x_r^2} + \nu_r\frac{\partial^2 w_r}{\partial y_r^2}\right)\right] = -\sum\left(\frac{\pi}{a_r}\right)\left(\frac{m\pi}{b}\right)^2\Big[K_{m\atop r}\left\{\theta_r^1(\xi_r)-\kappa_r'\phi_r^1(\xi_r)\right\}$$
$$+ L_{m\atop r}\left\{\theta_r^2(\xi_r)-\kappa_r'\phi_r^2(\xi_r)\right\} + M_{m\atop r}\left\{\theta_r^3(\xi_r)-\kappa_r'\phi_r^3(\xi_r)\right\}$$
$$+ N_{m\atop r}\left\{\theta_r^4(\xi_r)-\kappa_r'\phi_r^4(\xi_r)\right\}\Big]\sin m\pi\eta_r\,, \quad (4.7)$$

where $\kappa_r' = \varepsilon_r'\dfrac{\pi}{a_r}$, and the summations are taken about m. The notations of functions $f_r(\xi_r)$, $\theta_r(\xi_r)$, $\phi_r(\xi_r)$, $\varphi_r(\xi_r)$ in the above represent the various types of expressions as shown in Table 3, corresponding to each of such cases as tabulated in Table 1.

And then, substituting $\xi_r = 0$, or $\xi_r = 1$ into the above obtained expressions, they become as follows:

i) when $\xi_r = 0$,

$$w_r = \sum\left\{K_{m\atop r} + M_{m\atop r} f_r^3(0)\right\}\sin m\pi\eta_r\,;$$

$$\frac{\partial w_r}{\partial x_r} = \sum\frac{\pi}{a_r}\left\{L_{m\atop r}\theta_r^2(0) + M_{m\atop r}\theta_r^3(0) + N_{m\atop r}\theta_r^4(0)\right\}\sin m\pi\eta_r\,;$$

$$\frac{\partial^2 w_r}{\partial x_r^2} + \nu_r\frac{\partial^2 w_r}{\partial y_r^2} = \sum\left(\frac{\pi}{a_r}\right)^2\left\{K_{m\atop r}\phi_r^1(0) + M_{m\atop r}\phi_r^3(0) + N_{m\atop r}\phi_r^4(0)\right\}\sin m\pi\eta_r\,;$$

$$\frac{\partial^3 w_r}{\partial x_r^3} + (2-\nu_r)\frac{\partial^3 w_r}{\partial x_r\partial y_r^2} = \sum\left(\frac{\pi}{a_r}\right)^3\left\{L_{m\atop r}\varphi_r^2(0) + M_{m\atop r}\varphi_r^3(0) + N_{m\atop r}\varphi_r^4(0)\right\}\sin m\pi\eta_r\,;$$

$$\frac{\partial^2}{\partial y_r^2}\left(\frac{\partial w_r}{\partial x_r}\right) = -\sum\left(\frac{\pi}{a_r}\right)\left(\frac{m\pi}{b}\right)^2\left\{L_{m\atop r}\theta_r^2(0) + M_{m\atop r}\theta_r^3(0) + N_{m\atop r}\theta_r^4(0)\right\}\sin m\pi\eta_r\,;$$

$$\frac{\partial^2 w_r}{\partial y_r^2} = -\sum\left(\frac{m\pi}{b}\right)^2\left\{K_{m\atop r} + M_{m\atop r} f_r^3(0)\right\}\sin m\pi\eta_r\,;$$

$$\frac{\partial^4 w_r}{\partial y_r^4} = \sum\left(\frac{m\pi}{b}\right)^4\left\{K_{m\atop r} + M_{m\atop r} f_r^3(0)\right\}\sin m\pi\eta_r\,;$$

$$\frac{\partial^2}{\partial y_r^2}\left[\frac{\partial w_r}{\partial x_r} - \varepsilon_r'\left(\frac{\partial^2 w_r}{\partial x_r^2} + \nu_r\frac{\partial^2 w_r}{\partial y_r^2}\right)\right] = -\sum\left(\frac{\pi}{a_r}\right)\left(\frac{m\pi}{b}\right)^2\Big[-K_{m\atop r}\kappa_r'\phi_r^1(0) + L_{m\atop r}\theta_r^2(0)$$
$$+ M_{m\atop r}\left\{\theta_r^3(0) - \kappa_r'\phi_r^3(0)\right\} + N_{m\atop r}\left\{\theta_r^4(0) - \kappa_r'\phi_r^4(0)\right\}\Big]\sin m\pi\eta_r\,;$$

ii) when $\xi_r = 1$, there are no terms which vanish.

The expression of deflection of the r-th elementary plate has contained four unknown constants in the m-th term of the series. Hence, assuming that the composite plates are composed of k elementary plates, it will be seen that $4k$ unknown constants exist there concerning all of the m-th terms of their expressions. On the other hand, the total number of the boundary conditions in every case tabulated in Table 2 amounts to

$$4\left\{\sum_{s=1}^{4}(m_s + o_s) + \sum_{s=1}^{2} n_s + 1\right\}.$$

Considering now that the total number of the intermediate joints amounts to $\sum_{s=1}^{4}(m_s + o_s) + \sum_{s=1}^{2} n_s$, we can write

$$\sum_{s=1}^{4}(m_s + o_s) + \sum_{s=1}^{2} n_s + 1 = k.$$

From this, we instantly find that the total number of the boundary conditions and that of the unknown constants coincide with each other.

Next, observing the expressions obtained above (4.1) to (4.7), we can see that the equations obtained by substituting the expression (3) into the end side conditions and the intermediate joint conditions come to have a common factor $\sin m\pi\eta_r$ (in their m-th term) by the reason why all the boundary conditions are of homogeneous form.

And then, considering that the above mentioned equations must be valid without regard to η_r, it will be required that the equations obtained by omitting the common factor $\sin m\pi\eta_r$ hold, with respect to only the m-th terms. The groups of thus obtained equations constitute the system of homogeneous linear equations concerning the unknown constants, and also, regarding any m, the number of the unknown constants coincides with that of the equations as explained before.

Denoting, now, by Δ_m the determinant composed of the factors with which these unknown contants should be multiplied, our problem becomes to calculate P_r or Q_r from the following conditional equation:

$$\begin{vmatrix} \Delta_1 & 0 & 0 & 0 & \cdot & \cdot \\ 0 & \Delta_2 & 0 & 0 & \cdot & \cdot \\ 0 & 0 & \Delta_3 & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \ddots & & \\ \cdot & \cdot & \cdot & & \ddots & \\ \cdot & \cdot & \cdot & & & \Delta_\infty \end{vmatrix} = 0\,, \qquad (4)$$

Table 3.

case	$\theta_r^1(\xi_r)$	$\theta_r^2(\xi_r)$	$\theta_r^3(\xi_r)$	$\theta_r^4(\xi_r)$	Remarks
1	$\underset{r}{\alpha_1}\sinh\pi\underset{r}{\omega_1}\xi_r\cos\pi\underset{r}{\alpha_2}\xi_r$ $-\underset{r}{\alpha_2}\cosh\pi\underset{r}{\omega_1}\xi_r\sin\pi\underset{r}{\omega_2}\xi_r$	$\underset{r}{\omega_1}\cosh\pi\underset{r}{\omega_1}\xi_r\cos\pi\underset{r}{\alpha_2}\xi_r$ $-\underset{r}{\alpha_2}\sinh\pi\underset{r}{\omega_1}\xi_r\sin\pi\underset{r}{\alpha_2}\xi_r$	$\underset{r}{\omega_2}\cosh\pi\underset{r}{\omega_1}\xi_r\cos\pi\underset{r}{\alpha_2}\xi_r$ $+\underset{r}{\alpha_1}\sinh\pi\underset{r}{\omega_1}\xi_r\sin\pi\underset{r}{\omega_2}\xi_r$	$\underset{r}{\alpha_2}\sinh\pi\underset{r}{\omega_1}\xi_r\cos\pi\underset{r}{\alpha_2}\xi_r$ $+\underset{r}{\alpha_1}\cosh\pi\underset{r}{\alpha_1}\xi_r\sin\pi\underset{r}{\omega_2}\xi_r$	See remarks of Table 1
2	$\underset{r}{\lambda_1}\sinh\pi\underset{r}{\lambda_1}\xi_r$	$\underset{r}{\lambda_1}\cosh\pi\underset{r}{\lambda_1}\xi_r$	$-\underset{r}{\lambda_2}\sin\pi\underset{r}{\lambda_2}\xi_r$	$\underset{r}{\lambda_2}\cos\pi\underset{r}{\lambda_2}\xi_r$	〃
3	$\underset{r}{\lambda_1}\sinh\pi\underset{r}{\lambda_1}\xi_r$	$\underset{r}{\lambda_1}\cosh\pi\underset{r}{\lambda_1}\xi_r$	$\underset{r}{\lambda_2}\sinh\pi\underset{r}{\lambda_2}\xi_r$	$\underset{r}{\lambda_2}\cosh\pi\underset{r}{\lambda_2}\xi_r$	〃
4	$-\underset{r}{\lambda_1}\sin\pi\underset{r}{\lambda_1}\xi_r$	$\underset{r}{\lambda_1}\cos\pi\underset{r}{\lambda_1}\xi_r$	$-\underset{r}{\lambda_2}\sin\pi\underset{r}{\lambda_2}\xi_r$	$\underset{r}{\lambda_2}\cos\pi\underset{r}{\lambda_2}\xi_r$	〃
5	$\omega_r\sinh\pi\omega_r\xi_r$	$\cosh\pi\omega_r\xi_r$ $+\pi\omega_r\xi_r\sinh\pi\omega_r\xi_r$	$\omega_r\cosh\pi\omega_r\xi_r$	$\sinh\pi\omega_r\xi_r$ $+\pi\omega_r\xi_r\cosh\pi\omega_r\xi_r$	〃
6	$-\omega_r\sin\pi\omega_r\xi_r$	$\cos\pi\omega_r\xi_r$ $-\pi\omega_r\xi_r\sin\pi\omega_r\xi_r$	$\omega_r\cos\pi\omega_r\xi_r$	$\sin\pi\omega_r\xi_r$ $+\pi\omega_r\xi_r\cos\pi\omega_r\xi_r$	〃

case	$\phi_r^1(\xi_r)$	$\phi_r^2(\xi_r)$	$\phi_r^3(\xi_r)$	$\phi_r^4(\xi_r)$	Remarks
1	$\omega_r'\cosh\pi\underset{r}{\omega_1}\xi_r\cos\pi\underset{r}{\omega_2}\xi_r$ $-\omega_r''\sinh\pi\underset{r}{\omega_1}\xi_r\sin\pi\underset{r}{\omega_2}\xi_r$	$\omega_r'\sinh\pi\underset{r}{\omega_1}\xi_r\cos\pi\underset{r}{\omega_2}\xi_r$ $-\omega_r''\cosh\pi\underset{r}{\omega_1}\xi_r\sin\pi\underset{r}{\omega_2}\xi_r$	$\omega_r'\cosh\pi\underset{r}{\omega_1}\xi_r\sin\pi\underset{r}{\omega_2}\xi_r$ $+\omega_r''\sinh\pi\underset{r}{\omega_1}\xi_r\cos\pi\underset{r}{\omega_2}\xi_r$	$\omega_r'\sinh\pi\underset{r}{\omega_1}\xi_r\sin\pi\underset{r}{\omega_2}\xi_r$ $+\omega_r''\cosh\pi\underset{r}{\omega_1}\xi_r\cos\pi\underset{r}{\omega_2}\xi_r$	$\omega_r'=\left(\underset{r}{\omega_1^2}-\underset{r}{\omega_2^2}\right)-\nu_r\left(\frac{ma_r}{b}\right)^2=\left(\frac{a_r}{b}\right)^2\left\{(1-\nu_r)m^2-\frac{P_r}{2}\left(\frac{b}{a_r}\right)^2\right\}$ $\omega_r''=2\underset{r}{\omega_1}\underset{r}{\omega_2}=\left(\frac{a_r}{b}\right)^2\sqrt{-\left\{m^2-\frac{P_r}{2}\left(\frac{b}{a_r}\right)^2\right\}^2+(m^2-Q_r)m^2}$
2	$\underset{r}{\beta_1}\cosh\pi\underset{r}{\lambda_1}\xi_r$	$\underset{r}{\beta_1}\sinh\pi\underset{r}{\lambda_1}\xi_r$	$-\underset{r}{\beta_2}\cos\pi\underset{r}{\lambda_2}\xi_r$	$-\underset{r}{\beta_2}\sin\pi\underset{r}{\lambda_2}\xi_r$	$\underset{r}{\beta_1}=\underset{r}{\lambda_1^2}-\nu_r\left(\frac{ma_r}{b}\right)^2$ $\underset{r}{\beta_2}=\underset{r}{\lambda_2^2}+\nu_r\left(\frac{ma_r}{b}\right)^2$
3	$\underset{r}{\beta_1}\cosh\pi\underset{r}{\lambda_1}\xi_r$	$\underset{r}{\beta_1}\sinh\pi\underset{r}{\lambda_1}\xi_r$	$\underset{r}{\beta_2}\cosh\pi\underset{r}{\lambda_2}\xi_r$	$\underset{r}{\beta_2}\sinh\pi\underset{r}{\lambda_2}\xi_r$	$\underset{r}{\beta_1}=\underset{r}{\lambda_1^2}-\nu_r\left(\frac{ma_r}{b}\right)^2$ $\underset{r}{\beta_2}=\underset{r}{\lambda_2^2}-\nu_r\left(\frac{ma_r}{b}\right)^2$
4	$-\underset{r}{\beta_1}\cos\pi\underset{r}{\lambda_1}\xi_r$	$-\underset{r}{\beta_1}\sin\pi\underset{r}{\lambda_1}\xi_r$	$-\underset{r}{\beta_2}\cos\pi\underset{r}{\lambda_2}\xi_r$	$-\underset{r}{\beta_2}\sin\pi\underset{r}{\lambda_2}\xi_r$	$\underset{r}{\beta_1}=\underset{r}{\lambda_1^2}+\nu_r\left(\frac{ma_r}{b}\right)^2$ $\underset{r}{\beta_2}=\underset{r}{\lambda_2^2}+\nu_r\left(\frac{ma_r}{b}\right)^2$
5	$\beta_r\cosh\pi\omega_r\xi_r$	$2\omega_r\sinh\pi\omega_r\xi_r$ $+\pi\beta_r\xi_r\cosh\pi\omega_r\xi_r$	$\beta_r\sinh\pi\omega_r\xi_r$	$2\omega_r\cosh\pi\omega_r\xi_r$ $+\pi\beta_r\xi_r\sinh\pi\omega_r\xi_r$	$\beta_r=\omega_r^2-\nu_r\left(\frac{ma_r}{b}\right)^2$
6	$-\beta_r\cos\pi\omega_r\xi_r$	$-2\omega_r\sin\pi\omega_r\xi_r$ $-\pi\beta_r\xi_r\cos\pi\omega_r\xi_r$	$-\beta_r\sin\pi\omega_r\xi_r$	$2\omega_r\cos\pi\omega_r\xi_r$ $-\pi\beta_r\xi_r\sin\pi\omega_r\xi_r$	$\beta_r=\omega_r^2+\nu_r\left(\frac{ma_r}{b}\right)^2$

case	$\psi_r^1(\xi_r)$	$\psi_r^2(\xi_r)$	$\psi_r^3(\xi_r)$	$\psi_r^4(\xi_r)$	Remarks
1	$\bar{\omega}_r'\sinh\pi\underset{r}{\omega_1}\xi_r\cos\pi\underset{r}{\omega_2}\xi_r$ $+\bar{\omega}_r''\cosh\pi\underset{r}{\omega_1}\xi_r\sin\pi\underset{r}{\omega_2}\xi_r$	$\bar{\omega}_r'\cosh\pi\underset{r}{\omega_1}\xi_r\cos\pi\underset{r}{\omega_2}\xi_r$ $+\bar{\omega}_r''\sinh\pi\underset{r}{\omega_1}\xi_r\sin\pi\underset{r}{\omega_2}\xi_r$	$\bar{\omega}_r'\sinh\pi\underset{r}{\omega_1}\xi_r\sin\pi\underset{r}{\omega_2}\xi_r$ $-\bar{\omega}_r''\cosh\pi\underset{r}{\omega_1}\xi_r\cos\pi\underset{r}{\omega_2}\xi_r$	$\bar{\omega}_r'\cosh\pi\underset{r}{\omega_1}\xi_r\sin\pi\underset{r}{\omega_2}\xi_r$ $-\bar{\omega}_r''\sin\pi\underset{r}{\omega_1}\xi_r\cos\pi\underset{r}{\omega_2}\xi_r$	$\bar{\omega}_r'=\underset{r}{\omega_1}\left\{\left(\underset{r}{\omega_1^2}-3\,\underset{r}{\omega_2^2}\right)-(2-\nu_r)\left(\frac{ma_r}{b}\right)^2\right\}$ $\bar{\omega}_r''=\underset{r}{\omega_2}\left\{\left(\underset{r}{\omega_2^2}-3\,\underset{r}{\omega_1^2}\right)+(2-\nu_r)\left(\frac{ma_r}{b}\right)^2\right\}$
2	$\underset{r}{\gamma_1}\sinh\pi\underset{r}{\lambda_1}\xi_r$	$\underset{r}{\gamma_1}\cosh\pi\underset{r}{\lambda_1}\xi_r$	$\underset{r}{\gamma_2}\sin\pi\underset{r}{\lambda_2}\xi_r$	$-\underset{r}{\gamma_2}\cos\pi\underset{r}{\lambda_2}\xi_r$	$\underset{r}{\gamma_1}=\underset{r}{\lambda_1}\left\{\underset{r}{\lambda_1^2}-(2-\nu_r)\left(\frac{ma_r}{b}\right)^2\right\}$ $\underset{r}{\gamma_2}=\underset{r}{\lambda_2}\left\{\underset{r}{\lambda_2^2}+(2-\nu_r)\left(\frac{ma_r}{b}\right)^2\right\}$
3	$\underset{r}{\gamma_1}\sinh\pi\underset{r}{\lambda_1}\xi_r$	$\underset{r}{\gamma_1}\cosh\pi\underset{r}{\lambda_1}\xi_r$	$\underset{r}{\gamma_2}\sinh\pi\underset{r}{\lambda_2}\xi_r$	$\underset{r}{\gamma_2}\cosh\pi\underset{r}{\lambda_2}\xi_r$	$\underset{r}{\gamma_1}=\underset{r}{\lambda_1}\left\{\underset{r}{\lambda_1^2}-(2-\nu_r)\left(\frac{ma_r}{b}\right)^2\right\}$ $\underset{r}{\gamma_2}=\underset{r}{\lambda_2}\left\{\underset{r}{\lambda_2^2}-(2-\nu_r)\left(\frac{ma_r}{b}\right)^2\right\}$
4	$\underset{r}{\gamma_1}\sin\pi\underset{r}{\lambda_1}\xi_r$	$-\underset{r}{\gamma_1}\cos\pi\underset{r}{\lambda_1}\xi_r$	$\underset{r}{\gamma_2}\sin\pi\underset{r}{\lambda_2}\xi_r$	$-\underset{r}{\gamma_2}\cos\pi\underset{r}{\lambda_2}\xi_r$	$\underset{r}{\gamma_1}=\underset{r}{\lambda_1}\left\{\underset{r}{\lambda_1^2}+(2-\nu_r)\left(\frac{ma_r}{b}\right)^2\right\}$ $\underset{r}{\gamma_2}=\underset{r}{\lambda_2}\left\{\underset{r}{\lambda_2^2}+(2-\nu_r)\left(\frac{ma_r}{b}\right)^2\right\}$
5	$\underset{r}{\gamma_1}\sinh\pi\omega_r\xi_r$	$\underset{r}{\gamma_2}\cosh\pi\omega_r\xi_r$ $+\pi\underset{r}{\gamma_1}\xi_r\sinh\pi\omega_r\xi_r$	$\underset{r}{\gamma_1}\cosh\pi\omega_r\xi_r$	$\underset{r}{\gamma_2}\sinh\pi\omega_r\xi_r$ $+\pi\underset{r}{\gamma_1}\xi_r\cosh\pi\omega_r\xi_r$	$\underset{r}{\gamma_1}=\omega_r\left\{\omega_r^2-(2-\nu_r)\left(\frac{ma_r}{b}\right)^2\right\}$ $\underset{r}{\gamma_2}=3\omega_r^2-(2-\nu_r)\left(\frac{ma_r}{b}\right)^2$
6	$\underset{r}{\gamma_1}\sin\pi\omega_r\xi_r$	$-\underset{r}{\gamma_2}\cos\pi\omega_r\xi_r$ $+\pi\underset{r}{\gamma_1}\xi_r\sin\pi\omega_r\xi_r$	$-\underset{r}{\gamma_1}\cos\pi\omega_r\xi_r$	$-\underset{r}{\gamma_2}\sin\pi\omega_r\xi_r$ $-\pi\underset{r}{\gamma_1}\xi_r\cos\pi\omega_r\xi_r$	$\underset{r}{\gamma_1}=\omega_r\left\{\omega_r^2+(2-\nu_r)\left(\frac{ma_r}{b}\right)^2\right\}$ $\underset{r}{\gamma_2}=3\omega_r^2+(2-\nu_r)\left(\frac{ma_r}{b}\right)^2$

under which the homogeneous boundary value problems like this will be able to have solutions which are different from zero, in other words, the buckling can occur or $w_r \neq 0$.

Accordingly, from Eq.(4), we obtain

$$\varDelta_m = 0 . \tag{5}$$

This corresponds to the m-th mode of distortion in the y-direction. The least critical load of practical importance is generally obtained from Eq. (5) by putting $m = 1$.

The matrixes tabulated in Table-4 and -5 are prepared for making $\varDelta_m$ in practise. Then, taking from these tables the elements of the determinant corresponding to the various cases of boundary conditions, we can constitute $\varDelta_m$ for each of such cases without difficulty.

By Laplace's theoiem of expansion, $\varDelta_m$ is also expressed in the following form:

$$\Sigma (-1)^i \underset{(4)}{\varDelta} \cdot \underset{(4k-4)}{D} = 0 , \tag{6}$$

where,

$\underset{(4)}{\varDelta}$ = the minor determinant of fourth order in $\varDelta_m$, concerning only the end side conditions;

$(-1)^i \underset{(4k-4)}{D}$ = the cofactor of $\underset{(4)}{\varDelta}$ in $\varDelta_m$, concerning only the intermediate joint conditions.

As explained before, the critical loads aie required as the roots of Eq. (5) or (6), but these equations generally become the transendental equations of high order, and therefore their roots must graphically be found by so-called trial-and-error method. Some illustrations for the range which the least root is searched within will be shown in the numerical examples described in the later parts of this paper.

§ 5. A Consideration about the Conditional Equations.

Taking the forms of expression of the functions $f(\xi)$, $\theta(\xi)$, $\phi(\xi)$, $\psi(\xi)$ as shown in Table 1 and Table 3 according to the magnitude of P_r or Q_r, it is always possible to perform calculations in the domain of real guantities; but that is otherwise inconvenient to give the general expressions for calculation. Nevertheless, the differences among these various expressions of the functions except the singular cases of Case-5 and -6 shown in Table 1 are proved to be formal as follows:

For instance, let us take the expression of Case-3 in Table 1 as the general form of solution, that is

$$w_r = \sum \Big\{ \underset{r}{K_m} \bar{f}^1_r(\xi_r) + \underset{r}{L_m} \bar{f}^2_r(\xi_r) + \underset{r}{M_m} \bar{f}^3_r(\xi_r) + \underset{r}{N_m} \bar{f}^4_r(\xi_r) \Big\} \sin m\pi\eta_r ,$$

where

Table 4.—Matrix of end side conditions

Integration constants		K_1	L_1	M_1	N_1	$K_2 \ldots.$	K_k	L_k	M_k	N_k
1-st kind	Ⅰ)—1)$_1$	1	0	$f_1^3(0)$	0	0	0	0	0	0
	Ⅰ)—2)$_1$	$\phi_1^1(0)$	0	$\phi_1^3(0)$	$\phi_1^4(0)$	0	0	0	0	0
	Ⅰ)—1)$_k$	0	0	0	0	0	$f_k^1(1)$	$f_k^2(1)$	$f_k^3(1)$	$f_k^4(1)$
	Ⅰ)—2)$_k$	0	0	0	0	0	$\phi_k^1(1)$	$\phi_k^2(1)$	$\phi_k^3(1)$	$\phi_k^4(1)$
2-nd kind	Ⅱ)—1)$_1$	1	0	$f_1^3(0)$	0	0	0	0	0	0
	Ⅱ)—2)$_1$	0	$\theta_1^2(0)$	$\theta_1^3(0)$	$\theta_1^4(0)$	0	0	0	0	0
	Ⅰ)—1)$_k$	0	0	0	0	0	$f_k^1(1)$	$f_k^2(1)$	$f_k^3(1)$	$f_k^4(1)$
	Ⅰ)—2)$_k$	0	0	0	0	0	$\phi_k^1(1)$	$\phi_k^2(1)$	$\phi_k^3(1)$	$\phi_k^4(1)$
3-rd kind	Ⅱ)—1)$_1$	1	0	$f_1^3(0)$	0	0	0	0	0	0
	Ⅱ)—2)$_1$	0	$\theta_1^2(0)$	$\theta_1^3(0)$	$\theta_1^4(0)$	0	0	0	0	0
	Ⅱ)—1)$_k$	0	0	0	0	0	$f_k^1(1)$	$f_k^2(1)$	$f_k^3(1)$	$f_k^4(1)$
	Ⅱ)—2)$_k$	0	0	0	0	0	$\theta_k^1(1)$	$\theta_k^2(1)$	$\theta_k^3(1)$	$\theta_k^4(1)$
4-th kind	Ⅰ)—1)$_1$	1	0	$f_1^3(0)$	0	0	0	0	0	0
	Ⅰ)—2)$_1$	$\phi_1^1(0)$	0	$\phi_1^3(0)$	$\phi_1^4(0)$	0	0	0	0	0
	Ⅲ)—1)$_k$	0	0	0	0	0	$\phi_k^1(1)$	$\phi_k^2(1)$	$\phi_k^3(1)$	$\phi_k^4(1)$
	Ⅲ)—2)$_k$	0	0	0	0	0	$\psi_k^1(1)+P_k\theta_k^1(1)$	$\psi_k^2(1)+P_k\theta_k^2(1)$	$\psi_k^3(1)+P_k\theta_k^3(1)$	$\psi_k^4(1)+P_k\theta_k^4(1)$
5-th kind	Ⅱ)—1)$_1$	1	0	$f_1^3(0)$	0	0	0	0	0	0
	Ⅱ)—2)$_1$	0	$\theta_1^2(0)$	$\theta_1^3(0)$	$\theta_1^4(0)$	0	0	0	0	0
	Ⅲ)—1)$_k$	0	0	0	0	0	$\phi_k^1(1)$	$\phi_k^2(1)$	$\phi_k^3(1)$	$\phi_k^4(1)$
	Ⅲ)—2)$_k$	0	0	0	0	0	$\psi_k^1(1)+P_k\theta_k^1(1)$	$\psi_k^2(1)+P_k\theta_k^2(1)$	$\psi_k^3(1)+P_k\theta_k^3(1)$	$\psi_k^4(1)+P_k\theta_k^4(1)$
6-th kind	Ⅲ)—1)$_1$	$\phi_1^1(0)$	0	$\phi_1^3(0)$	$\phi_1^4(0)$	0	0	0	0	0
	Ⅲ)—2)$_1$	0	$\psi_1^2(0)+P_1\theta_1^2(0)$	$\psi_1^3(0)+P_1\theta_1^3(0)$	$\psi_1^4(0)+P_1\theta_1^4(0)$	0	0	0	0	0
	Ⅲ)—1)$_k$	0	0	0	0	0	$\phi_k^1(1)$	$\phi_k^2(1)$	$\phi_k^3(1)$	$\phi_k^4(1)$
	Ⅲ)—2)$_k$	0	0	0	0	0	$\psi_k^1(1)+P_k\theta_k^1(1)$	$\psi_k^2(1)+P_k\theta_k^2(1)$	$\psi_k^3(1)+P_k\theta_k^3(1)$	$\psi_k^4(1)+P_k\theta_k^4(1)$

Table 5.—Matrix of joining conditions

Integration constants			$\cdots\cdots N_{r-2}$	K_{r-1}	L_{r-1}	M_{r-1}	N_{r-1}	K_r	L_r	M_r	N_r	$K_{r+1}\cdots\cdots$
Rigidly conneted joints	a) No supporting	1)	…… 0	$f^1_{r-1}(1)$	$f^2_{r-1}(1)$	$f^3_{r-1}(1)$	$f^4_{r-1}(1)$	-1	0	$-f^3_r(0)$	0	0 ……
		2)	…… 0	$\theta^1_{r-1}(1)$	$\theta^2_{r-1}(1)$	$\theta^3_{r-1}(1)$	$\theta^4_{r-1}(1)$	0	$-\mu_r\theta^2_r(0)$	$-\mu_r\theta^3_r(0)$	$-\mu_r\theta^4_r(0)$	0 ……
		3)	…… 0	$\phi^1_{r-1}(1)$	$\phi^2_{r-1}(1)$	$\phi^3_{r-1}(1)$	$\phi^4_{r-1}(1)$	$-\mu'_r\phi^1_r(0)$	0	$-\mu'_r\phi^3_r(0)$	$-\mu'_r\phi^4_r(0)$	0 ……
		4)	…… 0	$\psi^1_{r-1}(1)$	$\psi^2_{r-1}(1)$	$\psi^3_{r-1}(1)$	$\psi^4_{r-1}(1)$	0	$-\mu''_r\psi^2_r(0)$	$-\mu''_r\psi^3_r(0)$	$-\mu''_r\psi^4_r(0)$	0 ……
	b) Simple supporting	$1)_{r-1}$	…… 0	$f^1_{r-1}(1)$	$f^2_{r-1}(1)$	$f^3_{r-1}(1)$	$f^4_{r-1}(1)$	0	0	0	0	0 ……
		$1)_r$	…… 0	0	0	0	0	1	0	$f^3_r(0)$	0	0 ……
		2	…… 0	$\theta^1_{r-1}(1)$	$\theta^2_{r-1}(1)$	$\theta^3_{r-1}(1)$	$\theta^4_{r-1}(1)$	0	$-\mu_r\theta^2_r(0)$	$-\mu_r\theta^3_r(0)$	$-\mu_r\theta^4_r(0)$	0 ……
		3	…… 0	$\phi^1_{r-1}(1)$	$\phi^2_{r-1}(1)$	$\phi^3_{r-1}(1)$	$\phi^4_{r-1}(1)$	$-\mu'_r\phi^1_r(0)$	0	$-\mu'_r\phi^3_r(0)$	$-\mu'_r\phi^4_r(0)$	0 ……
	c) Elastic supporting in 1-st way	1)	…… 0	$f^1_{r-1}(1)$	$f^2_{r-1}(1)$	$f^3_{r-1}(1)$	$f^4_{r-1}(1)$	-1	0	$-f^3_r(0)$	0	0 ……
		2)	…… 0	$\theta^1_{r-1}(1)$	$\theta^2_{r-1}(1)$	$\theta^3_{r-1}(1)$	$\theta^4_{r-1}(1)$	0	$-\mu_r\theta^2_r(0)$	$-\mu_r\theta^3_r(0)$	$-\mu_r\theta^4_r(0)$	0 ……
		3)	…… 0	$\phi^1_{r-1}(1)$	$\phi^2_{r-1}(1)$	$\phi^3_{r-1}(1)$	$\phi^4_{r-1}(1)$	$-\mu'_r\phi^1_r(1)$	$-\bar{\mu}_r\theta^2_r(0)$	$-(\mu'_r\phi^3_r(0)+\bar{\mu}_r\theta^3_r(0))$	$-(\mu'_r\phi^4_r(0)+\bar{\mu}_r\theta^4_r(0))$	0 ……
		4)	…… 0	$\psi^1_{r-1}(1)$	$\psi^2_{r-1}(1)$	$\psi^3_{r-1}(1)$	$\psi^4_{r-1}(1)$	$-\bar{\mu}'_r$	$-\mu''_r\psi^2_r(0)$	$-(\bar{\mu}'_r f^3_r(0)+\mu''_r\psi^3_r(0))$	$-\mu''_r\psi^4_r(0)$	0 ……
	d) Elastic supporting in 2-nd way	$1)_{r-1}$	…… 0	$f^1_{r-1}(1)$	$f^2_{r-1}(1)$	$f^3_{r-1}(1)$	$f^4_{r-1}(1)$	0	0	0	0	0 ……
		$1)_r$	…… 0	0	0	0	0	1	0	$f^3_r(0)$	0	0 ……
		2)	…… 0	$\theta^1_{r-1}(1)$	$\theta^2_{r-1}(1)$	$\theta^3_{r-1}(1)$	$\theta^4_{r-1}(1)$	0	$-\mu_r\theta^2_r(0)$	$-\mu_r\theta^3_r(0)$	$-\mu_r\theta^4_r(0)$	0 ……
		3)	…… 0	$\phi^1_{r-1}(1)$	$\phi^2_{r-1}(1)$	$\phi^3_{r-1}(1)$	$\phi^4_{r-1}(1)$	$-\mu'_r\phi^1_r(0)$	$-\bar{\mu}_r\theta^2_r(0)$	$-(\mu'_r\phi^3_r(0)+\bar{\mu}_r\theta^3_r(0))$	$-(\mu'_r\phi^4_r(0)+\bar{\mu}_r\theta^4_r(0))$	0 ……
Hinged joints	e) No supporting	1)	…… 0	$f^1_{r-1}(1)$	$f^2_{r-1}(1)$	$f^3_{r-1}(1)$	$f^4_{r-1}(1)$	-1	0	$-f^3_r(0)$	0	0 ……
		$2)_{r-1}$	…… 0	$\phi^1_{r-1}(1)$	$\phi^2_{r-1}(1)$	$\phi^3_{r-1}(1)$	$\phi^4_{r-1}(1)$	0	0	0	0	0 ……
		$2)_r$	…… 0	0	0	0	0	$\phi^1_r(1)$	0	$\phi^3_r(0)$	$\phi^4_r(0)$	0 ……
		3)	…… 0	$\psi^1_{r-1}(1)+P_{r-1}\theta^1_{r-1}(1)$	$\psi^2_{r-1}(1)+P_{r-1}\theta^2_{r-1}(1)$	$\psi^3_{r-1}(1)+P_{r-1}\theta^3_{r-1}(1)$	$\psi^4_{r-1}(1)+P_{r-1}\theta^4_{r-1}(1)$	0	$-\mu''_r(\psi^2_r(0)+P_r\theta^2_r(0))$	$-\mu''_r(\psi^3_r(0)+P_r\theta^3_r(0))$	$-\mu''_r(\psi^4_r(0)+P_r\theta^4_r(0))$	0 ……
	f) Elastic supporting in 1-st way	1)	…… 0	$f^1_{r-1}(1)$	$f^2_{r-1}(1)$	$f^3_{r-1}(1)$	$f^4_{r-1}(1)$	-1	0	$-f^3_r(0)$	0	0 ……
		$2)_{r-1}$	…… 0	$\phi^1_{r-1}(1)$	$\phi^2_{r-1}(1)$	$\phi^3_{r-1}(1)$	$\phi^4_{r-1}(1)$	0	0	0	0	0 ……
		$2)_r$	…… 0	0	0	0	0	$\phi^1_r(0)$	0	$\phi^3_r(0)$	$\phi^4_r(0)$	0 ……
		3)	…… 0	$\psi^1_{r-1}(1)+P_{r-1}\theta^1_{r-1}(1)$	$\psi^2_{r-1}(1)+P_{r-1}\theta^2_{r-1}(1)$	$\psi^3_{r-1}(1)+P_{r-1}\theta^3_{r-1}(1)$	$\psi^4_{r-1}(1)+P_{r-1}\theta^4_{r-1}(1)$	$-\bar{\mu}'_r$	$-\mu''_r(\psi^2_r(0)+P_r\theta^2_r(0))$	$\begin{bmatrix}-\bar{\mu}'_r f^3_r(0)\\ -\mu''_r(\psi^3_r(0)+P_r\theta^3_r(0))\end{bmatrix}$	$-\mu''_r(\psi^4_r(0)+P_r\theta^4_r(0))$	0 ……
Elastically built joints	g) No supporting	1)	…… 0	$f^1_{r-1}(1)$	$f^2_{r-1}(1)$	$f^3_{r-1}(1)$	$f^4_{r-1}(1)$	-1	0	$-f^3_r(0)$	0	0 ……
		2)	…… 0	$\theta^1_{r-1}(1)$	$\theta^2_{r-1}(1)$	$\theta^3_{r-1}(1)$	$\theta^4_{r-1}(1)$	$\mu_r\kappa'_r\phi^1_r(0)$	$-\mu_r\theta^2_r(0)$	$-\mu_r(\theta^3_r(0)-\kappa'_r\phi^3_r(0))$	$-\mu_r(\theta^4_r(0)-\kappa'_r\phi^4_r(0))$	0 ……
		3)	…… 0	$\phi^1_{r-1}(1)$	$\phi^2_{r-1}(1)$	$\phi^3_{r-1}(1)$	$\phi^4_{r-1}(1)$	$-\mu'_r\phi^1_r(0)$	0	$-\mu'_r\phi^3_r(0)$	$-\mu'_r\phi^4_r(0)$	0 ……
		4)	…… 0	$\psi^1_{r-1}(1)+P_{r-1}\theta^1_{r-1}(1)$	$\psi^2_{r-1}(1)+P_{r-1}\theta^2_{r-1}(1)$	$\psi^3_{r-1}(1)+P_{r-1}\theta^3_{r-1}(1)$	$\psi^4_{r-1}(1)+P_{r-1}\theta^4_{r-1}(1)$	0	$-\mu''_r(\psi^2_r(0)+P_r\theta^2_r(0))$	$-\mu''_r(\psi^3_r(0)+P_r\theta^3_r(0))$	$-\mu''_r(\psi^4_r(0)+P_r\theta^4_r(0))$	0 ……
	h) Simple supporting	$1)_{r-1}$	…… 0	$f^1_{r-1}(1)$	$f^2_{r-1}(1)$	$f^3_{r-1}(1)$	$f^4_{r-1}(1)$	0	0	0	0	0 ……
		$1)_r$	…… 0	0	0	0	0	1	0	$f^3_r(0)$	0	0 ……
		2)	…… 0	$\theta^1_{r-1}(1)$	$\theta^2_{r-1}(1)$	$\theta^3_{r-1}(1)$	$\theta^4_{r-1}(1)$	$\mu_r\kappa'_r\phi^1_r(0)$	$-\mu_r\theta^2_r(0)$	$-\mu_r(\theta^3_r(0)-\kappa'_r\phi^3_r(0))$	$-\mu_r(\theta^4_r(0)-\kappa'_r\phi^4_r(0))$	0 ……
		3)	…… 0	$\phi^1_{r-1}(1)$	$\phi^2_{r-1}(1)$	$\phi^3_{r-1}(1)$	$\phi^4_{r-1}(1)$	$-\mu'_r\phi^1_r(0)$	0	$-\mu'_r\phi^3_r(0)$	$-\mu'_r\phi^4_r(0)$	0 ……
	i) Elastic supporting in 1-st way	1)	…… 0	$f^1_{r-1}(1)$	$f^2_{r-1}(1)$	$f^3_{r-1}(1)$	$f^4_{r-1}(1)$	-1	0	$-f^3_r(0)$	0	0 ……
		2)	…… 0	$\theta^1_{r-1}(1)+\kappa_{r-1}\phi^1_{r-1}(1)$	$\theta^2_{r-1}(1)+\kappa_{r-1}\phi^2_{r-1}(1)$	$\theta^3_{r-1}(1)+\kappa_{r-1}\phi^3_{r-1}(1)$	$\theta^4_{r-1}(1)+\kappa_{r-1}\phi^4_{r-1}(1)$	$\mu_r\kappa'_r\phi^1_r(0)$	$-\mu_r\theta^2_r(0)$	$-\mu_r(\theta^3_r(0)-\kappa'_r\phi^3_r(0))$	$-\mu_r(\theta^4_r(0)-\kappa'_r\phi^4_r(0))$	0 ……
		3)	…… 0	$\phi^1_{r-1}(1)$	$\phi^2_{r-1}(1)$	$\phi^3_{r-1}(1)$	$\phi^4_{r-1}(1)$	$-\bar{\mu}''_r\phi^1_r(0)$	$-\bar{\mu}_r\theta^2_r(0)$	$-(\bar{\mu}''_r\phi^3_r(0)+\bar{\mu}_r\theta^3_r(0))$	$-(\bar{\mu}''_r\phi^4_r(0)+\bar{\mu}_r\theta^4_r(0))$	0 ……
		4)	…… 0	$\psi^1_{r-1}(1)+P_{r-1}\theta^1_{r-1}(1)$	$\psi^2_{r-1}(1)+P_{r-1}\theta^2_{r-1}(1)$	$\psi^3_{r-1}(1)+P_{r-1}\theta^3_{r-1}(1)$	$\psi^4_{r-1}(1)+P_{r-1}\theta^4_{r-1}(1)$	$-\bar{\mu}'_r$	$-\mu''_r(\psi^2_r(0)+P_r\theta^2_r(0))$	$\begin{bmatrix}-\bar{\mu}'_r f^3_r(0)\\ -\mu''_r(\psi^3_r(0)+P_r\theta^3_r(0))\end{bmatrix}$	$-\mu''_r(\psi^4_r(0)+P_r\theta^4_r(0))$	0 ……
	j) Elastic supporting in 2-nd way	$1)_{r-1}$	…… 0	$f^1_{r-1}(1)$	$f^2_{r-1}(1)$	$f^3_{r-1}(1)$	$f^4_{r-1}(1)$	0	0	0	0	0 ……
		$1)_r$	…… 0	0	0	0	0	1	0	$f^3_r(0)$	0	0 ……
		2)	…… 0	$\theta^1_{r-1}(1)+\kappa_{r-1}\phi^1_{r-1}(1)$	$\theta^2_{r-1}(1)+\kappa_{r-1}\phi^2_{r-1}(1)$	$\theta^3_{r-1}(1)+\kappa_{r-1}\phi^3_{r-1}(1)$	$\theta^4_{r-1}(1)+\kappa_{r-1}\phi^4_{r-1}(1)$	$\mu_r\kappa'_r\phi^1_r(0)$	$-\mu_r\theta^2_r(0)$	$-\mu_r(\theta^3_r(0)-\kappa'_r\phi^3_r(0))$	$-\mu_r(\theta^4_r(0)-\kappa'_r\phi^4_r(0))$	0 ……
		3)	…… 0	$\phi^1_{r-1}(1)$	$\phi^2_{r-1}(1)$	$\phi^3_{r-1}(1)$	$\phi^4_{r-1}(1)$	$-\bar{\mu}''_r\phi^1_r(0)$	$-\bar{\mu}_r\theta^2_r(0)$	$-(\bar{\mu}''_r\phi^3_r(0)+\bar{\mu}_r\theta^3_r(0))$	$-(\bar{\mu}''_r\phi^4_r(0)+\bar{\mu}_r\theta^4_r(0))$	0 ……

Remarks :

$$\mu_r=\frac{a_{r-1}}{a_r}\qquad \bar{\mu}_r=-\frac{C_{r-1}}{D_{r-1}}\left(\frac{\pi}{a_r}\right)\left(\frac{a_{r-1}}{a_r}\right)^2\left(\frac{ma_r}{b}\right)^2\qquad \kappa_r=\varepsilon_r\frac{\pi}{a_r}$$

$$\mu'_r=\frac{D_r}{D_{r-1}}\left(\frac{a_{r-1}}{a_r}\right)^2\qquad \bar{\mu}'_r=\left\{\frac{B_{r-1}}{D_{r-1}}\left(\frac{\pi}{a_r}\right)^2\left(\frac{ma_r}{b}\right)^2-\frac{Q'_{r-1}}{D_{r-1}}\right\}\left(\frac{a_{r-1}}{\pi}\right)\left(\frac{a_{r-1}}{a_r}\right)^2\left(\frac{ma_r}{b}\right)^2\qquad \kappa'_r=\varepsilon'_r\frac{\pi}{a_r}$$

$$\mu''_r=\frac{D_r}{D_{r-1}}\left(\frac{a_{r-1}}{a_r}\right)^3=\mu_r\mu'_r\qquad \bar{\mu}''_r=\frac{D_r}{D_{r-1}}\left(\frac{a_{r-1}}{a_r}\right)^2+\kappa'_r\frac{C_{r-1}}{D_{r-1}}\left(\frac{\pi}{a_r}\right)\left(\frac{a_{r-1}}{a_r}\right)^2\left(\frac{ma_r}{b}\right)^2=\mu'_r-\kappa'_r\bar{\mu}_r$$

$$\bar{f}_r^1(\xi_r) = \cosh \pi \underset{r}{\bar{\lambda}_1} \xi_r, \qquad \bar{f}_r^2(\xi_r) \sinh \pi \underset{r}{\bar{\lambda}_1} \xi_r,$$

$$\bar{f}_r^3(\xi_r) = \cosh \pi \underset{r}{\bar{\lambda}_2} \xi_r, \qquad \bar{f}_r^4(\xi_r) \sinh \pi \underset{r}{\bar{\lambda}_2} \xi_r,$$

and, the associate functions $\bar{\theta}_r$, $\bar{\phi}_r$, $\bar{\psi}_r$ are respectively expressed as shown in the line of Case-3 in Table 1 (the top bar is attached upon the notations for distinction).

Now, assuming, on trial, the value of P_r or Q_r within the limits of Case-1 in Table 1, $\underset{r}{\bar{\lambda}_1}$ and $\underset{r}{\bar{\lambda}_2}$ become the mutually conjugate complex quantities, that is

$$\underset{r}{\bar{\lambda}_1} = \underset{r}{\omega_1} + i \underset{r}{\omega_2}, \qquad \underset{r}{\bar{\lambda}_2} = \underset{r}{\omega_1} - i \underset{r}{\omega_2}.$$

Therefore, using the expressions of the functions in Case-1 shown in Table-1 and -3, we can see that the above functions are shown in the following forms:

$$\bar{f}_r^1(\xi_r) = f_r^1(\xi_r) + i f_r^4(\xi_r) \qquad \bar{f}_r^2(\xi_r) = f_r^2(\xi_r) + i f_r^3(\xi_r)$$

$$\bar{\theta}_r^1(\xi_r) = \theta_r^1(\xi_r) + i\,\theta_r^4(\xi_r) \qquad \bar{\theta}_r^2(\xi_r) = \theta_r^2(\xi_r) + i\,\theta_r^3(\xi_r)$$

$$\bar{\phi}_r^1(\xi_r) = \phi_r^1(\xi_r) + i\,\phi_r^4(\xi_r) \qquad \bar{\phi}_r^2(\xi_r) = \phi_r^2(\xi_r) + i\,\phi_r^3(\xi_r)$$

$$\bar{\psi}_r^1(\xi_r) = \psi_r^1(\xi_r) + i\psi_r^4(\xi_r) \qquad \bar{\psi}_r^2(\xi_r) = \psi_r^2(\xi_r) + i\,\psi_r^3(\xi_r)$$

$$\bar{f}_r^3(\xi_r) = f_r^1(\xi_r) - i f_r^4(\xi_r) \qquad \bar{f}_r^4(\xi_r) = f_r^2(\xi_r) - i f_r^3(\xi_r)$$

$$\bar{\theta}_r^3(\xi_r) = \theta_r^1(\xi_r) - i\,\theta_r^4(\xi_r) \qquad \bar{\theta}_r^4(\xi_r) = \theta_r^2(\xi_r) - i\,\theta_r^3(\xi_r)$$

$$\bar{\phi}_r^3(\xi_r) = \phi_r^1(\xi_r) - i\,\phi_r^4(\xi_r) \qquad \bar{\phi}_r^4(\xi_r) = \phi_r^2(\xi_r) - i\,\phi_r^3(\xi_r)$$

$$\bar{\psi}_r^3(\xi_r) = \psi_r^1(\xi_r) - i\,\psi_r^4(\xi_r) \qquad \bar{\psi}_r^4(\xi_r) = \psi_r^2(\xi_r) - i\,\psi_r^3(\xi_r)$$

Let us denote by $\bar{\Delta}_m$ the determinant from which the critical load is required in this case. Then the elements in the pair of the columns composed of the factors of $\underset{r}{K_m}$ and $\underset{r}{M_m}$ are conjugate with each other; and also, that is similar concerning those of $\underset{r}{L_m}$ and $\underset{r}{N_m}$. Further, the determinant obtained by replacing the elements in these pairs with the sum and the difference of the corresponding elements remains equivalent to the former determinant, owing to the theorem of determinant. Also, the new elements in thus obtained pair of the columns come to be expressed with the real part and the imaginary part of the former conjugate elements.

Accordingly, drawing out the determinant the common factors of these columns, such as 2 and $-i$, yielded by the above treatments, we find that this determinant coincides with what is produced with the expressions of the functions in Case-1 from the beginning. If P_r or Q_r exists within the limits of Case-2 or Case-4, in the former case the factors of $\underset{r}{N_m}$, and in the later case those of $\underset{r}{L_m}$ and $\underset{r}{N_m}$ become imaginary, and then drawing out the common factor i, we get the determinant produced by the tabulated expressions of the functions in each of such cases.

At any rate, we can generally conclude as follows:

$$\bar{\varDelta}_m = (-2)^s\, i^{s'}\, \varDelta_m\,.$$

That is to say, it has been cleared that the roots of $\varDelta_m = 0$ result from the zero of $\bar{\varDelta}_m$.

(B) GENERAL FORMULAE FOR THE CASES WHERE EVERY CONNECTING JOINT IS OF THE SAME KIND.

Taking up the proper elements from Table 4 and Table 5 respectively, the critical load in such a general case that a composite plate has various kinds of joints together, are obtained from Eq. (5) as already explained.

However, it seems that we are rather frequently confronted by such cases of practical importance that a composite plate has only the same kind of joints. Such cases will, therefore, be discussed in more detail. It is true these cases are yet solved in the same manner as before, but it is desirable to arrange the calculation process in a definite method so that we can perform it mechanically in compliance with the kinds of joints, because Eq. (5) generally becomes of high order and its computaion is tedious and liable to be mistaken.

From the reason why the connecting joints are of the same kind, in this case, we can produce the recurrence formulas for any number of the elementary plates, and from this we find that these formulas are reduced to so-called "Differenzengleichungen" when all the elementary plates are the same ones in points of dimension and material. Leaving the explanations of the above special cases into the later chapter of this paper, we begin with the derivation of the general recurrence formulas for each of the types of joints. Owing to such a fact as explained in the preceding section, the formulas will, hereafter, be given by using the expressions of Case-3 in Table 1.

§ 6. The Composite Plates Having Rigid Joints.

a) Case of No Supporting along Joints. i) *Conditions for Jointing.*— From Table 5, the conditions joining the $(r-1)$-th elementary plate to the r-th are written as follows:

$$\underset{r-1}{K_m}\cosh\pi\underset{r-1}{\lambda_1} + \underset{r-1}{L_m}\sinh\pi\underset{r-1}{\lambda_1} + \underset{r-1}{M_m}\cosh\pi\underset{r-1}{\lambda_2} + \underset{r-1}{N_m}\sinh\pi\underset{r-1}{\lambda_2} - \underset{r}{K_m} - \underset{r}{M_m} = 0\,; \quad (6,1)$$

$$\underset{r-1}{K_m}\underset{r-1}{\lambda_1}\sinh\pi\underset{r-1}{\lambda_1} + \underset{r-1}{L_m}\underset{r-1}{\lambda_1}\cosh\pi\underset{r-1}{\lambda_1} + \underset{r-1}{M_m}\underset{r-1}{\lambda_2}\sinh\pi\underset{r-1}{\lambda_2} + \underset{r-1}{N_m}\underset{r-1}{\lambda_2}\cosh\pi\underset{r-1}{\lambda_2} - \underset{r}{L_m}\mu_r\underset{r}{\lambda_1} - \underset{r}{N_m}\mu_r\underset{r}{\lambda_2} = 0\,; \tag{6.2}$$

$$\underset{r-1}{K_m}\underset{r-1}{\beta_1}\cosh\pi\underset{r-1}{\lambda_1} + \underset{r-1}{L_m}\underset{r-1}{\beta_1}\sinh\pi\underset{r-1}{\lambda_1} + \underset{r-1}{M_m}\underset{r-1}{\beta_2}\cosh\pi\underset{r-1}{\lambda_2} + \underset{r-1}{N_m}\underset{r-1}{\beta_2}\sinh\pi\underset{r-1}{\lambda_1} - \underset{r}{K_m}\mu_r'\underset{r}{\beta_1} - \underset{r}{M_m}\mu_r'\underset{r}{\beta_2} = 0\,; \tag{6.3}$$

$$\underset{r-1}{K_m}\underset{r-1}{\gamma_1}\sinh\pi\underset{r-1}{\lambda_1} + \underset{r-1}{L_m}\underset{r-1}{\gamma_1}\cosh\pi\underset{r-1}{\lambda_1} + \underset{r-1}{M_m}\underset{r-1}{\gamma_2}\sinh\pi\underset{r-1}{\lambda_2} + \underset{r-1}{N_m}\underset{r-1}{\gamma_2}\cosh\pi\underset{r-1}{\lambda_2} - \underset{r}{L_m}\mu_r''\underset{r}{\gamma_1} - \underset{r}{N_m}\mu_r''\underset{r}{\gamma_2} = 0\,, \tag{6.4}$$

where the notations λ_r, β_r, γ_r are expressed in such ways as shown in Case-3 of Table 1 and Table 3, that is

$$\left.\begin{matrix}\underset{r}{\lambda_1}\\ \underset{r}{\lambda_2}\end{matrix}\right\} = \frac{a_r}{b}\sqrt{\left\{m^2 - \frac{P_r}{2}\left(\frac{b}{a_r}\right)^2\right\} \pm \sqrt{\left\{m^2 - \frac{P_r}{2}\left(\frac{b}{a_r}\right)^2\right\}^2 - (m^2 - Q_r)\,m^2}}\,,$$

$$\underset{r}{\beta_1} = \underset{r}{\lambda_1}^2 - \nu_r\left(\frac{ma_r}{b}\right)^2, \qquad \underset{r}{\gamma_1} = \underset{r}{\lambda_1}\left\{\underset{r}{\lambda_1}^2 - (2-\nu_r)\left(\frac{ma_r}{b}\right)^2\right\},$$

$$\underset{r}{\beta_2} = \underset{r}{\lambda_2}^2 - \nu_r\left(\frac{ma_r}{b}\right)^2, \qquad \underset{r}{\gamma_2} = \underset{r}{\lambda_2}\left\{\underset{r}{\lambda_2}^2 - (2-\nu_r)\left(\frac{ma_r}{b}\right)^2\right\}.$$

Next, by the following operations:

$$(6.1)\times\underset{r-1}{\beta_1} - (6.3)\,;$$
$$(6.1)\times\underset{r-1}{\beta_2} - (6.3)\,;$$
$$(6.2)\times\underset{r-1}{\gamma_1} - (6.4)\times\underset{r-1}{\lambda_1}\,;$$
$$(6.2)\times\underset{r-1}{\gamma_2} - (6.4)\times\underset{r-1}{\lambda_2}\,,$$

we have

$$\underset{r-1}{M_m}(\underset{r-1}{\beta_1} - \underset{r-1}{\beta_2})\cosh\pi\underset{r-1}{\lambda_2} + \underset{r-1}{N_m}(\underset{r-1}{\beta_1} - \underset{r-1}{\beta_2})\sinh\pi\underset{r-1}{\lambda_2} - \underset{r}{K_m}(\underset{r-1}{\beta_1} - \mu_r'\underset{r}{\beta_1}) - \underset{r}{M_m}(\underset{r-1}{\beta_1} - \mu_r'\underset{r}{\beta_2}) = 0\,;$$

$$\underset{r-1}{K_m}(\underset{r-1}{\beta_1} - \underset{r-1}{\beta_2})\cosh\pi\underset{r-1}{\lambda_1} + \underset{r-1}{L_m}(\underset{r-1}{\beta_1} - \underset{r-1}{\beta_2})\sinh\pi\underset{r-1}{\lambda_1} + \underset{r}{K_m}(\underset{r-1}{\beta_2} + \mu_r'\underset{r}{\beta_1}) + \underset{r}{M_m}(\underset{r-1}{\beta_2} - \mu_r'\underset{r}{\beta_2}) = 0\,;$$

$$\underset{r-1}{M_m}(\underset{r-1}{\lambda_2}\underset{r-1}{\gamma_1} - \underset{r-1}{\lambda_1}\underset{r-1}{\gamma_2})\sinh\pi\underset{r-1}{\lambda_2} + \underset{r-1}{N_m}(\underset{r-1}{\lambda_2}\underset{r-1}{\gamma_1} - \underset{r-1}{\lambda_1}\underset{r-1}{\gamma_2})\cosh\pi\underset{r-1}{\lambda_2} - \underset{r}{L_m}\mu_r(\underset{r}{\lambda_1}\underset{r-1}{\gamma_1} - \mu_r'\underset{r-1}{\lambda_1}\underset{r}{\gamma_1}) - \underset{r}{N_m}\mu_r(\underset{r}{\lambda_2}\underset{r-1}{\gamma_1} - \mu_r'\underset{r-1}{\lambda_1}\underset{r}{\gamma_2}) = 0\,;$$

$$\underset{r-1}{K_m}(\underset{r-1}{\lambda_2}\underset{r-1}{\gamma_1} - \underset{r-1}{\lambda_1}\underset{r-1}{\gamma_2})\sinh\pi\underset{r-1}{\lambda_1} + \underset{r-1}{L_m}(\underset{r-1}{\lambda_2}\underset{r-1}{\gamma_1} - \underset{r-1}{\lambda_1}\underset{r-1}{\gamma_2})\cosh\pi\underset{r-1}{\lambda_1} + \underset{r}{L_m}\mu_r(\underset{r}{\lambda_1}\underset{r-1}{\gamma_2} - \mu_r'\underset{r-1}{\lambda_2}\underset{r}{\gamma_1}) + \underset{r}{N_m}\mu_r(\underset{r}{\lambda_2}\underset{r-1}{\gamma_2} - \mu_r'\underset{r-1}{\lambda_2}\underset{r}{\gamma_2}) = 0\,.$$

Now, the following notations are introduced in order to simplify the above expressions:

$$\left.\begin{aligned}
&\underset{r-1}{\varphi} = \underset{r-1}{\beta_1} - \underset{r-1}{\beta_2} = \underset{r-1}{\lambda_1^2} - \underset{r-1}{\lambda_2^2} = 2\left(\frac{a_{r-1}}{b}\right)^2 \sqrt{\left\{m^2 - \frac{P_{r-1}}{2}\left(\frac{b}{a_{r-1}}\right)^2\right\}^2 - (m^2 - Q_{r-1})m^2}\,,\\
&\text{and then,}\\
&\qquad \underset{r-1}{\lambda_2}\,\underset{r-1}{\gamma_1} - \underset{r-1}{\lambda_1}\,\underset{r-1}{\gamma_2} = \underset{r-1}{\lambda_1}\,\underset{r-1}{\lambda_2}\,\underset{r-1}{(\lambda_1^2 - \lambda_2^2)} = \underset{r-1}{\lambda_1}\,\underset{r-1}{\lambda_2}\,\underset{r-1}{\varphi}\,,\\
&\text{and,}\\
&\qquad \underset{r-1}{\tau_1} = \underset{r-1}{\beta_1} - \mu_r'\underset{r}{\beta_1}\,; \quad \underset{r-1}{\chi_1} = \mu_r(\underset{r}{\lambda_1}\,\underset{r-1}{\gamma_1} - \mu_r'\underset{r-1}{\lambda_1}\,\underset{r}{\gamma_1})\,;\\
&\qquad \underset{r-1}{\tau_1'} = \underset{r-1}{\beta_1} - \mu_r'\underset{r}{\beta_2}\,; \quad \underset{r-1}{\chi_1'} = \mu_r(\underset{r}{\lambda_2}\,\underset{r-1}{\gamma_1} - \mu_r'\underset{r-1}{\lambda_1}\,\underset{r}{\gamma_2})\,;\\
&\qquad \underset{r-1}{\tau_2} = \underset{r-1}{\beta_2} - \mu_r'\underset{r}{\beta_1}\,; \quad \underset{r-1}{\chi_2} = \mu_r(\underset{r}{\lambda_1}\,\underset{r-1}{\gamma_2} - \mu_r'\underset{r-1}{\lambda_2}\,\underset{r}{\gamma_1})\,;\\
&\qquad \underset{r-1}{\tau_2'} = \underset{r-1}{\beta_2} - \mu_r'\underset{r}{\beta_2}\,; \quad \underset{r-1}{\chi_2'} = \mu_r(\underset{r}{\lambda_2}\,\underset{r-1}{\gamma_2} - \mu_r'\underset{r-1}{\lambda_2}\,\underset{r}{\gamma_2})\,.
\end{aligned}\right\} \qquad (7)$$

Thus, the foregoing equations become

$$\underset{r-1}{M_m}\,\underset{r-1}{\varphi}\cosh\pi\underset{r-1}{\lambda_2} + \underset{r-1}{N_m}\,\underset{r-1}{\varphi}\sinh\pi\underset{r-1}{\lambda_2} - \underset{r}{K_m}\,\underset{r-1}{\tau_1} - \underset{r}{M_m}\,\underset{r-1}{\tau_1'} = 0\,; \qquad (6.1')$$

$$\underset{r-1}{K_m}\,\underset{r-1}{\varphi}\cosh\pi\underset{r-1}{\lambda_1} + \underset{r-1}{L_m}\,\underset{r-1}{\varphi}\sinh\pi\underset{r-1}{\lambda_1} + \underset{r}{K_m}\,\underset{r-1}{\tau_2} + \underset{r}{M_m}\,\underset{r-1}{\tau_2'} = 0\,; \qquad (6.2')$$

$$\underset{r-1}{M_m}\,\underset{r-1}{\lambda_1}\,\underset{r-1}{\lambda_2}\,\underset{r-1}{\varphi}\sinh\pi\underset{r-1}{\lambda_2} + \underset{r-1}{N_m}\,\underset{r-1}{\lambda_1}\,\underset{r-1}{\lambda_2}\,\underset{r-1}{\varphi}\cosh\pi\underset{r-1}{\lambda_2} - \underset{r}{L_m}\,\underset{r-1}{\chi_1} - \underset{r}{N_m}\,\underset{r-1}{\chi_1'} = 0\,; \qquad (6.3')$$

$$\underset{r-1}{K_m}\,\underset{r-1}{\lambda_1}\,\underset{r-1}{\lambda_2}\,\underset{r-1}{\varphi}\sinh\pi\underset{r-1}{\lambda_1} + \underset{r-1}{L_m}\,\underset{r-1}{\lambda_1}\,\underset{r-1}{\lambda_2}\,\underset{r-1}{\varphi}\cosh\pi\underset{r-1}{\lambda_1} + \underset{r}{L_m}\,\underset{r-1}{\chi_2} + \underset{r}{N_m}\,\underset{r-1}{\chi_2'} = 0\,. \qquad (6.4')$$

From the initial assumptions, $\underset{r-1}{\lambda_1} \neq \underset{r-1}{\lambda_2}$[10] and then $\underset{r-1}{\varphi} \neq 0$, moreover, such a singular case that either of $\underset{r-1}{\lambda_1}$ and $\underset{r-1}{\lambda_2}$ becomes zero is left out of consideration. Accordingly, by the operations:

$$(6.2') \times \cosh\pi\underset{r-1}{\lambda_1} - (6.4') \times \frac{\sinh\pi\underset{r-1}{\lambda_1}}{\underset{r-1}{\lambda_1}\,\underset{r-1}{\lambda_2}}\,;$$

$$(6.2') \times \sinh\pi\underset{r-1}{\lambda_1} - (6.4') \times \frac{\cosh\pi\underset{r-1}{\lambda_1}}{\underset{r-1}{\lambda_1}\,\underset{r-1}{\lambda_2}}\,;$$

$$(6.1') \times \cosh\pi\underset{r-1}{\lambda_2} - (6.3') \times \frac{\sinh\pi\underset{r-1}{\lambda_2}}{\underset{r-1}{\lambda_1}\,\underset{r-1}{\lambda_2}}\,;$$

$$(6.1') \times \sinh\pi\underset{r-1}{\lambda_2} - (6.3') \times \frac{\cosh\pi\underset{r-1}{\lambda_2}}{\underset{r-1}{\lambda_1}\,\underset{r-1}{\lambda_2}}\,,$$

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10) When $\underset{r}{\lambda_1} = \underset{r}{\lambda_2}$, the conditional equation is given as follows: $\lim_{\underset{r}{\lambda_2}\to\underset{r}{\lambda_1}}\left[\dfrac{\Delta m}{(\underset{r}{\lambda_1} - \underset{r}{\lambda_2})^2}\right] = 0.$

the following equations are obtained:

$$\left.\begin{aligned}
\underset{r-1}{K_m} &= -\underset{r}{K_m}\underset{r-1}{F_2} + \underset{r}{L_m}\underset{r-1}{G_2'} - \underset{r}{M_m}\underset{r-1}{H_2} + \underset{r}{N_m}\underset{r-1}{I_2'};\\
\underset{r-1}{L_m} &= \underset{r}{K_m}\underset{r-1}{F_2'} - \underset{r}{L_m}\underset{r-1}{G_2} + \underset{r}{M_m}\underset{r-1}{H_2'} - \underset{r}{N_m}\underset{r-1}{I_2};\\
\underset{r-1}{M_m} &= \underset{r}{K_m}\underset{r-1}{F_1} - \underset{r}{L_m}\underset{r-1}{G_1'} + \underset{r}{M_m}\underset{r-1}{H_1} - \underset{r}{N_m}\underset{r-1}{I_1'};\\
\underset{r-1}{N_m} &= -\underset{r}{K_m}\underset{r-1}{F_1'} + \underset{r}{L_m}\underset{r-1}{G_1} - \underset{r}{M_m}\underset{r-1}{H_1'} + \underset{r}{N_m}\underset{r-1}{I_1},
\end{aligned}\right\} \tag{6.5}$$

where

$$\left.\begin{aligned}
\underset{r-1}{F_2} &= \frac{\underset{r-1}{\tau_2}}{\underset{r-1}{\varphi}}\cosh\pi\underset{r-1}{\lambda_1}, & \underset{r-1}{G_2} &= \frac{\underset{r-1}{\chi_2}}{\underset{r-1}{\lambda_1}\,\underset{r-1}{\lambda_2}\,\underset{r-1}{\varphi}}\cosh\pi\underset{r-1}{\lambda_1},\\
\underset{r-1}{F_2'} &= \frac{\underset{r-1}{\tau_2}}{\underset{r-1}{\varphi}}\sinh\pi\underset{r-1}{\lambda_1}, & \underset{r-1}{G_2'} &= \frac{\underset{r-1}{\chi_2}}{\underset{r-r}{\lambda_1}\,\underset{r-1}{\lambda_2}\,\underset{r-1}{\varphi}}\sinh\pi\underset{r-1}{\lambda_1},\\
\underset{r-1}{F_1} &= \frac{\underset{r-1}{\tau_1}}{\underset{r-1}{\varphi}}\cosh\pi\underset{r-1}{\lambda_2}, & \underset{r-1}{G_1} &= \frac{\underset{r-1}{\chi_1}}{\underset{r-1}{\lambda_1}\,\underset{r-1}{\lambda_2}\,\underset{r-1}{\varphi}}\cosh\pi\underset{r-1}{\lambda_2},\\
\underset{r-1}{F_1'} &= \frac{\underset{r-1}{\tau_1}}{\underset{r-1}{\varphi}}\sinh\pi\underset{r-1}{\lambda_2}, & \underset{r-1}{G_1'} &= \frac{\underset{r-1}{\chi_1}}{\underset{r-1}{\lambda_1}\,\underset{r-1}{\lambda_2}\,\underset{r-1}{\varphi}}\sinh\pi\underset{r-1}{\lambda_2},\\
\underset{r-1}{H_2} &= \frac{\underset{r-1}{\tau_2'}}{\underset{r-1}{\varphi}}\cosh\pi\underset{r-1}{\lambda_1}, & \underset{r-1}{I_2} &= \frac{\underset{r-1}{\chi_2'}}{\underset{r-1}{\lambda_1}\,\underset{r-1}{\lambda_2}\,\underset{r-1}{\varphi}}\cosh\pi\underset{r-1}{\lambda_1},\\
\underset{r-1}{H_2'} &= \frac{\underset{r-1}{\tau_2'}}{\underset{r-1}{\varphi}}\sinh\pi\underset{r-1}{\lambda_1}, & \underset{r-1}{I_2'} &= \frac{\underset{r-1}{\chi_2'}}{\underset{r-1}{\lambda_1}\,\underset{r-1}{\lambda_2}\,\underset{r-1}{\varphi}}\sinh\pi\underset{r-1}{\lambda_1},\\
\underset{r-1}{H_1} &= \frac{\underset{r-1}{\tau_1'}}{\underset{r-1}{\varphi}}\cosh\pi\underset{r-1}{\lambda_2}, & \underset{r-1}{I_1} &= \frac{\underset{r-1}{\chi_1'}}{\underset{r-1}{\lambda_1}\,\underset{r-1}{\lambda_2}\,\underset{r-1}{\varphi}}\cosh\pi\underset{r-1}{\lambda_2},\\
\underset{r-1}{H_1'} &= \frac{\underset{r-1}{\tau_1'}}{\underset{r-1}{\varphi}}\sinh\pi\underset{r-1}{\lambda_2}, & \underset{r-1}{I_1'} &= \frac{\underset{r-1}{\chi_1'}}{\underset{r-1}{\lambda_1}\,\underset{r-1}{\lambda_2}\,\underset{r-1}{\varphi}}\sinh\pi\underset{r-1}{\lambda_2},
\end{aligned}\right\} \tag{8}$$

Now, putting as follows:

$$\left.\begin{aligned}
\underset{r}{K_m} &= A_m\underset{r}{U_2} + B_m\underset{r}{V_2};\\
\underset{r}{L_m} &= A_m\underset{r}{U_2'} + B_m\underset{r}{V_2'};\\
\underset{r}{M_m} &= A_m\underset{r}{U_1} + B_m\underset{r}{V_1};\\
\underset{r}{N_m} &= A_m\underset{r}{U_1'} + B_m\underset{r}{V_1'};
\end{aligned}\right\} \tag{6.6}$$

and substituting them into Eq. (6.5), we obtain

$$\left.\begin{aligned}
\underset{r-1}{K_m} &= A_m\Big[-\underset{r}{U_2}\underset{r-1}{F_2}+\underset{r}{U_2'}\underset{r-1}{G_2'}-\underset{r}{U_1}\underset{r-1}{H_2}+\underset{r}{U_1'}\underset{r-1}{I_2'}\Big]+B_m\Big[-\underset{r}{V_2}\underset{r-1}{F_2}+\underset{r}{V_2'}\underset{r-1}{G_2'}-\underset{r}{V_1}\underset{r-1}{H_2}+\underset{r}{V_1'}\underset{r-1}{I_2'}\Big];\\
\underset{r-1}{L_m} &= A_m\Big[\underset{r}{U_2}\underset{r-1}{F_2'}-\underset{r}{U_2'}\underset{r-1}{G_2}+\underset{r}{U_1}\underset{r-1}{H_2'}-\underset{r}{U_1'}\underset{r-1}{I_2}\Big]+B_m\Big[\underset{r}{V_2}\underset{r-1}{F_2'}-\underset{r}{V_2'}\underset{r-1}{G_2}+\underset{r}{V_1}\underset{r-1}{H_2'}-\underset{r}{V_1'}\underset{r-1}{I_2}\Big];\\
\underset{r-1}{M_m} &= A_m\Big[\underset{r}{U_2}\underset{r-1}{F_1}-\underset{r}{U_2'}\underset{r-1}{G_1'}+\underset{r}{U_1}\underset{r-1}{H_1}-\underset{r}{U_1'}\underset{r-1}{I_1'}\Big]+B_m\Big[\underset{r}{V_2}\underset{r-1}{F_1}-\underset{r}{V_2'}\underset{r-1}{G_1'}+\underset{r}{V_1}\underset{r-1}{H_1}-\underset{r}{V_1'}\underset{r-1}{I_1'}\Big];\\
\underset{r-1}{N_m} &= A_m\Big[-\underset{r}{U_2}\underset{r-1}{F_1'}+\underset{r}{U_2'}\underset{r-1}{G_1}-\underset{r}{U_1}\underset{r-1}{H_1'}+\underset{r}{U_1'}\underset{r-1}{I_1}\Big]+B_m\Big[-\underset{r}{V_2}\underset{r-1}{F_1'}+\underset{r}{V_2'}\underset{r-1}{G_1}-\underset{r}{V_1}\underset{r-1}{H_1'}+\underset{r}{V_1'}\underset{r-1}{I_1}\Big].
\end{aligned}\right\}\quad(6.7)$$

Hence, comparing (6.6) with (6.7), we finally obtain the following recurrence formulas for U_r and V_r:

$$\left.\begin{aligned}
\underset{r-1}{U_2} &= -\underset{r}{U_2}\underset{r-1}{F_2}+\underset{r}{U_2'}\underset{r-1}{G_2'}-\underset{r}{U_1}\underset{r-1}{H_2}+\underset{r}{U_1'}\underset{r-1}{I_2'}\,; &\quad \underset{r-1}{V_2} &= -\underset{r}{V_2}\underset{r-1}{F_2}+\underset{r}{V_2'}\underset{r-1}{G_2'}-\underset{r}{V_1}\underset{r-1}{H_2}+\underset{r}{V_1'}\underset{r-1}{I_2'}\,;\\
\underset{r-1}{U_2'} &= \underset{r}{U_2}\underset{r-1}{F_2'}-\underset{r}{U_2'}\underset{r-1}{G_2}+\underset{r}{U_1}\underset{r-1}{H_2'}-\underset{r}{U_1'}\underset{r-1}{I_2}\,; &\quad \underset{r-1}{V_2'} &= \underset{r}{V_2}\underset{r-1}{F_2'}-\underset{r}{V_2'}\underset{r-1}{G_2}+\underset{r}{V_1}\underset{r-1}{H_2'}-\underset{r}{V_2'}\underset{r-1}{I_2}\,;\\
\underset{r-1}{U_1} &= \underset{r}{U_2}\underset{r-1}{F_1}-\underset{r}{U_2'}\underset{r-1}{G_1'}+\underset{r}{U_1}\underset{r-1}{H_1}-\underset{r}{U_1'}\underset{r-1}{I_1'}\,; &\quad \underset{r-1}{V_1} &= \underset{r}{V_2}\underset{r-1}{F_1}-\underset{r}{V_2'}\underset{r-1}{G_1'}+\underset{r}{V_1}\underset{r-1}{H_1}-\underset{r}{V_1'}\underset{r-1}{I_1'}\,;\\
\underset{r-1}{U_1'} &= -\underset{r}{U_2}\underset{r-1}{F_1'}+\underset{r}{U_2'}\underset{r-1}{G_1}-\underset{r}{U_1}\underset{r-1}{H_1'}+\underset{r}{U_1'}\underset{r-1}{I_1}\,; &\quad \underset{r-1}{V_1'} &= -\underset{r}{V_2}\underset{r-1}{F_1'}+\underset{r}{V_2'}\underset{r-1}{G_1}-\underset{r}{V_1}\underset{r-1}{H_1'}+\underset{r}{V_1'}\underset{r-1}{I_1}\,.
\end{aligned}\right\}\quad(9)$$

Let us suppose that a composite plate has k elementary plates. If the values of $\underset{k}{U_2}$, $\underset{k}{U_2'}$, $\underset{k}{U_1}$, $\underset{k}{U_1'}$ and $\underset{k}{V_2}$, $\underset{k}{V_2'}$, $\underset{k}{V_1}$, $\underset{k}{V_1'}$ are originally given, we will be able to obtain the numerical values of $\underset{1}{U_2}$, $\underset{1}{U_2'}$, $\underset{1}{U_1}$, $\underset{1}{U_1'}$ and $\underset{1}{V_2}$, $\underset{1}{V_2'}$, $\underset{1}{V_1}$, $\underset{1}{V_1'}$, calculating peviously the values of the factors shown in (8) and using the recurrence formulas (9).

ii) *Conditions for End Side-k.*—Let us now investigate the expressions by which $\underset{k}{U_2}$, $\underset{k}{U_2'}$, $\underset{k}{U_1}$, $\underset{k}{U_1'}$ and $\underset{k}{V_2}$, $\underset{k}{V_2'}$, $\underset{k}{V_1}$, $\underset{k}{V_1'}$ must be represented at $\xi_k=1$ (it will be called *End side-k* hereafter), corresponding to types of the end side:

(1) Case of simply supported edge.—From Table 4, we have as the equations of the end side condition

$$\underset{k}{K_m}\cosh\pi\underset{k}{\lambda_1}+\underset{k}{L_m}\sinh\pi\underset{k}{\lambda_1}+\underset{k}{M_m}\cosh\pi\underset{k}{\lambda_2}+\underset{k}{N_m}\sinh\pi\underset{k}{\lambda_2}=0\,;$$

$$\underset{k}{K_m}\underset{k}{\beta_1}\cosh\pi\underset{k}{\lambda_1}+\underset{k}{L_m}\underset{k}{\beta_1}\sinh\pi\underset{k}{\lambda_1}+\underset{k}{M_m}\underset{k}{\beta_2}\cosh\pi\underset{k}{\lambda_2}+\underset{k}{N_m}\underset{k}{\beta_2}\sinh\pi\underset{k}{\lambda_2}=0\,.$$

From these,

$$\underset{k}{K_m}\cosh\pi\underset{k}{\lambda_1}+\underset{k}{L_m}\sinh\pi\underset{k}{\lambda_1}=0\,;$$

$$\underset{k}{M_m}\cosh\pi\underset{k}{\lambda_2}+\underset{k}{N_m}\sinh\pi\underset{k}{\lambda_2}=0\,.$$

Introducing A_m, B_m as new unknown constants, the above equations are satisfied by putting as follows:

$$\left.\begin{aligned} K_{\substack{m\\k}} &= A_m \sinh\pi\lambda_{\substack{1\\k}}, & M_{\substack{m\\k}} &= B_m \sinh\pi\lambda_{\substack{2\\k}}, \\ L_{\substack{m\\k}} &= -A_m \cosh\pi\lambda_{\substack{1\\k}}, & N_{\substack{m\\k}} &= -B_m \cosh\pi\lambda_{\substack{2\\k}}. \end{aligned}\right\} \tag{6.8}$$

Therefore, comparing (6.8) with (6.6), we finally obtain the following expressions:

$$\left.\begin{aligned} U_{\substack{2\\k}} &= \sinh\pi\lambda_{\substack{1\\k}}; & U_{\substack{1\\k}} &= 0; \\ U'_{\substack{2\\k}} &= -\cosh\pi\lambda_{\substack{1\\k}}; & U'_{\substack{1\\k}} &= 0; \\ V_{\substack{2\\k}} &= 0; & V_{\substack{1\\k}} &= \sinh\pi\lambda_{\substack{2\\k}}; \\ V'_{\substack{2\\k}} &= 0; & V'_{\substack{1\\k}} &= -\cosh\pi\lambda_{\substack{2\\k}}. \end{aligned}\right\} \tag{9.a}$$

(2) Case of fixed edge.—From Table 4

$$K_{\substack{m\\k}} \cosh\pi\lambda_{\substack{1\\k}} + L_{\substack{m\\k}} \sinh\pi\lambda_{\substack{1\\k}} + M_{\substack{m\\k}} \cosh\pi\lambda_{\substack{2\\k}} + N_{\substack{m\\k}} \sinh\pi\lambda_{\substack{2\\k}} = 0;$$

$$K_{\substack{m\\k}} \lambda_{\substack{1\\k}} \sinh\pi\lambda_{\substack{1\\k}} + L_{\substack{m\\k}} \lambda_{\substack{1\\k}} \cosh\pi\lambda_{\substack{1\\k}} + M_{\substack{m\\k}} \lambda_{\substack{2\\k}} \sinh\pi\lambda_{\substack{2\\k}} + N_{\substack{m\\k}} \lambda_{\substack{2\\k}} \cosh\pi\lambda_{\substack{2\\k}} = 0.$$

From both the above

$$M_{\substack{m\\k}} = K_{\substack{m\\k}} \left(\frac{\lambda_{\substack{1\\k}}}{\lambda_{\substack{2\\k}}} \sinh\pi\lambda_{\substack{1\\k}} \sinh\pi\lambda_{\substack{2\\k}} - \cosh\pi\lambda_{\substack{1\\k}} \cosh\pi\lambda_{\substack{2\\k}} \right) + L_{\substack{m\\k}} \left(\frac{\lambda_{\substack{1\\k}}}{\lambda_{\substack{2\\k}}} \cosh\pi\lambda_{\substack{1\\k}} \sinh\pi\lambda_{\substack{2\\k}} - \sinh\pi\lambda_{\substack{1\\k}} \cosh\pi\lambda_{\substack{2\\k}} \right);$$

$$N_{\substack{m\\k}} = -K_{\substack{m\\k}} \left(\frac{\lambda_{\substack{1\\k}}}{\lambda_{\substack{2\\k}}} \sinh\pi\lambda_{\substack{1\\k}} \cosh\pi\lambda_{\substack{2\\k}} - \cosh\pi\lambda_{\substack{1\\k}} \sinh\pi\lambda_{\substack{2\\k}} \right) - L_{\substack{m\\k}} \left(\frac{\lambda_{\substack{1\\k}}}{\lambda_{\substack{2\\k}}} \cosh\pi\lambda_{\substack{1\\k}} \cosh\pi\lambda_{\substack{2\\k}} - \sinh\pi\lambda_{\substack{1\\k}} \sinh\pi\lambda_{\substack{2\\k}} \right).$$

Let

$$K_{\substack{m\\k}} = A_m, \qquad L_{\substack{m\\k}} = B_m.$$

Substituting these in the above expressions and comparing with (6.6), we obtain

$$\left.\begin{aligned} U_{\substack{2\\k}} &= 1; & U_{\substack{1\\k}} &= \frac{\lambda_{\substack{1\\k}}}{\lambda_{\substack{2\\k}}} \sinh\pi\lambda_{\substack{1\\k}} \sinh\pi\lambda_{\substack{2\\k}} - \cosh\pi\lambda_{\substack{1\\k}} \cosh\pi\lambda_{\substack{2\\k}}; \\ U'_{\substack{2\\k}} &= 0; & U'_{\substack{1\\k}} &= -\frac{\lambda_{\substack{1\\k}}}{\lambda_{\substack{2\\k}}} \sinh\pi\lambda_{\substack{1\\k}} \cosh\pi\lambda_{\substack{2\\k}} + \cosh\pi\lambda_{\substack{1\\k}} \sinh\pi\lambda_{\substack{2\\k}}; \\ V_{\substack{2\\k}} &= 0; & V_{\substack{1\\k}} &= \frac{\lambda_{\substack{1\\k}}}{\lambda_{\substack{2\\k}}} \cosh\pi\lambda_{\substack{1\\k}} \sinh\pi\lambda_{\substack{2\\k}} - \sinh\pi\lambda_{\substack{1\\k}} \cosh\pi\lambda_{\substack{2\\k}}; \\ V'_{\substack{2\\k}} &= 1; & V'_{\substack{1\\k}} &= -\frac{\lambda_{\substack{1\\k}}}{\lambda_{\substack{2\\k}}} \cosh\pi\lambda_{\substack{1\\k}} \cosh\pi\lambda_{\substack{2\\k}} + \sinh\pi\lambda_{\substack{1\\k}} \sinh\pi\lambda_{\substack{2\\k}}. \end{aligned}\right\} \tag{9.b}$$

(3) Case of free edge.—From Table 4

$$\underset{k}{K_m}\underset{k}{\beta_1}\cosh\pi\underset{k}{\lambda_1}+\underset{k}{L_m}\underset{k}{\beta_1}\sinh\pi\underset{k}{\lambda_1}+\underset{k}{M_m}\underset{k}{\beta_2}\cosh\pi\underset{k}{\lambda_2}+\underset{k}{N_m}\underset{k}{\beta_2}\sinh\pi\underset{k}{\lambda_2}=0\,;$$

$$\underset{k}{K_m}(\underset{k}{\gamma_1}+P_k\underset{k}{\lambda_1})\sinh\pi\underset{k}{\lambda_1}+\underset{k}{L_m}(\underset{k}{\gamma_1}+P_k\underset{k}{\lambda_1})\cosh\pi\underset{k}{\lambda_1}+\underset{k}{M_m}(\underset{k}{\gamma_2}+P_k\underset{k}{\lambda_2})\sinh\pi\underset{k}{\lambda_2}+\underset{k}{N_m}(\underset{k}{\gamma_2}+P_k\underset{k}{\lambda_2})\cosh\pi\underset{k}{\lambda_2}=0,$$

and from both the above, we have

$$\begin{aligned}\underset{k}{M_m}=\underset{k}{K_m}&\left(\frac{\underset{k}{\gamma_1}+P_k\underset{k}{\lambda_1}}{\underset{k}{\gamma_2}+P_k\underset{k}{\lambda_2}}\sinh\pi\underset{k}{\lambda_1}\sinh\pi\underset{k}{\lambda_2}-\frac{\underset{k}{\beta_1}}{\underset{k}{\beta_2}}\cosh\pi\underset{k}{\lambda_1}\cosh\pi\underset{k}{\lambda_2}\right)\\&+\underset{k}{L_m}\left(\frac{\underset{k}{\gamma_1}+P_k\underset{k}{\lambda_1}}{\underset{k}{\gamma_2}+P_k\underset{k}{\lambda_2}}\cosh\pi\underset{k}{\lambda_1}\sinh\pi\underset{k}{\lambda_2}-\frac{\underset{k}{\beta_1}}{\underset{k}{\beta_2}}\sinh\pi\underset{k}{\lambda_1}\cosh\pi\underset{k}{\lambda_2}\right):\end{aligned}$$

$$\begin{aligned}\underset{k}{N_m}=-\underset{k}{K_m}&\left(\frac{\underset{k}{\gamma_1}+P_k\underset{k}{\lambda_1}}{\underset{k}{\gamma_2}+P_k\underset{k}{\lambda_2}}\sinh\pi\underset{k}{\lambda_1}\cosh\pi\underset{k}{\lambda_2}-\frac{\underset{k}{\beta_1}}{\underset{k}{\beta_2}}\cosh\pi\underset{k}{\lambda_1}\sinh\pi\underset{k}{\lambda_2}\right)\\&-\underset{k}{L_m}\left(\frac{\underset{k}{\gamma_1}+P_k\underset{k}{\lambda_1}}{\underset{k}{\gamma_2}+P_k\underset{k}{\lambda_2}}\cosh\pi\underset{k}{\lambda_1}\cosh\pi\underset{k}{\lambda_2}-\frac{\underset{k}{\beta_1}}{\underset{k}{\beta_2}}\sinh\pi\underset{k}{\lambda_1}\sinh\pi\underset{k}{\lambda_2}\right).\end{aligned}$$

Now, putting $\underset{k}{K_m}=A_m$, $\underset{k}{L_m}=B_m$ as before, we obtain by the comparison with (6.6)

$$\left.\begin{aligned}\underset{k}{U_2}=1\ ;\quad \underset{k}{U_1}&=\frac{\underset{k}{\gamma_1}+P_k\underset{k}{\lambda_1}}{\underset{k}{\gamma_2}+P_k\underset{k}{\lambda_2}}\sinh\pi\underset{k}{\lambda_1}\sinh\pi\underset{k}{\lambda_2}-\frac{\underset{k}{\beta_1}}{\underset{k}{\beta_2}}\cosh\pi\underset{k}{\lambda_1}\cosh\pi\underset{k}{\lambda_2}\,;\\\underset{k}{U_2'}=0\ ;\quad \underset{k}{U_1'}&=-\frac{\underset{k}{\gamma_1}+P_k\underset{k}{\lambda_1}}{\underset{k}{\gamma_2}+P_k\underset{k}{\lambda_2}}\sinh\pi\underset{k}{\lambda_1}\cosh\pi\underset{k}{\lambda_2}+\frac{\underset{k}{\beta_1}}{\underset{k}{\beta_2}}\cosh\pi\underset{k}{\lambda_1}\sinh\pi\underset{k}{\lambda_2}\,;\\\underset{k}{V_2}=0\ ;\quad \underset{k}{V_1}&=\frac{\underset{k}{\gamma_1}+P_k\underset{k}{\lambda_1}}{\underset{k}{\gamma_2}+P_k\underset{k}{\lambda_2}}\cosh\pi\underset{k}{\lambda_1}\sinh\pi\underset{k}{\lambda_2}-\frac{\underset{k}{\beta_1}}{\underset{k}{\beta_2}}\sinh\pi\underset{k}{\lambda_1}\cosh\pi\underset{k}{\lambda_2}\,;\\\underset{k}{V_2'}=1\ ;\quad \underset{k}{V_1'}&=-\frac{\underset{k}{\gamma_1}+P_k\underset{k}{\lambda_1}}{\underset{k}{\gamma_2}+P_k\underset{k}{\lambda_2}}\cosh\pi\underset{k}{\lambda_1}\cosh\pi\underset{k}{\lambda_2}+\frac{\underset{k}{\beta_1}}{\underset{k}{\beta_2}}\sinh\pi\underset{k}{\lambda_1}\sinh\pi\underset{k}{\lambda_2}\,.\end{aligned}\right\}\quad\textbf{(9.c)}$$

iii) *Conditional Equations for Determination of Critical Load.*—Since the expressions representing the end side conditions at $\xi_k=1$ are given in the foregoing paragraph, the conditional equations for determination of the critical load are produced respectively according to the types of the end side at $\xi_r=0$ (it will be called *End side*-1 hereafter) as follows:

(1) Case of simply supported edge.—From Table 4, we can write as the conditions for End side-1

$$\underset{1}{K_m}+\underset{1}{M_m}=0\,;$$

$$K_{\substack{m\\1}} \beta_{\substack{1\\1}} + M_{\substack{m\\1}} \beta_{\substack{2\\1}} = 0\,,$$

and then, we obtain

$$K_{\substack{m\\1}} = M_{\substack{m\\1}} = 0\,.$$

From these and (6.6), we can write down

$$K_{\substack{m\\1}} = A_m U_{\substack{2\\1}} + B_m V_{\substack{2\\1}} = 0\,;$$

$$M_{\substack{m\\1}} = A_m U_{\substack{1\\1}} + B_m V_{\substack{1\\1}} = 0.$$

We may finally conclude that buckling of the composite plate becomes possible when the above equations yield a solution for A_m and B_m different from zero. Hence the critical loads are found by setting the determinant of these two equations to zero. Thus

$$U_{\substack{1\\1}} V_{\substack{2\\1}} - V_{\substack{1\\1}} U_{\substack{2\\1}} = 0 \qquad \text{or} \qquad \frac{U_{\substack{1\\1}}}{U_{\substack{2\\1}}} - \frac{V_{\substack{1\\1}}}{V_{\substack{2\\1}}} = 0\,. \tag{9.d}$$

(2) Case of fixed edge.—From Table 4,

$$K_{\substack{m\\1}} + M_{\substack{m\\1}} = 0$$

$$L_{\substack{m\\1}} \lambda_{\substack{1\\1}} + N_{\substack{m\\1}} \lambda_{\substack{2\\1}} = 0$$

Substituting the expressions obtained from (6.6) for $r = 1$ in both the above, we have

$$A_m (U_{\substack{1\\1}} + U_{\substack{2\\1}}) + B_m (V_{\substack{1\\1}} + V_{\substack{2\\1}}) = 0\,;$$

$$A_m (U_{\substack{1\\1}}' \lambda_{\substack{2\\1}} + U_{\substack{2\\1}}' \lambda_{\substack{1\\1}}) + B_m (V_{\substack{1\\1}}' \lambda_{\substack{2\\1}} + V_{\substack{2\\1}}' \lambda_{\substack{1\\1}}) = 0\,.$$

Therefore, by elimination of A_m and B_m from these equations, we find the conditional equations as follows:

$$\left.\begin{aligned} &(U_{\substack{1\\1}} + U_{\substack{2\\1}})(V_{\substack{1\\1}}' \lambda_{\substack{2\\1}} + V_{\substack{2\\1}}' \lambda_{\substack{1\\1}}) - (V_{\substack{1\\1}} + V_{\substack{2\\1}})(U_{\substack{1\\1}}' \lambda_{\substack{2\\1}} + U_{\substack{2\\1}}' \lambda_{\substack{1\\1}}) = 0\,; \\ \text{or} \\ &\frac{U_{\substack{1\\1}} + U_{\substack{2\\1}}}{U_{\substack{1\\1}}' \lambda_{\substack{2\\1}} + U_{\substack{2\\1}}' \lambda_{\substack{1\\1}}} - \frac{V_{\substack{1\\1}} + V_{\substack{2\\1}}}{V_{\substack{1\\1}}' \lambda_{\substack{2\\1}} + V_{\substack{2\\1}}' \lambda_{\substack{1\\1}}} = 0\,. \end{aligned}\right\} \tag{9.e}$$

(3) Case of free edge.—From Table 4,

$$K_{\substack{m\\1}} \beta_{\substack{1\\1}} + M_{\substack{m\\1}} \beta_{\substack{2\\1}} = 0\,;$$

$$L_{\substack{m\\1}} (\gamma_{\substack{1\\1}} + P_1 \lambda_{\substack{1\\1}}) + N_{\substack{m\\1}} (\gamma_{\substack{2\\1}} + P_1 \lambda_{\substack{2\\1}}) = 0\,.$$

Substituting the expressions obtained from (6.6) for $r=1$ into both the above equations, we have

$$A_m\,(\underset{1}{U_1}\,\underset{1}{\beta_2}+\underset{1}{U_2}\,\underset{1}{\beta_1})+B_m\,(\underset{1}{V_1}\,\underset{1}{\beta_2}+\underset{1}{V_2}\,\underset{1}{\beta_1})=0\,;$$

$$A_m\left\{\underset{1}{U_1'}\,(\underset{1}{\gamma_2}+P_1\underset{1}{\lambda_2})+\underset{1}{U_2'}\,(\underset{1}{\gamma_1}+P_1\underset{1}{\lambda_1})\right\}+B_m\left\{\underset{1}{V_1'}(\underset{1}{\gamma_2}+P_1\underset{1}{\lambda_2})+\underset{1}{V_2'}(\underset{1}{\gamma_1}+P_1\underset{1}{\lambda_1})\right\}=0\,.$$

From these, the conditional equation for buckling becomes

$$\left.\begin{array}{l}(\underset{1}{U_1}\underset{1}{\beta_2}+\underset{1}{U_2}\underset{1}{\beta_1})\left\{\underset{1}{V_1'}(\underset{1}{\gamma_2}+P_1\underset{1}{\lambda_2})+\underset{1}{V_2'}(\underset{1}{\gamma_1}+P_1\underset{1}{\lambda_1})\right\}-(\underset{1}{V_1}\underset{1}{\beta_2}+\underset{1}{V_2}\underset{1}{\beta_1})\left\{\underset{1}{U_1'}(\underset{1}{\gamma_2}+P_1\underset{1}{\lambda_2})+\underset{1}{U_2'}(\underset{1}{\gamma_1}+P_1\underset{1}{\lambda_1})\right\}=0\\ \text{or}\\ \dfrac{\underset{1}{U_1}\,\underset{1}{\beta_2}+\underset{1}{U_2}\,\underset{1}{\beta_1}}{\underset{1}{U_1'}(\underset{1}{\gamma_2}+P_1\underset{1}{\lambda_2})+\underset{1}{U_2'}(\underset{1}{\gamma_1}+P_1\underset{1}{\lambda_1})}-\dfrac{\underset{1}{V_1}\,\underset{1}{\beta_2}+\underset{1}{V_2}\,\underset{1}{\beta_1}}{\underset{1}{V_1'}(\underset{1}{\gamma_2}+P_1\underset{1}{\lambda_2})+\underset{1}{V_2'}(\underset{1}{\gamma_1}+P_1\underset{1}{\lambda_1})}=0.\end{array}\right\}\quad(9.\mathrm{f})$$

Finally, we can see that each of the kinds shown in Table 4 is expressed by the various combinations of the end side conditions at $\xi_k=1$ and $\xi_1=0$ which are introduced respectively in the foregoing paragraphs ii) and iii). So, if it becomes possible to calculate the numerical values of $\underset{k}{U_1}$, $\underset{k}{U_1'}$, $\underset{k}{U_2}$ $\underset{k}{U_2'}$, and $\underset{k}{V_1}$. $\underset{k}{V_1'}$, $\underset{k}{V_2}$, $\underset{k}{V_2'}$ with the expressions in ii) for each case, the values of $\underset{1}{U_1}$, $\underset{1}{U_1'}$, $\underset{1}{U_2}$, $\underset{1}{U_2'}$ and $\underset{1}{V_1}$, $\underset{1}{V_1'}$, $\underset{1}{V_2}$, $\underset{1}{V_2'}$ be obtained, using the formulas (9), and then, the left hand side of a conditional equation such as each formula shown in the paragraph iii) can be computed.

Thus, the problem is completely settled, if we can find the least value of P_r or Q_r causing that the value of left member of the conditional equation is equal to zero. Such numerical examples will be illustrated in the later chapter.

b) Case of Rigid Supporting along the Joints. i) *Conditions for Jointing.*—From Table 5, the conditions joining the (r-1)-th elementary plate to the r-th are

$$\left.\begin{array}{r}\underset{r-1}{K_m}\cosh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{L_m}\sinh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{M_m}\cosh\pi\underset{r-1}{\lambda_2}+\underset{r-1}{N_m}\sinh\pi\underset{r-1}{\lambda_2}=0\,;\\ \underset{r}{K_m}+\underset{r}{M_m}\qquad\qquad=0\,;\\ \underset{r-1}{K_m}\underset{r-1}{\lambda_1}\sinh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{L_m}\underset{r-1}{\lambda_1}\cosh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{M_m}\underset{r-1}{\lambda_2}\sinh\pi\underset{r-1}{\lambda_2}+\underset{r-1}{N_m}\underset{r-1}{\lambda_2}\cosh\pi\underset{r-1}{\lambda_2}-\underset{r}{L_m}\mu_r\underset{r}{\lambda_1}-\underset{r}{N_m}\mu_r\underset{r}{\lambda_2}=0\,;\\ \underset{r-1}{K_m}\underset{r-1}{\beta_1}\cosh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{L_m}\underset{r-1}{\beta_1}\sinh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{M_m}\underset{r-1}{\beta_2}\cosh\pi\underset{r-1}{\lambda_2}+\underset{r-1}{N_m}\underset{r-1}{\beta_2}\sinh\pi\underset{r-1}{\lambda_2}-\underset{r}{K_m}\mu_r'\underset{r}{\beta_1}-\underset{r}{M_m}\mu_r'\underset{r}{\beta_2}=0\,.\end{array}\right\}\quad(6.9)$$

The constants accompained by the suffix r–1 are contained by only the three equations in (6.9). Then, in order to represent them with the constants accompanied by the suffix r, let us consider now the condition $|w_{r-1}|_{\underset{r-1}{\xi}=0}=0$ which holds at the neighbouring joint, or

$$\underset{r-1}{K_m} + \underset{r-1}{M_m} = 0\,; \tag{6.10}$$

together with the previous equations. That is, from (6,10)

$$\underset{r-1}{K_m} = -\underset{r-1}{M_m} = \underset{r-1}{B_m}\,,$$

and also, from the second equation of (6.9), we can put as follows;

$$\underset{r}{K_m} = -\underset{r}{M_m} = \underset{r}{B_m}\,. \tag{6.11}$$

The remaining three equations in (6.9), therefore, are rewritten as follows:

$$\left.\begin{aligned}
&\underset{r-1}{B_m}(\cosh\pi\underset{r-1}{\lambda_1} - \cosh\pi\underset{r-1}{\lambda_2}) + \underset{r-1}{L_m}\sinh\pi\underset{r-1}{\lambda_1} + \underset{r-1}{N_m}\sinh\pi\underset{r-1}{\lambda_2} = 0\,;\\
&\underset{r-1}{B_m}(\underset{r-1}{\lambda_1}\sinh\pi\underset{r-1}{\lambda_1} - \underset{r-1}{\lambda_2}\sinh\pi\underset{r-1}{\lambda_2}) + \underset{r-1}{L_m}\underset{r-1}{\lambda_1}\cosh\pi\underset{r-1}{\lambda_1} + \underset{r-1}{N_m}\underset{r-1}{\lambda_2}\cosh\pi\underset{r-1}{\lambda_2} - \underset{r}{L_m}\mu_r\underset{r}{\lambda_1} - \underset{r}{N_m}\mu_r\underset{r}{\lambda_2} = 0\,;\\
&\underset{r-1}{B_m}(\underset{r-1}{\beta_1}\cosh\pi\underset{r-1}{\lambda_1} - \underset{r-1}{\beta_2}\cosh\pi\underset{r-1}{\lambda_2}) + \underset{r-1}{L_m}\underset{r-1}{\beta_1}\sinh\pi\underset{r-1}{\lambda_1} + \underset{r-1}{N_m}\underset{r-1}{\beta_2}\sinh\pi\underset{r-1}{\lambda_2} - \underset{r}{B_m}\mu'_r(\underset{r}{\beta_1} - \underset{r}{\beta_2}) = 0.
\end{aligned}\right\} \tag{6.12}$$

The number of the constants with suffix $r-1$ coincides with that of the equations in the above. Therefore, solving these with respect to such constants, we have

$$\left.\begin{aligned}
\underset{r-1}{L_m} =\ & (\underset{r}{L_m}\underset{r}{\lambda_1} + \underset{r}{N_m}\underset{r}{\lambda_2})\mu_r \frac{\coth\pi\underset{r-1}{\lambda_1}}{\underset{r-1}{\lambda_1}\operatorname{cosech}\pi\underset{r-1}{\lambda_1} - \underset{r-1}{\lambda_2}\operatorname{cosech}\pi\underset{r-1}{\lambda_2}}\\
& - \underset{r}{B_m}\mu'_r\frac{\varphi_r}{\varphi_{r-1}}\cdot\frac{\left\{\underset{r-1}{\lambda_2}\left(\operatorname{cosech}\pi\underset{r-1}{\lambda_1}\operatorname{cosech}\pi\underset{r-1}{\lambda_2} - \coth\pi\underset{r-1}{\lambda_1}\coth\pi\underset{r-1}{\lambda_2}\right) + \underset{r-1}{\lambda_1}\right\}}{\underset{r-1}{\lambda_1}\operatorname{cosech}\pi\underset{r-1}{\lambda_1} - \underset{r-1}{\lambda_2}\operatorname{cosech}\pi\underset{r-1}{\lambda_2}}\,,\\
\underset{r-1}{N_m} =\ & -(\underset{r}{L_m}\underset{r}{\lambda_1} + \underset{r}{N_m}\underset{r}{\lambda_2})\mu_r \frac{\coth\pi\underset{r-1}{\lambda_2}}{\underset{r-1}{\lambda_1}\operatorname{cosech}\pi\underset{r-1}{\lambda_1} - \underset{r-1}{\lambda_2}\operatorname{cosech}\pi\underset{r-1}{\lambda_2}}\\
& - \underset{r}{B_m}\mu'_r\frac{\varphi_r}{\varphi_{r-1}}\cdot\frac{\left\{\underset{r-1}{\lambda_1}\left(\operatorname{cosech}\pi\underset{r-1}{\lambda_1}\operatorname{cosech}\pi\underset{r-1}{\lambda_2} - \coth\pi\underset{r-1}{\lambda_1}\coth\pi\underset{r-1}{\lambda_2}\right) + \underset{r-1}{\lambda_2}\right\}}{\underset{r-1}{\lambda_1}\operatorname{cosech}\pi\underset{r-1}{\lambda_1} - \underset{r-1}{\lambda_2}\operatorname{cosech}\pi\underset{r-1}{\lambda_2}}\,,\\
\underset{r-1}{B_m} =\ & -(\underset{r}{L_m}\underset{r}{\lambda_1} + \underset{r}{N_m}\underset{r}{\lambda_2})\mu_r \frac{1}{\underset{r-1}{\lambda_1}\operatorname{cosech}\pi\underset{r-1}{\lambda_1} - \underset{r-1}{\lambda_2}\operatorname{cosech}\pi\underset{r-1}{\lambda_2}}\\
& + \underset{r}{B_m}\mu'_r\frac{\varphi_r}{\varphi_{r-1}}\cdot\frac{\underset{r-1}{\lambda_1}\coth\pi\underset{r-1}{\lambda_1} - \underset{r-1}{\lambda_2}\coth\pi\underset{r-1}{\lambda_2}}{\underset{r-1}{\lambda_1}\operatorname{cosech}\pi\underset{r-1}{\lambda_1} - \underset{r-1}{\lambda_2}\operatorname{cosech}\pi\underset{r-1}{\lambda_2}}\,.
\end{aligned}\right\} \tag{6.13}$$

Multiplying both sides of the first expression by $\underset{r-1}{\lambda_1}$ and those of the second by $\underset{r-1}{\lambda_2}$ and adding both, we obtain

$$\underset{r-1}{L_m}\underset{r-1}{\lambda_1} + \underset{r-1}{N_m}\underset{r-1}{\lambda_2} = (\underset{r}{L_m}\underset{r}{\lambda_1} + \underset{r}{N_m}\underset{r}{\lambda_2})\,\mu_r\,\frac{\underset{r-1}{\lambda_1}\coth\pi\underset{r-1}{\lambda_1} - \underset{r-1}{\lambda_2}\coth\pi\underset{r-1}{\lambda_2}}{\underset{r-1}{\lambda_1}\operatorname{cosech}\pi\underset{r-1}{\lambda_1} - \underset{r-1}{\lambda_2}\operatorname{cosech}\pi\underset{r-1}{\lambda_2}}$$

$$-\underset{r}{B_m}\mu_r' \frac{\varphi_r}{\varphi_{r-1}} \cdot \frac{\left(\underset{r-1}{\lambda_1^2}+\underset{r-1}{\lambda_2^2}\right)+2\underset{r-1}{\lambda_1}\underset{r-1}{\lambda_2}\left(\mathrm{cosech}\pi\underset{r-1}{\lambda_1}\mathrm{cosech}\pi\underset{r-1}{\lambda_2}-\coth\pi\underset{r-1}{\lambda_1}\coth\pi\underset{r-1}{\lambda_2}\right)}{\underset{r-1}{\lambda_1}\,\mathrm{cosech}\pi\underset{r-1}{\lambda_1}-\underset{r-1}{\lambda_2}\,\mathrm{cosech}\pi\underset{r-1}{\lambda_2}}\,.$$

Further, denoting

$$\underset{r}{L_m}\underset{r}{\lambda_1}+\underset{r}{N_m}\underset{r}{\lambda_2}=\underset{r}{A_m}\,, \tag{6.14}$$

the above expression and the third in (6.13) are transformed as follows:

$$\left.\begin{aligned}\underset{r-1}{A_m}&=\quad\underset{r}{A_m}F_{r-1}-\underset{r}{B_m}G_{r-1}\,;\\ \underset{r-1}{B_m}&=-\underset{r}{A_m}F'_{r-1}+\underset{r}{B_m}G'_{r-1}\,,\end{aligned}\right\} \tag{6.15}$$

where

$$F_{r-1}=\mu_r\frac{T_{r-1}}{S_{r-1}}\,,\qquad G_{r-1}=\mu_r'\frac{\varphi_r}{\varphi_{r-1}}\frac{W_{r-1}}{S_{r-1}}\,,$$

$$F'_{r-1}=\mu_r\frac{1}{S_{r-1}}\,,\qquad G'_{r-1}=\mu_r'\frac{\varphi_r}{\varphi_{r-1}}\frac{T_{r-1}}{S_{r-1}}\,,$$

in which **(10)**

$$T_{r-1}=\underset{r-1}{\lambda_1}\coth\pi\underset{r-1}{\lambda_1}-\underset{r-1}{\lambda_2}\coth\pi\underset{r-1}{\lambda_2}\,,$$

$$S_{r-1}=\underset{r-1}{\lambda_1}\,\mathrm{cosech}\pi\underset{r-1}{\lambda_1}-\underset{r-1}{\lambda_2}\,\mathrm{cosech}\pi\underset{r-1}{\lambda_2}\,,$$

$$W_{r-1}=T^2_{r-1}-S^2_{r-1}=\underset{r-1}{\lambda_1^2}+\underset{r-1}{\lambda_2^2}+2\underset{r-1}{\lambda_1}\underset{r-1}{\lambda_2}\left(\mathrm{cosech}\pi\underset{r-1}{\lambda_1}\,\mathrm{cosech}\pi\underset{r-1}{\lambda_2}-\coth\pi\underset{r-1}{\lambda_1}\coth\pi\underset{r-1}{\lambda_2}\right).$$

Let us denote now

$$\left.\begin{aligned}\underset{r}{A_m}&=A_m U_r\,,\\ \underset{r}{B_m}&=A_m V_r\,,\end{aligned}\right\} \tag{6.16}$$

and substituting in (6.15), we get

$$\begin{aligned}\underset{r-1}{A_m}&=\quad A_m(U_rF_{r-1}-V_rG_{r-1})\,,\\ \underset{r-1}{B_m}&=-A_m(U_rF'_{r-1}-V_rG'_{r-1})\,.\end{aligned}$$

Then, by the comparison with (6.16), the following recurrence formulas are obtained

$$\left.\begin{aligned}U_{r-1}&=\quad U_rF_{r-1}-V_rG_{r-1}\,;\\ V_{r-1}&=-U_rF'_{r-1}+V_rG'_{r-1}\,.\end{aligned}\right\} \tag{11}$$

ii) *Conditions for End Side-k.*—These are given in the present case as follows:

(1) Case of simply supported edge.—In the same manner as the foregoing case the conditions at $x_k=a_k$ are represented by the equations

$$\underset{k}{K_m}\cosh\pi\underset{k}{\lambda_1}+\underset{k}{L_m}\sinh\pi\underset{k}{\lambda_1}=0\,;$$

$$M_{\substack{m\\k}} \cosh\pi\lambda_{\substack{2\\k}} + N_{\substack{m\\k}} \sinh\pi\lambda_{\substack{2\\k}} = 0 .$$

Observing (6.11), it follows that

$$\left.\begin{aligned} L_{\substack{m\\k}} &= - K_{\substack{m\\k}} \coth\pi\lambda_{\substack{1\\k}} = - B_{\substack{m\\k}} \coth\pi\lambda_{\substack{1\\k}} ; \\ N_{\substack{m\\k}} &= - M_{\substack{m\\k}} \coth\pi\lambda_{\substack{2\\k}} = B_{\substack{m\\k}} \coth\pi\lambda_{\substack{2\\k}} , \end{aligned}\right\}$$

and then, we can write the expression of $A_{\substack{m\\k}}$ by (6.14), *i.e.*,

$$A_{\substack{m\\k}} = - B_{\substack{m\\k}} (\lambda_{\substack{1\\k}} \coth\pi\lambda_{\substack{1\\k}} - \lambda_{\substack{2\\k}} \coth\pi\lambda_{\substack{2\\k}}).$$

Denoting $B_{\substack{m\\k}}$ by A_m, we obtain by comparing with (6.16)

$$\left.\begin{aligned} &U_k = - T_k ; \qquad V_k = 1 , \\ &\text{where} \\ &T_k = \lambda_{\substack{1\\k}} \coth\pi\lambda_{\substack{1\\k}} - \lambda_{\substack{2\\k}} \coth\pi\lambda_{\substack{2\\k}} . \end{aligned}\right\} \qquad (11.a)$$

(2) Case of fixed edge.—The end side conditions are expressed as

$$K_{\substack{m\\k}}\cosh\pi\lambda_{\substack{1\\k}} + L_{\substack{m\\k}}\sinh\pi\lambda_{\substack{1\\k}} + M_{\substack{m\\k}}\cosh\pi\lambda_{\substack{2\\k}} + N_{\substack{m\\k}}\sinh\pi\lambda_{\substack{2\\k}} = 0 ;$$

$$K_{\substack{m\\k}}\lambda_{\substack{1\\k}}\sinh\pi\lambda_{\substack{1\\k}} + L_{\substack{m\\k}}\lambda_{\substack{1\\k}}\cosh\pi\lambda_{\substack{1\\k}} + M_{\substack{m\\k}}\lambda_{\substack{2\\k}}\sinh\pi\lambda_{\substack{2\\k}} + N_{\substack{m\\k}}\lambda_{\substack{2\\k}}\cosh\pi\lambda_{\substack{2\\k}} = 0 .$$

Rewriting the above by (6.11), we have

$$B_{\substack{m\\k}} (\cosh\pi\lambda_{\substack{1\\k}} - \cosh\pi\lambda_{\substack{2\\k}}) + L_{\substack{m\\k}}\sinh\pi\lambda_{\substack{1\\k}} + N_{\substack{m\\k}}\sinh\pi\lambda_{\substack{2\\k}} = 0 ;$$

$$B_{\substack{m\\k}} (\lambda_{\substack{1\\k}}\sinh\pi\lambda_{\substack{1\\k}} - \lambda_{\substack{2\\k}}\sinh\pi\lambda_{\substack{2\\k}}) + L_{\substack{m\\k}}\lambda_{\substack{1\\k}}\cosh\pi\lambda_{\substack{1\\k}} + N_{\substack{m\\k}}\lambda_{\substack{2\\k}}\cosh\pi\lambda_{\substack{2\\k}} = 0 .$$

From these

$$L_{\substack{m\\k}} = - B_{\substack{m\\k}} \frac{\lambda_{\substack{2\\k}}(\operatorname{cosech}\pi\lambda_{\substack{1\\k}}\operatorname{cosech}\pi\lambda_{\substack{2\\k}} - \coth\pi\lambda_{\substack{1\\k}}\coth\pi\lambda_{\substack{2\\k}}) + \lambda_{\substack{1\\k}}}{\lambda_{\substack{1\\k}}\coth\pi\lambda_{\substack{1\\k}} - \lambda_{\substack{2\\k}}\coth\pi\lambda_{\substack{2\\k}}} ,$$

$$N_{\substack{m\\k}} = - B_{\substack{m\\k}} \frac{\lambda_{\substack{1\\k}}(\operatorname{cosech}\pi\lambda_{\substack{1\\k}}\operatorname{cosech}\pi\lambda_{\substack{2\\k}} - \coth\pi\lambda_{\substack{1\\k}}\coth\pi\lambda_{\substack{2\\k}}) + \lambda_{\substack{2\\k}}}{\lambda_{\substack{1\\k}}\coth\pi\lambda_{\substack{1\\k}} - \lambda_{\substack{2\\k}}\coth\pi\lambda_{\substack{2\\k}}} .$$

Producing the expression of $A_{\substack{m\\k}}$ by substitution of the above expressions in (6.14), we have for this case

$$A_{\substack{m\\k}} = - B_{\substack{m\\k}} \frac{(\lambda_{\substack{1\\k}}^2 + \lambda_{\substack{2\\k}}^2) + 2\lambda_{\substack{1\\k}}\lambda_{\substack{2\\k}}(\operatorname{cosech}\pi\lambda_{\substack{1\\k}}\operatorname{cosech}\pi\lambda_{\substack{2\\k}} - \coth\pi\lambda_{\substack{1\\k}}\coth\pi\lambda_{\substack{2\\k}})}{\lambda_{\substack{1\\k}} \coth\pi\lambda_{\substack{1\\k}} - \lambda_{\substack{2\\k}} \coth\pi\lambda_{\substack{2\\k}}} .$$

Denote again $B_{\substack{m\\k}}$ by A_m, and comparing the above with (6.16), we get the following expressions:

$$U_k = -\frac{W_k}{T_k}\,; \qquad V_k = 1\,,$$

in which

$$W_k = (\underset{k}{\lambda_1^2} + \underset{k}{\lambda_2^2}) + 2\underset{k}{\lambda_1}\underset{k}{\lambda_2}(\operatorname{cosech}\pi\underset{k}{\lambda_1}\operatorname{cosech}\pi\underset{k}{\lambda_2} - \coth\pi\underset{k}{\lambda_1}\coth\pi\underset{k}{\lambda_2})\,. \qquad (11.\text{b})$$

(3) Case of free edge.—The end side conditions are written in this case as follows:

$$\underset{k}{K_m}\underset{k}{\beta_1}\cosh\pi\underset{k}{\lambda_1} + \underset{k}{L_m}\underset{k}{\beta_1}\sinh\pi\underset{k}{\lambda_1} + \underset{k}{M_m}\underset{k}{\beta_2}\cosh\pi\underset{k}{\lambda_2} + \underset{k}{N_m}\underset{k}{\beta_2}\sinh\pi\underset{k}{\lambda_2} = 0\,;$$

$$\underset{k}{K_m}(\underset{k}{\gamma_1} + P_k\underset{k}{\lambda_1})\sinh\pi\underset{k}{\lambda_1} + \underset{k}{L_m}(\underset{k}{\gamma_1} + P_k\underset{k}{\lambda_1})\cosh\pi\underset{k}{\lambda_1} + \underset{k}{M_m}(\underset{k}{\gamma_2} + P_k\underset{k}{\lambda_2})\sinh\pi\underset{k}{\lambda_2} + \underset{k}{N_m}(\underset{k}{\gamma_2} + P_k\underset{k}{\lambda_2})\cosh\pi\underset{k}{\lambda_2} = 0\,.$$

By using (6.11), the above equations are rewritten as follows:

$$\underset{k}{B_m}(\underset{k}{\beta_1}\cosh\pi\underset{k}{\lambda_1} - \underset{k}{\beta_2}\cosh\pi\underset{k}{\lambda_2}) + \underset{k}{L_m}\underset{k}{\beta_1}\sinh\pi\underset{k}{\lambda_1} + \underset{k}{N_m}\underset{k}{\beta_2}\sinh\pi\underset{k}{\lambda_2} = 0\,;$$

$$\underset{k}{B_m}\left\{(\underset{k}{\gamma_1} + P_k\underset{k}{\lambda_1})\sinh\pi\underset{k}{\lambda_1} - (\underset{k}{\gamma_2} + P_k\underset{k}{\lambda_2})\sinh\pi\underset{k}{\lambda_2}\right\} + \underset{k}{L_m}(\underset{k}{\gamma_1} + P_k\underset{k}{\lambda_1})\cosh\pi\underset{k}{\lambda_1} + \underset{k}{N_m}(\underset{k}{\gamma_2} + P_k\underset{k}{\lambda_2})\cosh\pi\underset{k}{\lambda_2} = 0\,.$$

From these, the following expressions are obtained:

$$\left.\begin{aligned}
\underset{k}{L_m} &= -\underset{k}{B_m}\frac{(\underset{k}{\gamma_2} + P_k\underset{k}{\lambda_2})(\underset{k}{\beta_2}\operatorname{cosech}\pi\underset{k}{\lambda_1}\operatorname{cosech}\pi\underset{k}{\lambda_2} - \underset{k}{\beta_1}\coth\pi\underset{k}{\lambda_1}\coth\pi\underset{k}{\lambda_2}) + \underset{k}{\beta_2}(\underset{k}{\gamma_1} + P_k\underset{k}{\lambda_1})}{\underset{k}{\beta_2}(\underset{k}{\gamma_1} + P_k\underset{k}{\lambda_1})\coth\pi\underset{k}{\lambda_1} - \underset{k}{\beta_1}(\underset{k}{\gamma_2} + P_k\underset{k}{\lambda_2})\coth\pi\underset{k}{\lambda_2}}\,;\\
\underset{k}{N_m} &= -\underset{k}{B_m}\frac{(\underset{k}{\gamma_1} + P_k\underset{k}{\lambda_1})(\underset{k}{\beta_1}\operatorname{cosech}\pi\underset{k}{\lambda_1}\operatorname{cosech}\pi\underset{k}{\lambda_2} - \underset{k}{\beta_2}\coth\pi\underset{k}{\lambda_1}\coth\pi\underset{k}{\lambda_2}) + \underset{k}{\beta_1}(\underset{k}{\gamma_2} + P_k\underset{k}{\lambda_2})}{\underset{k}{\beta_2}(\underset{k}{\gamma_1} + P_k\underset{k}{\lambda_1})\coth\pi\underset{k}{\lambda_1} - \underset{k}{\beta_1}(\underset{k}{\gamma_2} + P_k\underset{k}{\lambda_2})\coth\pi\underset{k}{\lambda_2}}\,.
\end{aligned}\right\} \qquad (6.17)$$

Substituting in (6.14), we obtain

$$\underset{k}{A_m} = -\underset{k}{B_m}\frac{\{\underset{k}{\beta_2}\underset{k}{\lambda_1}(\underset{k}{\gamma_1} + P\underset{k}{\lambda_1}) + \underset{k}{\beta_1}\underset{k}{\lambda_2}(\underset{k}{\gamma_2} + P\underset{k}{\lambda_2})\} + \{\underset{k}{\beta_2}\underset{k}{\lambda_1}(\underset{k}{\gamma_2} + P\underset{k}{\lambda_2}) + \underset{k}{\beta_1}\underset{k}{\lambda_2}(\underset{k}{\gamma_1} + P\underset{k}{\lambda_1})\}\operatorname{cosech}\pi\underset{k}{\lambda_1}\operatorname{cosech}\pi\underset{k}{\lambda^2} - \{\underset{k}{\beta_1}\underset{k}{\lambda_1}(\underset{k}{\gamma_2} + P\underset{k}{\lambda_2}) + \underset{k}{\beta_2}\underset{k}{\lambda_2}(\underset{k}{\gamma_1} + P\underset{k}{\lambda_1})\}\coth\pi\underset{k}{\lambda_1}\coth\pi\underset{k}{\lambda_2}}{\underset{k}{\beta_2}(\underset{k}{\gamma_1} + P\underset{k}{\lambda_1})\coth\pi\underset{k}{\lambda_1} - \underset{k}{\beta_1}(\underset{k}{\gamma_2} + P\underset{k}{\lambda_2})\coth\pi\underset{k}{\lambda_2}}\,.$$

Denoting again $\underset{k}{B_m}$ by A_m and comparing with (6.16), we finally get

$$U_k = -\frac{\left\{\dfrac{\underset{k}{\lambda_1}(\underset{k}{\gamma_1} + P\underset{k}{\lambda_1})}{\underset{k}{\beta_1}} + \dfrac{\underset{k}{\lambda_2}(\underset{k}{\gamma_2} + P\underset{k}{\lambda_2})}{\underset{k}{\beta_2}}\right\} + \left\{\dfrac{\underset{k}{\lambda_1}(\underset{k}{\gamma_2} + P\underset{k}{\lambda_2})}{\underset{k}{\beta_1}} + \dfrac{\underset{k}{\lambda_2}(\underset{k}{\gamma_1} + P\underset{k}{\lambda_1})}{\underset{k}{\beta_2}}\right\}\operatorname{cosech}\pi\underset{k}{\lambda_1}\operatorname{cosech}\pi\underset{k}{\lambda_2} - \left\{\dfrac{\underset{k}{\lambda_2}(\underset{k}{\gamma_1} + P\underset{k}{\lambda_1})}{\underset{k}{\beta_1}} + \dfrac{\underset{k}{\lambda_1}(\underset{k}{\gamma_2} + P\underset{k}{\lambda_2})}{\underset{k}{\beta_2}}\right\}\coth\pi\underset{k}{\lambda_1}\coth\pi\underset{k}{\lambda_2}}{\dfrac{\underset{k}{\lambda_1} + P\underset{k}{\lambda_1}}{\underset{k}{\beta_1}}\coth\pi\underset{k}{\lambda_1} - \dfrac{\underset{k}{\lambda_2} + P\underset{k}{\lambda_2}}{\underset{k}{\beta_2}}\coth\pi\underset{k}{\lambda_2}} = -\frac{T_kT'_k - \bar{S}_kS''_k}{T'_k}\,;$$

$$\left.\begin{aligned} &V_k = 1\,, \\ &\text{where} \\ &T'_k = \frac{\gamma_{\underset{k}{1}} + P_k \lambda_{\underset{k}{1}}}{\beta_{\underset{k}{1}}} \coth\pi\lambda_{\underset{k}{1}} - \frac{\gamma_{\underset{k}{2}} + P_k \lambda_{\underset{k}{2}}}{\beta_{\underset{k}{2}}} \coth\pi\lambda_{\underset{k}{2}}\,, \\ &S_k = \frac{\lambda_{\underset{k}{1}}}{\beta_{\underset{k}{1}}} \operatorname{cosech}\pi\lambda_{\underset{k}{1}} - \frac{\lambda_{\underset{k}{2}}}{\beta_{\underset{k}{2}}} \operatorname{cosech}\pi\lambda_{\underset{k}{2}}\,, \\ &S''_k = (\gamma_{\underset{k}{1}} + P_k \lambda_{\underset{k}{1}}) \operatorname{cosech}\pi\lambda_{\underset{k}{1}} - (\gamma_{\underset{k}{2}} + P_k \lambda_{\underset{k}{2}}) \operatorname{cosech}\pi\lambda_{\underset{k}{2}}\,. \end{aligned}\right\} \quad (11.c)$$

iii) *Conditional Equations for Determination of Critical Load.*—We have used Eq. (6.10) for derivation of the formulas (11), and then, at such a case when the relation obtained from (6.10) by taking $r-1=1$ holds, in other words, the end side is simply supported or fixed at $x_1=0$, U_1 and V_1 can be calculated by means of formulas (11). Then the conditional equation for buckling are introduced as follows at End side-1 or $x_1=0$:

(1) Case of simply supported edge.—The end side conditions are

$$K_{\underset{1}{m}} + M_{\underset{1}{m}} = 0\,;$$

$$K_{\underset{1}{m}}\, \beta_{\underset{1}{1}} + M_{\underset{1}{m}}\, \beta_{\underset{1}{2}} = 0.$$

From these

$$K_{\underset{1}{m}} = -M_{\underset{1}{m}} = B_{\underset{1}{m}} = 0\,.$$

Therefore we obtain from the second equation in (6.16)

$$B_{\underset{1}{m}} = A_m V_1 = 0\,.$$

Thus, the conditional equation are given in the following from:

$$V_1 = 0\,. \qquad (11.d)$$

(2) Case of fixed edge.—The end side conditions are

$$K_{\underset{1}{m}} + M_{\underset{1}{m}} = 0\,;$$

$$L_{\underset{1}{m}}\, \lambda_{\underset{1}{1}} + N_{\underset{1}{m}}\, \lambda_{\underset{1}{2}} = 0\,.$$

Comparing the second equation of the above with (6.14), we have $A_{\underset{1}{m}}=0$. Thus, from the first equation in (6.16), the conditional equation is given as follows:

$$U_1 = 0\,. \qquad (11.e)$$

(3) Case of free edge.—Since $w_1 \neq 0$ at $x_1=0$ in this case, the formulas (11) are not applicable to calculate U_1 and V_1 from U_2 and V_2. Now, the end side

conditions are expressed as follows:

$$\left.\begin{aligned} K_{\underset{1}{m}}\beta_{\underset{1}{1}} + M_{\underset{1}{m}}\beta_{\underset{1}{2}} = 0: \\ L_{\underset{1}{m}}(\gamma_{\underset{1}{1}} + P_1\lambda_{\underset{1}{1}}) + N_{\underset{1}{m}}(\gamma_{\underset{1}{2}} + P_1\lambda_{\underset{1}{2}}) = 0. \end{aligned}\right\} \tag{6.18}$$

From the first equation

$$K_{\underset{1}{m}} = B_{\underset{1}{m}}\beta_{\underset{1}{2}}, \qquad M_{\underset{1}{m}} = -B_{\underset{1}{m}}\beta_{\underset{1}{1}}. \tag{6.19}$$

Accordingly, instead of (6.12) we obtain

$$\left.\begin{aligned} B_{\underset{1}{m}}(\beta_{\underset{1}{2}}\cosh\pi\lambda_{\underset{1}{1}} - \beta_{\underset{1}{1}}\cosh\pi\lambda_{\underset{1}{2}}) + L_{\underset{1}{m}}\sinh\pi\lambda_{\underset{1}{1}} + N_{\underset{1}{m}}\sinh\pi\lambda_{\underset{1}{2}} = 0; \\ B_{\underset{1}{m}}(\beta_{\underset{1}{2}}\lambda_{\underset{1}{1}}\sinh\pi\lambda_{\underset{1}{1}} - \beta_{\underset{1}{1}}\lambda_{\underset{1}{2}}\sinh\pi\lambda_{\underset{1}{2}}) + L_{\underset{1}{m}}\lambda_{\underset{1}{1}}\cosh\pi\lambda_{\underset{1}{1}} + N_{\underset{1}{m}}\lambda_{\underset{1}{2}}\cosh\pi\lambda_{\underset{1}{2}} - L_{\underset{2}{m}}\mu_2\lambda_{\underset{2}{1}} - N_{\underset{2}{m}}\mu_2\lambda_{\underset{2}{2}} = 0; \\ B_{\underset{1}{m}}(\cosh\pi\lambda_{\underset{1}{1}} - \cosh\pi\lambda_{\underset{1}{2}})\beta_{\underset{1}{1}}\beta_{\underset{1}{2}} + L_{\underset{1}{m}}\beta_{\underset{1}{1}}\sinh\pi\lambda_{\underset{1}{1}} + N_{\underset{1}{m}}\beta_{\underset{1}{2}}\sinh\pi\lambda_{\underset{1}{2}} - B_{\underset{2}{m}}\mu_2'(\beta_{\underset{2}{1}} - \beta_{\underset{2}{2}}) = 0. \end{aligned}\right\} \tag{6.20}$$

From these, the expressions of $L_{\underset{1}{m}}$ and $N_{\underset{1}{m}}$ are obtained in the following forms:

$$L_{\underset{1}{m}} = A_{\underset{2}{m}}\mu_2 \frac{\beta_{\underset{1}{2}}\coth\pi\lambda_{\underset{1}{1}}}{\lambda_{\underset{1}{1}}\beta_{\underset{1}{2}}\operatorname{cosech}\pi\lambda_{\underset{1}{1}} - \lambda_{\underset{1}{2}}\beta_{\underset{1}{1}}\operatorname{cosech}\pi\lambda_{\underset{1}{2}}}$$

$$- B_{\underset{2}{m}}\mu_2'\frac{\varphi_2}{\varphi_1} \cdot \frac{\{\lambda_{\underset{1}{2}}(\beta_{\underset{1}{1}}\operatorname{cosech}\pi\lambda_{\underset{1}{1}}\operatorname{cosech}\pi\lambda_{\underset{1}{2}} - \beta_{\underset{1}{2}}\coth\pi\lambda_{\underset{1}{1}}\coth\pi\lambda_{\underset{1}{2}}) + \lambda_{\underset{1}{1}}\beta_{\underset{1}{2}}\}}{\lambda_{\underset{1}{1}}\beta_{\underset{1}{2}}\operatorname{cosech}\pi\lambda_{\underset{1}{1}} - \lambda_{\underset{1}{2}}\beta_{\underset{1}{1}}\operatorname{cosech}\pi\lambda_{\underset{1}{2}}},$$

$$N_{\underset{1}{m}} = -A_{\underset{2}{m}}\mu_2 \frac{\beta_{\underset{1}{1}}\coth\pi\lambda_{\underset{1}{2}}}{\lambda_{\underset{1}{1}}\beta_{\underset{1}{2}}\operatorname{cosech}\pi\lambda_{\underset{1}{1}} - \lambda_{\underset{1}{2}}\beta_{\underset{1}{1}}\operatorname{cosech}\pi\lambda_{\underset{1}{2}}}$$

$$- B_{\underset{2}{m}}\mu_2'\frac{\varphi_2}{\varphi_1} \cdot \frac{\{\lambda_{\underset{1}{1}}(\beta_{\underset{1}{2}}\operatorname{cosech}\pi\lambda_{\underset{1}{1}}\operatorname{cosech}\pi\lambda_{\underset{1}{2}} - \beta_{\underset{1}{1}}\coth\pi\lambda_{\underset{1}{1}}\coth\pi\lambda_{\underset{1}{2}}) + \lambda_{\underset{1}{2}}\beta_{\underset{1}{1}}\}}{\lambda_{\underset{1}{1}}\beta_{\underset{1}{2}}\operatorname{cosech}\pi\lambda_{\underset{1}{1}} - \lambda_{\underset{1}{2}}\beta_{\underset{1}{1}}\operatorname{cosech}\pi\lambda_{\underset{1}{2}}}.$$

Let us substitute these in the second equation of (6.18), and denote as follows:

$$(A_{\underset{1}{m}}) = L_{\underset{1}{m}}(\gamma_{\underset{1}{1}} + P_1\lambda_{\underset{1}{1}}) + N_{\underset{1}{m}}(\gamma_{\underset{1}{2}} + P_1\lambda_{\underset{1}{2}}) = A_{\underset{2}{m}}\mu_2(F_1) - B_m\mu_2'\frac{\varphi_2}{\varphi_1}(G_1) = 0,$$

where

$$(F_1) = \frac{\dfrac{\gamma_{\underset{1}{1}} + P_1\lambda_{\underset{1}{1}}}{\beta_{\underset{1}{1}}}\coth\pi\lambda_{\underset{1}{1}} - \dfrac{\gamma_{\underset{1}{2}} + P_1\lambda_{\underset{1}{2}}}{\beta_{\underset{1}{2}}}\coth\pi\lambda_{\underset{1}{2}}}{\dfrac{\lambda_{\underset{1}{1}}}{\beta_{\underset{1}{1}}}\operatorname{cosech}\pi\lambda_{\underset{1}{1}} - \dfrac{\lambda_{\underset{1}{2}}}{\beta_{\underset{1}{2}}}\operatorname{cosech}\pi\lambda_{\underset{1}{2}}} = \frac{T_1'}{S_1},$$

$$
\left.
\begin{aligned}
(G_1) &= \frac{\left\{\dfrac{\underset{1}{\lambda_1}(\underset{1}{\gamma_1}+P_1\underset{1}{\lambda_1})}{\underset{1}{\beta_1}}+\dfrac{\underset{1}{\lambda_2}(\underset{1}{\gamma_2}+P_1\underset{1}{\lambda_2})}{\underset{1}{\beta_2}}\right\}+\left\{\dfrac{\underset{1}{\lambda_1}(\underset{1}{\gamma_2}+P_1\underset{1}{\lambda_2})}{\underset{1}{\beta_1}}+\dfrac{\underset{1}{\lambda_2}(\underset{1}{\gamma_1}+P_1\underset{1}{\lambda_1})}{\underset{1}{\beta_2}}\right\}\operatorname{cosech}\pi\underset{1}{\lambda_1}\operatorname{cosech}\pi\underset{1}{\lambda_2}}{\dfrac{\underset{1}{\lambda_1}}{\underset{1}{\beta_1}}\operatorname{cosech}\pi\underset{1}{\lambda_1}-\dfrac{\underset{1}{\lambda_2}}{\underset{1}{\beta_2}}\operatorname{cosech}\pi\underset{1}{\lambda_2}} \\
&\quad \frac{-\left\{\dfrac{\underset{1}{\lambda_2}(\underset{1}{\gamma_1}+P_1\underset{1}{\lambda_1})}{\underset{1}{\beta_1}}+\dfrac{\underset{1}{\lambda_1}(\underset{1}{\gamma_2}+P_1\underset{1}{\lambda_2})}{\underset{1}{\beta_2}}\right\}\coth\pi\underset{1}{\lambda_1}\coth\pi\underset{1}{\lambda_2}}{} = \frac{T_1\,T_1'-\bar{S}_1\,S_1''}{\bar{S}_1}\,,
\end{aligned}
\right\}\quad (11.f)
$$

in which

$$
\begin{aligned}
T_1 &= \underset{1}{\lambda_1}\coth\pi\underset{1}{\lambda_1}-\underset{1}{\lambda_2}\coth\pi\underset{1}{\lambda_2}\,,\\
T_1' &= \frac{\underset{1}{\gamma_1}+P_1\underset{1}{\lambda_1}}{\underset{1}{\beta_1}}\coth\pi\underset{1}{\lambda_1}-\frac{\underset{1}{\gamma_2}+P_1\underset{1}{\lambda_2}}{\underset{1}{\beta_2}}\coth\pi\underset{1}{\lambda_2}\,,\\
\bar{S}_1 &= \frac{\underset{1}{\lambda_1}}{\underset{1}{\beta_1}}\operatorname{cosech}\pi\underset{1}{\lambda_1}-\frac{\underset{1}{\lambda_2}}{\underset{1}{\beta_2}}\operatorname{cosech}\pi\underset{1}{\lambda_2}\,,\\
S_1'' &= (\underset{1}{\gamma_1}+P_1\underset{1}{\lambda_1})\operatorname{cosech}\pi\underset{1}{\lambda_1}-(\underset{1}{\gamma_2}+P_1\underset{1}{\lambda_2})\operatorname{cosech}\pi\underset{1}{\lambda_2}\,.
\end{aligned}
$$

Substituting again (6.16) in the above obtained equation, we have finally

$$
(\underset{1}{A_m}) = A_m\left\{U_2\mu_2(F_1)-V_2\mu_2'\,\frac{\varphi_2}{\varphi_1}\,(G_1)\right\}=0\,.
$$

To arrange the form of expression, we put also $(\underset{1}{A_m})=A_m\,U_1$ by referring to (6.16). Thus, the conditional equation for the present case becomes

$$
U_1=U_2\mu_2(F_1)-V_2\mu_2'\,\frac{\varphi_2}{\varphi_1}\,(G_1)=0\,. \qquad (11.g)
$$

We see now as before that the various combinations of both end side conditions given in the paragraphs ii) and iii) are able to represent each of the cases shown in Table 4 .

Let us calculate first U_k and V_k with the expressions in the paragraph ii) corresponding to each case. Knowing the values of U_k and V_k, the values of U_1 and V_1 are obtained by means of the recurrence formules (**11**). Therefore, the conditional eguation can be given by (**11.d**) or (**11.e**) for the case when the edge $x_1=0$ is simply supported or clamped. On the other hand, for the case when such an edge is free, U_2 and V_2 must be calculated previously by the formulas (**11**) in order to use the present conditional equation (**11.g**).

c) **Case of Elastic Supporting in 1-st Way.** i) *Conditions of Jointing*—

From Table 5, The conditions joining the $(r-1)$-th elementary plate to the r-th are

$$\underset{r-1}{K_m}\cosh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{L_m}\sinh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{M_m}\cosh\pi\underset{r-1}{\lambda_2}+\underset{r-1}{N_m}\sinh\pi\underset{r-1}{\lambda_2}-\underset{r}{K_m}-\underset{r}{M_m}=0\,;$$

$$\underset{r-1}{K_m}\underset{r-1}{\lambda_1}\sinh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{L_m}\underset{r-1}{\lambda_1}\cosh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{M_m}\underset{r-1}{\lambda_2}\sinh\pi\underset{r-1}{\lambda_2}+\underset{r-1}{N_m}\underset{r-1}{\lambda_2}\cosh\pi\underset{r-1}{\lambda_2}-\underset{r}{L_m}\mu_r\underset{r}{\lambda_1}-\underset{r}{N_m}\mu_r\underset{r}{\lambda_2}=0\,;$$

$$\begin{aligned}\underset{r-1}{K_m}\underset{r-1}{\beta_1}\cosh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{L_m}\underset{r-1}{\beta_1}\sinh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{M_m}\underset{r-1}{\beta_2}\cosh\pi\underset{r-1}{\lambda_2}+\underset{r-1}{N_m}\underset{r-1}{\beta_2}\sinh\pi\underset{r-1}{\lambda_2}-\underset{r}{K_m}\mu_r'\underset{r}{\beta_1}&\\ -\underset{r}{L_m}\bar{\mu}_r\underset{r}{\lambda_1}-\underset{r}{M_m}\mu_r'\underset{r}{\beta_2}-\underset{r}{N_m}\bar{\mu}_r\underset{r}{\lambda_2}&=0\,;\end{aligned}$$

$$\begin{aligned}\underset{r-1}{K_m}\underset{r-1}{\gamma_1}\sinh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{L_m}\underset{r-1}{\gamma_1}\cosh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{M_m}\underset{r-1}{\gamma_2}\sinh\pi\underset{r-1}{\lambda_2}+\underset{r-1}{N_m}\underset{r-1}{\gamma_2}\cosh\pi\underset{r-1}{\lambda_2}-\underset{r}{K_m}\bar{\mu}_r'&\\ -\underset{r}{L_m}\mu_r''\underset{r}{\gamma_1}-\underset{r}{M_m}\bar{\mu}_r'-\underset{r}{N_m}\mu_r''\underset{r}{\gamma_2}&=0\,.\end{aligned}$$

Proceeding as in the foregoing case a), we find again the equations (6.5) from the above equations by representing $\underset{r-1}{K_m}$, $\underset{r-1}{L_m}$, $\underset{r-1}{M_m}$, $\underset{r-1}{N_m}$ with $\underset{r}{K_m}$, $\underset{r}{L_m}$, $\underset{r}{M_m}$, $\underset{r}{N_m}$ but in this case, the formulas (6.5) must be accompanied by the expressions (12) given below instead of (8). The recurrence formulas for U_r anc V_r, again, become of the same form as (9).

$$\left.\begin{aligned}
\underset{r-1}{F_2}&=\frac{\underset{r-1}{\tau_2}}{\underset{r-1}{\varphi}}\cosh\pi\underset{r-1}{\lambda_1}+\frac{\bar{\mu}_r'}{\underset{r-1}{\lambda_1}\underset{r-1}{\varphi}}\sinh\pi\underset{r-1}{\lambda_1}\,; & \underset{r-1}{G_2}&=\frac{\underset{r-1}{\chi_2}}{\underset{r-1}{\lambda_1}\underset{r-1}{\lambda_2}\underset{r-1}{\varphi}}\cosh\pi\underset{r-1}{\lambda_1}+\frac{\bar{\mu}_r\underset{r}{\lambda_1}}{\underset{r-1}{\varphi}}\sinh\pi\underset{r-1}{\lambda_1}\,;\\
\underset{r-1}{F_2'}&=\frac{\underset{r-1}{\tau_2}}{\underset{r-1}{\varphi}}\sinh\pi\underset{r-1}{\lambda_1}+\frac{\bar{\mu}_r'}{\underset{r-1}{\lambda_1}\underset{r-1}{\varphi}}\cosh\pi\underset{r-1}{\lambda_1}\,; & \underset{r-1}{G_2'}&=\frac{\underset{r-1}{\chi_2}}{\underset{r-1}{\lambda_1}\underset{r-1}{\lambda_2}\underset{r-1}{\varphi}}\sinh\pi\underset{r-1}{\lambda_1}+\frac{\bar{\mu}_r\underset{r}{\lambda_1}}{\underset{r-1}{\varphi}}\cosh\pi\underset{r-1}{\lambda_1}\,;\\
\underset{r-1}{F_1}&=\frac{\underset{r-1}{\xi_1}}{\underset{r-1}{\varphi}}\cosh\pi\underset{r-1}{\lambda_2}+\frac{\bar{\mu}_r'}{\underset{r-1}{\lambda_2}\underset{r-1}{\varphi}}\sinh\pi\underset{r-1}{\lambda_2}\,; & \underset{r-1}{G_1}&=\frac{\underset{r-1}{\chi_1}}{\underset{r-1}{\lambda_1}\underset{r-1}{\lambda_2}\underset{r-1}{\varphi}}\cosh\pi\underset{r-1}{\lambda_2}+\frac{\bar{\mu}_r\underset{r}{\lambda_1}}{\underset{r-1}{\varphi}}\sinh\pi\underset{r-1}{\lambda_2}\,;\\
\underset{r-1}{F_1'}&=\frac{\underset{r-1}{\tau_1}}{\underset{r-1}{\varphi}}\sinh\pi\underset{r-1}{\lambda_2}+\frac{\bar{\mu}_r'}{\underset{r-1}{\lambda_2}\underset{r-1}{\varphi}}\cosh\pi\underset{r-1}{\lambda_2}\,; & \underset{r-1}{G_1'}&=\frac{\underset{r-1}{\chi_1}}{\underset{r-1}{\lambda_1}\underset{r-1}{\lambda_2}\underset{r-1}{\varphi}}\sinh\pi\underset{r-1}{\lambda_2}+\frac{\bar{\mu}_r\underset{r}{\lambda_1}}{\underset{r-1}{\varphi}}\cosh\pi\underset{r-1}{\lambda_2}\,;\\
\underset{r-1}{H_2}&=\frac{\underset{r-1}{\tau_2'}}{\underset{r-1}{\varphi}}\cosh\pi\underset{r-1}{\lambda_1}+\frac{\bar{\mu}_r'}{\underset{r-1}{\lambda_1}\underset{r-1}{\varphi}}\sinh\pi\underset{r-1}{\lambda_1}\,; & \underset{r-1}{I_2}&=\frac{\underset{r-1}{\chi_2'}}{\underset{r-1}{\lambda_1}\underset{r-1}{\lambda_2}\underset{r-1}{\varphi}}\cosh\pi\underset{r-1}{\lambda_1}+\frac{\bar{\mu}_r\underset{r}{\lambda_2}}{\underset{r-1}{\varphi}}\sinh\pi\underset{r-1}{\lambda_1}\,;\\
\underset{r-1}{H_2'}&=\frac{\underset{r-1}{\tau_2'}}{\underset{r-1}{\varphi}}\sinh\pi\underset{r-1}{\lambda_1}+\frac{\bar{\mu}_r'}{\underset{r-1}{\lambda_1}\underset{r-1}{\varphi}}\cosh\pi\underset{r-1}{\lambda_1}\,; & \underset{r-1}{I_2'}&=\frac{\underset{r-1}{\chi_2'}}{\underset{r-1}{\lambda_1}\underset{r-1}{\lambda_2}\underset{r-1}{\varphi}}\sinh\pi\underset{r-1}{\lambda_1}+\frac{\bar{\mu}_r\underset{r}{\lambda_2}}{\underset{r-1}{\varphi}}\cosh\pi\underset{r-1}{\lambda_1}\,;\\
\underset{r-1}{H_1}&=\frac{\underset{r-1}{\tau_1'}}{\underset{r-1}{\varphi}}\cosh\pi\underset{r-1}{\lambda_2}+\frac{\bar{\mu}_r'}{\underset{r-1}{\lambda_2}\underset{r-1}{\varphi}}\sinh\pi\underset{r-1}{\lambda_2}\,; & \underset{r-1}{I_1}&=\frac{\underset{r-1}{\chi_1'}}{\underset{r-1}{\lambda_1}\underset{r-1}{\lambda_2}\underset{r-1}{\varphi}}\cosh\pi\underset{r-1}{\lambda_2}+\frac{\bar{\mu}_r\underset{r}{\lambda_2}}{\underset{r-1}{\varphi}}\sinh\pi\underset{r-1}{\lambda_2}\,;\\
\underset{r-1}{H_1'}&=\frac{\underset{r-1}{\tau_1'}}{\underset{r-1}{\varphi}}\sinh\pi\underset{r-1}{\lambda_2}+\frac{\bar{\mu}_r'}{\underset{r-1}{\lambda_2}\underset{r-1}{\varphi}}\cosh\pi\underset{r-1}{\lambda_2}\,; & \underset{r-1}{I_1'}&=\frac{\underset{r-1}{\chi_1'}}{\underset{r-1}{\lambda_1}\underset{r-1}{\lambda_2}\underset{r-1}{\varphi}}\sinh\pi\underset{r-1}{\lambda_2}+\frac{\bar{\mu}_r\underset{r}{\lambda_2}}{\underset{r-1}{\varphi}}\cosh\pi\underset{r-1}{\lambda_2}\,.
\end{aligned}\right\}\quad(12)$$

ii) *Conditional Equations for Buckling.*—The expressions for End side-k and the conditional equation for the determination of critical load both are of the same form as in the case a). Then, we can finally conclude that all the procedures for calculation in the present case are established by replacing only (**8**) in those of the case a) with the above (**12**).

d) **Cases of Elastic Supporting in 2-nd Way.** i) *Conditions for Jointing.*—The conditions joining the $(r-1)$-th elementary plate and the r-th are written from Table 5 as follows:

$$\underset{r-1}{K_m}\cosh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{L_m}\sinh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{M_m}\cosh\pi\underset{r-1}{\lambda_2}+\underset{r-1}{N_m}\sinh\pi\underset{r-1}{\lambda_2}=0\,;$$

$$\underset{r}{K_m}+\underset{r}{M_m}=0\,;$$

$$\underset{r-1}{K_m}\underset{r-1}{\lambda_1}\sinh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{L_m}\underset{r-1}{\lambda_1}\cosh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{M_m}\underset{r-1}{\lambda_2}\sinh\pi\underset{r-1}{\lambda_2}+\underset{r-1}{N_m}\underset{r-1}{\lambda_2}\cosh\pi\underset{r-1}{\lambda_2}-\underset{r}{L_m}\mu_r\underset{r}{\lambda_1}-\underset{r}{N_m}\mu_r\underset{r}{\lambda_2}=0\,;$$

$$\underset{r-1}{K_m}\underset{r-1}{\beta_1}\cosh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{L_m}\underset{r-1}{\beta_1}\sinh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{M_m}\underset{r-1}{\beta_2}\cosh\pi\underset{r-1}{\lambda_2}+\underset{r-1}{N_m}\underset{r-1}{\beta_2}\sinh\pi\underset{r-1}{\lambda_2}$$

$$-\underset{r}{K_m}\mu_r'\underset{r}{\beta_1}-\underset{r}{L_m}\bar{\mu}_r\underset{r}{\lambda_1}-\underset{r}{M_m}\mu_r'\underset{r}{\beta_2}-\underset{r}{N_m}\bar{\mu}_r\underset{r}{\lambda_2}=0\,.$$

From these, we can arrive at the recurrence formulas (6.15) by such a treatement as in the foregoing case **b**). But each of the coeffcients written in (**10**) must be replaced by the following expressions:

$$\left.\begin{aligned}
F_{r-1}&=\mu_r\frac{T_{r-1}}{S_{r-1}}-\frac{\bar{\mu}_r}{\varphi_{r-1}}\frac{W_{r-1}}{S_{r-1}}, \qquad G_{r-1}=\mu_r'\frac{\varphi_r}{\varphi_{r-1}}\frac{W_{r-1}}{S_{r-1}},\\
F'_{r-1}&=\mu_r\frac{1}{S_{r-1}}-\frac{\bar{\mu}_r}{\varphi_{r-1}}\frac{T_{r-1}}{S_{r-1}}, \qquad G'_{r-1}=\mu_r'\frac{\varphi_r}{\varphi_{r-1}}\frac{T_{r-1}}{S_{r-1}},
\end{aligned}\right\}\tag{13}$$

where T_{r-1}, S_{r-1} and W_{r-1} are the same as shown in (**10**). Thus, the recurrence formulas for U_r and V_r are given by (**11**).

ii) *Conditional Equations for Buckling.*—The end side conditions at $\xi_k=1$ are expressed of the same form as in the case **b**). And also, the conditional equation becomes the same as in the case **b**), when the end $\xi_r=0$ is simply supported or fixed; but if the end side is free, the following equation is concluded in the same manner as in [**b**)-iii)-(3)][11]:

$$U_1=U_2\left\{\mu_2(F_1)-\frac{\bar{\mu}_2}{\varphi_1}(G_1)\right\}-V_2\mu_2'\frac{\varphi_2}{\varphi_1}(G_1)=0\,,\tag{13'}$$

where (F_1) and (G_1) are given in (**11.f**)

11) It indicates the paragraph such as [b) Case of Rigid Supporting-iii) Conditional Equations for Determination of Crit.cal Load-(3)Case of free edge], it is ditto hereafter.

§ 7. Composite Plates Having Hinged Joints.

e) Case of No Supporting. i) *Conditions for Jointing.*—The conditions joining the $(r-1)$-th elementary plate to the r-th are written from Table 5 as follows:

$$\underset{r-1}{K_m}\cosh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{L_m}\sinh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{M_m}\cosh\pi\underset{r-1}{\lambda_2}+\underset{r-1}{N_m}\sinh\pi\underset{r-1}{\lambda_2}-\underset{r}{K_m}-\underset{r}{M_m}=0\,;$$

$$\underset{r-1}{K_m}\underset{r-1}{\beta_1}\cosh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{L_m}\underset{r-1}{\beta_1}\sinh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{M_m}\underset{r-1}{\beta_2}\cosh\pi\underset{r-1}{\lambda_2}+\underset{r-1}{N_m}\underset{r-1}{\beta_2}\sinh\pi\underset{r-1}{\lambda_2}=0\,;$$

$$\underset{r}{K_m}\underset{r}{\beta_1}+\underset{r}{M_m}\underset{r}{\beta_2}=0\,;$$

$$\underset{r-1}{K_m}(\underset{r-1}{\gamma_1}+\underset{r-1}{P}\underset{r-1}{\lambda_1})\sinh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{L_m}(\underset{r-1}{\gamma_1}+\underset{r-1}{P}\underset{r-1}{\lambda_1})\cosh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{M_m}(\underset{r-1}{\gamma_2}+\underset{r-1}{P}\underset{r-1}{\lambda_2})\sinh\pi\underset{r-1}{\lambda_2}$$
$$+\underset{r-1}{N_m}(\underset{r-1}{\gamma_2}+\underset{r-1}{P}\underset{r-1}{\lambda_2})\cosh\pi\underset{r-1}{\lambda_2}-\underset{r}{L_m}\mu_r''(\underset{r}{\gamma_1}+P_r\underset{r}{\lambda_1})-\underset{r}{N_m}\mu_r''(\underset{r}{\gamma_2}+P_r\underset{r}{\lambda_2})=0.$$

Adding the equation obtained by replacing the suffix r in the third of the above equations with the suffix $r-1$ to the above equations, we can represent the constants having the suffix $r-1$ with those having the suffix r as in the foregoing case **b**). By introducing the new constant $\underset{r}{B'_m}$, we can put as follows:

$$\underset{r}{K_m}=\underset{r}{B'_m}\underset{r}{\beta_2}\,,\qquad \underset{r}{M_m}=-\underset{r}{B'_m}\underset{r}{\beta_1} \tag{7.1}$$

That is, the foregoing third equation are satisfied by the above expressions, and then the remaining three equations are transformed as follows:

$$\underset{r-1}{B'_m}(\underset{r-1}{\beta_2}\cosh\pi\underset{r-1}{\lambda_1}-\underset{r-1}{\beta_1}\cosh\pi\underset{r-1}{\lambda_2})+\underset{r-1}{L_m}\sinh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{N_m}\sinh\pi\underset{r-1}{\lambda_2}+\underset{r-1}{B'_m}\varphi_r=0\,;$$

$$\underset{r-1}{B'_m}\underset{r-1}{\beta_1}\underset{r-r}{\beta_2}(\cosh\pi\underset{r-1}{\lambda_1}-\cosh\pi\underset{r-1}{\lambda_2})+\underset{r-1}{L_m}\underset{r-1}{\beta_1}\sinh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{N_m}\underset{r-1}{\beta_2}\sinh\pi\underset{r-1}{\lambda_2}=0\,;$$

$$\underset{r-1}{B'_m}\left\{\underset{r-1}{\beta_2}(\underset{r-1}{\gamma_1}+\underset{r-1}{P}\underset{r-1}{\lambda_1})\sinh\pi\underset{r-1}{\lambda_1}-\underset{r-1}{\beta_1}(\underset{r-1}{\gamma_2}+\underset{r-1}{P}\underset{r-1}{\lambda_2})\sinh\pi\underset{r-1}{\lambda_2}\right\}+\underset{r-1}{L_m}(\underset{r-1}{\gamma_1}+\underset{r-1}{P}\underset{r-1}{\lambda_1})\cosh\pi\underset{r-1}{\lambda_1}$$
$$+\underset{r-1}{N_m}(\underset{r-1}{\gamma_2}+\underset{r-1}{P}\underset{r-1}{\lambda_2})\cosh\pi\underset{r-1}{\lambda_2}-\underset{r}{L_m}\mu_r''(\underset{r}{\gamma_1}+P_r\underset{r}{\lambda_1})-\underset{r}{N_m}\mu_r''(\underset{r}{\gamma_2}+P_r\underset{r}{\lambda_2})=0\,.$$

Solving the above equations with respect to the constants having the suffix $r-1$, we have

$$\underset{r-1}{L_m}=\left\{\underset{r}{L_m}(\underset{r}{\gamma_1}+P_r\underset{r}{\lambda_1})+\underset{r}{N_m}(\underset{r}{\gamma_2}+P_r\underset{r}{\lambda_2})\right\}\mu_r''\frac{\dfrac{1}{\underset{r-1}{\beta_1}}\coth\pi\underset{r-1}{\lambda_1}}{\dfrac{\underset{r-1}{\gamma_1}+\underset{r-1}{P}\underset{r-1}{\lambda_1}}{\underset{r-1}{\beta_1}}\operatorname{cosech}\pi\underset{r-1}{\lambda_1}-\dfrac{\underset{r-1}{\gamma_2}+\underset{r-1}{P}\underset{r-1}{\lambda_2}}{\underset{r-1}{\beta_2}}\operatorname{cosech}\pi\underset{r-1}{\lambda_2}}$$

$$-\underset{r}{B'_m}\,\frac{\varphi_r}{\varphi_{r-1}}\,\frac{\underset{r-1}{\beta_2}\left\{\frac{\underset{r-1}{\gamma_2}+\underset{r-1}{P}\,\underset{r-1}{\lambda_2}}{\underset{r-1}{\beta_2}}(\operatorname{cosech}\pi\underset{r-1}{\lambda_1}\operatorname{cosech}\pi\underset{r-1}{\lambda_2}-\coth\pi\underset{r-1}{\lambda_1}\coth\pi\underset{r-1}{\lambda_2})+\frac{\underset{r-1}{\gamma_1}+\underset{r-1}{P}\,\underset{r-1}{\lambda_1}}{\underset{r-1}{\beta_1}}\right\}}{\frac{\underset{r-1}{\gamma_1}+\underset{r-1}{P}\,\underset{r-1}{\lambda_1}}{\underset{r-1}{\beta_1}}\operatorname{cosech}\pi\underset{r-1}{\lambda_1}-\frac{\underset{r-1}{\gamma_2}+\underset{r-1}{P}\,\underset{r-1}{\lambda_2}}{\underset{r-1}{\beta_2}}\operatorname{cosech}\pi\underset{r-1}{\lambda_2}},$$

$$\underset{r-1}{N_m}=-\left\{\underset{r}{L_m}(\underset{r}{\gamma_1}+\underset{r}{P}\,\underset{r}{\lambda_1})+\underset{r}{N_m}(\underset{r}{\gamma_2}+\underset{r}{P}\,\underset{r}{\lambda_2})\right\}\mu''_r\,\frac{\frac{1}{\underset{r-1}{\beta_2}}\coth\pi\underset{r-1}{\lambda_2}}{\frac{\underset{r-1}{\gamma_1}+\underset{r-1}{P}\,\underset{r-1}{\lambda_1}}{\underset{r-1}{\beta_1}}\operatorname{cosech}\pi\underset{r-1}{\lambda_1}-\frac{\underset{r-1}{\gamma_2}+\underset{r-1}{P}\,\underset{r-1}{\lambda_2}}{\underset{r-1}{\beta_2}}\operatorname{cosech}\pi\underset{r-1}{\lambda_2}}$$

$$-\underset{r}{B'_m}\,\frac{\varphi_r}{\varphi_{r-1}}\,\frac{\underset{r-1}{\beta_1}\left\{\frac{\underset{r-1}{\gamma_1}+\underset{r-1}{P}\,\underset{r-1}{\lambda_1}}{\underset{r-1}{\beta_1}}(\operatorname{cosech}\pi\underset{r-1}{\lambda_1}\operatorname{cosech}\pi\underset{r-1}{\lambda_2}-\coth\pi\underset{r-1}{\lambda_1}\coth\pi\underset{r-1}{\lambda_2})+\frac{\underset{r-1}{\gamma_2}+\underset{r-1}{P}\,\underset{r-1}{\lambda_2}}{\underset{r-1}{\beta_2}}\right\}}{\frac{\underset{r-1}{\gamma_1}+\underset{r-1}{P}\,\underset{r-1}{\lambda_1}}{\underset{r-1}{\beta_1}}\operatorname{cosech}\pi\underset{r-1}{\lambda_1}-\frac{\underset{r-1}{\gamma_2}+\underset{r-1}{P}\,\underset{r-1}{\lambda_2}}{\underset{r-1}{\beta_2}}\operatorname{cosech}\pi\underset{r-1}{\lambda_2}},$$

$$\underset{r-1}{B'_m}=-\left\{\underset{r}{L_m}(\underset{r}{\gamma_1}+\underset{r}{P}\,\underset{r}{\lambda_1})+\underset{r}{N_m}(\underset{r}{\gamma_2}+\underset{r}{P}\,\underset{r}{\lambda_2})\right\}\mu''_r\,\frac{\frac{1}{\underset{r-1}{\beta_1}\,\underset{r-1}{\beta_2}}}{\frac{\underset{r-1}{\gamma_1}+\underset{r-1}{P}\,\underset{r-1}{\lambda_1}}{\underset{r-1}{\beta_1}}\operatorname{cosech}\pi\underset{r-1}{\lambda_1}-\frac{\underset{r-1}{\gamma_2}+\underset{r-1}{P}\,\underset{r-1}{\lambda_2}}{\underset{r-1}{\beta_2}}\operatorname{cosech}\pi\underset{r-1}{\lambda_2}}$$

$$+\underset{r}{B'_m}\,\frac{\varphi_r}{\varphi_{r-1}}\,\frac{\frac{\underset{r-1}{\gamma_1}+\underset{r-1}{P}\,\underset{r-1}{\lambda_1}}{\underset{r-1}{\beta_1}}\coth\pi\underset{r-1}{\lambda_1}-\frac{\underset{r-1}{\gamma_2}+\underset{r-1}{P}\,\underset{r-1}{\lambda_2}}{\underset{r-1}{\beta_2}}\coth\pi\underset{r-1}{\lambda_2}}{\frac{\underset{r-1}{\gamma_1}+\underset{r-1}{P}\,\underset{r-1}{\lambda_1}}{\underset{r-1}{\beta_1}}\operatorname{cosech}\pi\underset{r-1}{\lambda_1}-\frac{\underset{r-1}{\gamma_2}+\underset{r-1}{P}\,\underset{r-1}{\lambda_2}}{\underset{r-1}{\beta_2}}\operatorname{cosech}\pi\underset{r-1}{\lambda_2}}.$$

Let us multiply then both sides of the first expression by $\underset{r-1}{\gamma_1}+\underset{r-1}{P}\,\underset{r-1}{\lambda_1}$ and those of the second by $\underset{r-1}{\gamma_2}+\underset{r-1}{P}\,\underset{r-1}{\lambda_2}$ and add both.

Next, putting as follows:

$$\underset{r}{L_m}\,(\underset{r}{\gamma_1}+\underset{r}{P}\,\underset{r}{\lambda_1})\;+\;\underset{r}{N_m}\,(\underset{r}{\gamma_2}+\underset{r}{P}\,\underset{r}{\lambda_2})\;=\;\underset{r}{A_m}\,, \tag{7.2}$$

and denoting $\underset{r}{B'_m}\,\underset{r}{\beta_1}\,\underset{r}{\beta_2}$ by $\underset{r}{B_m}$ for shortness, we can obtain the following expressions, concerning A and B, of the same form as (6.15) by using together the third in the above expressions:

$$\left.\begin{aligned}\underset{r-1}{A_m}&=\quad\underset{r}{A_m}\,F_{r-1}-\underset{r}{B_m}\,G_{r-1}\,;\\ \underset{r-1}{B_m}&=-\,\underset{r}{A_m}\,F'_{r-1}+\underset{r}{B_m}\,G'_{r-1}\,,\end{aligned}\right\}$$

where

$$
\left.
\begin{aligned}
& F_{r-1}=\mu_r''\frac{T'_{r-1}}{S'_{r-1}}, \qquad G_{r-1}=\frac{\bar{\varphi}_r}{\bar{\varphi}_{r-1}}\frac{W'_{r-1}}{S'_{r-1}},\\
& F'_{r-1}=\mu_r''\frac{1}{S'_{r-1}}, \qquad G'_{r-1}=\frac{\bar{\varphi}_r}{\bar{\varphi}_{r-1}}\frac{T'_{r-1}}{S'_{r-1}},\\
& \text{in which}\\
& T'_{r-1}=\frac{\underset{r-1}{\gamma_1}+\underset{r-1}{P}\,\underset{r-1}{\lambda_1}}{\underset{r-1}{\beta_1}}\coth\pi\underset{r-1}{\lambda_1}-\frac{\underset{r-1}{\gamma_2}+\underset{r-1}{P}\,\underset{r-1}{\lambda_2}}{\underset{r-1}{\beta_2}}\coth\pi\underset{r-1}{\lambda_2},\\
& S'_{r-1}=\frac{\underset{r-1}{\gamma_1}+\underset{r-1}{P}\,\underset{r-1}{\lambda_1}}{\underset{r-1}{\beta_1}}\operatorname{cosech}\pi\underset{r-1}{\lambda_1}-\frac{\underset{r-1}{\gamma_2}+\underset{r-1}{P}\,\underset{r-1}{\lambda_2}}{\underset{r-1}{\beta_2}}\operatorname{cosech}\pi\underset{r-1}{\lambda_2},\\
& W'_{r-1}=T'^2_{r-1}-S'^2_{r-1}=\left(\frac{\underset{r-1}{\gamma_1}+\underset{r-1}{P}\,\underset{r-1}{\lambda_1}}{\underset{r-1}{\beta_1}}\right)^2+\left(\frac{\underset{r-1}{\gamma_2}+\underset{r-1}{P}\,\underset{r-1}{\lambda_2}}{\underset{r-1}{\beta_2}}\right)^2\\
& \qquad +2\frac{(\underset{r-1}{\gamma_1}+\underset{r-1}{P}\,\underset{r-1}{\lambda_1})(\underset{r-1}{\gamma_2}+\underset{r-1}{P}\,\underset{r-1}{\lambda_2})}{\underset{r-1}{\beta_1}\,\underset{r-1}{\beta_2}}\begin{pmatrix}\operatorname{cosech}\pi\underset{r-1}{\lambda_1}\operatorname{cosech}\pi\underset{r-1}{\lambda_2}\\ -\coth\pi\underset{r-1}{\lambda_1}\coth\pi\underset{r-1}{\lambda_2}\end{pmatrix},\\
& \bar{\varphi}_r=\frac{1}{\underset{r}{\beta_1}}-\frac{1}{\underset{r}{\beta_2}}, \qquad \bar{\varphi}_{r-1}=\frac{1}{\underset{r-1}{\beta_1}}-\frac{1}{\underset{r-1}{\beta_2}}.
\end{aligned}
\right\} \tag{14}
$$

Proceeding in the same manner as in the case **b**), we easily obtain

$$
\left.
\begin{aligned}
U_{r-1}&=\ \ U_r F_{r-1}-V_r G_{r-1};\\
V_{r-1}&=-U_r F'_{r-1}+V_r G'_{r-1}.
\end{aligned}
\right\} \tag{15}
$$

ii) *Conditions for End Side-k.*

(1) Case of simply supported edge.—At the end side $\xi_k=1$

$$\underset{k}{M_m}\cosh\pi\underset{k}{\lambda_2}+\underset{k}{N_m}\sinh\pi\underset{k}{\lambda_2}=0;$$

$$\underset{k}{K_m}\cosh\pi\underset{k}{\lambda_1}+\underset{k}{L_m}\sinh\pi\underset{k}{\lambda_1}=0.$$

Then, by using (7.1)

$$\underset{k}{L_m}=-\underset{k}{K_m}\coth\pi\underset{k}{\lambda_1}=-\underset{k}{B'_m}\underset{k}{\beta_2}\coth\pi\underset{k}{\lambda_1};$$

$$\underset{k}{N_m}=-\underset{k}{M_m}\coth\pi\underset{k}{\lambda_2}=\ \ \underset{k}{B'_m}\underset{k}{\beta_1}\coth\pi\underset{k}{\lambda_2}.$$

Again, using (7.2), we get

$$\underset{k}{A_m}=-\underset{k}{B'_m}\underset{k}{\beta_1}\underset{k}{\beta_2}\left\{\frac{\underset{k}{\gamma_1}+P_k\underset{k}{\lambda_1}}{\underset{k}{\beta_1}}\coth\pi\underset{k}{\lambda_1}-\frac{\underset{k}{\gamma_2}+P_k\underset{k}{\lambda_2}}{\underset{k}{\beta_2}}\coth\pi\underset{k}{\lambda_2}\right\}.$$

Denoting now $\underset{k}{B'_m}\,\underset{k}{\beta_1}\,\underset{k}{\beta_2}$ by $\underset{k}{B_m}$, the following expressions are obtained in the similar manner as in the case of (11.a):

$$\left.\begin{aligned} &U_k = -T'_k\,; \qquad V_k = 1\,,\\ &\text{where}\\ &T_k{}' = \frac{\underset{k}{\gamma_1} + P_k\,\underset{k}{\lambda_1}}{\underset{k}{\beta_1}}\coth\pi\underset{k}{\lambda_1} - \frac{\underset{k}{\gamma_2} + P_k\,\underset{k}{\lambda_2}}{\underset{k}{\beta_2}}\coth\pi\underset{k}{\lambda_2}\,. \end{aligned}\right\} \tag{15.a}$$

(2) Case of fixed edge.—In this case, the end side conditions yield

$$\underset{k}{K_m}\cosh\pi\underset{k}{\lambda_1} + \underset{k}{L_m}\sinh\pi\underset{k}{\lambda_1} + \underset{k}{M_m}\cosh\pi\underset{k}{\lambda_2} + \underset{k}{N_m}\sinh\pi\underset{k}{\lambda_2} = 0\;;$$

$$\underset{k}{K_m}\underset{k}{\lambda_1}\sinh\pi\underset{k}{\lambda_1} + \underset{k}{L_m}\underset{k}{\lambda_1}\cosh\pi\underset{k}{\lambda_1} + \underset{k}{M_m}\underset{k}{\lambda_2}\sinh\pi\underset{k}{\lambda_2} + \underset{k}{N_m}\underset{k}{\lambda_2}\cosh\pi\underset{k}{\lambda_2} = 0.$$

Rewriting these by (7.1), we have

$$\underset{k}{B'_m}(\underset{k}{\beta_2}\cosh\pi\underset{k}{\lambda_1} - \underset{k}{\beta_1}\cosh\pi\underset{k}{\lambda_2}) + \underset{k}{L_m}\sinh\pi\underset{k}{\lambda_1} + \underset{k}{N_m}\sinh\pi\underset{k}{\lambda_2} = 0\;;$$

$$\underset{k}{B'_m}(\underset{k}{\beta_2}\underset{k}{\lambda_1}\sinh\pi\underset{k}{\lambda_1} - \underset{k}{\beta_1}\underset{k}{\lambda_2}\sinh\pi\underset{k}{\lambda_2}) + \underset{k}{L_m}\underset{k}{\lambda_1}\cosh\pi\underset{k}{\lambda_1} + \underset{k}{N_m}\underset{k}{\lambda_2}\cosh\pi\underset{k}{\lambda_2} = 0\,.$$

Then, from both above, we get

$$\underset{k}{L_m} = -\underset{k}{B'_m}\,\frac{\underset{k}{\lambda_2}(\underset{k}{\beta_1}\operatorname{cosech}\pi\underset{k}{\lambda_1}\operatorname{cosech}\pi\underset{k}{\lambda_2} - \underset{k}{\beta_2}\coth\pi\underset{k}{\lambda_1}\coth\pi\underset{k}{\lambda_2}) + \underset{k}{\beta_2}\underset{k}{\lambda_1}}{\underset{k}{\lambda_1}\coth\pi\underset{k}{\lambda_1} - \underset{k}{\lambda_2}\coth\pi\underset{k}{\lambda_2}}\,,$$

$$\underset{k}{N_m} = -\underset{k}{B'_m}\,\frac{\underset{k}{\lambda_1}(\underset{k}{\beta_2}\operatorname{cosech}\pi\underset{k}{\lambda_1}\operatorname{cosech}\pi\underset{k}{\lambda_2} - \underset{k}{\beta_1}\coth\pi\underset{k}{\lambda_1}\coth\pi\underset{k}{\lambda_2}) + \underset{k}{\beta_1}\underset{k}{\lambda_2}}{\underset{k}{\lambda_1}\coth\pi\underset{k}{\lambda_1} - \underset{k}{\lambda_2}\coth\pi\underset{k}{\lambda_2}}\,.$$

Moreover, by using (7.2), $\underset{k}{A_m}$ is expressed as follows:

$$\underset{k}{A_m} = -\underset{k}{B'_m}\,\frac{\begin{aligned}&\{\underset{k}{\lambda_1}\underset{k}{\beta_2}(\underset{k}{\gamma_1}+P_k\underset{k}{\lambda_1}) + \underset{k}{\lambda_2}\underset{k}{\beta_1}(\underset{k}{\gamma_2}+P_k\underset{k}{\lambda_2})\} + \{\underset{k}{\lambda_1}\underset{k}{\beta_2}(\underset{k}{\gamma_2}+P_k\underset{k}{\lambda_2}) + \underset{k}{\lambda_2}\underset{k}{\beta_1}(\underset{k}{\gamma_1}+P_k\underset{k}{\lambda_1})\}\operatorname{cosech}\pi\underset{k}{\lambda_1}\operatorname{cosech}\pi\underset{k}{\lambda_2}\\ &\quad - \{\underset{k}{\lambda_1}\underset{k}{\beta_1}(\underset{k}{\gamma_2}+P_k\underset{k}{\lambda_2}) + \underset{k}{\lambda_2}\underset{k}{\beta_2}(\underset{k}{\gamma_1}+P_k\underset{k}{\lambda_1})\}\coth\pi\underset{k}{\lambda_1}\coth\pi\underset{k}{\lambda_2}\end{aligned}}{\underset{k}{\lambda_1}\coth\pi\underset{k}{\lambda_1} - \underset{k}{\lambda_2}\coth\pi\underset{k}{\lambda_2}}\,.$$

Denote $\underset{k}{B'_m}\,\underset{k}{\beta_1}\,\underset{k}{\beta_2}$ by $\underset{k}{B_m}$, then from this and in the same manner as in the case of (11.c), we obtain

$$\left.\begin{aligned} U_k = -\,&\frac{\begin{aligned}&\left\{\frac{\underset{k}{\lambda_1}(\underset{k}{\gamma_1}+P_k\underset{k}{\lambda_1})}{\underset{k}{\beta_1}} + \frac{\underset{k}{\lambda_2}(\underset{k}{\gamma_2}+P_k\underset{k}{\lambda_2})}{\underset{k}{\beta_2}}\right\} + \left\{\frac{\underset{k}{\lambda_1}(\underset{k}{\gamma_2}+P_k\underset{k}{\lambda_2})}{\underset{k}{\beta_1}} + \frac{\underset{k}{\lambda_2}(\underset{k}{\gamma_1}+P_k\underset{k}{\lambda_1})}{\underset{k}{\beta_2}}\right\}\operatorname{cosech}\pi\underset{k}{\lambda_1}\operatorname{cosech}\pi\underset{k}{\lambda_2}\\ &\quad - \left\{\frac{\underset{k}{\lambda_2}(\underset{k}{\beta_1}+P_k\underset{k}{\gamma_1})}{\underset{k}{\beta_1}} + \frac{\underset{k}{\lambda_1}(\underset{k}{\beta_2}+P_k\underset{k}{\lambda_2})}{\underset{k}{\beta_2}}\right\}\coth\pi\underset{k}{\lambda_1}\coth\pi\underset{k}{\lambda_2}\end{aligned}}{\underset{k}{\lambda_1}\coth\pi\underset{k}{\lambda_1} - \underset{k}{\lambda_2}\coth\pi\underset{k}{\lambda_2}}\\ = -&\frac{T_kT'_k - \bar{S}_kS''_k}{T_k}\,; \end{aligned}\right\} \tag{15.b}$$

$$V_k = 1 ,$$

in which, T_k , T_k', $\bar{S}_k$, S_k'' are the same as shown in (**11**.a) and (**11**.c).

(3) Case of free edge.—The conditions for End side-k are

$$K_{\underset{k}{m}}\beta_{\underset{k}{1}}\cosh\pi\lambda_{\underset{k}{1}} + L_{\underset{k}{m}}\beta_{\underset{k}{1}}\sinh\pi\lambda_{\underset{k}{1}} + M_{\underset{k}{m}}\beta_{\underset{k}{2}}\cosh\pi\lambda_{\underset{k}{2}} + N_{\underset{k}{m}}\beta_{\underset{k}{2}}\sinh\pi\lambda_{\underset{k}{2}} = 0 ;$$

$$K_{\underset{k}{m}}(\gamma_{\underset{k}{1}} + P_k\lambda_{\underset{k}{1}})\sinh\pi\lambda_{\underset{k}{1}} + L_{\underset{k}{m}}(\gamma_{\underset{k}{1}} + P_k\lambda_{\underset{k}{1}})\cosh\pi\lambda_{\underset{k}{1}} + M_{\underset{k}{m}}(\gamma_{\underset{k}{2}} + P_k\lambda_{\underset{k}{2}})\sinh\pi\lambda_{\underset{k}{2}} + N_{\underset{k}{m}}(\gamma_{\underset{k}{2}} + P_k\lambda_{\underset{k}{2}})\cosh\pi\lambda_{\underset{k}{2}} = 0 .$$

Using (7.1), these become

$$B'_{\underset{k}{m}}\beta_{\underset{k}{1}}\beta_{\underset{k}{2}}(\cosh\pi\lambda_{\underset{k}{1}} - \cosh\pi\lambda_{\underset{k}{2}}) + L_{\underset{k}{m}}\beta_{\underset{k}{1}}\sinh\pi\lambda_{\underset{k}{1}} + N_{\underset{k}{m}}\beta_{\underset{k}{2}}\sinh\pi\lambda_{\underset{k}{2}} = 0 ;$$

$$B'_{\underset{k}{m}}\left\{\beta_{\underset{k}{2}}(\gamma_{\underset{k}{1}} + P_k\lambda_{\underset{k}{1}})\sinh\pi\lambda_{\underset{k}{1}} - \beta_{\underset{k}{1}}(\gamma_{\underset{k}{2}} + P_k\lambda_{\underset{k}{2}})\sinh\pi\lambda_{\underset{k}{2}}\right\} + L_{\underset{k}{m}}(\gamma_{\underset{k}{1}} + P_k\lambda_{\underset{k}{1}})\cosh\pi\lambda_{\underset{k}{1}} + N_{\underset{k}{m}}(\gamma_{\underset{k}{2}} + P_k\lambda_{\underset{k}{2}})\cosh\pi\lambda_{\underset{k}{2}} = 0 .$$

Then, from these, we have

$$L_{\underset{k}{m}} = -B'_{\underset{k}{m}} \frac{\beta_{\underset{k}{2}}\left\{\beta_{\underset{k}{1}}(\gamma_{\underset{k}{2}} + P_k\lambda_{\underset{k}{2}})(\operatorname{cosech}\pi\lambda_{\underset{k}{1}}\operatorname{cosech}\pi\lambda_{\underset{k}{2}} - \coth\pi\lambda_{\underset{k}{1}}\coth\pi\lambda_{\underset{k}{2}}) + \beta_{\underset{k}{2}}(\gamma_{\underset{k}{1}} + P_k\lambda_{\underset{k}{1}})\right\}}{\beta_{\underset{k}{2}}(\gamma_{\underset{k}{1}} + P_k\lambda_{\underset{k}{1}})\coth\pi\lambda_{\underset{k}{1}} - \beta_{\underset{k}{1}}(\gamma_{\underset{k}{2}} + P_k\lambda_{\underset{k}{2}})\coth\pi\lambda_{\underset{k}{2}}} ,$$

$$N_{\underset{k}{m}} = -B'_{\underset{k}{m}} \frac{\beta_{\underset{k}{1}}\left\{\beta_{\underset{k}{2}}(\gamma_{\underset{k}{1}} + P_k\lambda_{\underset{k}{1}})(\operatorname{cosech}\pi\lambda_{\underset{k}{1}}\operatorname{cosech}\pi\lambda_{\underset{k}{2}} - \coth\pi\lambda_{\underset{k}{1}}\coth\pi\lambda_{\underset{k}{2}}) + \beta_{\underset{k}{1}}(\gamma_{\underset{k}{2}} + P_k\lambda_{\underset{k}{2}})\right\}}{\beta_{\underset{k}{2}}(\gamma_{\underset{k}{1}} + P_k\lambda_{\underset{k}{1}})\coth\pi\lambda_{\underset{k}{1}} - \beta_{\underset{k}{1}}(\gamma_{\underset{k}{2}} + P_k\lambda_{\underset{k}{2}})\coth\pi\lambda_{\underset{k}{2}}} .$$

Referring to (7.2), we can produce the expression for $A_{\underset{k}{m}}$ by using both above, and also denote $B'_{\underset{k}{m}}\beta_{\underset{k}{1}}\beta_{\underset{k}{2}}$ by $B_{\underset{k}{m}}$, thus, U_k and V_k are expressed in the following forms:

$$\left.\begin{aligned} U_k &= -\frac{T_k'^2 - S_k'^2}{T_k'} = -\frac{W_k'}{T_k'} ; \\ V_k &= 1 . \end{aligned}\right\} \qquad (\mathbf{15}.\mathrm{c})$$

iii) *Conditional Equations for Buckling.*—Since the expressions (7.1) are used for derivation of the Formula (**15**), in such a case that the relations (7.1) are kept by taking $r-1=1$, *i. e.*, End side-1 are simply supported or free, U_1 and V_1 can be calculated by the formula (**15**). From this, we obtain the conditional equations as given below, but in the case of fixed edge, some other considerations are necessary.

(1) Case of simply supported edge.—The conditions for the end side are

$$K_{\underset{1}{m}} + M_{\underset{1}{m}} = 0 ;$$

$$K_{\underset{1}{m}}\beta_{\underset{1}{1}} + M_{\underset{1}{m}}\beta_{\underset{1}{2}} = 0 .$$

Hence, in the same manner as in the paragraph [b)-iii)-(1)], the following equation results as the conditional equation for buckling:

$$V_1 = 0 . \qquad (\mathbf{15}.\mathrm{d})$$

(2) Case of fixed edge.—At the end side

$$\underset{1}{K_m} + \underset{1}{M_m} = 0\,;$$

$$\underset{1}{L_m}\,\underset{1}{\lambda_1} + \underset{1}{N_m}\,\underset{1}{\lambda_2} = 0\,.$$

From these, we can see that the expressions (7.1) are not applicable to the case when $r=1$. Therefore, referring to the first equation of the above, we can put

$$\underset{1}{K_m} = -\underset{1}{M_m} = \underset{1}{B'_m}\,.$$

Then, the conditions joining the 1-st elementary plate to the 2-nd are written as follows:

$$\underset{1}{B_m'}(\cosh\pi\underset{1}{\lambda_1}-\cosh\pi\underset{1}{\lambda_2})+\underset{1}{L_m}\sinh\pi\underset{1}{\lambda_1}+\underset{1}{N_m}\sinh\pi\underset{1}{\lambda_2}+\underset{2}{B_m}\varphi_2=0\,;$$

$$\underset{1}{B_m'}(\underset{1}{\beta_1}\cosh\pi\underset{1}{\lambda_1}-\beta_2\cosh\pi\underset{1}{\lambda_2})+\underset{1}{L_m}\underset{1}{\beta_1}\sinh\pi\underset{1}{\lambda_1}+\underset{1}{N_m}\underset{1}{\beta_2}\sinh\pi\underset{1}{\lambda_2}=0\,;$$

$$\underset{1}{B_m'}\left\{(\gamma_1+P_1\underset{1}{\lambda_1})\sinh\pi\underset{1}{\lambda_1}-(\underset{1}{\gamma_2}+P_1\underset{1}{\lambda_2})\sinh\pi\underset{1}{\lambda_2}\right\}+\underset{1}{L_m}(\underset{1}{\gamma_1}+P_1\underset{1}{\lambda_1})\cosh\pi\underset{1}{\lambda_1}+\underset{1}{N_m}(\underset{1}{\gamma_2}+P_1\underset{1}{\lambda_2})\cosh\pi\underset{1}{\lambda_2}-\underset{2}{A_m}\mu_2''=0.$$

Hence, on solving these about $\underset{1}{L_m}$ and $\underset{1}{N_m}$

$$\underset{1}{L_m}=\underset{2}{A_m}\mu_2''\frac{\coth\pi\underset{1}{\lambda_1}}{(\underset{1}{\gamma_1}+P_1\underset{1}{\lambda_1})\operatorname{cosech}\pi\underset{1}{\lambda_1}-(\underset{1}{\gamma_2}+P_1\underset{1}{\lambda_2})\operatorname{cosech}\pi\underset{1}{\lambda_2}}$$

$$-\underset{2}{B'_m}\frac{\varphi_2}{\varphi_1}\frac{\left\{(\underset{1}{\gamma_2}+P_1\underset{1}{\lambda_2})(\underset{1}{\beta_2}\operatorname{cosech}\pi\underset{1}{\lambda_1}\operatorname{cosech}\pi\underset{1}{\lambda_2}-\underset{1}{\beta_1}\coth\pi\underset{1}{\lambda_1}\coth\pi\underset{1}{\lambda_2})+\underset{1}{\beta_2}(\underset{1}{\gamma_1}+P_1\underset{1}{\lambda_1})\right\}}{(\underset{1}{\gamma_1}+P_1\underset{1}{\lambda_1})\operatorname{cosech}\pi\underset{1}{\lambda_1}-(\underset{1}{\gamma_2}+P_1\underset{1}{\lambda_2})\operatorname{cosech}\pi\underset{1}{\lambda_2}}\,,$$

$$\underset{1}{N_m}=-\underset{2}{A_m}\mu_2''\frac{\coth\pi\underset{1}{\lambda_2}}{(\underset{1}{\gamma_1}+P_1\underset{1}{\lambda_1})\operatorname{cosech}\pi\underset{1}{\lambda_1}-(\underset{1}{\gamma_2}+P_1\underset{1}{\lambda_2})\operatorname{cosech}\pi\underset{1}{\lambda_2}}$$

$$-\underset{2}{B'_m}\frac{\varphi_2}{\varphi_1}\frac{\left\{(\underset{1}{\gamma_1}+P_1\underset{1}{\lambda_1})(\underset{1}{\beta_1}\operatorname{cosech}\pi\underset{1}{\lambda_1}\operatorname{cosech}\pi\underset{1}{\lambda_2}-\underset{1}{\beta_2}\coth\pi\underset{1}{\lambda_1}\coth\pi\underset{1}{\lambda_2})+\underset{1}{\beta_1}(\underset{1}{\gamma_2}+P_1\underset{1}{\lambda_2})\right\}}{(\underset{1}{\gamma_1}+P_1\underset{1}{\lambda_1})\operatorname{cosech}\pi\underset{1}{\lambda_1}-(\underset{1}{\gamma_2}+P_1\underset{1}{\lambda_2})\operatorname{cosech}\pi\underset{1}{\lambda_2}}\,.$$

Substitute these in the second of the initial equations and denote $\underset{2}{B_m'}\underset{2}{\beta_1}\underset{2}{\beta_2}$ by $\underset{2}{B_m}$, and moreover set as follows:

$$(\underset{1}{A_m})=\underset{1}{L_m}\,\underset{1}{\lambda_1}+\underset{1}{N_m}\,\underset{1}{\lambda_2}=\underset{2}{A_m}\,\mu_2''(F_1)-\underset{2}{B_m}\frac{\varphi_2}{\varphi_1}(G_1)=0\,,$$

where

$$(F_1)=\frac{\underset{1}{\lambda_1}\coth\pi\underset{1}{\lambda_1}-\underset{1}{\lambda_2}\coth\pi\underset{1}{\lambda_2}}{(\underset{1}{\gamma_1}+P_1\underset{1}{\lambda_1})\operatorname{cosech}\pi\underset{1}{\lambda_1}-(\underset{1}{\gamma_2}+P_1\underset{1}{\lambda_2})\operatorname{cosech}\pi\underset{1}{\lambda_2}}=\frac{T_1}{S_1''}\,,$$

$$(G_1)=\frac{\left.\left\{\dfrac{\underset{1}{\lambda_1}(\underset{1}{\gamma_1}+P_1\underset{1}{\lambda_1})}{\underset{1}{\beta_1}}+\dfrac{\underset{1}{\lambda_2}(\underset{1}{\gamma_2}+P_1\underset{1}{\lambda_2})}{\underset{1}{\beta_2}}\right\}+\left\{\dfrac{\underset{1}{\lambda_1}(\underset{1}{\gamma_2}+P_1\underset{1}{\lambda_2})}{\underset{1}{\beta_1}}+\dfrac{\underset{1}{\lambda_2}(\underset{1}{\gamma_1}+P_1\underset{1}{\lambda_1})}{\underset{1}{\beta_2}}\right\}\mathrm{cosech}\,\pi\underset{1}{\lambda_1}\,\mathrm{cosech}\,\pi\underset{1}{\lambda_2}\right.}{(\underset{1}{\gamma_1}+P_1\underset{1}{\lambda_1})\mathrm{cosech}\,\pi\underset{1}{\lambda_1}-(\underset{1}{\gamma_2}+P_1\underset{1}{\lambda_2})\mathrm{cosech}\,\pi\underset{1}{\lambda_2}}$$

$$\frac{-\left\{\dfrac{\underset{1}{\lambda_2}(\underset{1}{\gamma_1}+P_1\underset{1}{\lambda_1})}{\underset{1}{\beta_1}}+\dfrac{\underset{1}{\lambda_1}(\underset{1}{\gamma_2}+P_1\underset{1}{\lambda_2})}{\underset{1}{\beta_2}}\right\}\coth\pi\underset{1}{\lambda_1}\,\coth\pi\underset{1}{\lambda_2}}{}=\frac{T_1T_1'-\bar{S}_1S_1''}{S_1''}\,, \qquad \textbf{(15.e)}$$

where T_1, T_1', $\bar{S}$, S'' are the same as in (**11**.f). Then, putting

$$(\underset{1}{A_m})=A_mU_1=0\,,$$

the conditional equation for buckling is obtained in the following form:

$$U_1=U_2\mu_2''(F_1)-V_2\frac{\bar{\varphi}_2}{\varphi_1}(G_1)=0\,. \qquad \textbf{(15.f)}$$

(3) Case of free edge.—At the end side

$$\underset{1}{K_m}\underset{1}{\beta_1}+\underset{1}{M_m}\underset{1}{\beta_2}=0\,;$$

$$\underset{1}{L_m}(\underset{1}{\gamma_1}+P_1\underset{1}{\lambda_1})+\underset{1}{N_m}(\underset{1}{\gamma_2}+P_1\underset{1}{\lambda_2})=0.$$

With reference to (7.2), the second equation in the above gives $\underset{1}{A_m}=0$. Therefore, the conditional equation in this case results in

$$U_1=0\,. \qquad \textbf{(15.g)}$$

Finally we see that the proper combination of each case in the articles ii) and iii) can represent every kind shown in Table 4 similarly as before.

f) Case of Elastic Supporting in 1-st Way. i) *Conditions for jointing.*—Referring to Table 5, the conditions joining the $(r-1)$-th elementary plate to the r-th become

$$\underset{r-1}{K_m}\cosh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{L_m}\sinh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{M_m}\cosh\pi\underset{r-1}{\lambda_2}+\underset{r-1}{N_m}\sinh\pi\underset{r-1}{\lambda_2}-\underset{r}{K_m}-\underset{r}{M_m}=0\,;$$

$$\underset{r-1}{K_m}\underset{r-1}{\beta_1}\cosh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{L_m}\underset{r-1}{\beta_1}\sinh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{M_m}\underset{r-1}{\beta_2}\cosh\pi\underset{r-1}{\lambda_2}+\underset{r-1}{N_m}\underset{r-1}{\beta_2}\sinh\pi\underset{r-1}{\lambda_2}=0\,;$$

$$\underset{r}{K_m}\underset{r}{\beta_1}+\underset{r}{M_m}\underset{r}{\beta_2}=0;$$

$$\begin{aligned}&\underset{r-1}{K_m}(\underset{r-1}{\gamma_1}+\underset{r-1}{P}\underset{r-1}{\lambda_1})\sinh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{L_m}(\underset{r-1}{\gamma_1}+\underset{r-1}{P}\underset{r-1}{\lambda_1})\cosh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{M_m}(\underset{r-1}{\gamma_2}+\underset{r-1}{P}\underset{r-1}{\lambda_2})\sinh\pi\underset{r-1}{\lambda_2}\\&\quad+\underset{r-1}{N_m}(\underset{r-1}{\gamma_2}+\underset{r-1}{P}\underset{r-1}{\lambda_2})\cosh\pi\underset{r-1}{\lambda_2}-\underset{r}{K_m}\bar{\mu}_r'-\underset{r}{L_m}\mu_r''(\underset{r}{\gamma_1}+P_r\underset{r}{\lambda_1})-\underset{r}{M_m}\bar{\mu}_r'-\underset{r}{N_m}\mu_r''(\underset{r}{\gamma_2}+P_r\underset{r}{\lambda_2})=0\,.\end{aligned}$$

From these, with such a similar treatment as in the previous case e), the formulas (**15**) are also obtained. But, in the present case, their coefficients (**14**) must be replaced by the following expressions:

$$\left.\begin{aligned} F_{r-1}&=\mu_r''\frac{T'_{r-1}}{S'_{r-1}}, & G_{r-1}&=\frac{\bar{\varphi}_r}{\bar{\varphi}_{r-1}}\frac{W'_{r-1}}{S'_{r-1}}-\bar{\mu}_r'\bar{\varphi}_r\frac{T'_{r-1}}{S'_{r-1}},\\ F'_{r-1}&=\mu_r''\frac{1}{S'_{r-1}}, & G'_{r-1}&=\frac{\bar{\varphi}_r}{\bar{\varphi}_{r-1}}\frac{T'_{r-1}}{S'_{r-1}}-\bar{\mu}_r'\bar{\varphi}_r\frac{1}{S'_{r-1}},\end{aligned}\right\} \tag{16}$$

in which $\bar{\varphi}_{r-1}$, $\bar{\varphi}_r$, T'_{r-1}, S'_{r-1}, $W'_{r-1}=T'^2_{r-1}-S'^2_{r-1}$ are similarly expressed as in (**14**).

ii) *Conditional Equations for Buckling.*—The conditions for End side-k are the same as in the previous case **e**). Moreover, the conditional equations for buckling become of the same form as in the case **e**), when the other end side, *i. e.*, End side-1 is simply supported or free. Especially in the case where the end side $\xi_1=0$ is fixed, by the similar discussion as in [**e**)-iii)-(2)], we obtain the conditional equation given below:

$$U_1=U_2\mu_2''(F_1)-V_2\left\{\frac{\bar{\varphi}_2}{\bar{\varphi}_1}(G_1)-\bar{\mu}_r'\bar{\varphi}_2(F_1)\right\}=0, \tag{16'}$$

where (F_1), (G_1) are the same as in (**15.e**).

§ 8. Composite Plates Elastically Built in Joints.

g) **Case of No Supporting** i) *Conditions for jointing.*—Referring to Table 5, the conditions joining the $(r-1)$-th elementary plate to the r-th are written as follows:

$$\underset{r-1}{K_m}\cosh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{L_m}\sinh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{M_m}\cosh\pi\underset{r-1}{\lambda_2}+\underset{r-1}{N_m}\sinh\pi\underset{r-1}{\lambda_2}-\underset{r}{K_m}-\underset{r}{M_m}=0;$$

$$\begin{aligned}&\underset{r-1}{K_m}\underset{r-1}{\lambda_1}\sinh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{L_m}\underset{r-1}{\lambda_1}\cosh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{M_m}\underset{r-1}{\lambda_2}\sinh\pi\underset{r-1}{\lambda_2}+\underset{r-1}{N_m}\underset{r-1}{\lambda_2}\cosh\pi\underset{r-1}{\lambda_2}\\&\qquad+\underset{r}{K_m}\mu_r\kappa_r'\underset{r}{\beta_1}-\underset{r}{L_m}\mu_r\underset{r}{\lambda_1}+\underset{r}{M_m}\mu_r\kappa_r'\underset{r}{\beta_2}-\underset{r}{N_m}\mu_r\underset{r}{\lambda_2}=0;\end{aligned}$$

$$\underset{r-1}{K_m}\underset{r-1}{\beta_1}\cosh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{L_m}\underset{r-1}{\beta_1}\sinh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{M_m}\underset{r-1}{\beta_2}\cosh\pi\underset{r-1}{\lambda_2}+\underset{r-1}{N_m}\underset{r-1}{\beta_2}\sinh\pi\underset{r-1}{\lambda_2}-\underset{r}{K_m}\mu_r'\underset{r}{\beta_1}-\underset{r}{M_m}\mu_r'\underset{r}{\beta_2}=0;$$

$$\begin{aligned}&\underset{r-1}{K_m}(\underset{r-1}{\gamma_1}+\underset{r-1}{P}\underset{r-1}{\lambda_1})\sinh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{L_m}(\underset{r-1}{\gamma_1}+\underset{r-1}{P}\underset{r-1}{\lambda_1})\cosh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{M_m}(\underset{r-1}{\gamma_2}+\underset{r-1}{P}\underset{r-1}{\lambda_2})\sinh\pi\underset{r-1}{\lambda_2}\\&\qquad+\underset{r-1}{N_m}(\underset{r-1}{\gamma_2}+\underset{r-1}{P}\underset{r-1}{\lambda_2})\cosh\pi\underset{r-1}{\lambda_2}-\underset{r}{L_m}\mu_r''(\underset{r}{\gamma_1}+P_r\underset{r}{\lambda_1})-\underset{r}{N_m}\mu_r''(\underset{r}{\gamma_2}+P_r\underset{r}{\lambda_2})=0.\end{aligned}$$

From these, by such a treatment as discussed in the case e.), the recurrence formulas (9) are again obtained ; but their coefficients must be replaced by the following expressions, instead of (8) :

$$\begin{aligned}\underset{r-1}{F_2}&=\frac{\underset{r-1}{\tau_2}}{\underset{r-1}{\varphi}}\cosh\pi\underset{r-1}{\lambda_1}+\mu_r\kappa_r'\frac{\underset{r}{\beta_1}(\underset{r-1}{\gamma_2}+\underset{r-1}{P}\underset{r-1}{\lambda_2})}{\underset{r-1}{\lambda_1}\underset{r-1}{\lambda_2}\underset{r-1}{\varphi}}\sinh\pi\underset{r-1}{\lambda_1}, & \underset{r-1}{G_2}&=\frac{\underset{r-1}{\chi_2}}{\underset{r-1}{\lambda_1}\underset{r-2}{\lambda_2}\underset{r-1}{\varphi}}\cosh\pi\underset{r-1}{\lambda_1},\\ \underset{r-1}{F_2'}&=\frac{\underset{r-1}{\tau_2}}{\underset{r-1}{\varphi}}\sinh\pi\underset{r-1}{\lambda_1}+\mu_r\kappa_r'\frac{\underset{r}{\beta_1}(\underset{r-1}{\gamma_2}+\underset{r-1}{P}\underset{r-1}{\lambda_2})}{\underset{r-1}{\lambda_1}\underset{r-1}{\lambda_2}\underset{r-1}{\varphi}}\cosh\pi\underset{r-1}{\lambda_1}, & \underset{r-1}{G_2'}&=\frac{\underset{r-1}{\chi_2}}{\underset{r-1}{\lambda_1}\underset{r-1}{\lambda_2}\underset{r-1}{\varphi}}\sinh\pi\underset{r-1}{\lambda_2},\end{aligned}$$

$$
\left.
\begin{aligned}
&\underset{r-1}{F_1}=\frac{\underset{r-1}{\tau_1}}{\underset{r-1}{\varphi}}\cosh\pi\underset{r-1}{\lambda_2}+\mu_r\kappa_r'\frac{\underset{r}{\beta_1}(\underset{r-1}{\gamma_1}+\underset{r-1}{P}\,\underset{r-1}{\lambda_1})}{\underset{r-1}{\lambda_1}\,\underset{r-1}{\lambda_2}\,\underset{r-1}{\varphi}}\sinh\pi\underset{r-1}{\lambda_2}\,,\quad \underset{r-1}{G_1}=\frac{\underset{r-1}{\chi_1}}{\underset{r-1}{\lambda_1}\,\underset{r-1}{\lambda_2}\,\underset{r-1}{\varphi}}\cosh\pi\underset{r-1}{\lambda_2}\,,\\
&\underset{r-1}{F_1'}=\frac{\underset{r-1}{\tau_1}}{\underset{r-1}{\varphi}}\sinh\pi\underset{r-1}{\lambda_2}+\mu_r\kappa_r'\frac{\underset{r}{\beta_1}(\underset{r-1}{\gamma_1}+\underset{r-1}{P}\,\underset{r-1}{\lambda_1})}{\underset{r-1}{\lambda_1}\,\underset{r-1}{\lambda_2}\,\underset{r-1}{\varphi}}\cosh\pi\underset{r-1}{\lambda_2}\,,\quad \underset{r-1}{G_1'}=\frac{\underset{r-1}{\chi_1}}{\underset{r-1}{\lambda_1}\,\underset{r-1}{\lambda_2}\,\underset{r-1}{\varphi}}\sinh\pi\underset{r-1}{\lambda_2}\,,\\
&\underset{r-1}{H_2}=\frac{\underset{r-1}{\tau_2'}}{\underset{r-1}{\varphi}}\cosh\pi\underset{r-1}{\lambda_1}+\mu_r\kappa_r'\frac{\underset{r}{\beta_2}(\underset{r-1}{\gamma_2}+\underset{r-1}{P}\,\underset{r-1}{\lambda_2})}{\underset{r-1}{\lambda_1}\,\underset{r-1}{\lambda_2}\,\underset{r-1}{\varphi}}\sinh\pi\underset{r-1}{\lambda_1}\,,\quad \underset{r-1}{I_2}=\frac{\underset{r-1}{\chi_2'}}{\underset{r-1}{\lambda_1}\,\underset{r-1}{\lambda_2}\,\underset{r-1}{\varphi}}\cosh\pi\underset{r-1}{\lambda_1}\,,\\
&\underset{r-1}{H_2'}=\frac{\underset{r-1}{\tau_2'}}{\underset{r-1}{\varphi}}\sinh\pi\underset{r-1}{\lambda_1}+\mu_r\kappa_r'\frac{\underset{r}{\beta_2}(\underset{r-1}{\gamma_2}+\underset{r-1}{P}\,\underset{r-1}{\lambda_2})}{\underset{r-1}{\lambda_1}\,\underset{r-1}{\lambda_2}\,\underset{r-1}{\varphi}}\cosh\pi\underset{r-1}{\lambda_1}\,,\quad \underset{r-1}{I_2'}=\frac{\underset{r-1}{\chi_2'}}{\underset{r-1}{\lambda_1}\,\underset{r-1}{\lambda_2}\,\underset{r-1}{\varphi}}\sinh\pi\underset{r-1}{\lambda_1}\,,\\
&\underset{r-1}{H_1}=\frac{\underset{r-1}{\tau_1'}}{\underset{r-1}{\varphi}}\cosh\pi\underset{r-1}{\lambda_2}+\mu_r\kappa_r'\frac{\underset{r}{\beta_2}(\underset{r-1}{\gamma_1}+\underset{r-1}{P}\,\underset{r-1}{\lambda_1})}{\underset{r-1}{\lambda_1}\,\underset{r-1}{\lambda_2}\,\underset{r-1}{\varphi}}\sinh\pi\underset{r-1}{\lambda_2}\,,\quad \underset{r-1}{I_1}=\frac{\underset{r-1}{\chi_1'}}{\underset{r-1}{\lambda_1}\,\underset{r-1}{\lambda_2}\,\underset{r-1}{\varphi}}\cosh\pi\underset{r-1}{\lambda_2}\,,\\
&\underset{r-1}{H_1'}=\frac{\underset{r-1}{\tau_1'}}{\underset{r-1}{\varphi}}\sinh\pi\underset{r-1}{\lambda_2}+\mu_r\kappa_r'\frac{\underset{r}{\beta_2}(\underset{r-1}{\gamma_1}+\underset{r-1}{P}\,\underset{r-1}{\lambda_1})}{\underset{r-1}{\lambda_1}\,\underset{r-1}{\lambda_2}\,\underset{r-1}{\varphi}}\cosh\pi\underset{r-1}{\lambda_2}\,,\quad \underset{r-1}{I_1'}=\frac{\underset{r-1}{\chi_1'}}{\underset{r-1}{\lambda_1}\,\underset{r-1}{\lambda_2}\,\underset{r-1}{\varphi}}\sinh\pi\underset{r-1}{\lambda_2}\,,\\
&\qquad\qquad\text{where}\qquad \kappa_r'=\varepsilon_r'\frac{\pi}{a_r}\,.
\end{aligned}
\right\}\qquad(17)
$$

ii) *Conditional Equations for Buckling.*—The conditions for End side-k and the conditional equation are similarly expressed as in the case **a**).

h) Case of Rigid Supporting. i) *Conditions for jointing.*—Referring to Table 5 gives

$$\underset{r-1}{K_m}\cosh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{L_m}\sinh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{M_m}\cosh\pi\underset{r-1}{\lambda_2}+\underset{r-1}{N_m}\sinh\pi\underset{r-1}{\lambda_2}=0\,;$$

$$\underset{r}{K_m}+\underset{r}{M_m}=0\,;$$

$$\underset{r-1}{K_m}\,\underset{r-1}{\lambda_1}\sinh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{L_m}\,\underset{r-1}{\lambda_1}\cosh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{M_m}\,\underset{r-1}{\lambda_2}\sinh\pi\underset{r-1}{\lambda_2}+\underset{r-1}{N_m}\,\underset{r-1}{\lambda_2}\cosh\pi\underset{r-1}{\lambda_2}$$
$$+\underset{r}{K_m}\mu_r\kappa_r'\underset{r}{\beta_1}-\underset{r}{L_m}\mu_r\underset{r}{\lambda_1}+\underset{r}{M_m}\mu_r\kappa_r'\underset{r}{\beta_2}-\underset{r}{N_m}\mu_r\underset{r}{\lambda_2}=0\,;$$

$$\underset{r-1}{K_m}\,\underset{r-1}{\beta_1}\cosh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{L_m}\,\underset{r-1}{\beta_1}\sinh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{M_m}\,\underset{r-1}{\beta_2}\cosh\pi\underset{r-1}{\lambda_2}+\underset{r-1}{N_m}\,\underset{r-1}{\beta_2}\sinh\pi\underset{r-1}{\lambda_2}-\underset{r}{K_m}\mu_r'\underset{r}{\beta_1}-\underset{r}{M_m}\mu_r'\underset{r}{\beta_2}=0\,.$$

From the second equation of the above, it can be taken that

$$\underset{r}{K_m}=-\underset{r}{M_m}=\underset{r}{B_m}$$

Then, the remaining three equations are rearranged as follows:

$$\underset{r-1}{B_m}(\cosh\pi\underset{r-1}{\lambda_1}-\cosh\pi\underset{r-1}{\lambda_2})+\underset{r-1}{L_m}\sinh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{N_m}\sinh\pi\underset{r-1}{\lambda_2}=0\,;$$

$$\underset{r-1}{B_m}(\underset{r-1}{\lambda_1}\sinh\pi\underset{r-1}{\lambda_1}-\underset{r-1}{\lambda_2}\sinh\pi\underset{r-1}{\lambda_2})+\underset{r-1}{L_m}\,\underset{r-1}{\lambda_1}\cosh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{N_m}\,\underset{r-1}{\lambda_2}\cosh\pi\underset{r-1}{\lambda_2}$$

$$-L_{m}\mu_{r}\lambda_{1} - N_{m}\mu_{r}\lambda_{2} + B_{m}\mu_{r}\kappa_{r}'\varphi_{r} = 0\,;$$

$$B_{m}(\beta_{1}\cosh\pi\lambda_{1} - \beta_{2}\cosh\pi\lambda_{2}) + L_{m}\beta_{1}\sinh\pi\lambda_{1} + N_{m}\beta_{2}\sinh\pi\lambda_{2} - B_{m}\mu_{r}'\varphi_{r} = 0\,.$$

Comparing with (6.12), we see the seconds equation has one more term (the last in the left member) than that of Eqs. (6.12) But, the similar treatments as before give the equation of such a form as (6.15), while it is necessary to notice that each of the coefficients are replaced by the following:

$$\left.\begin{aligned} F_{r-1} &= \mu_{r}\frac{T_{r-1}}{S_{r-1}}, & G_{r-1} &= \mu_{r}'\frac{\varphi_{r}}{\varphi_{r-1}}\frac{W_{r-1}}{S_{r-1}} + \mu_{r}\kappa_{r}'\varphi_{r}\frac{T_{r-1}}{S_{r-1}}, \\ F'_{r-1} &= \mu_{r}\frac{1}{S_{r-1}}, & G'_{r-1} &= \mu_{r}'\frac{\varphi_{r}}{\varphi_{r-1}}\frac{T_{r-1}}{S_{r-1}} + \mu_{r}\kappa_{r}'\varphi_{r}\frac{1}{S_{r-1}}, \end{aligned}\right\} \quad (\mathbf{18})$$

where T_{r-1}, S_{r-1}, W_{r-1} are the same as in (**10**). Therefore, taking account of the above, we arrived at the same recurrence formula as (**11**).

ii) *Conditional Equations for Buckling.*—The conditions for End side-k again are the same as in **b**). And also the conditional equations for buckling become of the same form as in **b**)', when the other end side is simply supported or fixed. On the other hand, when the other end side $\xi_1 = 0$ is free, the similar treatments as in [§6-**b**)-iii)-(3)] give the conditional equation in the following form:

$$U_{1} = U_{2}\mu_{2}(F_{1}) - V_{2}\left\{\mu_{2}'\frac{\varphi_{2}}{\varphi_{1}}(G_{1}) + \mu_{2}\kappa_{2}'\varphi_{2}(F_{1})\right\} = 0\,, \quad (\mathbf{18'})$$

where (F_1) and (G_1) are similarly expressed as in (**11.f**)

i) Case of Elastic Supporting in 1-st Way. i) *Conditions for Jointing.*— On referring to Table 5, at the present case

$$K_{m}\cosh\pi\lambda_{1} + L_{m}\sinh\pi\lambda_{1} + M_{m}\cosh\pi\lambda_{2} + N_{m}\sinh\pi\lambda_{2} - K_{m} - M_{m} = 0\,; \quad (8.1)$$

$$K_{m}(\lambda_{1}\sinh\pi\lambda_{1} + \kappa\beta_{1}\cosh\pi\lambda_{1}) + L_{m}(\lambda_{1}\cosh\pi\lambda_{1} + \kappa\beta_{1}\sinh\pi\lambda_{1}) + M_{m}(\lambda_{2}\sinh\pi\lambda_{2} + \kappa\beta_{2}\cosh\pi\lambda_{2})$$
$$+ N_{m}(\lambda_{2}\cosh\pi\lambda_{2} + \kappa\beta_{2}\sinh\pi\lambda_{2}) + K_{m}\mu_{r}\kappa_{r}'\beta_{1} - L_{m}\mu_{r}\lambda_{1} + M_{m}\mu_{r}\kappa_{r}'\beta_{2} - N_{m}\mu_{r}\lambda_{2} = 0\,; \quad (8.2)$$

$$K_{m}\beta_{1}\cosh\pi\lambda_{1} + L_{m}\beta_{1}\sinh\pi\lambda_{1} + M_{m}\beta_{2}\cosh\pi\lambda_{2} + N_{m}\beta_{2}\sinh\pi\lambda_{2}$$
$$- K_{m}\bar{\mu}_{r}''\beta_{1} - L_{m}\bar{\mu}_{r}\lambda_{1} - M_{m}\bar{\mu}_{r}''\beta_{2} - N_{m}\bar{\mu}_{r}\lambda_{2} = 0\,; \quad (8.3)$$

$$K_{m}(\gamma_{1} + P\lambda_{1})\sinh\pi\lambda_{1} + L_{m}(\gamma_{1} + P\lambda_{1})\cosh\pi\lambda_{1} + M_{m}(\gamma_{2} + P\lambda_{2})\sinh\pi\lambda_{2} + N_{m}(\gamma_{2} + P\lambda_{2})\cosh\pi\lambda_{2}$$
$$- K_{m}\bar{\mu}_{r}' - L_{m}\mu_{r}''(\gamma_{1} + P_{r}\lambda_{1}) - M_{m}\bar{\mu}_{r}' - N_{m}\mu_{r}''(\gamma_{2} + P_{r}\lambda_{2}) = 0\,, \quad (8.4)$$

Hence, the operation (8.2) − (8.3) × κ_{r-1} gives

$$\underset{r-1}{K_m}\,\underset{r-1}{\lambda_1}\sinh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{L_m}\,\underset{r-1}{\lambda_1}\cosh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{M_m}\,\underset{r-1}{\lambda_2}\sinh\pi\underset{r-1}{\lambda_2}+\underset{r-1}{N_m}\,\underset{r-1}{\lambda_2}\cosh\pi\underset{r-1}{\lambda_2}$$

$$+\underset{r}{K_m}(\mu_r\kappa_r'+\bar{\mu}_r''\kappa_{r-1})\underset{r}{\beta_1}+\underset{r}{M_m}(\mu_r\kappa_r'+\bar{\mu}_r''\kappa_{r-1})\underset{r}{\beta_2}-\underset{r}{L_m}(\mu_r-\bar{\mu}_r\kappa_{r-1})\underset{r}{\lambda_1}-\underset{r}{N_m}(\mu_r-\bar{\mu}_r\kappa_{r-1})\underset{r}{\lambda_2}=0\,; \qquad (8.2')$$

In the next place, by the following operations:

$$(8.1)\times\underset{r-1}{\beta_1}-(8.3)\,;$$
$$(8.1)\times\underset{r-1}{\beta_2}-(3.3)\,;$$
$$(8.2')\times(\underset{r-1}{\gamma_1}+\underset{r-1}{P}\,\underset{r-1}{\lambda_1})-(8.4)\times\underset{r-1}{\lambda_1}\,;$$
$$(8.2')\times(\underset{r-1}{\gamma_2}+\underset{r-1}{P}\,\underset{r-1}{\lambda_2})-(8.4)\times\underset{r-1}{\lambda_2}\,,$$

we obtain

$$\underset{r-1}{M_m}\,\underset{r-1}{\varphi}\cosh\pi\underset{r-1}{\lambda_2}+\underset{r-1}{N_m}\,\underset{r-1}{\varphi}\sinh\pi\underset{r-1}{\lambda_2}-\underset{r}{K_m}\underset{r-1}{\bar{\tau}_1}-\underset{r}{M_m}\underset{r-1}{\bar{\tau}_1'}+\underset{r}{L_m}\bar{\mu}_r\underset{r}{\lambda_1}+\underset{r}{N_m}\bar{\mu}_r\underset{r}{\lambda_2}=0\,; \qquad (8.5)$$

$$\underset{r-1}{K_m}\,\underset{r-1}{\varphi}\cosh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{L_m}\,\underset{r-1}{\varphi}\sinh\pi\underset{r-1}{\lambda_1}+\underset{r}{K_m}\underset{r-1}{\bar{\tau}_2}+\underset{r}{M_m}\underset{r-1}{\bar{\tau}_2'}-\underset{r}{L_m}\bar{\mu}_r\underset{r}{\lambda_1}-\underset{r}{N_m}\bar{\mu}_r\underset{r}{\lambda_2}=0\,; \qquad (8.6)$$

$$\underset{r-1}{M_m}\,\underset{r-1}{\lambda_1}\,\underset{r-1}{\lambda_2}\,\underset{r-1}{\varphi}\sinh\pi\underset{r-1}{\lambda_2}+\underset{r-1}{N_m}\,\underset{r-1}{\lambda_1}\,\underset{r-1}{\lambda_2}\,\underset{r-1}{\varphi}\cosh\pi\underset{r-1}{\lambda_2}+\underset{r}{K_m}\underset{r-1}{(\chi)_1}+\underset{r}{M_m}\underset{r-1}{(\chi')_1}-\underset{r}{L_m}\underset{r-1}{\bar{\chi}_1}-\underset{r}{N_m}\underset{r-1}{\bar{\chi}_1'}=0\,; \qquad (8.7)$$

$$\underset{r-1}{K_m}\,\underset{r-1}{\lambda_1}\,\underset{r-1}{\lambda_2}\,\underset{r-1}{\varphi}\sinh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{L_m}\,\underset{r-1}{\lambda_1}\,\underset{r-1}{\lambda_2}\,\underset{r-1}{\varphi}\cosh\pi\underset{r-1}{\lambda_1}-\underset{r}{K_m}\underset{r-1}{(\chi)_2}-\underset{r}{M_m}\underset{r-1}{(\chi')_2}+\underset{r}{L_m}\underset{r-1}{\bar{\chi}_2}+\underset{r}{N_m}\underset{r-1}{\bar{\chi}_2'}=0\,, \qquad (8.8)$$

where

$$\left.\begin{aligned}
&\underset{r-1}{\bar{\tau}_1}=\underset{r-1}{\beta_1}-\bar{\mu}_r''\underset{r}{\beta_1}\,, \qquad \underset{r-1}{\bar{\tau}_2}=\underset{r-1}{\beta_2}-\bar{\mu}_r''\underset{r}{\beta_1}\,,\\
&\underset{r-1}{\bar{\tau}_1'}=\underset{r-1}{\beta_1}-\bar{\mu}_r''\underset{r}{\beta_2}\,, \qquad \underset{r-1}{\bar{\tau}_2'}=\underset{r-1}{\beta_2}-\bar{\mu}_r''\underset{r}{\beta_2}\,,\\
&\underset{r-1}{(\chi)_1}=(\mu_r\kappa_r'+\bar{\mu}_r''\kappa_{r-1})\underset{r}{\beta_1}(\underset{r-1}{\gamma_1}+\underset{r-1}{P}\,\underset{r-1}{\lambda_1})+\bar{\mu}_r'\underset{r-1}{\lambda_1}\,, \quad \underset{r-1}{\bar{\chi}_1}=(\mu_r-\bar{\mu}_r\kappa_{r-1})\underset{r}{\lambda_1}(\underset{r-1}{\gamma_1}+\underset{r-1}{P}\,\underset{r-1}{\lambda_1})-\mu_r''\underset{r-1}{\lambda_1}(\underset{r}{\gamma_1}+P_r\underset{r}{\lambda_1})\,,\\
&\underset{r-1}{(\chi')_1}=(\mu_r\kappa_r'+\bar{\mu}_r''\kappa_{r-1})\underset{r}{\beta_2}(\underset{r-1}{\gamma_1}+\underset{r-1}{P}\,\underset{r-1}{\lambda_1})+\bar{\mu}_r'\underset{r-1}{\lambda_1}\,, \quad \underset{r-1}{\bar{\chi}_1'}=(\mu_r-\bar{\mu}_r\kappa_{r-1})\underset{r}{\lambda_2}(\underset{r-1}{\gamma_1}+\underset{r-1}{P}\,\underset{r-1}{\lambda_1})-\mu_r''\underset{r-1}{\lambda_1}(\underset{r}{\gamma_2}+P_r\underset{r}{\lambda_2})\,,\\
&\underset{r-1}{(\chi)_2}=(\mu_r\kappa_r'+\bar{\mu}_r''\kappa_{r-1})\underset{r}{\beta_1}(\underset{r-1}{\gamma_2}+\underset{r-1}{P}\,\underset{r-1}{\lambda_2})+\bar{\mu}_r'\underset{r-1}{\lambda_2}\,, \quad \underset{r-1}{\bar{\chi}_2}=(\mu_r-\bar{\mu}_r\kappa_{r-1})\underset{r}{\lambda_1}(\underset{r-1}{\gamma_2}+\underset{r-1}{P}\,\underset{r-1}{\lambda_2})-\mu_r''\underset{r-1}{\lambda_2}(\underset{r}{\gamma_1}+P_r\underset{r}{\lambda_1})\,,\\
&\underset{r-1}{(\chi')_2}=(\mu_r\kappa_r'+\bar{\mu}_r''\kappa_{r-1})\underset{r}{\beta_2}(\underset{r-1}{\gamma_2}+\underset{r-1}{P}\,\underset{r-1}{\lambda_2})+\bar{\mu}_r'\underset{r-1}{\lambda_2}\,, \quad \underset{r-1}{\bar{\chi}_2'}=(\mu_r-\bar{\mu}_r\kappa_{r-1})\underset{r}{\lambda_2}(\underset{r-1}{\gamma_2}+\underset{r-1}{P}\,\underset{r-1}{\lambda_2})-\mu_r''\underset{r-1}{\lambda_2}(\underset{r}{\gamma_2}+P_r\underset{r}{\lambda_2})\,,\\
&\text{in which} \qquad \kappa_{r-1}=\varepsilon_{r-1}\frac{\pi}{a_{r-1}}\,, \qquad \kappa_r'=\varepsilon_r'\frac{\pi}{a_r}\,.
\end{aligned}\right\} \quad (19)$$

Furthermore, let us perform the following operations:

$$(8.6)\times\cosh\pi\underset{r-1}{\lambda_1}-(8.8)\times\frac{\sinh\pi\underset{r-1}{\lambda_1}}{\underset{r-1}{\lambda_1}\,\underset{r-1}{\lambda_2}}\,;$$

$$(8.6)\times\sinh\pi\underset{r-1}{\lambda_1}-(8.8)\times\frac{\cosh\pi\underset{r-1}{\lambda_1}}{\underset{r-1}{\lambda_1}\,\underset{r-1}{\lambda_2}}\,;$$

$$(8.5)\times\cosh\pi\underset{r-1}{\lambda_2}-(8.7)\times\frac{\sinh\pi\underset{r-1}{\lambda_2}}{\underset{r-1}{\lambda_1}\;\underset{r-1}{\lambda_2}}\,;$$

$$(8.5)\times\sinh\pi\underset{r-1}{\lambda_2}-(8.7)\times\frac{\cosh\pi\underset{r-1}{\lambda_2}}{\underset{r-1}{\lambda_1}\;\underset{r-1}{\lambda_2}}\,.$$

Obtaining then the equations of the same kind as (6.5), we can easily arrive at the recurrence formula (**9**) in which the coefficients must be represented by (**20**) given below instead of (**8**).

$$\left.\begin{aligned}
\underset{r-1}{F_2}&=\frac{\underset{r-1}{\tau_2}}{\underset{r-1}{\varphi}}\cosh\pi\underset{r-1}{\lambda_1}+\frac{\underset{r-1}{(\chi)_2}}{\underset{r-1}{\lambda_1}\underset{r-1}{\lambda_2}\underset{r-1}{\varphi}}\sinh\pi\underset{r-1}{\lambda_1}, &
\underset{r-1}{G_2}&=-\frac{\underset{r-1}{\bar\chi_2}}{\underset{r-1}{\lambda_1}\underset{r-1}{\lambda_2}\underset{r-1}{\varphi}}\cosh\pi\underset{r-1}{\lambda_1}+\frac{\bar\mu_r\underset{r}{\lambda_1}}{\underset{r-1}{\varphi}}\sinh\pi\underset{r-1}{\lambda_1},\\
\underset{r-1}{F_2'}&=\frac{\underset{r-1}{\bar\tau_2}}{\underset{r-1}{\varphi}}\sinh\pi\underset{r-1}{\lambda_1}+\frac{\underset{r-1}{(\chi)_2}}{\underset{r-1}{\lambda_1}\underset{r-1}{\lambda_2}\underset{r-1}{\varphi}}\cosh\pi\underset{r-1}{\lambda_1}, &
\underset{r-1}{G_2'}&=\frac{\underset{r-1}{\bar\chi_2}}{\underset{r-1}{\lambda_1}\underset{r-1}{\lambda_2}\underset{r-1}{\varphi}}\sinh\pi\underset{r-1}{\lambda_1}+\frac{\bar\mu_r\underset{r}{\lambda_1}}{\underset{r-1}{\varphi}}\cosh\pi\underset{r-1}{\lambda_1},\\
\underset{r-1}{F_1}&=\frac{\underset{r-1}{\bar\tau_1}}{\underset{r-1}{\varphi}}\cosh\pi\underset{r-1}{\lambda_2}+\frac{\underset{r-1}{(\chi)_1}}{\underset{r-1}{\lambda_1}\underset{r-1}{\lambda_2}\underset{r-1}{\varphi}}\sinh\pi\underset{r-1}{\lambda_2}, &
\underset{r-1}{G_1}&=\frac{\underset{r-1}{\bar\chi_1}}{\underset{r-1}{\lambda_1}\underset{r-1}{\lambda_2}\underset{r-1}{\varphi}}\cosh\pi\underset{r-1}{\lambda_2}+\frac{\bar\mu_r\underset{r}{\lambda_1}}{\underset{r-1}{\varphi}}\sinh\pi\underset{r-1}{\lambda_2},\\
\underset{r-1}{F_1'}&=\frac{\underset{r-1}{\bar\tau_1}}{\underset{r-1}{\varphi}}\sinh\pi\underset{r-1}{\lambda_2}+\frac{\underset{r-1}{(\chi)_1}}{\underset{r-1}{\lambda_1}\underset{r-1}{\lambda_2}\underset{r-1}{\varphi}}\cosh\pi\underset{r-1}{\lambda_2}, &
\underset{r-1}{G_1'}&=\frac{\underset{r-1}{\bar\chi_1}}{\underset{r-1}{\lambda_1}\underset{r-1}{\lambda_2}\underset{r-1}{\varphi}}\sinh\pi\underset{r-1}{\lambda_2}+\frac{\bar\mu_r\underset{r}{\lambda_1}}{\underset{r-1}{\varphi}}\cosh\pi\underset{r-1}{\lambda_2},\\
\underset{r-1}{H_2}&=\frac{\underset{r-1}{\bar\tau_2'}}{\underset{r-1}{\varphi}}\cosh\pi\underset{r-1}{\lambda_1}+\frac{\underset{r-1}{(\chi')_2}}{\underset{r-1}{\lambda_1}\underset{r-1}{\lambda_2}\underset{r-1}{\varphi}}\sinh\pi\underset{r-1}{\lambda_1}, &
\underset{r-1}{I_2}&=\frac{\underset{r-1}{\bar\chi_2'}}{\underset{r-1}{\lambda_1}\underset{r-1}{\lambda_2}\underset{r-1}{\varphi}}\cosh\pi\underset{r-1}{\lambda_1}+\frac{\bar\mu_r\underset{r}{\lambda_2}}{\underset{r-1}{\varphi}}\sinh\pi\underset{r-1}{\lambda_1},\\
\underset{r-1}{H_2'}&=\frac{\underset{r-1}{\bar\tau_2'}}{\underset{r-1}{\varphi}}\sinh\pi\underset{r-1}{\lambda_1}+\frac{\underset{r-1}{(\chi')_2}}{\underset{r-1}{\lambda_1}\underset{r-1}{\lambda_2}\underset{r-1}{\varphi}}\cosh\pi\underset{r-1}{\lambda_1}, &
\underset{r-1}{I_2'}&=\frac{\underset{r-1}{\bar\chi_2'}}{\underset{r-1}{\lambda_1}\underset{r-1}{\lambda_2}\underset{r-1}{\varphi}}\sinh\pi\underset{r-1}{\lambda_1}+\frac{\bar\mu_r\underset{r}{\lambda_2}}{\underset{r-1}{\varphi}}\cosh\pi\underset{r-1}{\lambda_1},\\
\underset{r-1}{H_1}&=\frac{\underset{r-1}{\bar\tau_1'}}{\underset{r-1}{\varphi}}\cosh\pi\underset{r-1}{\lambda_2}+\frac{\underset{r-1}{(\chi')_1}}{\underset{r-1}{\lambda_1}\underset{r-1}{\lambda_2}\underset{r-1}{\varphi}}\sinh\pi\underset{r-1}{\lambda_2}, &
\underset{r-1}{I_1}&=\frac{\underset{r-1}{\bar\chi_1'}}{\underset{r-1}{\lambda_1}\underset{r-1}{\lambda_2}\underset{r-1}{\varphi}}\cosh\pi\underset{r-1}{\lambda_2}+\frac{\bar\mu_r\underset{r}{\lambda_2}}{\underset{r-1}{\varphi}}\sinh\pi\underset{r-1}{\lambda_2},\\
\underset{r-1}{H_1'}&=\frac{\underset{r-1}{\bar\tau_1'}}{\underset{r-1}{\varphi}}\sinh\pi\underset{r-1}{\lambda_2}+\frac{\underset{r-1}{(\chi')_1}}{\underset{r-1}{\lambda_1}\underset{r-1}{\lambda_2}\underset{r-1}{\varphi}}\cosh\pi\underset{r-1}{\lambda_2}, &
\underset{r-1}{I_1'}&=\frac{\underset{r-1}{\bar\chi_1'}}{\underset{r-1}{\lambda_1}\underset{r-1}{\lambda_2}\underset{r-1}{\varphi}}\sinh\pi\underset{r-1}{\lambda_2}+\frac{\bar\mu_r\underset{r}{\lambda_2}}{\underset{r-1}{\varphi}}\cosh\pi\underset{r-1}{\lambda_2}.
\end{aligned}\right\}\quad(\mathbf{20})$$

It is obvious that, putting $\kappa_{r-1}=\kappa_r'=0$ in the above, these coincide with (**12**) in the case c).

ii) *Conditional Equations for Buckling.*—The conditions for End side-k and the conditional equation in this case are similarly expressed as in the case a).

j) Case of Elastic Supporting in 2-nd Way. i) *Conditions for Jointing.*—On referring to Table 5,

$$\underset{r-1}{K_m}\cosh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{L_m}\sinh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{M_m}\cosh\pi\underset{r-1}{\lambda_2}+\underset{r-1}{N_m}\sinh\pi\underset{r-1}{\lambda_2}=0\,; \quad (8.9)$$

$$\underset{r}{K_m}+\underset{r}{M_m}=0\,; \quad (8.10)$$

$$\begin{aligned}&\underset{r-1}{K_m}(\underset{r-1}{\lambda_1}\sinh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{\kappa}\,\underset{r-1}{\beta_1}\cosh\pi\underset{r-1}{\lambda_1})+\underset{r-1}{L_m}(\underset{r-1}{\lambda_1}\cosh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{\kappa}\,\underset{r-1}{\beta_1}\sinh\pi\underset{r-1}{\lambda_1})+\underset{r-1}{M_m}(\underset{r-1}{\lambda_2}\sinh\pi\underset{r-1}{\lambda_2}\\&\quad+\underset{r-1}{\kappa}\,\underset{r-1}{\beta_2}\cosh\pi\underset{r-1}{\lambda_2})+\underset{r-1}{N_m}(\underset{r-1}{\lambda_2}\cosh\pi\underset{r-1}{\lambda_2}+\underset{r-1}{\kappa}\,\underset{r-1}{\beta_2}\sinh\pi\underset{r-1}{\lambda_2})+\underset{r}{K_m}\mu_r\kappa_r'\underset{r}{\beta_1}-\underset{r}{L_m}\mu_r\underset{r}{\lambda_1}\\&\quad+\underset{r}{M_m}\mu_r\kappa_r'\underset{r}{\beta_2}-\underset{r}{N_m}\mu_r\underset{r}{\lambda_2}=0\,; \quad (8.11)\end{aligned}$$

$$\begin{aligned}&\underset{r-1}{K_m}\underset{r-1}{\beta_1}\cosh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{L_m}\underset{r-1}{\beta_1}\sinh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{M_m}\underset{r-1}{\beta_2}\cosh\pi\underset{r-1}{\lambda_2}+\underset{r-1}{N_m}\underset{r-1}{\beta_2}\sinh\pi\underset{r-1}{\lambda_2}\\&\quad-\underset{r}{K_m}\bar{\mu}_r''\underset{r}{\beta_1}-\underset{r}{L_m}\bar{\mu}_r\underset{r}{\lambda_1}-\underset{r}{M_m}\bar{\mu}_r''\underset{r}{\beta_2}-\underset{r}{N_m}\bar{\mu}_r\underset{r}{\lambda_2}=0\,, \quad (8.12)\end{aligned}$$

then, by the operation (8.11)$-$(8.12)$\times\kappa_{r-1}$, we have

$$\begin{aligned}&\underset{r-1}{K_m}\underset{r-1}{\lambda_1}\sinh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{L_m}\underset{r-1}{\lambda_1}\cosh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{M_m}\underset{r-1}{\lambda_2}\sinh\pi\underset{r-1}{\lambda_2}+\underset{r-1}{N_m}\underset{r-1}{\lambda_2}\cosh\pi\underset{r-1}{\lambda_2}\\&+\underset{r}{K_m}(\mu_r\kappa_r'+\bar{\mu}_r''\kappa_{r-1})\underset{r}{\beta_1}+\underset{r}{M_m}(\mu_r\kappa_r'+\bar{\mu}_r''\kappa_{r-1})\underset{r}{\beta_2}-\underset{r}{L_m}(\mu_r-\bar{\mu}_r\kappa_{r-1})\underset{r}{\lambda_1}-\underset{r}{N_m}(\mu_r-\bar{\mu}_r\kappa_{r-1})\underset{r}{\lambda_2}=0. \quad (8.11')\end{aligned}$$

Now, since we can put $\underset{r}{K_m}=-\underset{r}{M_m}=\underset{r}{B_m}$ from Eq. (8.10), the remaining three equations, *i. e.*, (8.9), (8.11′), and (8.12) are rewritten as follows:

$$\underset{r-1}{B_m}(\cosh\pi\underset{r-1}{\lambda_1}-\cosh\pi\underset{r-1}{\lambda_2})+\underset{r-1}{L_m}\sinh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{N_m}\sinh\pi\underset{r-1}{\lambda_2}=0\,;$$

$$\begin{aligned}&\underset{r-1}{B_m}(\underset{r-1}{\lambda_1}\sinh\pi\underset{r-1}{\lambda_1}-\underset{r-1}{\lambda_2}\sinh\pi\underset{r-1}{\lambda_2})+\underset{r-1}{L_m}\underset{r-1}{\lambda_1}\cosh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{N_m}\underset{r-1}{\lambda_2}\cosh\pi\underset{r-1}{\lambda_2}\\&\quad+\underset{r}{B_m}(\mu_r\kappa_r'+\bar{\mu}_r''\kappa_{r-1})\varphi_r-\underset{r}{A_m}(\mu_r-\bar{\mu}_r\kappa_{r-1})=0\,;\end{aligned}$$

$$\underset{r-1}{B_m}(\underset{r-1}{\beta_1}\cosh\pi\underset{r-1}{\lambda_1}-\underset{r-1}{\beta_2}\cosh\pi\underset{r-1}{\lambda_2})+\underset{r-1}{L_m}\underset{r-1}{\beta_1}\sinh\pi\underset{r-1}{\lambda_1}+\underset{r-1}{N_m}\underset{r-1}{\beta_2}\sinh\pi\underset{r-1}{\lambda_2}-\underset{r}{B_m}\bar{\mu}_r''\varphi_r-\underset{r}{A_m}\bar{\mu}_r=0\,,$$

where

$$\underset{r}{A_m}=\underset{r}{L_m}\underset{r}{\lambda_1}+\underset{r}{N_m}\underset{r}{\lambda_2}\,.$$

Solving the above three equations with respect to the constants having the suffix $r-1$ and producing the expression of $\underset{r-1}{A_m}$, we can obtain the similar equations as (6.15) in the case **b**). And then the recurrence formulas (**11**) are obtained again, but the following expressions must be used as their coefficients instead of (**10**):

$$\left.\begin{aligned}F_{r-1}&=(\mu_r-\bar{\mu}_r\kappa_{r-1})\frac{T_{r-1}}{S_{r-1}}-\frac{\bar{\mu}_r}{\varphi_{r-1}}\frac{W_{r-1}}{S_{r-1}}\,, & G_{r-1}&=\bar{\mu}_r''\frac{\varphi_r}{\varphi_{r-1}}\frac{W_{r-1}}{S_{r-1}}+(\mu_r\kappa_r'+\bar{\mu}_r''\kappa_{r-1})\varphi_r\frac{T_{r-1}}{S_{r-1}}\,,\\ F'_{r-1}&=(\mu_r-\bar{\mu}_r\kappa_{r-1})\frac{1}{S_{r-1}}-\frac{\bar{\mu}_r}{\varphi_{r-1}}\frac{T_{r-1}}{S_{r-1}}\,; & G'_{r-1}&=\bar{\mu}_r''\frac{\varphi_r}{\varphi_{r-1}}\frac{T_{r-1}}{S_{r-1}}+(\mu_r\kappa_r'+\bar{\mu}_r''\kappa_{r-1})\varphi_r\frac{1}{S_{r-1}}\,,\end{aligned}\right\}\ (\mathbf{21})$$

where T_{r-1}, S_{r-1} and W_{r-1} are the same as in **(10)**. Putting $\kappa_{r-1}=\kappa_r'=0$ in the above, these come to coincide with **(13)** and also it is obvious that the present case becomes the foregoing case **d**).

ii) *Conditional Equations for Buckling.*—The conditions for End side-k are the same as in the case **b**), and the conditional equations in this case come to coincide with those of the case **b**), when it is simply supported or fixed at the other end side where $x_1=0$. If the end side $x_1=0$ is free, in the same manner as in [**b**)-iii)-(3)] the following considerations become necessary.

That is, the relations (8.10) are not applicable to the case when $r=1$. In the present case, the end side conditions are given by (6.18), *i. e.*,

$$K_{\underset{1}{m}}\,\underset{1}{\beta_1} + M_{\underset{1}{m}}\,\underset{1}{\beta_2} = 0$$

$$L_{\underset{1}{m}}\,(\underset{1}{\gamma_1} + P_1\,\underset{1}{\lambda_1}) + N_{\underset{1}{m}}\,(\underset{1}{\gamma_2} + P_1\,\underset{1}{\lambda_2}) = 0$$

From the first equation, we can write

$$K_{\underset{1}{m}} = B_{\underset{1}{m}}\,\underset{1}{\beta_2}\,, \qquad M_{\underset{1}{m}} = -\,B_{\underset{1}{m}}\,\underset{1}{\beta_1}\,.$$

Using these and representing, in the same manner as in the case **b**), $L_{\underset{1}{m}}$ and $N_{\underset{1}{m}}$ with $A_{\underset{2}{m}}$ and $B_{\underset{2}{m}}$ by the conditions joining the 1-st elementary plate to the 2nd, and next substituting them in the second equation of the above, we finally obtain

$$L_{\underset{1}{m}}(\underset{1}{\gamma_1}+P_1\underset{1}{\lambda_1})+N_{\underset{1}{m}}(\underset{1}{\gamma_2}+P_1\underset{1}{\lambda_2})=A_{\underset{2}{m}}\left\{(\mu_2-\bar{\mu}_2\kappa_1)(F_1)-\frac{\bar{\mu}_2}{\varphi_1}(G^1)\right\}$$
$$-B_{\underset{2}{m}}\left\{\bar{\mu}_2''\frac{\varphi_2}{\varphi_1}(G_1)+(\mu_2\kappa_2'+\bar{\mu}_2''\kappa_1)\,\varphi_2\,(F_1)\right\}=0.$$

Hence, it can readily be concluded that the conditional equation is written as follows:

$$U_1=U_2\left\{(\mu_2-\bar{\mu}_2\kappa_1)(F_1)-\frac{\bar{\mu}_2}{\varphi_1}(G_1)\right\}-V_2\left\{\bar{\mu}_2''\frac{\varphi_2}{\varphi_1}(G_1)+(\mu_2\kappa_2'+\bar{\mu}_2''\kappa_1)\varphi_2(F_1)\right\}=0, \qquad (21')$$

where (F_1) and (G_1) are the same as in (11.f). Taking again $\kappa_1=\kappa_2'=0$ in the above, we will obtain **(13′)** in the foregoing case **d**). That is a matter of course, since $\kappa_1=\kappa_2'=0$ means that the supporting beams vanish.

It is finally summarized that the every formula enumerated in this chapter is of similar form, and then the directions for use of them are also similar to each other, therefore the procedures explained in detail about the representative cases **a**) and **b**) are generally applicable to the others.

CHAPTER III

NUMERICAL ILLUSTRATIONS FOR GENERAL FORMULAE

The numerical illustrations about the foregoing formulas will be made here by each of the examples in various cases. Now, the characteristic values in so-called "Eigenwertproblem", such as the critical load in problems of elastic instability, the natural frequency in vibration problems and so on, are generally reduced to the roots of determinants which become of complex form in many practical cases.

And then, it is general that their values must be found by tedious calculations by a trial-and-error method. Although, in the last chapter, those calculating processes have been arranged in such a definite way that regular treatings are possible, yet we may be subjected to considerably laborious task for calculation. And here, in the first instance, it may be desired to foreknow, as closely as possible, the limits which a characteristic value is searched within.

In many cases, we may set without much difficulty an inequality to hold between the unknown characteristic value of a certain system and the already known of another similar systems by comparing their physical conditions with each other. And moreover we can suppose that the more similar both the physical conditions are, the nearer their characteristic values are. Therefore, if we obtain more adequate numerical results concerning the characteristic values of some systems, by using these, we will be able to set the limits more closely which the unknown characteristic value of another similar system lies within. Thus, the numerical results in this chapter are desired to be used as references for other cases.

In the following examples, only such cases where either one of loads p and q_r is acting are worked because it may be considered that such cases are more usual and important than the cases where both of p and q_r are acting. No special differences on the calculating process will be found in the latter cases too.

(A) THE COMPOSITE PLATES WITH A RIGID JOINT AND VARIOUS EDGE CONDITIONS ALONG TWO OPPOSITE END SIDES.

Let us observe first the cases where a composite plate, composed of two elementary plates rigidly joined and having no support along the joint, *i. e.*, as in [§ 6. a) The Case of Rigid Jointing, and No Supporting along the Joint], is supported in various ways along the end edges.

Furthermore, the theoretical surfaces on which the composite plate *will* be distorted in the first stage of buckling will be investigated in order to verify the foregoing formulas and their numerical results.

§ 9. Some Inspections on the Formulae to be valid for Single Plates.

When those two elementary plates have a common thickness and the common elastic constants, the composite plates as above specified becomes a single plate. If so, the general formulas shown in § 6 must coincide with those of single plates. Let us investigate this in the following: we can denote now

$$D_1 = D_2 = D, \qquad q_1 = q_2 = q,$$

therefore

$$P_r = \frac{pa_r^2}{D\pi^2}, \qquad Q_r = \frac{qb^2}{D\pi^2},$$

and then $\underset{r}{\lambda_1}$ and $\underset{r}{\lambda_2}$ become

$$\left.\begin{matrix} \underset{r}{\lambda_1} \\ \\ \underset{r}{\lambda_2} \end{matrix}\right\} = \frac{a_r}{b}\sqrt{\left\{m^2 - \frac{pb^2}{2D\pi^2}\right\} \pm \sqrt{\left\{m^2 - \frac{pb^2}{2D\pi^2}\right\}^2 - \left\{m^2 - \frac{qb^2}{D\pi^2}\right\} m^2}}\,.$$

From this, we have the following relations:

$$\mu_2 \underset{2}{\lambda_1} = \underset{1}{\lambda_1}, \qquad \mu_2 \underset{2}{\lambda_2} = \underset{1}{\lambda_2}, \qquad \mu_2' \varphi_2 = \mu_2^2 \varphi_2 = \varphi_1 \qquad (\text{where } \mu_2' = \mu_2^2)\,.$$

Accordingly, the expressions in (7) are simplified as follows:

$$\left.\begin{aligned}
\underset{1}{\tau_1} &= \underset{1}{\beta_1} - \mu_2' \underset{2}{\beta_1} = \underset{1}{\lambda_1^2} - \mu_2' \underset{2}{\lambda_1^2} = 0, \\
\underset{1}{\tau_1'} &= \underset{1}{\beta_1} - \mu_2' \underset{2}{\beta_2} = \underset{1}{\lambda_1^2} - \mu_2' \underset{2}{\lambda_2^2} = \underset{1}{\lambda_1^2} - \underset{1}{\lambda_2^2} = \varphi_1, \\
\underset{1}{\tau_2} &= \underset{1}{\beta_2} - \mu_2' \underset{2}{\beta_1} = \underset{1}{\lambda_2^2} - \mu_2' \underset{2}{\lambda_1^2} = \underset{1}{\lambda_2^2} - \underset{1}{\lambda_1^2} = -\underset{1}{\tau_1'} = -\varphi_1, \\
\underset{1}{\tau_2'} &= \underset{1}{\beta_2} - \mu_2' \underset{2}{\beta_2} = \underset{1}{\lambda_2^2} - \mu_2' \underset{2}{\lambda_2^2} = 0,
\end{aligned}\right\}$$

$$\left.\begin{aligned}
\underset{1}{\chi_1} &= \mu_2 \left\{ \underset{2}{\lambda_1} \underset{1}{\gamma_1} - \mu_2' \underset{1}{\lambda_1} \underset{2}{\gamma_1} \right\} = 0, \\
\underset{1}{\chi_1'} &= \mu_2 \left\{ \underset{2}{\lambda_2} \underset{1}{\gamma_1} - \mu_2' \underset{1}{\lambda_1} \underset{2}{\gamma_2} \right\} = \mu_2 \underset{1}{\lambda_1} \underset{2}{\lambda_2} \varphi_1 = \underset{1}{\lambda_1} \underset{1}{\lambda_2} \varphi_1, \\
\underset{1}{\chi_2} &= \mu_2 \left\{ \underset{2}{\lambda_1} \underset{1}{\gamma_2} - \mu_2' \underset{1}{\lambda_2} \underset{2}{\gamma_1} \right\} = -\mu_2 \underset{2}{\lambda_1} \underset{1}{\lambda_2} \varphi_1 = -\underset{1}{\lambda_1} \underset{1}{\lambda_2} \varphi_1 = -\underset{1}{\chi_1'}, \\
\underset{1}{\chi_2'} &= \mu_2 \left\{ \underset{2}{\lambda_2} \underset{1}{\gamma_2} - \mu_2' \underset{1}{\lambda_2} \underset{2}{\gamma_2} \right\} = 0,
\end{aligned}\right\}$$

and again those in (8) are written as

$$
\left.
\begin{aligned}
&\underset{1}{F_2}=\frac{\underset{1}{\tau_2}}{\varphi_1}\cosh\pi\underset{1}{\lambda_1}=-\cosh\pi\underset{1}{\lambda_1}, \quad \underset{1}{G_2}=\frac{\underset{1}{\chi_2}}{\underset{1}{\lambda_1}\,\underset{1}{\lambda_2}\,\varphi_1}\cosh\pi\underset{1}{\lambda_1}=-\cosh\pi\underset{1}{\lambda_1}, \quad \underset{1}{H_2}=0, \quad \underset{1}{I_2}=0,\\
&\underset{1}{F_2'}=\frac{\underset{1}{\tau_2}}{\varphi_1}\sinh\pi\underset{1}{\lambda_1}=-\sinh\pi\underset{1}{\lambda_1}, \quad \underset{1}{G_2'}=\frac{\underset{1}{\chi_2}}{\underset{1}{\lambda_1}\,\underset{1}{\lambda_2}\,\varphi_1}\sinh\pi\underset{1}{\lambda_1}=-\sinh\pi\underset{1}{\lambda_1}, \quad \underset{1}{H_2'}=0, \quad \underset{1}{I_2'}=0,\\
&\underset{1}{F_1}=0, \quad \underset{1}{G_1}=0, \quad \underset{1}{H_1}=\frac{\underset{1}{\tau_1'}}{\varphi_1}\cosh\pi\underset{1}{\lambda_2}=\cosh\pi\underset{1}{\lambda_2}, \quad \underset{1}{I_1}=\frac{\underset{1}{\chi_1'}}{\underset{1}{\lambda_1}\,\underset{1}{\lambda_2}\,\varphi_1}\cosh\pi\underset{1}{\lambda_2}=\cosh\pi\underset{1}{\lambda_2},\\
&\underset{1}{F_1'}=0, \quad \underset{1}{G_1'}=0. \quad \underset{1}{H_1'}=\frac{\underset{1}{\tau_1'}}{\varphi_1}\sinh\pi\underset{1}{\lambda_2}=\sinh\pi\underset{1}{\lambda_2}, \quad \underset{1}{I_1'}=\frac{\underset{1}{\chi_1'}}{\underset{1}{\lambda_1}\,\underset{1}{\lambda_2}\,\varphi_1}\sinh\pi\underset{1}{\lambda_2}=\sinh\pi\underset{1}{\lambda_2},
\end{aligned}
\right\}
$$

Further, if both the end sides are simply supported, at End side-2 where $\xi_2=1$ the plate has the conditions which, with reference to (9.a) in § 6, are written in the following expressions:

$$
\left.
\begin{aligned}
&\underset{2}{U_2}=\ \ \sinh\pi\underset{2}{\lambda_1}; &&\underset{2}{U_1}=\ 0;\\
&\underset{2}{U_2'}=-\cosh\pi\underset{2}{\lambda_1}; &&\underset{2}{U_1'}=\ 0;\\
&\underset{2}{V_2}=\ 0; &&\underset{2}{V_1}=\ \sinh\pi\underset{2}{\lambda_2};\\
&\underset{2}{V_2'}=\ 0; &&\underset{2}{V_1'}=-\cosh\pi\underset{2}{\lambda_2}.
\end{aligned}
\right\}
$$

Therefore, by using the recurrence formulas (**9**), we have

$$
\left.
\begin{aligned}
&\underset{1}{U_2}=-\underset{2}{U_2}\,\underset{1}{F_2}+\underset{2}{U_2'}\,\underset{1}{G_2'}=\ \cosh\pi\underset{1}{\lambda_1}\sinh\pi\underset{2}{\lambda_1}+\sinh\pi\underset{1}{\lambda_1}\cosh\pi\underset{2}{\lambda_1}; \quad \underset{1}{V_2}=0;\\
&\underset{1}{U_2'}=\ \underset{2}{U_2}\underset{1}{F_2'}-\underset{2}{U_2'}\,\underset{1}{G_2}=-\sinh\pi\underset{1}{\lambda_1}\sinh\pi\underset{2}{\lambda_1}-\cosh\pi\underset{1}{\lambda_1}\cosh\pi\underset{2}{\lambda_1}; \quad \underset{1}{V_2'}=0;\\
&\underset{1}{U_1}=0; \quad \underset{1}{V_1}=\ \underset{2}{V_1}\underset{1}{H_1}-\underset{2}{V_1'}\underset{1}{I_1'}=\ \sinh\pi\underset{2}{\lambda_2}\cosh\pi\underset{1}{\lambda_2}+\cosh\pi\underset{2}{\lambda_2}\sinh\pi\underset{1}{\lambda_2};\\
&\underset{1}{U_1'}=0; \quad \underset{1}{V_1'}=-\underset{2}{V_1}\underset{1}{H_1'}+\underset{2}{V_1'}\underset{1}{I_1}=-\sinh\pi\underset{2}{\lambda_2}\sinh\pi\underset{1}{\lambda_2}-\cosh\pi\underset{2}{\lambda_2}\cosh\pi\underset{1}{\lambda_2}.
\end{aligned}
\right\} \tag{9.1}
$$

On the other hand, the conditional equation for buckling in this case is given by (**9.d**), That is

$$
\underset{1}{U_1}\,\underset{1}{V_2}-\underset{1}{V_1}\,\underset{1}{U_2}=0.
$$

Substituting (9.1) in the above, we obtain

$$
(\sinh\pi\underset{2}{\lambda_2}\cosh\pi\underset{1}{\lambda_2}+\cosh\pi\underset{2}{\lambda_2}\sinh\pi\underset{1}{\lambda_2})(\cosh\pi\underset{1}{\lambda_1}\sinh\pi\underset{2}{\lambda_1}+\sinh\pi\underset{1}{\lambda_1}\cosh\pi\underset{2}{\lambda_1})=0.
$$

From this, we have

$$
\sinh\pi(\underset{1}{\lambda_2}+\underset{2}{\lambda_2})\sinh\pi(\underset{1}{\lambda_1}+\underset{2}{\lambda_1})=0. \tag{9.2}
$$

Now, observing that the following relations can be written:

$$\left.\begin{aligned}\lambda_1&=\underset{1}{\lambda_1}+\underset{2}{\lambda_1}\\ \lambda_2&=\underset{1}{\lambda_2}+\underset{2}{\lambda_2}\end{aligned}\right\}=\frac{a_1+a_2}{b}\sqrt{\left\{m^2-\frac{pb^2}{2D\pi^2}\right\}\pm\sqrt{\left\{m^2-\frac{pb^2}{2D\pi^2}\right\}^2-\left\{m^2-\frac{qb^2}{D\pi^2}\right\}m^2}}\,,$$

it follows that, by (9.2)

$$\sinh\pi\begin{cases}(\underset{1}{\lambda_1}+\underset{2}{\lambda_1})\\ (\underset{1}{\lambda_2}+\underset{2}{\lambda_2})\end{cases}=i\sin\pi\,\frac{a}{b}\sqrt{-\left\{m^2-\frac{pb^2}{2D\pi^2}\right\}\mp\sqrt{\left\{m^2-\frac{pb^2}{2D\pi^2}\right\}^2-\left\{m^2-\frac{qb^2}{D\pi^2}\right\}m^2}}=0\,,$$

where $$a=a_1+a_2\,.$$

Then, we have

$$\frac{a}{b}\sqrt{-\left\{m^2-\frac{pb^2}{2D\pi^2}\right\}\mp\sqrt{\left\{m^2-\frac{pb^2}{2D\pi^2}\right\}^2-\left\{m^2-\frac{qb^2}{D\pi^2}\right\}m^2}}=n\,,$$

$$n=0,\ 1,\ 2,\ 3,\ldots\ldots\ldots\ldots$$

Squaring both sides of the above expression and writing as $\frac{pa^2}{D\pi^2}=P$, $\frac{qb^2}{D\pi^2}=Q$, we finally obtain

$$\left(n^2+\frac{a^2}{b^2}m^2\right)^2-P\,n^2-Q\frac{a^4}{b^4}m^2=0\,,$$

or

$$P=\frac{\left(n^2+\dfrac{a^2}{b^2}m^2\right)^2}{n^2+\dfrac{qa^2}{pb^2}m^2} \tag{9.3}$$

This is nothing else but the formula of a single plate having simply supported edges.[12]

In the next place, let us observe the case where both of the end sides are fixed. From **(9.b)**, the conditions at End side-2 are expressed as follows:

$$\left.\begin{aligned}&\underset{2}{U_2}=1\,;\quad \underset{2}{U_1}=\frac{\underset{2}{\lambda_1}}{\underset{2}{\lambda_2}}\sinh\pi\underset{2}{\lambda_1}\sinh\pi\underset{2}{\lambda_2}-\cosh\pi\underset{2}{\lambda_1}\cosh\pi\underset{2}{\lambda_2}\,;\\ &\underset{2}{U_2'}=0\,;\quad \underset{2}{U_1'}=-\frac{\underset{2}{\lambda_1}}{\underset{2}{\lambda_2}}\sinh\pi\underset{2}{\lambda_1}\cosh\pi\underset{2}{\lambda_2}+\cosh\pi\underset{2}{\lambda_1}\sinh\pi\underset{2}{\lambda_2}\,;\end{aligned}\right\}$$

12) Compare with the expression (47) in the paper by S.Iguchi: "Die Eigenwertprobleme für die elastische rechteckige Platte," Memoirs of the Faculty of Eng., Hokkaido Imperial University, Vol. 4, No. 4, 1938 s.41.

$$\left.\begin{array}{l}\underset{2}{V_2}=0\,;\quad \underset{2}{V_1}=\dfrac{\underset{2}{\lambda_1}}{\underset{2}{\lambda_2}}\cosh\pi\underset{2}{\lambda_1}\sinh\pi\underset{2}{\lambda_2}-\sinh\pi\underset{2}{\lambda_1}\cosh\pi\underset{2}{\lambda_2}\,;\\[2ex] \underset{2}{V_2'}=1\,;\quad \underset{2}{V_1'}=-\dfrac{\underset{2}{\lambda_1}}{\underset{2}{\lambda_2}}\cosh\pi\underset{2}{\lambda_1}\cosh\pi\underset{2}{\lambda_2}+\sinh\pi\underset{2}{\lambda_1}\sinh\pi\underset{2}{\lambda_2}\,.\end{array}\right\}$$

Therefore, by using the formulas (**9**), we have

$$\left.\begin{array}{l}\underset{1}{U_2}=-\underset{1}{F_2}=\cosh\pi\underset{1}{\lambda_1}\,;\\ \underset{1}{U_2'}=\underset{1}{F_2'}=-\sinh\pi\underset{1}{\lambda_1}\,;\\ \underset{1}{U_1}=\underset{2}{U_1}\underset{1}{H_1}-\underset{2}{U_1'}\underset{1}{I_1'}=\dfrac{\underset{2}{\lambda_1}}{\underset{2}{\lambda_2}}\sinh\pi\underset{2}{\lambda_1}\sinh\pi\lambda_2-\cosh\pi\underset{2}{\lambda_1}\cosh\pi\lambda_2\,;\\ \underset{1}{U_1'}=-\underset{2}{U_1}\underset{1}{H_1'}+\underset{2}{U_1'}\underset{1}{I_1}=-\dfrac{\underset{2}{\lambda_1}}{\underset{2}{\lambda_2}}\sinh\pi\underset{2}{\lambda_1}\cosh\pi\lambda_2+\cosh\pi\underset{2}{\lambda_1}\sinh\pi\lambda_2\,;\\ \underset{1}{V_2}=\underset{1}{G_2'}=-\sinh\pi\underset{1}{\lambda_1}\,;\\ \underset{1}{V_2'}=-\underset{1}{G_2}=\cosh\pi\underset{1}{\lambda_1}\,;\\ \underset{1}{V_1}=\underset{2}{V_1}\underset{1}{H_1}-\underset{2}{V_1'}\underset{1}{I_1'}=\dfrac{\underset{2}{\lambda_1}}{\underset{2}{\lambda_2}}\cosh\pi\underset{2}{\lambda_1}\sinh\pi\lambda_2-\sinh\pi\underset{2}{\lambda_1}\cosh\pi\lambda_2\,;\\ \underset{1}{V_1'}=-\underset{2}{V_1}\underset{1}{H_1'}+\underset{2}{V_1'}\underset{1}{I_1}=-\dfrac{\underset{2}{\lambda_1}}{\underset{2}{\lambda_2}}\cosh\pi\underset{2}{\lambda_1}\cosh\pi\lambda_2+\sinh\pi\underset{2}{\lambda_1}\sinh\pi\lambda_2\,.\end{array}\right\}\quad(9.4)$$

Now, the conditional equation for buckling in the present case is given by (**9.e**), that is

$$(\underset{1}{U_1}+\underset{1}{U_2})\,(\underset{1}{V_1'}\underset{1}{\lambda_2}+\underset{1}{V_2'}\underset{1}{\lambda_1})-(\underset{1}{V_1}+\underset{1}{V_2})\,(\underset{1}{U_1'}\underset{1}{\lambda_2}+\underset{1}{U_2'}\underset{1}{\lambda_1})=0\text{-}$$

Substituting (9.4) in this

$$\underset{2}{\lambda_2}\sinh\pi\lambda_1\sinh\pi\lambda_2\left[(\underset{1}{\lambda_1}\underset{2}{\lambda_1}+\underset{1}{\lambda_2}\underset{2}{\lambda_2})-(\underset{2}{\lambda_1}\underset{1}{\lambda_2}+\underset{1}{\lambda_1}\underset{2}{\lambda_2})\,(\coth\pi\lambda_1\coth\pi\lambda_2-\operatorname{cosech}\pi\lambda_1\operatorname{cosech}\pi\lambda_2)\right]=0,$$

where $\lambda_1=\underset{1}{\lambda_1}+\underset{2}{\lambda_1}\,,\qquad \lambda_2=\underset{1}{\lambda_2}+\underset{2}{\lambda_2}\,.$

Equating the factor outside the square brackets in the above to zero, the formula of such a case that both the end sides are simply supported is obtained and then it results that the plate in this case remains plane. But, since it is trivial in this case, the expressions in the square brackets must be equated to zero.

In the next place, multiplying both the sides of thus obtained equation by $2+\dfrac{a_1}{a_2}+\dfrac{a_2}{a_1}$ and observing that the following relations, in the present case, hold:

$$\underset{1}{\lambda_1}\,\underset{2}{\lambda_1}\frac{a_1}{a_2}=\underset{1}{\lambda_1^2}\,;\quad \underset{1}{\lambda_2}\,\underset{2}{\lambda_2}\frac{a_1}{a_2}=\underset{1}{\lambda_2^2}\,;\quad \underset{1}{\lambda_1}\,\underset{2}{\lambda_2}\frac{a_1}{a_2}=\underset{1}{\lambda_1}\,\underset{1}{\lambda_2}\,;$$

$$\underset{1}{\lambda_1}\,\underset{2}{\lambda_1}\frac{a_2}{a_1}=\underset{2}{\lambda_1^2}\,;\quad \underset{1}{\lambda_2}\,\underset{2}{\lambda_2}\frac{a_2}{a_1}=\underset{2}{\lambda_2^2}\,;\quad \underset{2}{\lambda_1}\,\underset{1}{\lambda_2}\frac{a_2}{a_1}=\underset{2}{\lambda_1}\,\underset{2}{\lambda_2}\,,$$

we easily find

$$\lambda_1^2+\lambda_2^2-2\lambda_1\lambda_2(\coth\pi\lambda_1\coth\pi\lambda_2-\operatorname{cosech}\pi\lambda_1\operatorname{cosech}\pi\lambda_2)=0\,.$$

The above equation is transformed moreover as follows:

$$\begin{aligned}&\lambda_1^2+\lambda_2^2-2\lambda_1\lambda_2(\coth\pi\lambda_1\coth\pi\lambda_2-\operatorname{cosech}\pi\lambda_1\operatorname{cosech}\pi\lambda_2)\\&=(\lambda_1\coth\pi\lambda_1-\lambda_2\coth\pi\lambda_2)^2-(\lambda_1\operatorname{cosech}\pi\lambda_1-\lambda_2\operatorname{cosech}\pi\lambda_2)^2\\&=\left\{\lambda_1(\coth\pi\lambda_1-\operatorname{cosech}\pi\lambda_1)-\lambda_2(\coth\pi\lambda_2-\operatorname{cosech}\pi\lambda_2)\right\}\left\{\lambda_1(\coth\pi\lambda_1+\operatorname{cosech}\pi\lambda_1)-\lambda_2(\coth\pi\lambda_2+\operatorname{cosech}\pi\lambda_2)\right\}\\&=\left(\lambda_1\tanh\frac{\pi}{2}\lambda_1-\lambda_2\tanh\frac{\pi}{2}\lambda_2\right)\left(\lambda_1\coth\frac{\pi}{2}\lambda_1-\lambda_2\coth\frac{\pi}{2}\lambda_2\right)=0\,.\end{aligned}$$

Accordingly, we get

$$\lambda_1\tanh\frac{\pi}{2}\lambda_1-\lambda_2\tanh\frac{\pi}{2}\lambda_2=0\,;\qquad \lambda_1\coth\frac{\pi}{2}\lambda_1-\lambda_2\coth\frac{\pi}{2}\lambda_2=0\,.\quad (9.5)$$

The first equation corresponds to symmetric distortion with respect to the center line of the plate parallel to y-axis, and the second corresponds to reversely symmetric distortion[13].

Thus, it may be understood that the general formulas can include a single plate as a special case.

In the following examples, the composite plates shall always be specified as follows:

1) The composite plate is composed of two elementary plates whose lengths are taken as $a_1=a_2=\dfrac{b}{2}$, and therefore that becomes square.
2) Each of the elementary plates has the common constants of elasticity and the *Poisson's ratio* $\nu_r=0.3$.
3) And again the thickness ratio of both the elementary plates is assumed as $h_1:h_2=1:1.5$.

(22)

By the way, from the above specifications 2) and 3), the following relation results:

13) See paper by S. Iguchi, p. 44, *loc. cit.* p. 209.

$$D_1 : D_2 = 1 : (1.5)^3. \tag{23}$$

The above relation can also be obtained when $h_1 = h_2$, $E_1 = E_2/(1.5)^3$ and $\nu_1 = \nu_2$. Since in the foregoing discussions the constants h_r and E_r are always contained in only the flexural rigidity D_r, then, it can be seen that, as far as the relation (**23**) holds, the manner in which the constant h_r or E_r of each elementary plate varies from those of the others is out of the question for calculation. For instance, in the case when only the load p is acting, the same value of P should be obtained in spite of h_r or E_r, as far as (**23**) holds. But when the load q_r acts in the same time, the next condition due to § 3 must be considered together.

$$\frac{q_1}{E_1 \, h_1} = \frac{q_2}{E_2 \, h_2}. \tag{9.6}$$

It is obvious that the above facts are applicable to such cases that the number of the elementary plates is more than two.

§ 10. Case when Both End Sides are Simply Supported.

As in the previous section, the conditions at End side-2 are expressed as follows:

$$\left.\begin{aligned}
&\underset{2}{U_2} = \sinh\pi\underset{2}{\lambda_1}\,; && \underset{2}{U_1} = 0\,;\\
&\underset{2}{U_2'} = -\cosh\pi\underset{2}{\lambda_1}\,; && \underset{2}{U_1'} = 0\,;\\
&\underset{2}{V_2} = 0\,; && \underset{2}{V_1} = \sinh\pi\underset{2}{\lambda_2}\,;\\
&\underset{2}{V_2'} = 0\,; && \underset{2}{V_1'} = -\cosh\pi\underset{2}{\lambda_2}\,.
\end{aligned}\right\} \tag{10.1}$$

And from the recurrence formulas (**9**), we have

$$\left.\begin{aligned}
\underset{1}{U_2} &= -\underset{2}{U_2}\,\underset{1}{F_2} + \underset{2}{U_2'}\,\underset{1}{G_2'} = -\underset{1}{F_2}\sinh\pi\underset{2}{\lambda_1} - \underset{1}{G_2'}\cosh\pi\underset{2}{\lambda_1}\,;\\
\underset{1}{U_1} &= \underset{2}{U_2}\,\underset{1}{F_1} - \underset{2}{U_2'}\,\underset{1}{G_1'} = \underset{1}{F_1}\sinh\pi\underset{2}{\lambda_1} + \underset{1}{G_1'}\cosh\pi\underset{2}{\lambda_1}\,;\\
\underset{1}{V_2} &= -\underset{2}{V_1}\,\underset{1}{H_2} + \underset{2}{V_1'}\,\underset{1}{I_2'} = -\underset{1}{H_2}\sinh\pi\underset{2}{\lambda_2} - \underset{1}{I_2'}\cosh\pi\underset{2}{\lambda_2}\,;\\
\underset{1}{V_1} &= \underset{2}{V_1}\,\underset{1}{H_1} - \underset{2}{V_1'}\,\underset{1}{I_1'} = \underset{1}{H_1}\sinh\pi\underset{2}{\lambda_2} + \underset{1}{I_1'}\cosh\pi\underset{2}{\lambda_2}\,.
\end{aligned}\right\} \tag{10.2}$$

Referring to (**9.d**), the conditional equation for buckling is

$$\underset{1}{U_1}\,\underset{1}{V_2} - \underset{1}{U_2}\,\underset{1}{V_1} = 0\,. \tag{10.3}$$

(1) *The case where only the load q_r is acting.*—In this case, the following relation must be taken into account as described before, that is

$$\frac{q_1}{E_1 \, h_1} = \frac{q_2}{E_2 \, h_2},$$

then, observing $E_1 = E_2$, we have

$$\frac{q_1}{q_2} = \frac{h_1}{h_2} = \frac{1}{1.5} \tag{10.4}$$

and again by (**23**)

$$\frac{D_2}{D_1} = (1.5)^3 = 3.375 \tag{10.5}$$

Therefore, with $a_1 = a_2$, the following expressions can be obtained:

$$\mu_2 = 1, \qquad \mu_2' = \mu_2'' = 3.375 \tag{10.6}$$

Using now the foregoing (10.4) and (10.5), the expressions Q_1 and Q_2 are related to each other as follows:

$$Q_2 = \frac{q_2}{D_2}\frac{b^2}{\pi^2} = \frac{1}{(1.5)^2}\frac{q_1}{D_1}\frac{b^2}{\pi^2} = \frac{Q_1}{(1.5)^2} = \frac{Q_1}{2.25}\,. \tag{10.7}$$

Let us denote by q' the critical load under which buckling would occur, if the thickness were h_1 all over the composite plate, in other words, the composite plate were the single plate having the thickness h_1.

In the next place, denote by q''[14] the critical load under which buckling would occur, if the thickness were h_2 all over the composite plate. Then, we can easily suppose that the critical loads q_1 and q_2 in the present case where the composite plate has two kinds of thickness lie within the following limits:

$$q' < q_1 < q'', \qquad q' < q_2 < q''.$$

On the other hand, it is well known that the critical value Q of a simply supported square plate is 4[15]. Therefore, we can write

$$\frac{q'\,b^2}{D_1\,\pi^2} = 4\,; \qquad \frac{q''\,b^2}{D_2\,\pi^2} = 4\,.$$

From these

$$Q_1 = \frac{q_1\,b^2}{D_1\,\pi^2} > \frac{q'\,b^2}{D_1\,\pi^2} = 4, \qquad Q_2 = \frac{q_2\,b^2}{D_2\,\pi^2} < \frac{q''\,b^2}{D_2\,\pi^2} = 4.$$

Considering (10.7), we can claim $\dfrac{Q_1}{(1.5)^2} = Q_2 < 4$ and then $Q_1 < 9$. And again

$$Q_2\,(1.5)^2 = Q_1 > 4, \qquad \text{then} \qquad Q_2 > 1.77777$$

Accordingly, the following inequalties are concluded:

---

14) q' and q'' are the load intensities per unit length distributed along the outside edges like q_1 and q_2.

15) Putting $q=0$, and $m=n=1$ in (9.3), we obtain $P=4$. Since the plate is square, this value is considered as that of Q in this case.

$$4<Q_1<9, \qquad 1.778<Q_2<4. \tag{10.8}$$

The above described method for estimation of the limits which the least value of Q must be searched within, can be also applicable to the case when the number of the elementary plates is more than two. That is to say, those limits can be estimated as before, by referring to the two single plates of the same size as the composite plate:—one having the minimum thickness and the other having the maximum one among those of the elementary plates. In more general cases when each of the elementary plates has h_r and E_r different from the others, the same consideration as above is possible concerning the flexural rigidities.

Now, in this example, Q_1 (or Q_2) which satisfies the formula (10.3) must be searched within the limits (10.8). Yet, since Eq. (10.3) is a transcendental equation of high order, it is difficult to solve that in analytic manner, and then the graphical method must be applied.

Let us trace first, within the foregoing limits, the curve representing the value of the left side of Eq. (10.3) as the function of Q_1 (or Q_2). Then we may find the desired value of Q_1 (or Q_2) by observing the position at which that curve intersects with zero line. Though the above described procedures are able to be performed mechanically, it may be desired to transform the computation formulas as simply as possible in order to save labour for calculation and prevent accumulations of the various kinds of errors which are apt to be caused by many repeatings of calculation (such as those by counting 5 and higher fractions as units and disregarding the rest).

For references, this will be shown below. Considering that the critical load in this case corresponds to $m=1$ and also Q_1, $Q_2>1$ from (10.8), we have

$$\left.\begin{aligned}
&\underset{1}{\lambda_1}=\frac{1}{2}\sqrt{\sqrt{Q_1}+1}, && \underset{2}{\lambda_1}=\frac{1}{2}\sqrt{\sqrt{Q_2}+1},\\
&\underset{1}{\lambda_2}=i\underset{1}{\lambda_2}, \quad \underset{1}{\lambda_2}=\frac{1}{2}\sqrt{\sqrt{Q_1}-1}; && \underset{2}{\lambda_2}=i\underset{2}{\lambda_2}, \quad \underset{2}{\lambda_2}=\sqrt{\sqrt{Q_2}-1},\\
&\varphi_1=\underset{1}{\lambda_1^2}-\underset{1}{\lambda_2^2}=\sqrt{Q_1}\,/2, && \underset{1}{\lambda_1}\underset{1}{\lambda_2}\varphi_1=i\left(\frac{1}{2}\right)^3\sqrt{Q_1(Q_1-1)}.
\end{aligned}\right\} \tag{10.9}$$[16]

And, knowing that we can write as follows:

---

16) Referring to Table 1, we can easily judge which of $\underset{r}{\lambda_1}$ and $\underset{r}{\lambda_2}$ take real value, corresponding to the values of P_r or Q_r. Namely, when P_r or Q_r lies within the limit of Case-1 in Table 1, $\underset{r}{\lambda_1}$ and $\underset{r}{\lambda_2}$ become the conjugate complex numbers. P_r or Q_r lying within the limit of Case-2, $\underset{r}{\lambda_1}$ is real and $\underset{r}{\lambda_2}$ imaginary. And, at Case-3, both of them are real, and on the other hand, at Case-4, both are imaginary.

$$\left.\begin{aligned}
&\underset{1}{\beta_1}=\left(\frac{1}{2}\right)^2(0.7+\sqrt{Q_1}), && \underset{1}{\gamma_1}=-\underset{1}{\lambda_1}\,\underset{1}{\beta_2},\\
&\underset{1}{\beta_2}=\left(\frac{1}{2}\right)^2(0.7-\sqrt{Q_1}), && \underset{1}{\gamma_2}=-\underset{1}{\lambda_2}\,\underset{1}{\beta_1}=-i\underset{1}{\lambda_2}\,\underset{1}{\beta_1},\\
&\underset{2}{\beta_1}=\left(\frac{1}{2}\right)^2(0.7+\sqrt{Q_2}), && \underset{2}{\gamma_1}=-\underset{2}{\lambda_1}\,\underset{2}{\beta_2},\\
&\underset{2}{\beta_2}=\left(\frac{1}{2}\right)^2(0.7-\sqrt{Q_2}), && \underset{2}{\gamma_2}=-\underset{2}{\lambda_2}\,\underset{2}{\beta_1}=-i\underset{2}{\lambda_2}\,\underset{2}{\beta_1},
\end{aligned}\right\}$$

the next expressions are obtained:

$$\left.\begin{aligned}
&\underset{1}{\tau_1}=-\left(\frac{1}{2}\right)^2(1.6625+1.25\sqrt{Q_1}), && \underset{1}{\chi_1}=-\underset{1}{\lambda_1}\,\underset{2}{\lambda_1}\,\underset{1}{\tau_2'},\\
&\underset{1}{\tau_1'}=-\left(\frac{1}{2}\right)^2(1.6625-3.25\sqrt{Q_1}), && \underset{1}{\chi_1'}=-\underset{1}{\lambda_1}\,\underset{2}{\lambda_2}\,\underset{1}{\tau_2}=-i\underset{1}{\lambda_1}\,\underset{2}{\lambda_2}\,\underset{1}{\tau_2},\\
&\underset{1}{\tau_2}=-\left(\frac{1}{2}\right)^2(1.6625+3.25\sqrt{Q_1}), && \underset{1}{\chi_2}=-\underset{2}{\lambda_1}\,\underset{1}{\lambda_2}\,\underset{1}{\tau_1'}=-i\underset{2}{\lambda_1}\,\underset{1}{\lambda_2}\,\underset{1}{\tau_1'},\\
&\underset{1}{\tau_2'}=-\left(\frac{1}{2}\right)^2(1.6625-1.25\sqrt{Q_1}), && \underset{1}{\chi_2'}=-\underset{1}{\lambda_2}\,\underset{2}{\lambda_2}\,\underset{1}{\tau_1}=\ \underset{1}{\lambda_2}\,\underset{2}{\lambda_2}\,\underset{1}{\tau_1},
\end{aligned}\right\}\quad(10.10)$$

Hence, the factors reguired for (10.2) are expressed in the following forms by the reference of (8) [§ 6].

$$\left.\begin{aligned}
&\underset{1}{F_2}=\frac{\underset{1}{\tau_2}}{\varphi_1}\cosh\pi\underset{1}{\lambda_1}, && \underset{1}{H_2}=\frac{\underset{1}{\tau_2'}}{\varphi_1}\cosh\pi\underset{1}{\lambda_1},\\
&\underset{1}{F_1}=\frac{\underset{1}{\tau_1}}{\varphi_1}\cosh\pi\underset{1}{\lambda_2}=\frac{\underset{1}{\tau_1}}{\varphi_1}\cos\pi\underset{1}{\lambda_2}, && \underset{1}{H_1}=\frac{\underset{1}{\tau_1'}}{\varphi_1}\cosh\pi\underset{1}{\lambda_2}=\frac{\underset{1}{\tau_1'}}{\varphi_1}\cos\pi\underset{1}{\lambda_2},\\
&\underset{1}{G_2'}=-\frac{\underset{2}{\lambda_1}\underset{1}{\tau_1'}}{\underset{1}{\lambda_1}\varphi_1}\sinh\pi\underset{1}{\lambda_1}, && \underset{1}{I_2'}=-\frac{\underset{2}{\lambda_2}\underset{1}{\tau_1}}{\underset{1}{\lambda_1}\varphi_1}\sinh\pi\underset{1}{\lambda_1}=-i\frac{\underset{2}{\lambda_2}\underset{1}{\tau_1}}{\underset{1}{\lambda_1}\varphi_1}\sinh\pi\underset{1}{\lambda_1},\\
&\underset{1}{G_1'}=-\frac{\underset{2}{\lambda_1}\underset{1}{\tau_2'}}{\underset{1}{\lambda_2}\varphi_1}\sinh\pi\underset{1}{\lambda_2}=-\frac{\underset{2}{\lambda_1}\underset{1}{\tau_2'}}{\underset{1}{\lambda_2}\varphi_1}\sin\pi\underset{1}{\lambda_2}, && \underset{1}{I_1'}=-\frac{\underset{2}{\lambda_2}\underset{1}{\tau_2}}{\underset{1}{\lambda_2}\varphi_1}\sinh\pi\underset{1}{\lambda_2}=-i\frac{\underset{2}{\lambda_2}\underset{1}{\tau_2}}{\underset{1}{\lambda_2}\varphi_1}\sin\pi\underset{1}{\lambda_2}.
\end{aligned}\right\}\quad(10\cdot11)$$

Substituting the above in (10.2) and arranging them, we have

$$\left.\begin{aligned}
&\underset{1}{U_2}=-\frac{\sinh\pi\underset{1}{\lambda_1}\,\sinh\pi\underset{2}{\lambda_1}}{\underset{1}{\lambda_1}\,\varphi_1}R; && \underset{1}{V_2}=-i\frac{\sin\pi\underset{2}{\lambda_2}\,\sinh\pi\underset{1}{\lambda_1}}{\underset{1}{\lambda_1}\,\varphi_1}Z;\\
&\underset{1}{U_1}=\frac{\sinh\pi\underset{2}{\lambda_1}\,\sin\pi\underset{1}{\lambda_2}}{\underset{1}{\lambda_2}\,\varphi_1}R'; && \underset{1}{V_1}=i\frac{\sin\pi\underset{1}{\lambda_2}\,\sin\pi\underset{2}{\lambda_2}}{\underset{1}{\lambda_2}\,\varphi_1}Z',
\end{aligned}\right\}\quad(10.12)$$

where

$$
\left.\begin{aligned}
R &= \underset{1}{\tau_2}\,\underset{1}{\lambda_1}\coth\pi\underset{1}{\lambda_1} - \underset{1}{\tau_1'}\,\underset{2}{\lambda_1}\coth\pi\underset{2}{\lambda_1}\,,\\
R' &= \underset{1}{\tau_1}\,\underset{1}{\lambda_2}\cot\pi\underset{1}{\lambda_2} - \underset{1}{\tau_2'}\,\underset{2}{\lambda_1}\coth\pi\underset{2}{\lambda_1}\,,\\
Z &= \underset{1}{\tau_2'}\,\underset{1}{\lambda_1}\coth\pi\underset{1}{\lambda_1} - \underset{1}{\tau_1}\,\underset{2}{\lambda_2}\cot\pi\underset{2}{\lambda_2}\,,\\
Z' &= \underset{1}{\tau_1'}\,\underset{1}{\lambda_2}\cot\pi\underset{1}{\lambda_2} - \underset{1}{\tau_2}\,\underset{2}{\lambda_2}\cot\pi\underset{2}{\lambda_2}\,.
\end{aligned}\right\}
$$

Therefore, we see that the conditional equation can be written by (10.3) as

$$RZ' - R'Z = 0\,.$$

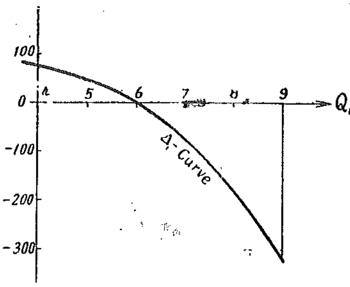


Fig. 4.

Plotting the left side of the above equation as a function of Q_1 in the interval $4<Q_1<9$ as described before, its general form of curve appears as shown in Fig.4 From this, we can know that the curve intersects with the zero line near the point $Q_1=6$. Then, investigating the curve in more detail near that point, we obtain the numerical results in Table 6. Accordingly, by linear interpolation [Fig. 5.], the desired least root Q_1 is determined as follows:

$$Q_1 \doteq 5.988$$

Table 6.

Q_1	5.95	6.00	6.05	6.10
Δ_1	2.324	-0.724	-0.383	-6.955

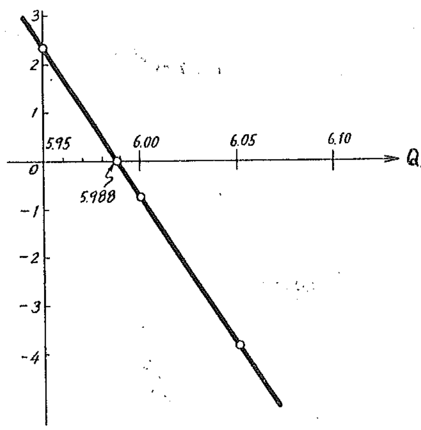


Fig. 5.

Using (10.7), the value of Q_2 is obtained also as below:

$$Q_2 = 2.661$$

Thus, the least characterisitic value Q_1 (or Q_2), corresponding to the first mode of buckling, has been found and then the critical load q_r or the critical stress is easily determined, if the values of D_r, b, and a_r are actually given.

In the next place, let us observe the configuration of the composite plate buckled in order to gain admission by our common ideas. Referring to the column "The limits of P_r or Q_r" in Table 1 and taking into account $m=1$, $P_r=0$, we have

$$1>(1-Q_1)\,, \qquad 1>(1-Q_2)\,,$$

or

$$1<Q_1\,, \qquad 1<Q_2.$$

From these, it can be seen that the configuration of the buckled plate should be represented by the expression (3) composed of the terms which take the forms of

Case-2 in Table 1. As explained in [§ 6-a)-ii)-(1)], the conditions at $\xi_2=1$ are

$$\underset{2}{K_m}\cosh\pi\underset{2}{\lambda_1}+\underset{2}{L_m}\sinh\pi\underset{2}{\lambda_1}=0\,;$$

$$\underset{2}{M_m}\cos\pi\underset{2}{\lambda_2}+\underset{2}{N_m}\sin\pi\underset{2}{\lambda_2}=0\,.$$

The above equations are satisfied by putting as follows:

$$\underset{2}{K_m}=A_m\sinh\pi\underset{2}{\lambda_1},\quad \underset{2}{L_m}=-A_m\cosh\pi\underset{2}{\lambda_1},\quad \underset{2}{M_m}=B_m\sin\pi\underset{2}{\lambda_2},\quad \underset{2}{N_m}=-B_m\cos\pi\underset{2}{\lambda_2}.$$

On the other hand, from [§ 6-a)-iii)-(1)]

$$\underset{1}{K_m}=\underset{1}{M_m}=0\,.$$

Finally, by referring to Table 5, we can write the conditions of joining in the following forms[17]:

$$\left.\begin{aligned}
\underset{1}{L_m}\sinh\pi\underset{1}{\lambda_1}+\underset{1}{N_m}\sin\pi\underset{1}{\lambda_2}-A_m\sinh\pi\underset{2}{\lambda_1}-B_m\sin\pi\underset{2}{\lambda_2}&=0\,;\\
\underset{1}{L_m}\underset{1}{\lambda_1}\cosh\pi\underset{1}{\lambda_1}+\underset{1}{N_m}\underset{1}{\lambda_2}\cos\pi\underset{1}{\lambda_2}+A_m\underset{2}{\lambda_1}\cosh\pi\underset{2}{\lambda_1}+B_m\underset{2}{\lambda_2}\cos\pi\underset{2}{\lambda_2}&=0\,;\\
\underset{1}{L_m}\underset{1}{\beta_1}\sinh\pi\underset{1}{\lambda_1}-\underset{1}{N_m}\underset{1}{\beta_2}\sin\pi\underset{1}{\lambda_2}-A_m\,3.375\,\underset{2}{\beta_1}\sinh\pi\underset{2}{\lambda_1}+B_m\,3.375\,\underset{2}{\beta_2}\sin\pi\underset{2}{\lambda_2}&=0\,;\\
\underset{1}{L_m}\underset{1}{\gamma_1}\cosh\pi\underset{1}{\lambda_1}-\underset{1}{N_m}\underset{1}{\gamma_2}\cos\pi\underset{1}{\lambda_2}+A_m\,3.375\,\underset{2}{\gamma_1}\cosh\pi\underset{2}{\lambda_1}-B_m\,3.375\,\underset{2}{\gamma_2}\cos\pi\underset{2}{\lambda_2}&=0\,.
\end{aligned}\right\}\quad(10.13)$$

Now, since the formerly obtained values of Q_1 and Q_2 satisfy $\Delta_1=0$, then $\Delta_m=0$ can not generally hold for such values of them when $m\neq1$. From this, in the case $m\neq1$, all of the constant coefficients in (10.13) must be zero and then the m-th term ($m\neq1$) of the series (3) identically vanishes. In this way, the configuration of the buckled composite plate is expressed as follows: In the portion of the 1-st elementary plate

$$w_1=\left\{\underset{1}{L_1}\sinh\pi\underset{1}{\lambda_1}\xi_1+\underset{1}{N_1}\sin\pi\underset{1}{\lambda_2}\xi_1\right\}\sin\pi\eta_1\,.$$

In the portion of the 2-nd elementary plate

$$\begin{aligned}
w_2&=\left\{A_1\sinh\pi\underset{2}{\lambda_1}\cosh\pi\underset{2}{\lambda_1}\xi_2-A_1\cosh\pi\underset{2}{\lambda_1}\sinh\pi\underset{2}{\lambda_1}\xi_2+B_1\sin\pi\underset{2}{\lambda_2}\cos\pi\underset{2}{\lambda_2}\xi_2-B_1\cos\pi\underset{2}{\lambda_2}\sin\pi\underset{2}{\lambda_2}\xi_2\right\}\sin\pi\eta_2\\
&=\left\{A_1\sinh\pi\underset{2}{\lambda_1}(1-\xi_2)+B_1\sin\pi\underset{2}{\lambda_2}(1-\xi_2)\right\}\sin\pi\eta_2\,.
\end{aligned}$$

The constant coefficients in the above are determined by solving the equations which are obtained from (10.13) by putting $m=1$. But, (10.13) is homogenious with respect to these coefficients and only the ratios between them, therefore, are obtained, and the absolute values of w_1 and w_2 must remain indefinite. It seems to be an usual

17) λ_r, β_r and γ_r are expressed here in such forms as are shown in the remarks column in Table 1 and Table 3.

thing in linear buckling theory.[18)]

Now let $\frac{\underset{1}{N_1}}{\underset{1}{L_1}}=\alpha_1,\ \frac{\underset{1}{A_1}}{\underset{1}{L_1}}=\alpha_2,\ \frac{\underset{1}{B_1}}{\underset{1}{L_1}}=\alpha_3$, and substitute the numerical data into each term of (10.13). Then we obtain the equations

$$
\begin{aligned}
0.949\,63\,\alpha_1 - 6.351\,71\,\alpha_2 - 0.948\,37\,\alpha_3 &= -\ \ 9.209\,96\\
0.188\,48\,\alpha_1 - 5.215\,07\,\alpha_2 - 0.126\,00\,\alpha_3 &= \ \ \ \ 8.599\,94\\
1.659\,04\,\alpha_1 + 49.975\,24\,\alpha_2 - 2.980\,69\,\alpha_3 &= \ \ \ \ 28.984\,11\\
1.186\,35\,\alpha_1 + 32.781\,84\,\alpha_2 - 1.982\,81\,\alpha_3 &= -\ 30.048\,81
\end{aligned}
$$

Though the coefficients α_1, α_2, α_3 must be obtained from any three equations in the above, it is general that these four equations can not simultaneously hold without somewhat of errors because the obtained values of Q_1 and Q_2 contain more or less errors caused by the graphical method. Therefore, considering the above simultaneous equations as a kind of observation equations and determining α_1, α_2 and α_3 by the method of least squares, we may obtain the more reasonable results.[19)]. Hence, by making the normal equations from the foregoing and solving them, the most probable values of α are determined as follows:

$$\alpha_1 = -\ 661.645\,50, \qquad \alpha_2 = -\ 11.710\,49, \qquad \alpha_3 = -\ 574.335\,23$$

Using the above obtained values, the deflection surfaces (the shape functions) are finally expressed as

$$
\begin{aligned}
w_1 &= \underset{1}{L_1}\, w_1' ,\\
w_1' &= -\left\{\sinh \pi \underset{1}{\lambda_1}\, \xi_1 - 661.645\,50\ \sin \pi \underset{1}{\lambda_2}\, \xi_1\right\} \sin \pi \eta_1 ;\\
w_2 &= \underset{1}{L_1}\, w_2' ,\\
w_2' &= \left\{11.710\,49 \sinh \pi \underset{2}{\lambda_1}\,(1-\xi_2) + 574.335\,23 \sin \pi \underset{2}{\lambda_2}\,(1-\xi_2)\right\} \sin \pi \eta_2 .
\end{aligned}
$$

From these, we can draw their configurations, considering $\underset{1}{L_1}$ as a constant of ratio (in strictly speaking, the magnitude of $\underset{1}{L_1}$ may be considered such as not zero but infinitesimally small). The numerical results by the above expressions are given in Table 7 and their graphic representations in Fig 6. From this, we can recognize such a proper result that a larger deflection occurs in the thinner portion, *i. e.*, in the 1-st elementary plate.

18) This fact is discussed in S. Timoshenko, "Theory of Elastic Stability," 1936, p. 74.

19) See S. Iguchi, "Die Eigenschwingungen und Klangfiguren des vierseitig freien Platte," Memoirs of the Faculty of Eng., H. I. U., Vol. 6, No. 5, p.16.

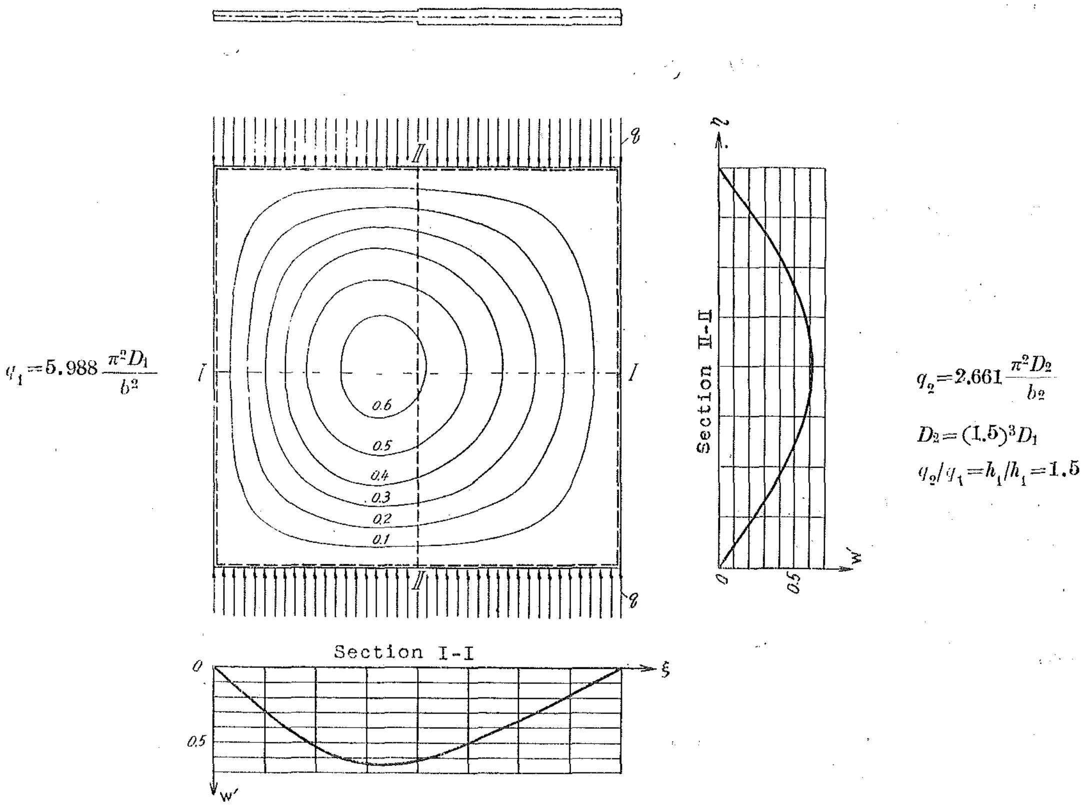


Fig. 6.—Configuration of buckled composite plate.

Table 7.[20]

ξ_1, ξ_2 / η	$w_1'/1000$				$w_2'/1000$			
	0.25	0.50	0.75	1.00	0	0.25	0.50	0.75
0.125	0.1149	0.2042	0.2486	0.2369	0.2370	0.1914	0.1358	0.0705
0.250	0.2123	0.3773	0.4594	0.4378	0.4379	0.3537	0.2509	0.1303
0.375	0.2774	0.4930	0.6003	0.5720	0.5722	0.4621	0.3279	0.1703
0.500	0.3002	0.5336	0.6497	0.6191	0.6193	0.5002	0.3549	0.1843

(2) *The case where only the load p is acting.*—Denote by p' what would be the critical load if h_1 were a thickness all over the composite plate, and again, denote it by p'', if h_2 were so. Then we can suppose that the critical load in this case lies between those of the extreme cases as above considered. That is to say

$$p' < p < p''.$$

On the other hand, $\frac{p' a^2}{D_1 \pi^2} = 4\,;\quad \frac{p'' a^2}{D_2 \pi^2} = 4\,,$ where $a_1 + a_2 = a = b\,.$

20) $w_1' = w_2'$ is necessary at $\xi_1 = 1$ and $\xi_2 = 0$, and then their numerical results are given in Table 7 in order to show the precision of computations.

Then,

$$4P_1=4\,\frac{p\,a_1^2}{D_1\pi^2}=\frac{p\,a^2}{D_1\pi^2}>\frac{p'\,a^2}{D_1\pi^2}=4\,,$$

$$4P_2=4\,\frac{p\,a_2^2}{D_2\pi^2}=\frac{p\,a^2}{D_2\pi^2}<\frac{p''\,a^2}{D_2\pi^2}=4\,.$$

From these, we have

$$P_1>1\,,\qquad P_2<1\,.$$

And observing

$$\frac{P_1}{P_2}=\frac{p\,a_1^2}{D_1\pi^2}\Big/\frac{p\,a_2^2}{D_2\pi^2}=\frac{D_2}{D_1}=(1.5)^3=3.375,$$

we have

$$P_1=(1.5)^3\,P_2=3.375\,P_2. \tag{10.14}$$

Accordingly, the following inequalities with respect to the least of the characteristic values are obtained:

$$1<P_1<3.375,\qquad 0.296<P_2<1\,. \tag{10.15}$$

Now, for $m=1$

$$\left.\begin{aligned}
&\underset{1}{\lambda_1}=i\underset{1}{\lambda_1}\,,\quad \underset{1}{\lambda_1}=\frac{1}{2}\sqrt{2P_1-1-2\sqrt{P_1(P_1-1)}}\,,\qquad \underset{2}{\lambda_1}=\frac{1}{2}\left(\sqrt{1-P_2}+i\sqrt{P_2}\right),\\
&\underset{1}{\lambda_2}=i\underset{1}{\lambda_2}\,,\quad \underset{1}{\lambda_2}=\frac{1}{2}\sqrt{2P_1-1+2\sqrt{P_1(P_1-1)}}\,,\qquad \underset{2}{\lambda_2}=\frac{1}{2}\left(\sqrt{1-P_2}-i\sqrt{P_2}\right)=\underset{2}{\bar{\lambda}_1},^{21)}\\
&\qquad\underset{1}{\lambda_1}\underset{1}{\lambda_2}\,\varphi_1=-\left(\frac{1}{2}\right)^2\sqrt{P_1(P_1-1)}\,,\qquad \varphi_1=\sqrt{P_1(P_1-1)}\,,
\end{aligned}\right\} \tag{10.16}$$

and then

$$\begin{aligned}
&\underset{1}{\beta_1}=-\left(\underset{1}{\lambda_1^2}+0.075\right), && \underset{1}{\gamma_1}=-i\underset{1}{\gamma_1}\,, && \underset{1}{\gamma_1}=\underset{1}{\lambda_1}\left(\underset{1}{\lambda_1^2}+0.425\right),\\
&\underset{1}{\beta_2}=-\left(\underset{1}{\lambda_2^2}+0.075\right), && \underset{1}{\gamma_2}=-i\underset{1}{\gamma_2}\,, && \underset{1}{\gamma_2}=\underset{1}{\lambda_2}\left(\underset{1}{\lambda_2^2}+0.425\right),\\
&\underset{2}{\beta_1}=\left(\underset{2}{\lambda_1^2}-0.075\right), && \underset{2}{\gamma_1}=\underset{2}{\lambda_1}\left(\underset{2}{\lambda_1^2}-0.425\right), &&\\
&\underset{2}{\beta_2}=\left(\underset{2}{\bar\lambda_1^2}-0.075\right)=\underset{2}{\bar\beta_1}\,, && \underset{2}{\gamma_2}=\underset{2}{\bar\lambda_1}\left(\underset{2}{\bar\lambda_1^2}-0.425\right)=\underset{2}{\bar\gamma_1}\,. &&
\end{aligned}$$

Again

$$\underset{1}{\tau_1}=\underset{1}{\beta_1}-\mu_2'\underset{2}{\beta_1}=-\left(\frac{1}{2}\right)^2\left\{1.6625-2\sqrt{P_1(P_1-1)}+i2\mu_2'\sqrt{P_2(1-P_2)}\right\},$$

$$\underset{1}{\tau_1'}=\underset{1}{\beta_1}-\mu_2'\underset{2}{\beta_2}=\underset{1}{\beta_1}-\mu_2'\underset{2}{\bar\beta_1}=\underset{1}{\bar\tau_1}\,,$$

---

21) $\underset{2}{\bar\lambda_1}$ denotes the conjugate complex number of $\underset{2}{\lambda_1}$ and hereafter, the head line will be used as such meaning unless special notes are given.

$$
\left.
\begin{aligned}
&\tau_{\substack{2\\1}} = \beta_{\substack{2\\1}} - \mu_2' \beta_{\substack{1\\2}} = -\left(\frac{1}{2}\right)^2 \left\{1.6625 + 2\sqrt{P_1(P_1-1)} + i2\mu_2' \sqrt{P_2(1-P_2)}\right\},\\
&\tau_{\substack{2\\1}}' = \beta_{\substack{2\\1}} - \mu_2' \beta_{\substack{2\\2}} = \beta_{\substack{2\\1}} - \mu_2' \bar{\beta}_{\substack{1\\2}} = \bar{\tau}_{\substack{2\\1}},\\
&\chi_{\substack{1\\1}} = -i\,(\lambda_{\substack{1\\2}} \gamma_{\substack{1\\1}} + \mu_2' \lambda_{\substack{1\\1}} \gamma_{\substack{1\\2}}) = \left(\frac{1}{2}\right)^3 \lambda_{\substack{1\\1}} (-\mathfrak{r} + i\mathfrak{s}),\\
&\chi_{\substack{1\\1}}' = -i\,(\bar{\lambda}_{\substack{1\\2}} \gamma_{\substack{1\\1}} + \mu_2' \lambda_{\substack{1\\1}} \bar{\gamma}_{\substack{1\\2}}) = -\bar{\chi}_{\substack{1\\1}},\\
&\chi_{\substack{2\\1}} = -i\,(\lambda_{\substack{1\\2}} \gamma_{\substack{2\\1}} + \mu_2' \lambda_{\substack{2\\1}} \gamma_{\substack{1\\2}}) = \left(\frac{1}{2}\right)^3 \lambda_{\substack{2\\1}} (-\mathfrak{r}' + i\mathfrak{s}'),\\
&\chi_{\substack{2\\1}}' = -i\,(\bar{\lambda}_{\substack{1\\2}} \gamma_{\substack{2\\1}} + \mu_2' \lambda_{\substack{2\\1}} \bar{\gamma}_{\substack{1\\2}}) = -\bar{\chi}_{\substack{2\\1}},
\end{aligned}
\right\} \quad (10.17)
$$

in which

$$
\left.
\begin{aligned}
\mathfrak{r} &= \sqrt{P_2}\left\{-5.0875 + 2P_1 + 2\sqrt{P_1(P_1-1)}\right\},\\
\mathfrak{r}' &= \sqrt{P_2}\left\{-5.0875 + 2P_1 - 2\sqrt{P_1(P_1-1)}\right\},\\
\mathfrak{s} &= \sqrt{1-P_2}\left\{1.6625 + 2P_1 + 2\sqrt{P_1(P_1-1)}\right\},\\
\mathfrak{s}' &= \sqrt{1-P_2}\left\{1.6625 + 2P_1 - 2\sqrt{P_1(P_1-1)}\right\}.
\end{aligned}
\right\} \quad (10.18)
$$

Therefore, the coefficients of (10.2) are expressed as follows:

$$
\left.
\begin{aligned}
&F_{\substack{2\\1}} = \frac{\tau_{\substack{2\\1}}}{\varphi_1} \cos\pi\lambda_{\substack{1\\1}}, &&H_{\substack{2\\1}} = \frac{\bar{\tau}_{\substack{2\\1}}}{\varphi_1} \cos\pi\lambda_{\substack{1\\1}} = \bar{F}_{\substack{2\\1}},\\
&F_{\substack{1\\1}} = \frac{\tau_{\substack{1\\1}}}{\varphi_1} \cos\pi\lambda_{\substack{2\\1}}, &&H_{\substack{1\\1}} = \frac{\bar{\tau}_{\substack{1\\1}}}{\varphi_1} \cos\pi\lambda_{\substack{2\\1}} = \bar{F}_{\substack{1\\1}},\\
&G_{\substack{2\\1}}' = i\,\frac{\chi_{\substack{2\\1}}}{\lambda_{\substack{1\\1}} \lambda_{\substack{2\\1}} \varphi_1} \sin\pi\lambda_{\substack{1\\1}}, &&I_{\substack{2\\1}}' = -i\,\frac{\bar{\chi}_{\substack{2\\1}}}{\lambda_{\substack{1\\1}} \lambda_{\substack{2\\1}} \varphi_1} \sin\pi\lambda_{\substack{1\\1}} = G_{\substack{2\\1}}',\\
&G_{\substack{1\\1}}' = i\,\frac{\chi_{\substack{1\\1}}}{\lambda_{\substack{1\\1}} \lambda_{\substack{2\\1}} \varphi_1} \sin\pi\lambda_{\substack{2\\1}}, &&I_{\substack{1\\1}}' = -i\,\frac{\bar{\chi}_{\substack{1\\1}}}{\lambda_{\substack{1\\1}} \lambda_{\substack{2\\1}} \varphi_1} \sin\pi\lambda_{\substack{2\\1}} = \bar{G}_{\substack{1\\1}}'.
\end{aligned}
\right\} \quad (10.19)
$$

Using these, (10.2) becomes

$$
\left.
\begin{aligned}
U_{\substack{2\\1}} &= -F_{\substack{2\\1}} \sinh\pi\lambda_{\substack{1\\2}} - G_{\substack{2\\1}}' \cosh\pi\lambda_{\substack{1\\2}} = \frac{\sinh\pi\lambda_{\substack{1\\1}} \sinh\pi\lambda_{\substack{1\\2}}}{4\varphi_1}(R + iZ);\\
U_{\substack{1\\1}} &= F_{\substack{1\\1}} \sinh\pi\lambda_{\substack{1\\2}} + G_{\substack{1\\1}}' \cosh\pi\lambda_{\substack{1\\2}} = -\frac{\sinh\pi\lambda_{\substack{2\\1}} \sinh\pi\lambda_{\substack{1\\2}}}{4\varphi_1}(R' + iZ');
\end{aligned}
\right\} \quad (10.20)
$$

$$\left.\begin{aligned} \underset{1}{V_2} &= -\underset{1}{\bar{F}_2}\sinh\pi\underset{2}{\bar{\lambda}_1} - \underset{1}{\bar{G}_2'}\cosh\pi\underset{2}{\bar{\lambda}_1} = \underset{1}{\bar{U}_2}\,; \\ \underset{1}{V_1} &= \underset{1}{\bar{F}_1}\sinh\pi\underset{2}{\bar{\lambda}_1} + \underset{1}{\bar{G}'_1}\cosh\pi\underset{2}{\bar{\lambda}_1} = \underset{1}{\bar{U}_1}\,. \end{aligned}\right|$$

where

$$\left.\begin{aligned} R' &= 2\mu_2'\sqrt{P_2(1-P_2)}\cot\pi\underset{1}{\lambda_2} - 2\underset{1}{\lambda_1}\frac{\mathfrak{r}\sinh\pi\sqrt{1-P_2}-\mathfrak{s}\sin\pi\sqrt{P_2}}{\cosh\pi\sqrt{1-P_2}-\cos\pi\sqrt{P_2}}\,, \\ R &= 2\mu_2'\sqrt{P_2(1-P_2)}\cot\pi\underset{1}{\lambda_1} - 2\underset{1}{\lambda_2}\frac{\mathfrak{r}'\sinh\pi\sqrt{1-P_2}-\mathfrak{s}'\sin\pi\sqrt{P_2}}{\cosh\pi\sqrt{1-P_2}-\cos\pi\sqrt{P_2}}\,, \\ Z' &= -\left\{1.6625-2\sqrt{P_1(P_1-1)}\right\}\cot\pi\underset{1}{\lambda_2} + 2\underset{1}{\lambda_1}\frac{\mathfrak{r}\sin\pi\sqrt{P_2}+\mathfrak{s}\sinh\pi\sqrt{1-P_2}}{\cosh\pi\sqrt{1-P_2}-\cos\pi\sqrt{P_2}}\,, \\ Z &= -\left\{1.6625+2\sqrt{P_1(P_1-1)}\right\}\cot\pi\underset{1}{\lambda_1} + 2\underset{1}{\lambda_2}\frac{\mathfrak{r}'\sin\pi\sqrt{P_2}+\mathfrak{s}'\sinh\pi\sqrt{1-P_2}}{\cosh\pi\sqrt{1-P_2}-\cos\pi\sqrt{P_2}}\,. \end{aligned}\right\} \quad (10.21)$$

Hence the conditional equation (10.3) are reduced to the following equation:

$$RZ' - R'Z = 0\,.$$

By proceeding in the same manner as in the previous example, we obtain two pairs of the characteristic values within the limits as formerly obtained. That is

$$\begin{cases} P_1 = 1.5114 \\ P_2 = 0.4478 \end{cases} \qquad \begin{cases} P_1 = 2.8392 \\ P_2 = 0.8412 \end{cases}$$

It is obvious that the former in the above gives the critical load for buckling.

In the next place, we see that the expression of the deflection surface for the 1-st elementary plate must be composed of terms of such forms as shown in Case-4 [Table 1], because the value of P_1 lies within the limits of such a case. On the other hand, the expression for the 2-nd elementary plate must be composed of terms whose forms are shown in Case-1 [Table 1], because the value of P_2 lies within such a limit. The constant coefficients in these expressions can be determined by the same way as before. The further calculations give, for the least of the characteristic values, *viz.*, the buckling load,

$$w_1 = \underset{1}{L_1}\,w_1'\,,$$

$$w_1' = \left\{\sin\pi\underset{1}{\lambda_1}\,\xi_1 + 0.480\,093\sin\pi\underset{1}{\lambda_2}\,\xi_1\right\}\sin\pi\eta_1\,;$$

$$w_2 = \underset{1}{L_1}\,w_2'\,,$$

$$w_2' = \Big\{0.764\,567 \cosh\pi\omega_{\substack{1\\2}}\xi_2 \cos\pi\omega_{\substack{2\\2}}\xi_2 - 0.687154 \sinh\pi\omega_{\substack{1\\2}}\xi_2 \cos\pi\omega_{\substack{2\\2}}\xi_2$$
$$-0.095\,609 \cosh\pi\omega_{\substack{1\\2}}\xi_2 \sin\pi\omega_{\substack{2\\2}}\xi_2 - 0.021\,992 \sinh\pi\omega_{\substack{1\\2}}\xi_2 \sin\pi\omega_{\substack{2\\2}}\xi_2\Big\} \sin\pi\eta_2.^{22)}$$

The numerical results by the above are given in Table 8 and their illustrations are Fig. 7.

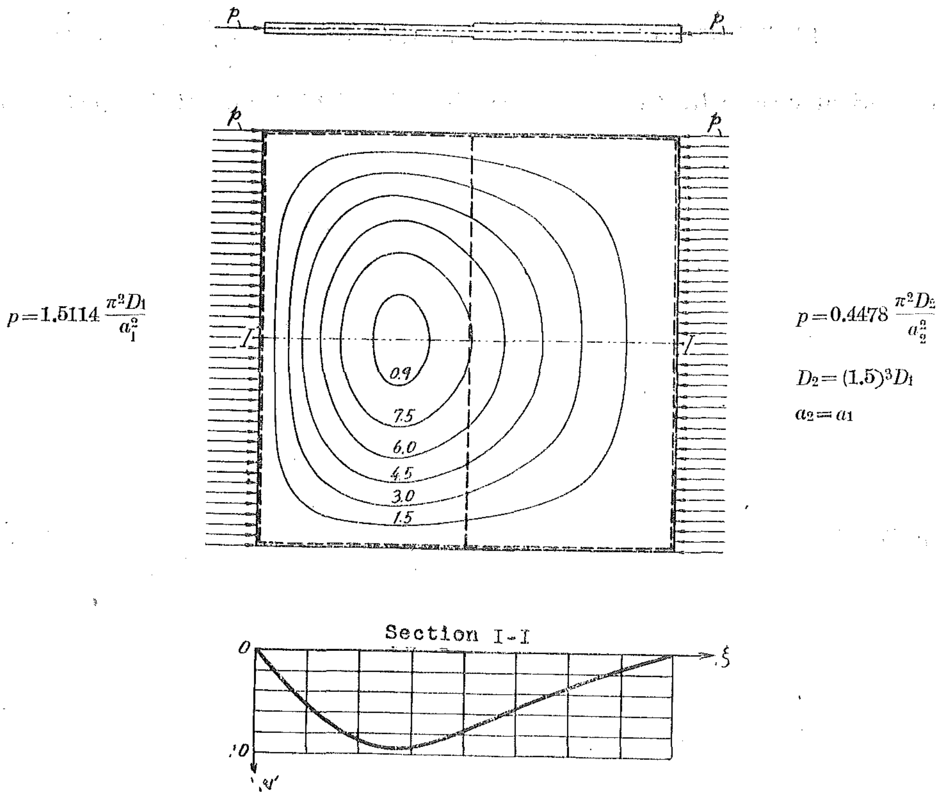


Fig. 7.

Table 8.

η \ ξ_1, ξ	w_1' 0.25	0.50	0.75	1.00	w_2' 0	0.25	0.50	0.75
0.125	0.203 81	0.333 94	0.356 04	0.292 58	0.292 58	0.208 93	0.132 83	0.064 20
0.250	0.376 59	0.617 06	0.657 89	0.540 63	0.540 63	0.386 07	0.245 45	0.118 62
0.375	0.492 04	0.806 22	0.859 57	0.706 36	0.706 36	0.504 42	0.320 69	0.154 99
0.500	0.532 58	0.872 65	0.930 39	0.764 56	0.764 56	0.545 98	0.347 11	0.167 76

In the next place, for the other of the characteristic values previously obtained which corresponds to second mode of buckling

22) $\lambda_{\substack{1\\1}}$, $\lambda_{\substack{2\\1}}$ and $\omega_{\substack{1\\2}}$, $\omega_{\substack{2\\2}}$ are respectively given in the corresponding line of Table I.

$$w_1 = \underset{1}{L_1}\, w_1' ,$$

$$w_1' = \left\{\sin\pi\underset{1}{\lambda_1}\,\xi_1 - 0.651\,599\,\sin\pi\underset{1}{\lambda_2}\,\xi_1\right\}\sin\pi\eta_1 \,:$$

$$w_2 = \underset{1}{L_1}\, w_2' ,$$

$$w_2' = \Big\{1.144\,088\cosh\pi\underset{2}{\omega_1}\,\xi_2\cos\pi\underset{2}{\omega_2}\,\xi_2 - 0.738\,258\sinh\pi\underset{2}{\omega_1}\,\xi_2\cos\pi\underset{2}{\omega_2}\,\xi_2$$
$$+0.492\,985\cosh\pi\underset{2}{\omega_1}\,\xi_2\sin\pi\underset{2}{\omega_2}\,\xi_2 - 1.060\,840\sinh\pi\underset{2}{\omega_1}\,\xi_2\sin\pi\underset{2}{\omega_2}\,\xi_2\Big\}\sin\pi\eta_2 \,.$$

The numerical results by the above are given in Table 9 and their illustrations are Fig. 8.

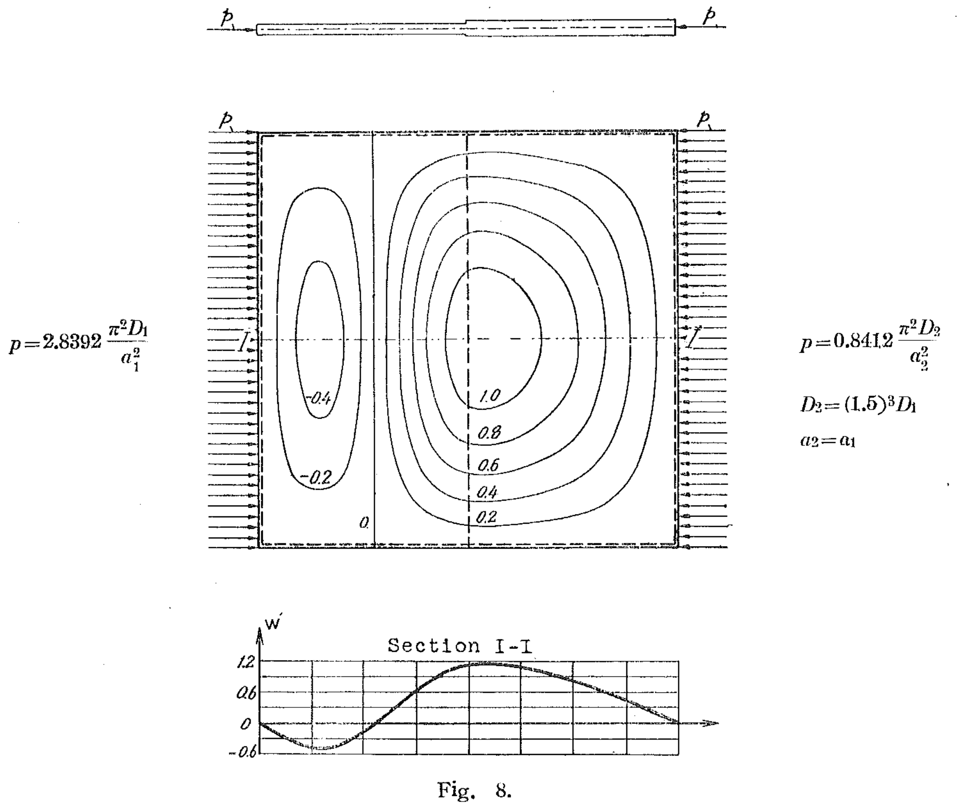


Fig. 8.

Table 9.

η \ $\xi_1,\ \xi_2$	w_1' 0.25	w_1' 0.50	w_1' 0.75	w_1' 1.00	w_2' 0	w_2' 0.25	w_2' 0.50	w_2' 0.75
0.125	−0.182 65	−0.072 79	0.251 04	0.437 82	0.437 82	0.418 03	0.323 15	0.175 25
0.250	−0.337 51	−0.134 50	0.463 87	0.808 99	0.808 99	0.772 43	0.597 12	0.323 82
0.375	−0.440 98	−0.175 73	0.606 07	1.057 00	1.057 00	1.009 22	0.780 17	0.423 10
0.500	−0.477 31	−0.190 21	0.656 01	1.144 09	1.144 08	1.092 37	0.844 45	0.457 96

§ 11 Case when One End Side is Clamped and the Other is Simply Supported

Let End side-2 is simply supported. Then this condition is expressed by (10.1) as in the previous section. On the other hand, the other end side in this case comes to be fixed. The conditional equation for buckling, therefore, must be written as (**9**.e). Thus, the following functions reduced by the recurrence formulas are necessary besides (10.2). That is,

$$\left.\begin{aligned}
U_{\substack{2\\1}}' &= U_{\substack{2\\2}} F_{\substack{2\\1}}' - U_{\substack{2\\2}}' G_{\substack{2\\1}} = F_{\substack{2\\1}}' \sinh\pi\lambda_{\substack{1\\2}} + G_{\substack{2\\1}} \cosh\pi\lambda_{\substack{1\\2}} ;\\
U_{\substack{1\\1}}' &= -U_{\substack{2\\2}} F_{\substack{1\\1}}' + U_{\substack{2\\2}}' G_{\substack{1\\1}} = -F_{\substack{1\\1}}' \sinh\pi\lambda_{\substack{1\\2}} - G_{\substack{1\\1}} \cosh\pi\lambda_{\substack{1\\2}} ;\\
V_{\substack{2\\1}}' &= V_{\substack{1\\2}} H_{\substack{2\\1}}' - V_{\substack{1\\2}}' I_{\substack{2\\1}} = H_{\substack{2\\1}}' \sinh\pi\lambda_{\substack{2\\2}} + I_{\substack{2\\1}} \cosh\pi\lambda_{\substack{2\\2}} ;\\
V_{\substack{1\\1}}' &= -V_{\substack{1\\2}} H_{\substack{1\\1}}' + V_{\substack{1\\2}}' I_{\substack{1\\1}} = -H_{\substack{1\\1}}' \sinh\pi\lambda_{\substack{2\\2}} - I_{\substack{1\\1}} \cosh\pi\lambda_{\substack{2\\2}} .
\end{aligned}\right\} \tag{11.1}$$

Referring to (**9**.e), the conditional equation is given by

$$(U_{\substack{1\\1}} + U_{\substack{2\\1}})\,(V_{\substack{1\\1}}'\lambda_{\substack{2\\1}} + V_{\substack{2\\1}}'\lambda_{\substack{1\\1}}) - (V_{\substack{1\\1}} + V_{\substack{2\\1}})\,(U_{\substack{1\\1}}'\lambda_{\substack{2\\1}} + U_{\substack{2\\1}}'\lambda_{\substack{1\\1}}) = 0 . \tag{11.2}$$

(1) *The case where only the load* q_r *is acting*.—This case is different from the case (1) in the previous section at only the point that a fixed end side exists. Therefore, we can easily suppose that the composite plate has somewhat larger rigidity than the previous plate. From this fact, the critical load may be larger, *i.e.*,

$$Q_1 > 5.988 , \qquad Q_2 > 2.661 .$$

Next we must require the upper limit since the lower limits have been obtained above. But, since it seems that convenient examples can not be found in a place close by, we will refer to a single square plate whose *Poisson's ratio* ν is 0.25. Denoting by q'' the critical load under which buckling occurs when such a plate has the uniform thickness h_2 all over the plate, the following result is known :[23]

$$\frac{q''b^2}{D_2\pi^2} = 7.69$$

Taking into account that the effects of *Poisson's ratio* are usually very small,[24] we may not fault by putting as follows:

$$q_2 < q'' ,$$

from which
$$Q_2 = \frac{q_2 b^2}{D_2\pi^2} < \frac{q''b^2}{D_2\pi^2} = 7.69$$

23) See S. Timoshenko. "Theory of Elastic Stabilty," 1936, p. 345, Table 35.

24) See, for instance, Section E in the paper by S. Iguchi, *loc. cit.* p. 218.

Since the relation (10.7) holds also in this case, we get

$$Q_1 = 2.25\, Q_2 < 2.25 \times 7.69 = 17.3025$$

Consequently, the upper limit has been obtained, but if we know a more adequate example for comparison, such limits can be set more closely. Finally, we can now suppose as

$$5.988 < Q_1 < 17.302\,, \qquad 2.661 < Q_2 < 7.69 \tag{11.3}$$

From these, (10.9) is again used for λ_r and then (10.10) is applicable for τ_r, τ_r', χ_r, χ_r'. Therefore, F_r, G_r', H_r, I_r' become such as (10.11) and U_r, V_r for $r=1$ can be expressed in the same way as in (10.12). Besides them, (11.1) demanded for this case is rewritten in the following manner. In the first instance, the necessary factors F_r', G_r, H_r', I_r become as follows by (8):

$$\left.\begin{aligned}
&F'_{\substack{2\\1}} = \frac{\tau_{\substack{2\\1}}}{\varphi_1}\sinh\pi\lambda_{\substack{1\\1}}\,, && H'_{\substack{2\\1}} = \frac{\tau'_{\substack{2\\1}}}{\varphi_1}\sinh\pi\lambda_{\substack{1\\1}}\,,\\
&F'_{\substack{1\\1}} = \frac{\tau_{\substack{1\\1}}}{\varphi_1}\sinh\pi\lambda_{\substack{2\\1}} = i\,\frac{\tau_{\substack{1\\1}}}{\varphi_1}\sin\pi\lambda_{\substack{2\\1}}, && H'_{\substack{1\\1}} = \frac{\tau'_{\substack{1\\1}}}{\varphi_1}\sinh\pi\lambda_{\substack{2\\1}} = i\,\frac{\tau'_{\substack{1\\1}}}{\varphi_1}\sin\pi\lambda_{\substack{2\\1}},\\
&G_{\substack{2\\1}} = -\frac{\lambda_{\substack{1\\2}}\tau'_{\substack{1\\1}}}{\lambda_{\substack{1\\1}}\varphi_1}\cosh\pi\lambda_{\substack{1\\1}}\,, && I_{\substack{2\\1}} = -\frac{\lambda_{\substack{2\\2}}\tau_{\substack{1\\1}}}{\lambda_{\substack{1\\1}}\varphi_1}\cosh\pi\lambda_{\substack{1\\1}} = -i\,\frac{\lambda_{\substack{2\\2}}\tau_{\substack{1\\1}}}{\lambda_{\substack{1\\1}}\varphi_1}\cosh\pi\lambda_{\substack{1\\1}}\,,\\
&G_{\substack{1\\1}} = -\frac{\lambda_{\substack{1\\2}}\tau'_{\substack{2\\1}}}{\lambda_{\substack{2\\1}}\varphi_1}\cosh\pi\lambda_{\substack{2\\1}} = i\,\frac{\lambda_{\substack{1\\2}}\tau'_{\substack{2\\1}}}{\lambda_{\substack{2\\1}}\varphi_1}\cos\pi\lambda_{\substack{2\\1}}\,, && I_{\substack{1\\1}} = -\frac{\lambda_{\substack{2\\2}}\tau_{\substack{2\\1}}}{\lambda_{\substack{2\\1}}\varphi_1}\cosh\pi\lambda_{\substack{2\\1}} = -\frac{\lambda_{\substack{2\\2}}\tau_{\substack{2\\1}}}{\lambda_{\substack{2\\1}}\varphi_1}\cos\pi\lambda_{\substack{2\\1}}\,.
\end{aligned}\right\} \tag{11.4}$$

Substituting these in (11.1) and arranging them, we have

$$\left.\begin{aligned}
&U'_{\substack{2\\1}} = \frac{\cosh\pi\lambda_{\substack{1\\1}}\sinh\pi\lambda_{\substack{1\\2}}}{\lambda_{\substack{1\\1}}\varphi_1}R''\,; && V'_{\substack{2\\1}} = i\,\frac{\cosh\pi\lambda_{\substack{1\\1}}\sin\pi\lambda_{\substack{2\\2}}}{\lambda_{\substack{1\\1}}\varphi_1}Z''\,;\\
&U'_{\substack{1\\1}} = -i\,\frac{\cos\pi\lambda_{\substack{2\\1}}\sinh\pi\lambda_{\substack{1\\2}}}{\lambda_{\substack{2\\1}}\varphi_1}R'''\,; && V'_{\substack{1\\1}} = \frac{\cos\pi\lambda_{\substack{2\\1}}\sin\pi\lambda_{\substack{2\\2}}}{\lambda_{\substack{2\\1}}\varphi_1}Z'''\,,\\
&\text{in which} &&\\
&R'' = \tau_{\substack{2\\1}}\lambda_{\substack{1\\1}}\tanh\pi\lambda_{\substack{1\\1}} - \tau'_{\substack{1\\1}}\lambda_{\substack{1\\2}}\coth\pi\lambda_{\substack{1\\2}}\,, &&\\
&R''' = \tau_{\substack{1\\1}}\lambda_{\substack{2\\1}}\tan\pi\lambda_{\substack{2\\1}} + \tau'_{\substack{2\\1}}\lambda_{\substack{1\\2}}\coth\pi\lambda_{\substack{1\\2}}\,, &&\\
&Z'' = \tau'_{\substack{2\\1}}\lambda_{\substack{1\\1}}\tanh\pi\lambda_{\substack{1\\1}} - \tau_{\substack{1\\1}}\lambda_{\substack{2\\2}}\cot\pi\lambda_{\substack{2\\2}}\,, &&\\
&Z''' = \tau'_{\substack{1\\1}}\lambda_{\substack{2\\1}}\tan\pi\lambda_{\substack{2\\1}} + \tau_{\substack{2\\1}}\lambda_{\substack{2\\2}}\cot\pi\lambda_{\substack{2\\2}}\,. &&
\end{aligned}\right\} \tag{11.5}$$

Then

$$\underset{1}{U_1'}\,\underset{1}{\lambda_2}+\underset{1}{U_2'}\,\underset{1}{\lambda_1}=\frac{\sinh\pi\underset{2}{\lambda_1}}{\varphi_1}\left(R'''\cos\pi\underset{1}{\lambda_2}+R''\cosh\pi\underset{1}{\lambda_1}\right);$$

$$\underset{1}{V_1'}\,\underset{1}{\lambda_2}+\underset{1}{V_2'}\,\underset{1}{\lambda_1}=i\,\frac{\sin\pi\underset{2}{\lambda_2}}{\varphi_1}\left(Z'''\cos\pi\underset{1}{\lambda_2}+Z''\cosh\pi\underset{1}{\lambda_1}\right).$$

And also by (10.12), it can be written that

$$\underset{1}{U_1}+\underset{1}{U_2}=\frac{\sinh\pi\underset{2}{\lambda_1}}{\varphi_1}\left(R'\frac{\sin\pi\underset{1}{\lambda_2}}{\underset{1}{\lambda_2}}-R\frac{\sinh\pi\underset{1}{\lambda_1}}{\underset{1}{\lambda_1}}\right);$$

$$\underset{1}{V_1}+\underset{1}{V_2}=i\frac{\sin\pi\underset{2}{\lambda_2}}{\varphi_1}\left(Z'\frac{\sin\pi\underset{1}{\lambda_2}}{\underset{1}{\lambda_2}}-Z\frac{\sinh\pi\underset{1}{\lambda_1}}{\underset{1}{\lambda_1}}\right).$$

Finally, the conditional equation (11.2) is transformed as

$$\left(R\frac{\sinh\pi\underset{1}{\lambda_1}}{\underset{1}{\lambda_1}}-R'\frac{\sin\pi\underset{1}{\lambda_2}}{\underset{1}{\lambda_2}}\right)\left(Z''\cosh\pi\underset{1}{\lambda_1}+Z'''\cos\pi\underset{1}{\lambda_2}\right)$$
$$-\left(Z\frac{\sinh\pi\underset{1}{\lambda_1}}{\underset{1}{\lambda_1}}-Z'\frac{\sin\pi\underset{1}{\lambda_2}}{\underset{1}{\lambda_2}}\right)\left(R''\cosh\pi\underset{1}{\lambda_1}+R'''\cos\pi\underset{1}{\lambda_2}\right)=0. \tag{11.6}$$

Proceeding as in the previous example, the least root, inside the limits (11.3), is found as follows:

$$\begin{cases} Q_1=8.5118 \\ Q_2=3.7830 \end{cases}$$

And, since Q_1 and Q_2 lie within the limits of Case-2 in Table 1, their deflection surfaces are expressed in the following form by using the expressions for such a case:

$$w_1=\underset{1}{K_1}\,w_1',$$

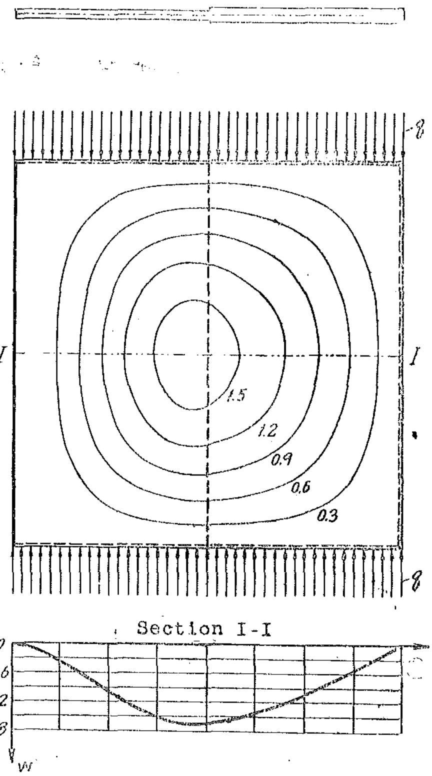
$$w_1'=\left\{(\cosh\pi\underset{1}{\lambda_1}\,\xi_1-\cos\pi\underset{1}{\lambda_2}\,\xi_1)-1.014\,204\left(\sinh\pi\underset{1}{\lambda_1}\,\xi_1-\frac{\underset{1}{\lambda_1}}{\underset{1}{\lambda_2}}\sin\pi\underset{1}{\lambda_2}\,\xi_1\right)\right\}\sin\pi\eta_1;$$

$$w_2=\underset{1}{K_1}\,w_2',$$

$$w_2'=\left\{0.026\,881\sinh\pi\underset{2}{\lambda_1}(1-\xi_2)+1.450\,140\sin\pi\underset{2}{\lambda_2}(1-\xi_2)\right\}\sin\pi\eta_2.$$

The numerical results by the above are given in Table 10 and then Fig. 9 is drawn.

$q_1 = 8.5118 \dfrac{\pi^2 D_1}{b^2}$



$q_2 = 3.7830 \dfrac{\pi^2 D_2}{b^2}$

$D_2 = (1.5)^3 D_1$

$q_2/q_1 = h_2/h_1 = 1.5$

Fig. 9.

Table 10.

	w_1'				w_2'			
η \ ξ_2, ξ_1	0.25	0.50	0.75	1.00	0	0.25	0.50	0.75
0.125	0.130 77	0.381 99	0.586 30	0.630 26	0.630 26	0.543 20	0.402 55	0.214 20
0.250	0.241 63	0.705 84	1.083 36	1.164 59	1.164 59	1.003 72	0.743 84	0.395 80
0.375	0.315 71	0.922 22	1.415 48	1.521 61	1.521 60	1.311 42	0.971 87	0.517 14
0.500	0.341 72	0.998 20	1.532 10	1.646 97	1.646 97	1.419 47	1.051 94	0.559 75

(2) *The case where only the load p is acting.*—Since the load conditions are the same as in the case (2) of the previous section, by considering the differences for the end side conditions, we can assume

$$P_1 > 1.5114, \qquad P_2 > 0.4478$$

Now, for the upper limit, it is desirable to use the numerical results of the single plate which has the same load conditions and the same boundary conditions, but those seems not to be found in a place close by. Then let us use the numerical results concerning such a single plate as is clamped along the two opposite sides and simply supported along the remaining two, and subjected to the loads acting in the direction perpendicular to the clamped edges. Denoting by p'' the critical load of the above referred plate in which $\nu=0.25$ and the thickness is h_2, the following relation is known.[25]

$$\frac{p''a^2}{D_2\pi^2}=6.74 \qquad (2a_1=2a_2=a=b)$$

We can easily assume the next inequality by considering the differences among the thicknesses and the boundary conditions:

$$p_2<p'',$$

from which

$$P_2=\frac{p_2a_2^2}{D_2\pi^2}=\frac{1}{4}\frac{p_2a^2}{D_2\pi^2}<\frac{1}{4}\frac{p''a^2}{D_2\pi^2}=1.685$$

Since the relation (10.14) holds also in this case,

$$P_1=3.375\,P_2<3.375\times 1.685=5.6868$$

Finally, we have

$$1.512<P_1<5.686\,, \qquad 0.448<P_2<1.685$$

Especially, the partial intervals

$$1.512<P_1<3.375\,, \qquad 0.448<P_2<1 \tag{11.7}$$

are included in those of the foregoing (10.15), and then in such ranges λ_r, τ_r, χ_r can be expressed by (10.16) and (10.17). Therefore, we see that F_r, G_r', H_r, I_r' are written such as (10.19) and U_r, V_r for $r=1$ as (10.20).

Besides the above, (11.1) is demanded in this case. The necessary coefficients are written by (8) as follows:

$$\left.\begin{aligned}
&F'_{\substack{2\\1}}=i\,\frac{\tau_{\substack{2\\1}}}{\varphi_1}\sin\pi\lambda_{\substack{1\\1}}\,, && H'_{\substack{2\\1}}=i\,\frac{\bar{\tau}_{\substack{2\\1}}}{\varphi_1}\sin\pi\lambda_{\substack{1\\1}}=-\bar{F}'_{\substack{2\\1}}\,,\\
&F'_{\substack{1\\1}}=i\,\frac{\tau_{\substack{1\\1}}}{\varphi_1}\sin\pi\lambda_{\substack{2\\1}}\,, && H'_{\substack{1\\1}}=i\,\frac{\bar{\tau}_{\substack{1\\1}}}{\varphi_1}\sin\pi\lambda_{\substack{2\\1}}=-\bar{F}'_{\substack{1\\1}}\,,\\
&G_{\substack{2\\1}}=\frac{\chi_{\substack{2\\1}}}{\lambda_{\substack{1\\1}}\lambda_{\substack{2\\1}}\varphi_1}\cos\pi\lambda_{\substack{1\\1}}\,, && I_{\substack{2\\1}}=-\frac{\bar{\chi}_{\substack{2\\1}}}{\lambda_{\substack{1\\1}}\lambda_{\substack{2\\1}}\varphi_1}\cos\pi\lambda_{\substack{1\\1}}=-\bar{G}_{\substack{2\\1}}\,,\\
&G_{\substack{1\\1}}=\frac{\chi_{\substack{1\\1}}}{\lambda_{\substack{1\\1}}\lambda_{\substack{2\\1}}\varphi_1}\cos\pi\lambda_{\substack{2\\1}}\,, && I_{\substack{1\\1}}=-\frac{\bar{\chi}_{\substack{1\\1}}}{\lambda_{\substack{1\\1}}\lambda_{\substack{2\\1}}\varphi_1}\cos\pi\lambda_{\substack{2\\1}}=-\bar{G}_{\substack{1\\1}}\,.
\end{aligned}\right\} \tag{11.8}$$

25) See S. Timoshenko, "Theory of Elastic Stability," 1936, p. 364, Table 38.

Substituting these in (11.1) and arranging them, we get

$$\left.\begin{aligned}
U_{\substack{2\\1}}' &= \quad F_{\substack{2\\1}}' \sinh\pi\lambda_{\substack{1\\2}} + G_{\substack{2\\1}} \cosh\pi\lambda_{\substack{1\\2}} = \quad \frac{\cos\pi\lambda_{\substack{1\\1}} \sinh\pi\lambda_{\substack{1\\2}}}{4\varphi_1}(R''-iZ'')\,;\\
U_{\substack{1\\1}}' &= -F_{\substack{1\\1}}' \sinh\pi\lambda_{\substack{1\\2}} - G_{\substack{1\\1}} \cosh\pi\lambda_{\substack{1\\2}} = -\frac{\cos\pi\lambda_{\substack{2\\1}} \sinh\pi\lambda_{\substack{1\\2}}}{4\varphi_1}(R'''-iZ''');\\
V_{\substack{2\\1}}' &= -\bar{F}_{\substack{2\\1}}' \sinh\pi\bar{\lambda}_{\substack{1\\2}} - \bar{G}_{\substack{2\\1}} \cosh\pi\bar{\lambda}_{\substack{1\\2}} = -\bar{U}_{\substack{2\\1}}'\,;\\
V_{\substack{1\\1}}' &= \quad \bar{F}_{\substack{1\\1}}' \sinh\pi\bar{\lambda}_{\substack{1\\2}} + \bar{G}_{\substack{1\\1}} \cosh\pi\bar{\lambda}_{\substack{1\\2}} = -\bar{U}_{\substack{1\\1}}'\,,
\end{aligned}\right\} \tag{11.9}$$

where

$$\left.\begin{aligned}
R''' &= 2\mu_2' \sqrt{P_2(1-P_2)}\,\tan\pi\lambda_{\substack{2\\1}} + 2\lambda_{\substack{1\\1}} \frac{\mathfrak{r}\sinh\pi\sqrt{1-P_2} - \mathfrak{s}\sin\pi\sqrt{P_2}}{\cosh\pi\sqrt{1-P_2} - \cos\pi\sqrt{P_2}}\,,\\
R'' &= 2\mu_2' \sqrt{P_2(1-P_2)}\,\tan\pi\lambda_{\substack{1\\1}} + 2\lambda_{\substack{2\\1}} \frac{\mathfrak{r}'\sinh\pi\sqrt{1-P_2} - \mathfrak{s}'\sin\pi\sqrt{P_2}}{\cosh\pi\sqrt{1-P_2} - \cos\pi\sqrt{P_2}}\,,\\
Z''' &= \left\{1.6625 - 2\sqrt{P_1(P_1-1)}\right\} \tan\pi\lambda_{\substack{2\\1}} + 2\lambda_{\substack{1\\1}} \frac{\mathfrak{r}\sin\pi\sqrt{P_2} + \mathfrak{s}\sinh\pi\sqrt{1-P_2}}{\cosh\pi\sqrt{1-P_2} - \cos\pi\sqrt{P_2}}\,,\\
Z'' &= \left\{1.6625 + 2\sqrt{P_1(P_1-1)}\right\} \tan\pi\lambda_{\substack{1\\1}} + 2\lambda_{\substack{2\\1}} \frac{\mathfrak{r}'\sin\pi\sqrt{P_2} + \mathfrak{s}'\sinh\pi\sqrt{1-P_2}}{\cosh\pi\sqrt{1-P_2} - \cos\pi\sqrt{P_2}}\,,
\end{aligned}\right\} \tag{11.10}$$

in which $\mathfrak{r}$, $\mathfrak{s}$, $\mathfrak{r}'$, $\mathfrak{s}'$, are the same as in (10.18). Therefore, we have

$$\begin{aligned}
U_{\substack{1\\1}}' \lambda_{\substack{2\\1}} + U_{\substack{2\\1}}' \lambda_{\substack{1\\1}} = i\,(U_{\substack{1\\1}}' \lambda_{\substack{2\\1}} + U_{\substack{2\\1}}' \lambda_{\substack{1\\1}}) = i\,\frac{\sinh\pi\lambda_{\substack{1\\2}}}{4\varphi_1}\Big\{&(R''\,\lambda_{\substack{1\\1}} \cos\pi\lambda_{\substack{1\\1}} - R'''\lambda_{\substack{2\\1}} \cos\pi\lambda_{\substack{2\\1}})\\
&- i\,(Z''\,\lambda_{\substack{1\\1}} \cos\pi\lambda_{\substack{1\\1}} - Z'''\,\lambda_{\substack{2\\1}} \cos\pi\lambda_{\substack{2\\1}})\Big\}\,;
\end{aligned}$$

$$V_{\substack{1\\1}}' \lambda_{\substack{2\\1}} + V_{\substack{2\\1}}' \lambda_{\substack{1\\1}} = -i\,(\bar{U}_{\substack{1\\1}}' \lambda_{\substack{2\\1}} + \bar{U}_{\substack{2\\1}}' \lambda_{\substack{1\\1}}) = \overline{(U_{\substack{1\\1}}' \lambda_{\substack{2\\1}} + U_{\substack{2\\1}}' \lambda_{\substack{1\\1}})}\,.$$

Again, using (10.20)

$$U_{\substack{1\\1}} + U_{\substack{2\\1}} = i\,\frac{\sinh\pi\lambda_{\substack{1\\2}}}{4\varphi_1}\left\{(R\sin\pi\lambda_{\substack{1\\1}} - R'\sin\pi\lambda_{\substack{2\\1}}) + i\,(Z\sin\pi\lambda_{\substack{1\\1}} - Z'\sin\pi\lambda_{\substack{2\\1}})\right\};$$

$$V_{\substack{1\\1}} + V_{\substack{2\\1}} = \bar{U}_{\substack{1\\1}} + \bar{U}_{\substack{2\\1}} = \overline{(U_{\substack{1\\1}} + U_{\substack{2\\1}})}\,.$$

Finally the conditional equation (11.2), in the present case, is transformed into the following form:

$$(R'' \underset{1}{\lambda_1} \cos\pi\underset{1}{\lambda_1} - R''' \underset{1}{\lambda_2} \cos\pi\underset{1}{\lambda_2}) \; (Z \sin\pi\underset{1}{\lambda_1} - Z' \sin\pi\underset{1}{\lambda_2})$$
$$+ (Z'' \underset{1}{\lambda_1} \cos\pi\underset{1}{\lambda_1} - Z''' \underset{1}{\lambda_2} \cos\pi\underset{1}{\lambda_2}) \; (R \sin\pi\underset{1}{\lambda_1} - R' \sin\pi\underset{1}{\lambda_2}) = 0\,.$$

Searching the root of this equation in the limits (11.7), we obtain the following:

$$\begin{cases} P_1 = 2.2547 \\ P_2 = 0.6681 \end{cases}$$

Next, since the above obtained values of P_1 and P_2 lie respectively within the limits of Case-4 and of Case-1 in Table 1, each deflection surface of the elementary plates can be written by using the expressions for such cases in that table. Accordingly,

$$w_1 = K_1 w_1',$$

$$w_1' = \left\{(\cos\pi\underset{1}{\lambda_1}\,\xi_1 - \cos\pi\underset{1}{\lambda_2}\,\xi_1) + 1.951\,938\,(\sin\pi\underset{1}{\lambda_1}\,\xi_1 - \frac{\underset{1}{\lambda_1}}{\underset{1}{\lambda_2}} \sin\pi\underset{1}{\lambda_2}\,\xi_1)\right\} \sin\pi\eta_1\,;$$

$$w_2 = K_1 w_2',$$

$$w_2' = \Big\{2.721\,767 \cosh\pi\underset{2}{\omega_1}\,\xi_2 \cos\pi\underset{2}{\omega_2}\,\xi_2 - 2.065\,730 \sinh\pi\underset{2}{\omega_1}\,\xi_2 \cos\pi\underset{2}{\omega_2}\,\xi_2$$
$$- 0.202\,575 \cosh\pi\underset{2}{\omega_1}\,\xi_2 \sin\pi\underset{2}{\omega_2}\,\xi_2 - 0.225\,955 \sinh\pi\underset{2}{\omega_1}\,\xi_2 \sin\pi\underset{2}{\omega_2}\,\xi_2\Big\} \sin\pi\eta_2.$$

The numerical results by these expressions are given in Table 11 and their graphical representations give Fig 10.

With the above, it becomes that the least characteristic value of practical importance has been obtained.

By the way, let us search yet in the remained intervals

$$3.375 < P_1 < 5.687\,, \qquad 1 < P_2 < 1.685 \tag{11.11}$$

for references.

In the above limits, the expressions of $\underset{1}{\lambda_1}$ and $\underset{1}{\lambda_2}$ do not vary from the preceding [see (10.16) § 10], but those of $\underset{2}{\lambda_1}$ and $\underset{2}{\lambda_2}$ must be rewritten as follows:

$$\left.\begin{aligned} \underset{2}{\lambda_1} &= i\underset{2}{\lambda_1}\,, & \underset{2}{\lambda_1} &= \frac{1}{2}\sqrt{2P_2 - 1 - 2\sqrt{P_2(P_2-1)}}\,, \\ \underset{2}{\lambda_2} &= i\underset{2}{\lambda_2}\,, & \underset{2}{\lambda_2} &= \frac{1}{2}\sqrt{2P_2 - 1 + 2\sqrt{P_2(P_2-1)}}\,. \end{aligned}\right\} \tag{11.12}$$

Then, all of $\underset{1}{\lambda_1}$, $\underset{1}{\lambda_2}$, $\underset{2}{\lambda_1}$ and $\underset{2}{\lambda_2}$ become imaginary.

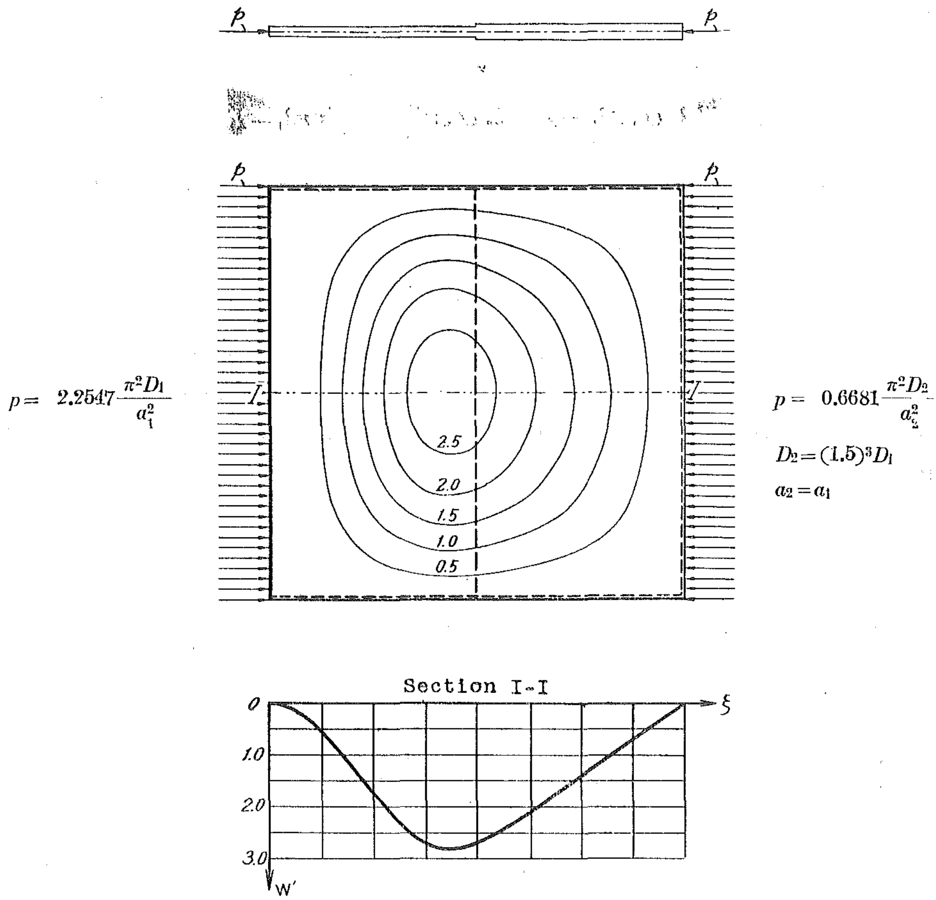


Fig. 10.

Table 11.

η \ ξ_1, ξ_2	w_1' 0.25	w_1' 0.50	w_1' 0.75	w_1' 1.00	w_2' 0	w_2' 0.25	w_2' 0.50	w_2' 0.75
0.125	0.199 56	0.669 61	1.045 55	1.041 56	1.041 56	0.811 30	0.549 23	0.275 87
0.250	0.368 74	1.237 31	1.931 96	1.924 58	1.924 58	1.499 10	1.014 86	0.509 76
0.375	0.481 78	1.616 61	2.524 22	2.514 58	2.514 58	1.958 66	1.325 97	0.666 03
0.500	0.521 48	1.749 81	2.732 19	2.721 76	2.721 76	2.120 04	1.435 22	0.720 91

Therefore it follows that

$$\left.\begin{array}{l}
\left.\begin{matrix}\underset{1}{\tau_1}\\ \underset{1}{\tau_1'}\end{matrix}\right\} = -\left(\frac{1}{2}\right)^2\left\{1.6625-2\sqrt{P_1(P_1-1)}\pm 2\mu_2\sqrt{P_2(P_2-1)}\right\}, \quad \begin{matrix}\underset{1}{\chi_1} = \underset{1}{\lambda_1}\,\underset{2}{\lambda_1}\,\underset{1}{\tau_2'}\,,\\ \underset{1}{\chi_1'} = \underset{1}{\lambda_1}\,\underset{2}{\lambda_2}\,\underset{1}{\tau_2}\,,\end{matrix}\\
\left.\begin{matrix}\underset{1}{\tau_2}\\ \underset{1}{\tau_2'}\end{matrix}\right\} = -\left(\frac{1}{2}\right)^2\left\{1.6625+2\sqrt{P_1(P_1-1)}\pm 2\mu_2\sqrt{P_2(P_2-1)}\right\}, \quad \begin{matrix}\underset{1}{\chi_2} = \underset{1}{\lambda_2}\,\underset{2}{\lambda_1}\,\underset{1}{\tau_1'}\,,\\ \underset{1}{\chi_2'} = \underset{1}{\lambda_2}\,\underset{2}{\lambda_2}\,\underset{1}{\tau_1}\,.\end{matrix}
\end{array}\right\} \qquad (11.13)$$

Comparing the present case with the previous case in this section, we can find that the foregoing conditional equation (11.6) is also applicable to this case, provided that we use these $i\underset{1}{\lambda_1}$ and $i\underset{2}{\lambda_1}$ instead of those $\underset{1}{\lambda_1}$ and $\underset{2}{\lambda_1}$, and replace the foregoing (10.10) by the present (11.13), because the other notations are used in a similar way in both cases as far as both boundary conditions are similar with each other. Therefore, searching the root of the conditional equation within the limits (11.11), the second root is obtained as follows:

$$\begin{cases} P_1 = 3.7394 \\ P_2 = 1.1080 \end{cases}$$

Since the above P_1 and P_2 lie in the limits of Case-4 in Table 1, the following expressions of the deflection surfaces are obtained by taking the terms of such a case:

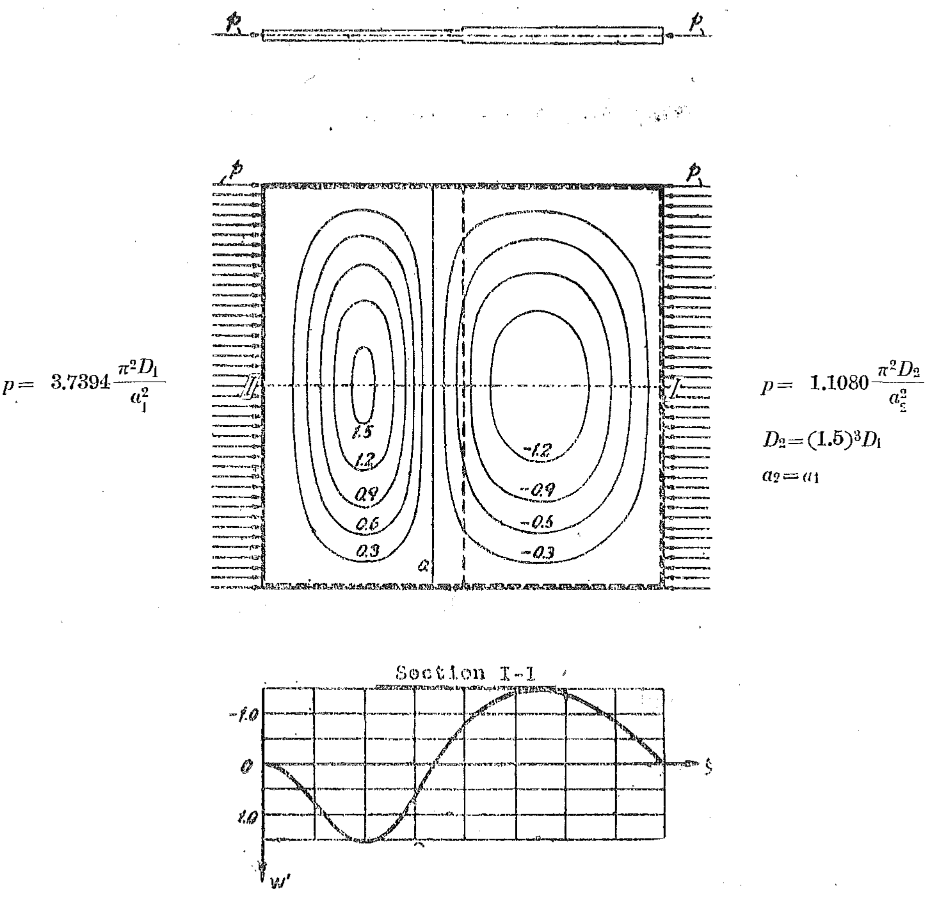


Fig. 11

$$w_1 = K_1\, w_1',$$

$$w_1' = \left\{ (\cos\pi \underset{1}{\lambda_1}\, \xi_1 - \cos\pi \underset{1}{\lambda_2}\, \xi_1) - 1.876\,703\, (\sin\pi \underset{1}{\lambda_1}\, \xi_1 - \frac{\underset{1}{\lambda_1}}{\underset{1}{\lambda_2}} \sin\pi \underset{1}{\lambda_2}\, \xi_1) \right\} \sin\pi\eta_1\,;$$

$$w_2 = K_1\, w_2'\,,$$

$$w_2' = \left\{ 1.382\,478 \sin\pi \underset{2}{\lambda_1}\, (1-\xi_2) - 2.458\,218 \sin \underset{2}{\lambda_2}\, (1-\xi_2) \right\} \sin\pi\eta_2\,.$$

The numerical results are given in Table 12 and then Fig. 11 is drawn.

Table 12.

$\xi_1,\ \xi_2$ / η	w_1'				w_2'			
	0.25	0.50	0.75	1.00	0	0.25	0.50	0.75
0.125	0.295 48	0.598 23	0.259 56	−0.296 95	−0.296 95	−0.540 72	−0.546 88	−0.337 17
0.250	0.545 98	1.105 40	0.479 62	−0 548 70	−0.548 71	−0.999 13	−1.010 52	−0.623 03
0.375	0.713 35	1.444 27	0.626 65	−0.716 91	−0.716 92	−1.305 42	−1.320 31	−0.814 02
0.500	0.772 13	1.563 27	0.678 29	−0.775 98	−0.775 98	−1.412 98	−1.429 09	−0.881 09

§ 12. Case when Both End Sides are Clamped

From (9.b) [§ 6], the conditions at End side-2 are

$$\left.\begin{aligned}
U_{\substack{2\\2}} &= 1\,; & U_{\substack{1\\2}} &= \frac{\underset{2}{\lambda_1}}{\underset{2}{\lambda_2}} \sinh\pi\underset{2}{\lambda_1} \sinh\pi\underset{2}{\lambda_2} - \cosh\pi\underset{2}{\lambda_1} \cosh\pi\underset{2}{\lambda_2}\,;\\
U_{\substack{2\\2}}' &= 0\,; & U_{\substack{1\\2}}' &= -\frac{\underset{2}{\lambda_1}}{\underset{2}{\lambda_2}} \sinh\pi\underset{2}{\lambda_1} \cosh\pi\underset{2}{\lambda_2} + \cosh\pi\underset{2}{\lambda_1} \sinh\pi\underset{2}{\lambda_2}\,;\\
V_{\substack{2\\2}} &= 0\,; & V_{\substack{1\\2}} &= \frac{\underset{2}{\lambda_1}}{\underset{2}{\lambda_2}} \cosh\pi\underset{2}{\lambda_1} \sinh\pi\underset{2}{\lambda_2} - \sinh\pi\underset{2}{\lambda_1} \cosh\pi\underset{2}{\lambda_2}\,;\\
V_{\substack{2\\2}}' &= 1\,; & V_{\substack{1\\2}}' &= -\frac{\underset{2}{\lambda_1}}{\underset{2}{\lambda_2}} \cosh\pi\underset{2}{\lambda_1} \cosh\pi\underset{2}{\lambda_2} + \sinh\pi\underset{2}{\lambda_1} \sinh\pi\underset{2}{\lambda_2}\,.
\end{aligned}\right\} \tag{12.1}$$

Then, the recurrence formulas (9) become

$$\left.\begin{aligned}
U_{\substack{2\\1}} &= -F_{\substack{2\\1}} - U_{\substack{1\\2}} H_{\substack{2\\1}} + U_{\substack{1\\2}}' I_{\substack{2\\1}}'\,; & V_{\substack{2\\1}} &= G_{\substack{2\\1}}' - V_{\substack{1\\2}} H_{\substack{2\\1}} + V_{\substack{1\\2}}' I_{\substack{2\\1}}'\,;\\
U_{\substack{2\\1}}' &= F_{\substack{2\\1}}' + U_{\substack{1\\2}} H_{\substack{2\\1}}' - U_{\substack{1\\2}}' I_{\substack{2\\1}}\,; & V_{\substack{2\\1}}' &= -G_{\substack{2\\1}} + V_{\substack{1\\2}} H_{\substack{2\\1}}' - V_{\substack{1\\2}}' I_{\substack{2\\1}}\,;\\
U_{\substack{1\\1}} &= F_{\substack{1\\1}} + U_{\substack{1\\2}} H_{\substack{1\\1}} - U_{\substack{1\\2}}' I_{\substack{1\\1}}'\,; & V_{\substack{1\\1}} &= -G_{\substack{1\\1}}' + V_{\substack{1\\2}} H_{\substack{1\\1}} - V_{\substack{1\\2}}' I_{\substack{1\\1}}'\,;\\
U_{\substack{1\\1}}' &= -F_{\substack{1\\1}}' - U_{\substack{1\\2}} H_{\substack{1\\1}}' + U_{\substack{1\\2}}' I_{\substack{1\\1}}\,; & V_{\substack{1\\1}}' &= G_{\substack{1\\1}} - V_{\substack{1\\2}} H_{\substack{1\\1}}' + V_{\substack{1\\2}}' I_{\substack{1\\1}}\,.
\end{aligned}\right\} \tag{12.2}$$

The conditional equation remains in the same form as the foregoing (11.2):

$$(U_1+U_2)\,(V_1'\lambda_2+V_2'\lambda_1)-(V_1+V_2)\,(U_1'\lambda_2+U_2'\lambda_1)=0\,.$$

(1) *The case where only the load q_r is acting.*—Comparing the present case with the case (1) in the previous section, it can be supposed that the rigidity of the present plate is larger than that in the previous case due to the difference between both the boundary conditions. Then, we can assume

$$Q_1 > 8.512\,, \qquad Q_2 > 3.783$$

For the upper limit, the example which has been referred for comparison in the previous case (1) can be used once more. Accordingly, we easily obtain the following inequality:

$$8.512 < Q_1 < 17.302 \qquad 3.783 < Q_2 < 7.69 \tag{12.3}$$

Since these are wholly included within the intervals of (11.3), the expressions of λ_r, τ_r, χ_r become the same as in the previous case (1),[26] and then, also, those of F_r, G_r, H_r, I_r and F_r', G_r', H_r', I_r' become the same.[27] Therefore, substituting these expressions and (12.1)[28] in (12.2), it is easy to write the reduced equation of (11.2) in detail. But, since such an equation, in this case, will become rather tedious, then it is advantageous that the numerical computations are performed step by step with respect to each of the factors.[29]

Proceeding as above, the least root can be obtained in the limits (12.3). That is

$$\begin{cases} Q_1 = 12.6805 \\ Q_2 = \ \ 5.6358 \end{cases}$$

Considering that both of Q_1 and Q_2 lie within the limits of Case-2 in Table 1, we can write the expressions of the deflection surfaces by using the forms of the functions in such a case, *i. e.*,

$$w_1 = K_1\, w_1',$$

$$w_1' = \left\{ \cosh\pi\lambda_1\,\xi_1 - 1.000\,753\,\sinh\pi\lambda_1\,\xi_1 - \cos\pi\lambda_2\,\xi_1 + 1.335\,529\,\sin\pi\lambda_2\,\xi_1 \right\} \sin\pi\eta_1\,;$$

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26) See (10.9) and (10.10)

27) See (10.11) and (11.4)

28) In which the following expressions are used:

$$U_{\substack{1\\2}} = \frac{\lambda_{\substack{1\\2}}}{\lambda_{\substack{2\\2}}}\sinh\pi\lambda_{\substack{1\\2}}\sin\pi\lambda_{\substack{2\\2}} - \cosh\pi\lambda_{\substack{1\\2}}\cos\pi\lambda_{\substack{2\\2}}\,, \qquad V_{\substack{1\\2}} = \frac{\lambda_{\substack{1\\2}}}{\lambda_{\substack{2\\2}}}\cosh\pi\lambda_{\substack{1\\2}}\sin\pi\lambda_{\substack{2\\2}} - \sinh\pi\lambda_{\substack{1\\2}}\cos\pi\lambda_{\substack{2\\2}}\,,$$

$$U_{\substack{1\\2}}' = i\left(\frac{\lambda_{\substack{1\\2}}}{\lambda_{\substack{2\\2}}}\sinh\pi\lambda_{\substack{1\\2}}\cos\pi\lambda_{\substack{2\\2}} + \cosh\pi\lambda_{\substack{1\\2}}\sin\pi\lambda_{\substack{2\\2}}\right), \qquad V_{\substack{1\\2}}' = i\left(\frac{\lambda_{\substack{1\\2}}}{\lambda_{\substack{2\\2}}}\cosh\pi\lambda_{\substack{1\\2}}\cos\pi\lambda_{\substack{2\\2}} + \sinh\pi\lambda_{\substack{1\\2}}\sin\pi\lambda_{\substack{2\\2}}\right),$$

29) This process can not be avoided when the number of the elementary plates is large.

$q_1 = 12.6805 \dfrac{\pi^2 D_1}{b^2}$

$q_2 = 5.6358 \dfrac{\pi^2 D_2}{b^2}$

$D_2 = (1.5)^3 D_1$

$q_2/q_1 = 1.5$

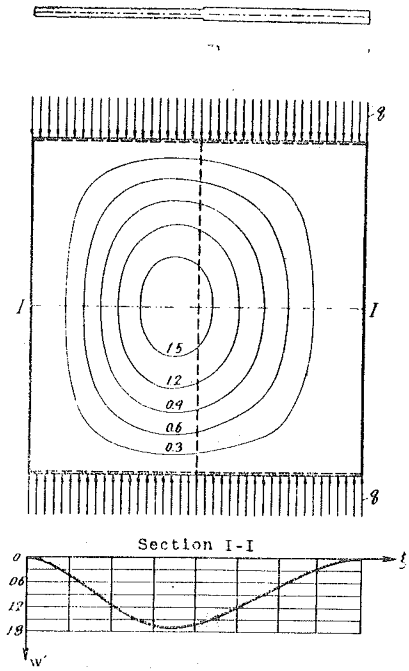


Fig. 12.

$$w_2 = K_1 w_2',$$

$$w_2' = \Big\{ 0.292\,306 \cosh \pi\lambda_{\substack{1\\2}} \xi_2 - 0.207\,917 \sinh \pi\lambda_{\substack{1\\2}} \xi_2 + 1.325\,631 \cos \pi\lambda_{\substack{2\\2}} \xi_2 - 0.431\,144 \sin \pi\lambda_{\substack{2\\2}} \xi_2 \Big\} \sin \pi\eta_2.$$

Their numerical results are given in Table 13 and then Fig. 12 is obtained.

Table 13.

ξ_1, ξ_2 / η	w_1'				w_2'			
	0.25	0.50	0.75	1.00	0	0.25	0.50	0.75
0.125	0.156 04	0.438 70	0.633 53	0.619 15	0.619 15	0.460 94	0.266 70	0.085 27
0.250	0.288 33	0.810 63	1.170 64	1.144 06	1.144 05	0.851 71	0.492 80	0.157 57
0.375	0.376 72	1.059 13	1.529 51	1.494 78	1.494 77	1.112 82	0.643 88	0.205 88
0.500	0.407 76	1.146 39	1.655 53	1.617 94	1.617 93	1.204 50	0.696 93	0.222 84

(2) *The case where only the load p is acting.*—It has been known that the least characteristic value of a square single plate becomes 6.743, if the conditions of load and boundary are the same ones. Similarly proceeding in the way how (10.15) is obtained in [case (2), § 10], we have

$$1.686 < P_1 < 5.689, \qquad 0.499 < P_2 < 1.685$$

Besides, we can easily suppose that P in this case is larger than the one of [case (2), § 11] and then the above inequality can be more closely modified, *i. e.*,

$$2.255 < P_1 < 5.689, \qquad 0.668 < P_2 < 1.685 \tag{12.4}$$

Especially in these, the partial intervals

$$2.255 < P_1 < 3.375, \qquad 0.668 < P_2 < 1 \tag{12.5}$$

are included within the limits (11.7) of the case (2) in § 11, and then λ_r, τ_r, χ_r, F_r,

G_r, H_r, I_r and F_r', G_r', H_r', I_r' become of the same form as those in the foregoing case; in other words the expressions (10.16), (10.17), (10.19) and (11.8) can also be used in this case. Referring to (12.1), U_r, U_r', V_r, V_r' for $r=2$ which need to be used for the recurrence formulas (12.2) are now written as follows:

$$U_{\frac{1}{2}} = -\left\{P_2\cosh\pi\sqrt{1-P_2}+(1-P_2)\cos\pi\sqrt{P_2}\right\}+i\sqrt{P_2(1-P_2)}\left\{\cosh\pi\sqrt{1-P_2}-\cos\pi\sqrt{P_2}\right\};$$

$$U_{\frac{1}{2}}' = \left\{P_2\sinh\pi\sqrt{1-P_2}+\sqrt{P_2(1-P_2)}\sin\pi\sqrt{P_2}\right\}$$
$$-i\left\{(1-P_2)\sin\pi\sqrt{P_2}+\sqrt{P_2(1-P_2)}\sinh\pi\sqrt{1-P_2}\right\};$$

$$V_{\frac{1}{2}} = -\left\{P_2\sinh\pi\sqrt{1-P_2}-\sqrt{P_2(1-P_2)}\sin\pi\sqrt{P_2}\right\}$$
$$-i\left\{(1-P_2)\sin\pi\sqrt{P_2}-\sqrt{P_2(1-P_2)}\sinh\pi\sqrt{1-P_2}\right\};$$

$$V_{\frac{1}{2}}' = \left\{P_2\cosh\pi\sqrt{1-P_2}-(1-P_2)\cos\pi\sqrt{P_2}\right\}+i\sqrt{P_2(1-P_2)}\left\{\cosh\pi\sqrt{1-P_2}+\cos\pi\sqrt{P_2}\right\}.$$

Substituting these in (12.2), we can rewrite the conditional equation (11.2) in a detailed form. But, thus obtained expressions may become of a rather complex form, and then it will, also in this case, be advantageous to promote calculations step by step with respect to each of the factors. Searching the least root of (11.2) within the limits (12.5), we obtain

$$\begin{cases} P_1 = 2.5819 \\ P_2 = 0.7650 \end{cases}$$

Observing that the above P_1 lies within the limits of Case-4 and the P_2 lies within those of Case-1 respectively in Tabe 1, the deflection surfaces are expressed by using the proper functions tabulated in Table 1 as follows:

$$w_1 = K_1 w_1',$$

$$w_1' = \left\{(\cos\pi\lambda_{\frac{1}{1}}\xi_1-\cos\pi\lambda_{\frac{2}{1}}\xi_1)+2.078\,457\,(\sin\pi\lambda_{\frac{1}{1}}\xi_1-\frac{\lambda_{\frac{1}{1}}}{\lambda_{\frac{2}{1}}}\sin\pi\lambda_{\frac{2}{1}}\xi_1)\right\}\sin\pi\eta_1;$$

$$w_2 = K_1 w_2',$$

$$w_2' = \left\{2.395\,588\cosh\pi\omega_{\frac{1}{2}}\xi_2\cos\pi\omega_{\frac{2}{2}}\xi_2-3.758\,034\sinh\pi\omega_{\frac{1}{2}}\xi_2\cos\pi\omega_{\frac{2}{2}}\xi_2\right.$$
$$\left.-0.443\,412\cosh\pi\omega_{\frac{1}{2}}\xi_2\sin\pi\omega_{\frac{2}{2}}\xi_2+0.695\,973\sinh\pi\omega_{\frac{1}{2}}\xi_2\sin\pi\omega_{\frac{2}{2}}\xi_2\right\}\sin\pi\eta_2.$$

By these, the numerical results in Table 14 and the illustration of Fig. 13 are obtained.

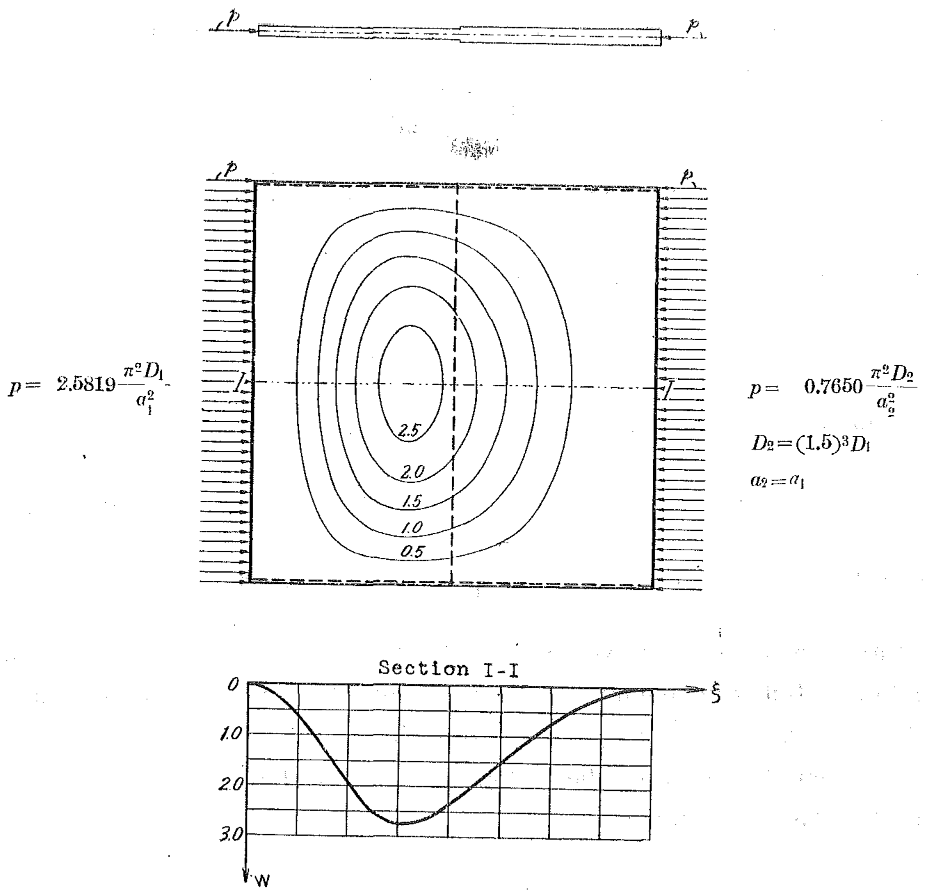


Fig. 13.

Table 14.

ξ_1, ξ_2 / η	w_1'				w_2'			
	0.25	0.50	0.75	1.00	0	0.25	0.50	0.75
0.125	0.234 64	0.748 42	1.062 89	0.916 74	0.916 74	0.578 54	0.277 52	0.072 52
0.250	0.433 57	1.382 92	1.964 00	1.693 94	1.693 94	1.069 03	0.512 81	0.134 01
0.375	0.566 49	1.806 86	2.566 08	2.213 23	2.213 23	1.396 75	0.670 01	0.175 09
0.500	0.613 16	1.955 73	2.777 50	2.395 58	2.395 58	1.511 83	0.725 22	0.189 51

And yet, let us investigate within the remained intervals:

$$3.375 < P_1 < 5.689\,, \qquad 1 < P_2 < 1.685$$

for references. These searching ranges coincide with those in [case (2), § 11], *i.e.*, (11.11), and therefore the expressions of λ_r, τ_r, χ_r, F_r, G_r, H_r, I_r and F_r', G_r', H_r', I_r' will remain as before. Then, using the above described expressions and

(12.1),[30] we can compute (12.2) and from this the numerical values of the left side of (11.2) are obtained. These procedures are similar to those in the preceding examples. Hence we get

$$\begin{cases} P_1 = 4.2814 \\ P_2 = 1.2686 \end{cases}$$

Since both P_1 and P_2 lie within the limits of Case-4 in Table 1, then the deflection surfaces are expressed by using the functions of that case as follows:

$$w_1 = K_1\, w_1' ,$$

$$w_1' = \left\{(\cos\pi\underset{1}{\lambda_1}\,\xi_1 - \cos\pi\underset{1}{\lambda_2}\,\xi_1) - 2.937\,336\,(\sin\pi\underset{1}{\lambda_1}\,\xi_1 - \frac{\underset{1}{\lambda_1}}{\underset{1}{\lambda_2}}\sin\pi\underset{1}{\lambda_2}\,\xi_1)\right\}\sin\pi\eta_1 \,;$$

$$w_2 = K_1\, w_2' ,$$

$$w_2' = \left\{0.380\,189\cos\pi\underset{2}{\lambda_1}\,\xi_2 - 2.187\,220\sin\pi\underset{2}{\lambda_1}\,\xi_2 - 1.636\,503\cos\pi\underset{2}{\lambda_2}\,\xi_2 \right.$$

$$\left. + 0.336\,294\sin\pi\underset{2}{\lambda_2}\,\xi_2\right\}\sin\pi\eta_2 \,.$$

From the above, Table 15 and Fig. 14 are obtained.

Table 15.

ξ_1, ξ_2 / η	w_1'				w_2'			
	0.25	0.50	0.75	1.00	0	0.25	0.50	0.75
0.125	0.323 79	0.536 91	0.008 88	−0.480 76	−0.480 76	−0.479 29	−0.304 36	−0.095 26
0.250	0.598 30	0.992 09	0.016 42	−0.888 34	−0.888 35	−0.885 63	−0.562 40	−0.176 02
0.375	0.781 71	1.296 22	0.021 46	−1.160 67	−1.160 68	−1.157 13	−0.734 81	−0.229 98
0.500	0.846 12	1.403 02	0.023 22	−1.256 30	−1.256 31	−1.252 47	−0.795 35	−0.248 93

Furthermore, for the third root, the fourth and so on beyond the foregoing limits, the similar expressions of factors as for the second root can be used for computation.

For instance, the third root is acquired as

$$\begin{cases} P_1 = 8.1215 \\ P_2 = 2.4064 \end{cases}$$

30) In these expressions,

$$\underset{2}{U_1} = -\left(\frac{\underset{2}{\lambda_1}}{\underset{2}{\lambda_2}}\sin\pi\underset{2}{\lambda_1}\sin\pi\underset{2}{\lambda_2} + \cos\pi\underset{2}{\lambda_1}\cos\pi\underset{2}{\lambda_2}\right), \qquad \underset{2}{V_1} = i\left(\frac{\underset{2}{\lambda_1}}{\underset{2}{\lambda_2}}\cos\pi\underset{2}{\lambda_1}\sin\pi\underset{2}{\lambda_2} - \sin\pi\underset{2}{\lambda_1}\cos\pi\underset{2}{\lambda_2}\right),$$

$$\underset{2}{U_1}' = -i\left(\frac{\underset{2}{\lambda_1}}{\underset{2}{\lambda_2}}\sin\pi\underset{2}{\lambda_1}\cos\pi\underset{2}{\lambda_2} - \cos\pi\underset{2}{\lambda_1}\sin\pi\underset{2}{\lambda_2}\right), \qquad \underset{2}{V_1}' = -\left(\frac{\underset{2}{\lambda_1}}{\underset{2}{\lambda_2}}\cos\pi\underset{2}{\lambda_1}\cos\pi\underset{2}{\lambda_2} + \sin\pi\underset{2}{\lambda_1}\sin\pi\underset{2}{\lambda_2}\right),$$

are used.

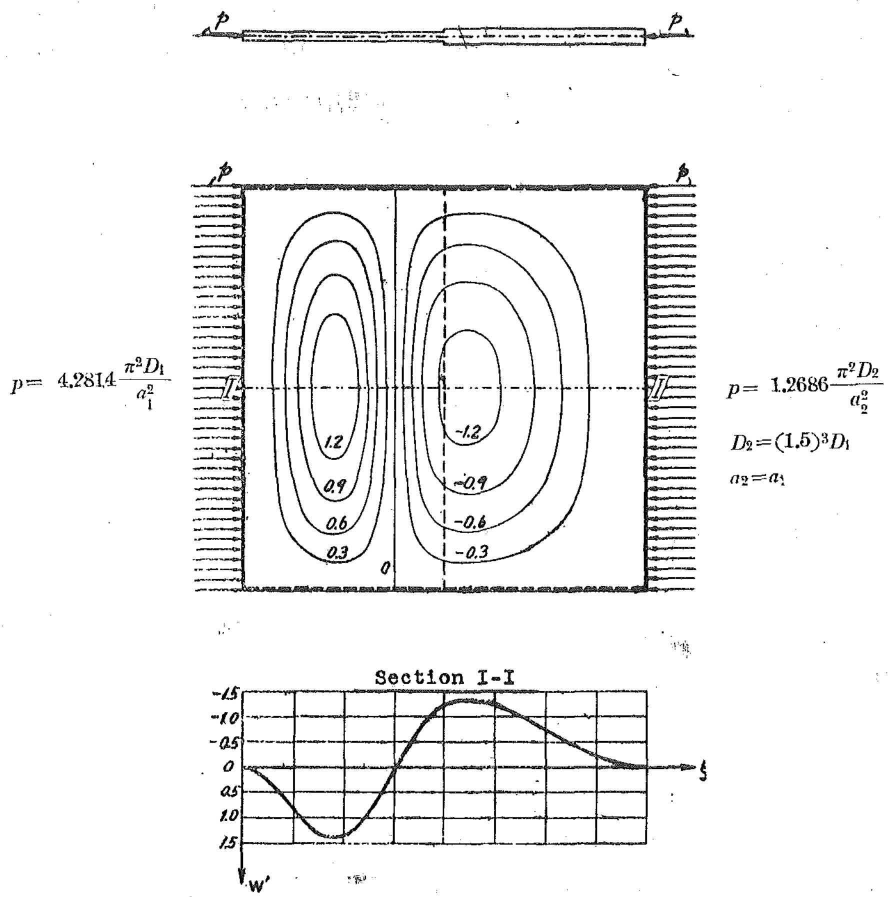


Fig. 14.

and the fourth root is

$$\begin{cases} P_1 = 10.7575 \\ P_2 = \ \ 3.1874 \end{cases}$$

The deflection surfaces for these are expressed, by using the functions of Case-4 in Table 1, in the following forms respectively.

That is, for the third root

$$w_1 = K_1 w_1' ,$$

$$w_1' = \left\{(\cos\pi\underset{1}{\lambda_1}\xi_1 - \cos\pi\underset{1}{\lambda_2}\xi_1) + 2.409\,083(\sin\pi\underset{1}{\lambda_1}\xi_1 - \frac{\underset{1}{\lambda_1}}{\underset{1}{\lambda_2}}\sin\pi\underset{1}{\lambda_2}\xi_1)\right\}\sin\pi\eta_1 \ ;$$

$$w_2 = K_1 w_2' ,$$

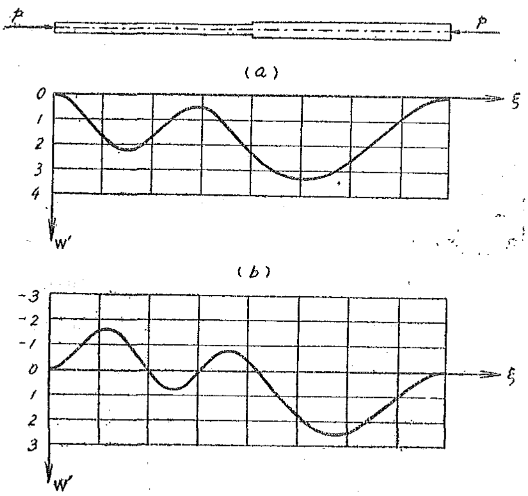
$$w_2' = \Big\{1.590\,001\cos\pi\underset{2}{\lambda_1}\xi_2 + 0.676\,857\sin\pi\underset{2}{\lambda_1}\xi_2 + 0.719\,264\cos\pi\underset{2}{\lambda_2}\xi_2$$

$$+ 1.543\,869\sin\pi\underset{2}{\lambda_2}\xi_2\Big\}\sin\pi\eta_2 \,.$$

By these, requiring the deflection line along $\eta=0.5$, Table 16 and Fig. 15 (a) are obtained.

Table 16.

ξ	0.125	0.250	0.375	0.500	0.625	0.750	0.875	1.000
w_1'	0.546 95	1.664 87	2.253 65	1.774 37	0.822 48	0.494 28	1.220 78	2.309 26
w_2'	3.042 65	3.370 58	—	2.615 28	—	0.882 56	0.236 67	0



$p=8.1215\dfrac{\pi^2 D_1}{a_1^2}$ $\qquad$ $p=2.4064\dfrac{\pi^2 D_2}{a_2^2}$

$D_2=(1.5)^3 D_1$

$a_2=a_1$

$p=10.7575\dfrac{\pi^2 D_1}{a_1^2}$ $\qquad$ $p=3.1874\dfrac{\pi^2 D_2}{a_2^2}$

$D_2=(1.5)^3 D_1$

$a_2=a_1$

Fig. 15.

For the fourth root

$$w_1=K_1\, w_1',$$

$$w_1'=\left\{(\cos\pi\underset{1}{\lambda_1}\,\xi_1-\cos\pi\underset{1}{\lambda_2}\,\xi_1)-5.273\,009\,\left(\sin\pi\underset{1}{\lambda_1}\,\xi_1-\frac{\underset{1}{\lambda_1}}{\underset{1}{\lambda_2}}\sin\pi\underset{1}{\lambda_2}\,\xi_1\right)\right\}\sin\pi\eta_1\,;$$

$$w_2=K_1\, w_2',$$

$$w_2'=\left\{-0.407\,752\cos\pi\underset{2}{\lambda_1}\,\xi_2-2.653\,156\sin\pi\underset{2}{\lambda_1}\,\xi_2+0.826\,894\cos\pi\underset{2}{\lambda_2}\,\xi_2\right.$$

$$\left.-1.372\,524\sin\pi\underset{2}{\lambda_2}\,\xi_2\right\}\sin\pi\eta_2\,.$$

By these, requiring the deflection line along $\eta=0.5$, Table 17 and Fig. 15 (b) are obtained.

Table 17.

ξ	0.125	0.250	0.375	0.500	0.625	0.750	0.875	1.000
w_1'	0.651 84	1.560 78	1.243 38	−0.086 52	−0.816 55	−0.164 09	0.740 88	0.419 07
w_2'	−0.724 14	−1.803 28	−2.450 78	−2.470 53	—	−1.060 10	−0.303 92	0

§ 13. Case when One End Side is Free and the Other is Simply Supported.

Let us assume that End side-2 is simply supported, then the boundary conditions in this case are nothing else those obtained by replacing the clamped edge in the cases taken in § 11 with a free edge. From this, we can see that the expressions (11.1) are applicable to this case, but as the conditional equation, the following equation must be used, *i. e.*, from (9.f)

$$(\underset{1}{U_1}\,\underset{1}{\beta_2}+\underset{1}{U_2}\,\underset{1}{\beta_1})\Big\{\underset{1}{V_1}'(\underset{1}{\gamma_2}+P_1\,\underset{1}{\lambda_2})+\underset{1}{V_2}'(\underset{1}{\gamma_1}+P_1\,\underset{1}{\lambda_1})\Big\}$$
$$-(\underset{1}{V_1}\,\underset{1}{\beta_2}+\underset{1}{V_2}\,\underset{1}{\beta_1})\Big\{\underset{1}{U_1}'(\underset{1}{\gamma_2}+P_1\,\underset{1}{\lambda_2})+\underset{1}{U_2}'(\underset{1}{\gamma_1}+P_1\,\underset{1}{\lambda_1})\Big\}=0\,. \tag{13.1}$$

Let us consider next about the case where only the load q_r is acting. It is already known that the least characteristic value Q of a single plate whose *Poisson's ratio* ν is 0.25, under the similar load and boundary conditions, is 1.440.[31] Then, by using this for comparison, we obtain the limits for the least root. That is

$$1.440 < Q_1 < 3.240\,, \qquad 0.640 < Q_2 < 1.440 \tag{13.2}$$

Observing these, we can write as

$$\left.\begin{aligned}
&\underset{1}{\lambda_1}=\frac{1}{2}\sqrt{\sqrt{Q_1}+1}\,, && \underset{2}{\lambda_1}=\frac{1}{2}\sqrt{\sqrt{Q_2}+1}\,,\\
&\underset{1}{\lambda_2}=i\underset{1}{\lambda_2},\quad \underset{1}{\lambda_2}=\frac{1}{2}\sqrt{\sqrt{Q_1}-1}\,, && \underset{2}{\lambda_2}=\begin{cases}\dfrac{1}{2}\sqrt{1-\sqrt{Q_2}} & \text{when } 0.64<Q_2<1,\\[2ex] i\underset{1}{\lambda_2},\quad \underset{1}{\lambda_2}=\dfrac{1}{2}\sqrt{\sqrt{Q_2}-1} & \text{when } 1<Q_2<1.44.\end{cases}
\end{aligned}\right\} \tag{13.3}$$

In the first place, let us investigate within the following partial intervals:

$$1.440 < Q_1 < 2.250\,, \qquad 0.640 < Q_2 < 1 \tag{13.4}$$

Compare the expressions of λ in the above with those[32] in the case (1) in § 11, then we find a difference only at the point where $\underset{2}{\lambda_2}$ becomes real quantity in this case. Therefore, from the already obtained expressions of β_r, γ_r, τ_r, τ_r', χ_r, χ_r', and F_r, G_r, H_r, I_r, F_r', G_r', H_r', I_r' and U_r, V_r, U_r', V_r' for $r=1$, we can obtain those for the present case by the substitution $\underset{2}{\lambda_2}=\dfrac{\underset{2}{\lambda_2}}{i}$. Thus, The final equation (13.1) is rewritten as follows:

$$\left(R\,\frac{\underset{1}{\beta_1}}{\underset{1}{\lambda_1}}\sinh\pi\underset{1}{\lambda_1}-R'\,\frac{\underset{1}{\beta_2}}{\underset{1}{\lambda_2}}\sin\pi\underset{1}{\lambda_2}\right)\left(Z''\,\frac{\cosh\pi\underset{1}{\lambda_1}}{\underset{1}{\beta_1}}+Z'''\,\frac{\cos\pi\underset{1}{\lambda_2}}{\underset{1}{\beta_2}}\right)$$

---

31) See S. Timoshenko, "Theory of Elastic Stability," 1936. p.339, Table 32.

32) See (10.9).

$$-\left(Z\frac{\beta_{\substack{1\\1}}}{\lambda_{\substack{1\\1}}}\sinh\pi\lambda_{\substack{1\\1}}-Z'\frac{\beta_{\substack{2\\1}}}{\lambda_{\substack{2\\1}}}\sin\pi\lambda_{\substack{2\\1}}\right)\left(R''\frac{\cosh\pi\lambda_{\substack{1\\1}}}{\beta_{\substack{1\\1}}}+R'''\frac{\cos\pi\lambda_{\substack{2\\1}}}{\beta_{\substack{2\\1}}}\right)=0\,.$$

Searching the root of this equation within the limits (13.4), we have

$$\begin{cases} Q_1 = 1.9811 \\ Q_2 = 0.8805 \end{cases}$$

The deflection surfaces, by using the expressions of Case-2 in Table 1 for w_1 and those[33] of Case-3 for w_2, are respectively expressed in the following forms:

$$w_1 = K_1 w_1'\,,$$

$$w_1' = \left\{\left(\cosh\pi\lambda_{\substack{1\\1}}\xi_1+\frac{\beta_{\substack{1\\1}}}{\beta_{\substack{2\\1}}}\cos\pi\lambda_{\substack{2\\1}}\xi_1\right)-0.915\,471\left(\sinh\pi\lambda_{\substack{1\\1}}\xi_1+\frac{\gamma_{\substack{1\\1}}}{\gamma_{\substack{2\\1}}}\sin\pi\lambda_{\substack{2\\1}}\xi_1\right)\right\}\sin\pi\eta_1\,;$$

$$w_2 = K_1 w'_2\,,$$

$$w_2' = \left\{0.062\,651\sinh\pi\lambda_{\substack{1\\2}}(1-\xi_2)+3.160\,924\sinh\pi\lambda_{\substack{2\\2}}(1-\xi_2)\right\}\sin\pi\eta_2\,.$$

The numerical results are given in Table 18 and then Fig. 16 is obtained.

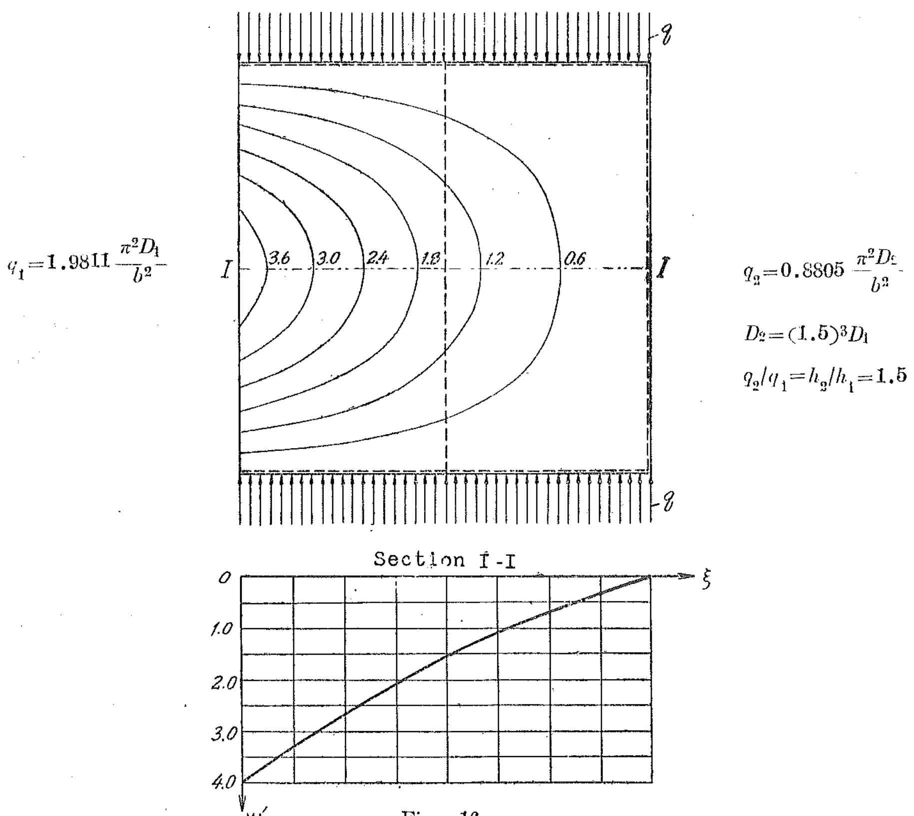


Fig. 16.

33) For β_r and γ_r, the proper expressions must be picked out from Table 3. It is ditto hereafter.

Table 18

η \ ξ_1, ξ_2	w_1' 0	0.25	0.50	0.75	1.00	w_2' 0	0.25	0.50	0.75
0.125	1.522 60	1.262 40	1.025 29	0.796 72	0.589 39	0.589 37	0.419 12	0.269 16	0.132 96
0.250	2.813 43	2.332 64	1 894 53	1.472 17	1 089 08	1.089 03	0 774 44	0.497 36	0.245 68
0 375	3.675 92	3.047 73	2.475 31	1.923 48	1.422 94	1.422 89	1.011 85	0.649 83	0.321 00
0.500	3.978 78	3.298 84	2.679 25	2.081 96	1.540 18	1.540 12	1.095 22	0.703 37	0 347 45

§ 14. Case when One End Side is Free and the Other is Clamped.

Let us assume that End side-2 is clamped, then the boundary conditions in this case are nothing else those obtained by replacing End side-1 in the cases studied in § 12 with a free edge. The formulas (12.1) and (12.2) are applicable to this case and again Eq. (13.1) is used as the conditional equation of this case.

Now, let us consider the case where only the load q_r is acting. Taking the single plate whose *Poisson's ratio* ν is 0.25 for comparison, it is already known that its least characteristic value Q is 1.70,[34] if the load and boundary conditions are the same as those of the present case. From this, we can obtain the following limits for the least root.

$$1.70 < Q_1 < 3.825\,, \qquad 0.755 < Q_2 < 1.70$$

And yet, let us compare the present case with that of § 13. Then, since it is obviously considered that the least root in this case is larger than the one in § 13, the above obtained limits are more closely modified as follows:

$$1.981 < Q_1 < 3.825\,, \qquad 0.881 < Q_2 < 1.70$$

Observing that the above intervals have large overlap with (13.2), we see that the foregoing (13.3) can be used also as the expression of λ_r in this case. Therefore $\underset{1}{\lambda_1}$, $\underset{2}{\lambda_1}$ and $\underset{2}{\lambda_2}$ become real and $\underset{1}{\lambda_2}$ imaginary, when

$$1.981 < Q_1 < 2.25\,, \qquad 0.881 < Q_2 < 1 \tag{14.1}$$

Again, $\underset{1}{\lambda_1}$ and $\underset{2}{\lambda_1}$ become real and $\underset{2}{\lambda_1}$ and $\underset{2}{\lambda_2}$ imaginary, when

$$2.25 < Q_1 < 3.825 \qquad 1 < Q_2 < 1.70 \tag{14.2}$$

Compare the present case with case (1) in § 12, then the expressions of each factors up to U, U', V, V' are used in the same form as before because the load condition and the boundary condition at End side-2 are the same for both cases. But, in the intervals (14.1), we must take the substitution $\underset{2}{\lambda_2} = \dfrac{\underset{2}{\lambda_2}}{i}$ in the former expressions,

34) See S. Timoshenko, "Theory of Elastic Stability," 1936, p.342, Table 33.

since λ_2 is imaginary in case (1) in § 12. On the other hand, in the intervals (14.2) the expressions in both cases become the same ones. In this case too, since to write the final equation (13.1) in detail becomes tedious, the calculations should be performed step by step with respect to each factor.

Now, we find that no roots exist within the intervals (14.1), but within (14.2) the following root is obtained:

$$\begin{cases} Q_1 = 2.3485 \\ Q_2 = 1.0438 \end{cases}$$

Accordingly, by using the functions of Case-2 in Table 1, we obtain the following expressions of the deflection surfaces:

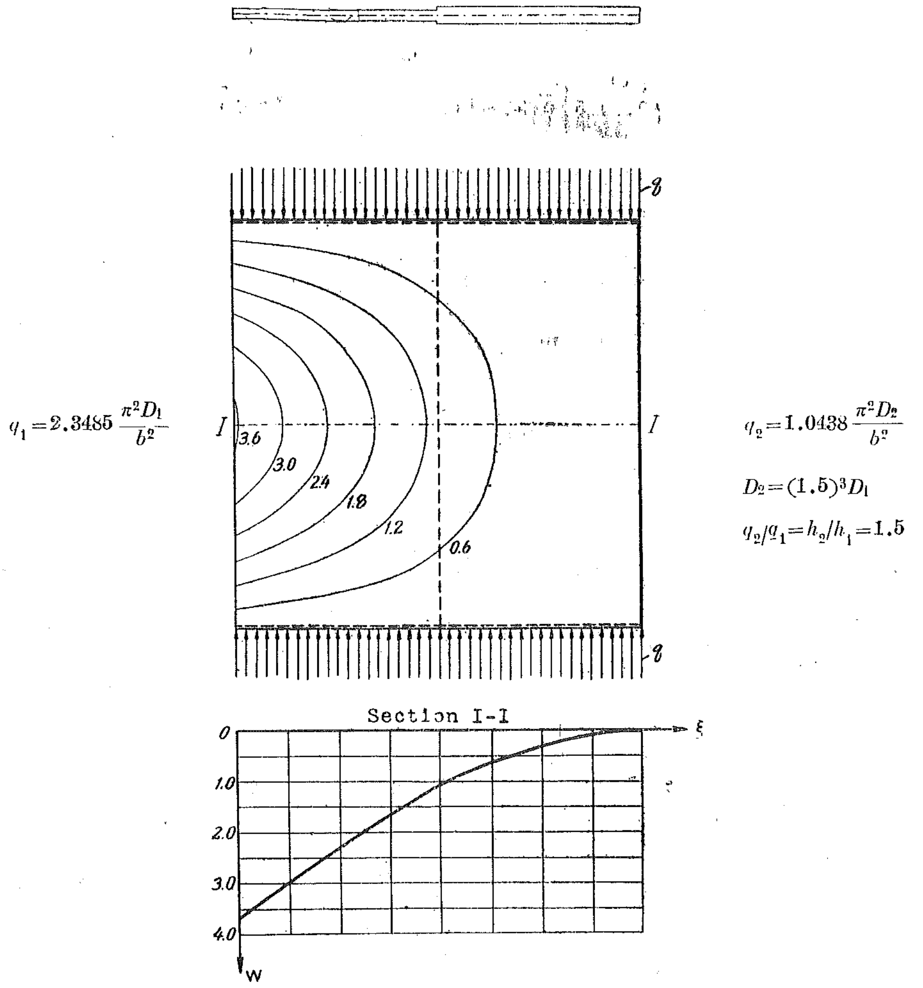


Fig. 17.

$$w_1 = K_1 w_1' ,$$

$$w_1' = \left\{ \left(\cosh\pi\lambda_{\substack{1\\1}} \xi_1 + \frac{\beta_{\substack{1\\1}}}{\beta_{\substack{2\\1}}} \cos\pi\lambda_{\substack{2\\1}} \xi_1\right) - 0.904\,262 \left(\sinh\pi\lambda_{\substack{1\\1}} \xi_1 + \frac{\gamma_{\substack{1\\1}}}{\gamma_{\substack{2\\1}}} \sin\pi\lambda_{\substack{2\\1}} \xi_1\right) \right\} \sin\pi\eta_1 ;$$

$$w_2 = K_1 w_2' ,$$

$$w_2' = \left\{ 0.334\,074 \cosh\pi\lambda_{\substack{1\\2}} \xi_2 - 0.187\,055 \sinh\pi\lambda_{\substack{1\\2}} \xi_2 + 0.761\,741 \cos\pi\lambda_{\substack{2\\2}} \xi_2 - 6.354\,424 \sin\pi\lambda_{\substack{2\\2}} \xi_2 \right\} \sin\pi\eta_2 .$$

By these, Table 19 and Fig. 17 are obtained.

Table 19.

ξ_1, ξ_2 / η	w_1'					w_2'			
	0	0.25	0.50	0.75	1.00	0	0.25	0.50	0.75
0.125	1.408 93	1.137 81	0.880 75	0.631 94	0.419 34	0.419 34	0.256 81	0.127 77	0.036 65
0.250	2.603 40	2.102 42	1.627 44	1.167 69	0.774 86	0.774 86	0.474 54	0.236 10	0.067 72
0.375	3.401 49	2.746 94	2.126 35	1.525 66	1.012 40	1.012 40	0.620 02	0.308 49	0.088 48
0.500	3.681 75	2.973 26	2.301 55	1.651 36	1.095 81	1.095 81	0.671 10	0.333 90	0.095 78

§ 15. Case when Both End Sides are Free

From (**9.c**) in § 6, the conditions at End side-2 are written as follows:

$$\left.\begin{aligned}
U_{\substack{2\\2}} &= 1 ; & U_{\substack{1\\2}} &= \frac{\gamma_{\substack{1\\2}} + P_2 \lambda_{\substack{1\\2}}}{\gamma_{\substack{2\\2}} + P_2 \lambda_{\substack{2\\2}}} \sinh\pi\lambda_{\substack{1\\2}} \sinh\pi\lambda_{\substack{2\\2}} - \frac{\beta_{\substack{1\\2}}}{\beta_{\substack{2\\2}}} \cosh\pi\lambda_{\substack{1\\2}} \cosh\pi\lambda_{\substack{2\\2}} ; \\
U_{\substack{2\\2}}' &= 0 ; & U_{\substack{1\\2}}' &= -\frac{\gamma_{\substack{1\\2}} + P_2 \lambda_{\substack{1\\2}}}{\gamma_{\substack{2\\2}} + P_2 \lambda_{\substack{2\\2}}} \sinh\pi\lambda_{\substack{1\\2}} \cosh\pi\lambda_{\substack{2\\2}} + \frac{\beta_{\substack{1\\2}}}{\beta_{\substack{2\\2}}} \cosh\pi\lambda_{\substack{1\\2}} \sinh\pi\lambda_{\substack{2\\2}} ; \\
V_{\substack{2\\2}} &= 0 ; & V_{\substack{1\\2}} &= \frac{\gamma_{\substack{1\\2}} + P_2 \lambda_{\substack{1\\2}}}{\gamma_{\substack{2\\2}} + P_2 \lambda_{\substack{2\\2}}} \cosh\pi\lambda_{\substack{1\\2}} \sinh\pi\lambda_{\substack{2\\2}} - \frac{\beta_{\substack{1\\2}}}{\beta_{\substack{2\\2}}} \sinh\pi\lambda_{\substack{1\\2}} \cosh\pi\lambda_{\substack{2\\2}} ; \\
V_{\substack{2\\2}}' &= 1 ; & V_{\substack{1\\2}}' &= -\frac{\gamma_{\substack{1\\2}} + P_2 \lambda_{\substack{1\\2}}}{\gamma_{\substack{2\\2}} + P_2 \lambda_{\substack{2\\2}}} \cosh\pi\lambda_{\substack{1\\2}} \cosh\pi\lambda_{\substack{2\\2}} + \frac{\beta_{\substack{1\\2}}}{\beta_{\substack{2\\2}}} \sinh\pi\lambda_{\substack{1\\2}} \sinh\pi\lambda_{\substack{2\\2}} .
\end{aligned}\right\} \quad (15.1)$$

In the next place, paying our attention only to differences about $U_{\substack{1\\2}}$, $U_{\substack{1\\2}}'$, $V_{\substack{1\\2}}$, $V_{\substack{1\\2}}'$, we see that the recurrence formulas (12.2) used in the last example where End side-2 is clamped are applicable to this case. And again the conditional equation is similarly given by (13.1) as in § 13 and § 14.

Now, let us consider the case where only the load q_r is acting. It is already known that the critical value of a single plate with the *Poisson's ratio* $\nu = 0.3$ and the same boundary conditions is 0.9524.[35] Therefore, by referring to this value, we can establish the following limits for the least characteristic value:

$$0.952 < Q_1 < 2.143\,, \qquad 0.423 < Q_2 < 0.952$$

Within the above limits, $\lambda_{\substack{1\\1}}$, $\lambda_{\substack{1\\2}}$ and $\lambda_{\substack{2\\2}}$ always are real but $\lambda_{\substack{2\\1}}$ becomes real or imaginary, corresponding to whether Q_1 is smaller or larger than unit. That is,

$$\text{when} \qquad 0.952 < Q_1 < 1\,, \qquad 0.423 < Q_2 < 0.444 \tag{15.2}$$

all of λ_r become real, but

$$\text{when} \qquad 1 < Q_1 < 2.143\,, \qquad 0.444 < Q_2 < 0.952 \tag{15.3}$$

only $\lambda_{\substack{2\\1}}$ becomes imaginary.

Now, let us compare λ_r in the case (1) in § 11 with the ones in this case. The formers are given by (10.9); *i. e.*, $\lambda_{\substack{1\\1}}$, $\lambda_{\substack{1\\2}}$ are real and $\lambda_{\substack{2\\1}}$, $\lambda_{\substack{2\\2}}$ imaginary. Considering this, we can use, also in this case, the former expressions, if, when (15.2) holds, we put $\lambda_{\substack{2\\1}} = \dfrac{\lambda_{\substack{2\\1}}}{i}$, $\lambda_{\substack{2\\2}} = \dfrac{\lambda_{\substack{2\\2}}}{i}$ in each factor (from λ_r to F_r, G_r, H_r, I_r and F_r', G_r', H_r', I_r') of the former case, or when (15.3) holds, only $\lambda_{\substack{2\\2}}$ is replaced as such. Therefore, using thus obtained expressions and (15.1) subjected to the similar replacements, the recurrence formulas (12.2) can be calculated in real domain. Then the conditional equation (13.1) comes to be computed. The actual computations give no roots within (15.2) but the following results within (15.3).

$$\begin{cases} Q_1 = 1.5045 \\ Q_2 = 0.6687 \end{cases}$$

From these, by using the functions of Case-2 in Table 1 for w_1 and those of Case-3 for w_2, we can express the deflection surfaces as follows:

$$w_1 = K_1\, w_1',$$

$$w_1' = \left\{(\cosh\pi\lambda_{\substack{1\\1}}\,\xi_1 + \frac{\beta_{\substack{1\\1}}}{\beta_{\substack{2\\1}}}\cos\pi\lambda_{\substack{2\\1}}\,\xi_1) - 0.904\ 420\,(\sinh\pi\lambda_{\substack{1\\1}}\,\xi_1 + \frac{\gamma_{\substack{1\\1}}}{\gamma_{\substack{2\\1}}}\sin\pi\lambda_{\substack{1\\2}}\,\xi_1)\right\}\sin\pi\eta_1\,;$$

$$w_2 = K_1\, w_2'\,,$$

$$w_2' = \left\{0.197\ 852\cosh\pi\lambda_{\substack{1\\2}}\,\xi_2 - 0.164\ 236\sinh\pi\lambda_{\substack{1\\2}}\,\xi_2 + 2.547\ 359\cosh\pi\lambda_{\substack{2\\2}}\,\xi_2 - 1.467\ 802\sinh\pi\lambda_{\substack{2\\2}}\,\xi_2\right\}\sin\pi\eta_2\,.$$

35) C. Haragúchi, "Studies on Bending of the Rectangular Plate simply supported along two Opposite Sides and rested on Elastic Foundation and its some other problems," Jour. Civ. Eng. Soc., Vol. 29, No. 12, 1943, p.6, Table 1.

Accordingly, Table 20 and Fig. 18 are obtained.

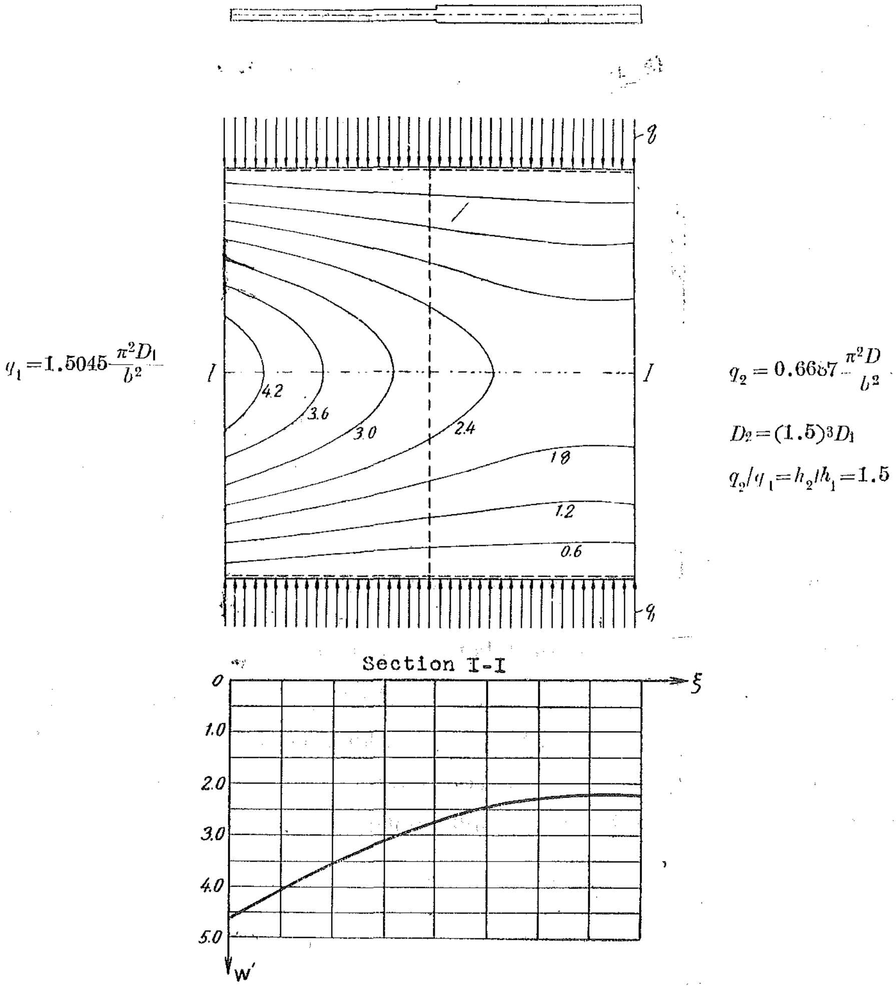


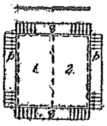
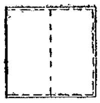
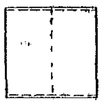
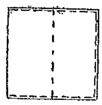
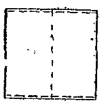
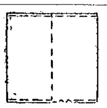
Fig. 18.

Table 20.

η \ ξ_1, ξ_2	w_1' 0	0.25	0.50	0.75	1.00	w_2' 0	0.25	0.50	0.75	1.00
0.125	1.782 78	1.556 27	1.367 00	1.196 85	1.050 55	1.050 53	0.945 66	0.880 81	0.851 80	0.858 60
0.250	3.294 19	2.875 65	2.525 93	2.211 57	1.941 19	1.941 16	1.747 38	1.627 55	1.573 95	1.586 52
0.375	4.304 05	3.757 20	3.300 28	2.889 48	2.536 28	2.536 24	2.283 05	2.126 49	2.056 46	2.072 88
0.500	4.658 67	4.066 77	3.572 19	3.127 55	2.745 25	2.745 21	2.471 15	2.301 70	2.275 89	2.243 67

The foregoing explanations for transformation of expressions are intended to perform calculations as much in real domain as possible and save labour. However, the calculating processes will be entirely mechanical, if it is not minded that those become more or less tedious without the above considerations.

Table 21.-Summary of the numerical results.

End side conditions	$Q_r = \frac{q_r b^2}{D_r \pi^2}$, $(p=0)$	$P_r = \frac{p\, a_r^2}{D_r \pi^2}$, $(q_r=0)$			
	Least root	Least root	2-nd root	3-rd root	4-th root
	$Q_1 = 5.988$ $Q_2 = 2.661$	$P_1 = 1.5114$ $P_2 = 0.4478$	$P_1 = 2.8392$ $P_2 = 0.8412$		
	$Q_1 = 8.5118$ $Q_2 = 3.7830$	$P_1 = 2.2547$ $P_2 = 0.6681$	$P_1 = 3.7394$ $P_2 = 1.1080$		
	$Q_1 = 12.6805$ $Q_2 = 5.6358$	$P_1 = 2.5819$ $P_2 = 0.7650$	$P_1 = 4.2814$ $P_2 = 1.2686$	$P_1 = 8.1215$ $P_2 = 2.4064$	$P_1 = 10.7575$ $P_2 = 3.1874$
	$Q_1 = 1.9811$ $Q_2 = 0.8805$				
	$Q_1 = 2.3485$ $Q_2 = 1.0438$				
	$Q_1 = 1.5045$ $Q_2 = 0.6687$				

(B) COMPOSITE PLATES WITH VARIOUS CONNECTING JOINTS.

In the foregoing examples, we have observed about some cases where the composite plates rigidly connected along the intermediate joint have the various kinds of end edge conditions. Now, we will investigate about some cases where the connecting manners along the intermediate joint are various too. Let us consider also these cases are specified such as (22).

§ 16. Plate Having a Rigid Joint and Simply Supported along the Joint.

It is in the same manner as before that the recurrence formulas in this case are given by (**11**), and the conditions at End side-2 by (**11.a**), (**11.b**) or (**11.c**) and also the conditional equation for buckling by (**11.d**), (**11.e**) or (**11.f**), corresponding to each of the various end side conditions.

(1) *The case where both end sides are simply supported and only the load p is acting.*—From (**11.a**), the conditions at End side-2 are written as follows:

$$U_2 = -T_2, \qquad V_2 = 1.$$

On the other hand, the conditional equation is, by (**11.d**),

$$V_1 = 0 .$$

Then, by the recurrence formulas (11) and (10)

$$V_1 = \mu_2 \frac{T_2}{S_1} + \mu_2' \frac{\varphi_2}{\varphi_1} \frac{T_1}{S_1} = 0 .$$

And, considering now $\mu_2 = 1$ and $\mu_2' = 3.375$, we have

$$T_2 + 3.375 \frac{\varphi_2}{\varphi_1} T_1 = 0 . \tag{16.1}$$

The above is the conditional equation in the present case and its form is considerably simple. Now it is already known that the least critical value of a single plate, which is specified as $\frac{a}{b} = \frac{1}{2}$ and $\nu = 0.3$, under the same load condition is 1.5625.[36)]

Supposing the case where the thicknesses of both elementary plates become equal to each other, each elementary plate should buckle as a simply supported single plate owing to the symmetrical circumstances. In the same time, the critical value P of the above supposed plates should become 1.5625. From this, denoting by p' or p'' respectively the critical load when both thicknesses are h_1 or h_2, we can write

$$\frac{p' a_1^2}{D_1 \pi^2} = 1.5625 , \qquad \frac{p'' a_2^2}{D_1 \pi^2} = 1.5625 \qquad \left(a_1 = a_2 = \frac{b}{2} \right) .$$

Next, denoting the desired critical load in this case by p, we can easily assume as follows:

$$p' < p < p'' .$$

Therefore, we have

$$P_1 = \frac{p\, a_1^2}{D_1 \pi^2} > \frac{p' a_1^2}{D_1 \pi^2} = 1.5625 , \qquad P_2 = \frac{p\, a_2^2}{D_2 \pi^2} < \frac{p'' a_2^2}{D_2 \pi^2} = 1.5625$$

And also observing $P_1 = (1.5)^3 P_2$ from (10.14), we finally obtain

$$1.563 < P_1 < 5.273 , \qquad 0.463 < P_2 < 1.562$$

The following partial intervals within the above are involved within the limits (10.15) [case (2), § 10]:

$$1.563 < P_1 < 3.375 , \qquad 0.463 < P_2 < 1 , \tag{16.2}$$

---

36) We find the result $k = \frac{N_x b^2}{D \pi^2} = 6.25$ for the case $\frac{a}{b} = 0.5$ in Table 31 in S. Timoshenko's "Theory of Elastic Stability," 1936, p. 332. Now, if this result is rewritten for advantage of comparison, it follows that $P = \frac{p\, a^2}{D \pi^2} = 1.5625$

and then (10.16) can be used also in this case as the expressions of λ_r. So;

$$\varphi_1 = \sqrt{P_1(P_1-1)}, \qquad \varphi_2 = i\sqrt{P_2(1-P_2)},$$

$$T_2 = \lambda_{\frac{1}{2}}\coth\pi\lambda_{\frac{1}{2}} - \bar{\lambda}_{\frac{1}{2}}\coth\pi\bar{\lambda}_{\frac{1}{2}} = i\,\frac{\sqrt{P_2}\sinh\pi\sqrt{1-P_2} - \sqrt{1-P_2}\sin\pi\sqrt{P_2}}{\cosh\pi\sqrt{1-P_2} - \cos\pi\sqrt{P_2}},$$

$$T_1 = \lambda_{\frac{1}{1}}\cot\pi\lambda_{\frac{1}{1}} - \lambda_{\frac{2}{1}}\cot\pi\lambda_{\frac{2}{1}}.$$

Hence the conditional equation (16.1) is rewritten in the following form:

$$\frac{1}{\cosh\pi\sqrt{1-P_2} - \cos\pi\sqrt{P_2}}\left(\frac{\sinh\pi\sqrt{1-P_2}}{\sqrt{1-P_2}} - \frac{\sin\pi\sqrt{P_2}}{\sqrt{P_2}}\right) + 3.375\,\frac{\lambda_{\frac{1}{1}}\cot\pi\lambda_{\frac{1}{1}} - \lambda_{\frac{2}{1}}\cot\pi\lambda_{\frac{2}{1}}}{\sqrt{P_1(P_1-1)}} = 0.$$

Searching the least root of the above within the limits (16.2), we obtain

$$\begin{cases} P_1 = 2.2716 \\ P_2 = 0.6731 \end{cases}$$

In the next place, the deflection w_1 or w_2 should be expressed respectively by using the functions of Case-4 or Case-1 in Table 1. But, let us proceed temporarily with the expressions of Case-3, since treating the expressions of Case-1 will become more or less tedious. Then, after some operations, the following are obtained:

$$w_1 = A\,w_1', \quad w_1' = \left\{\frac{\sin\pi\lambda_{\frac{1}{1}}\xi_1}{\sin\pi\lambda_{\frac{1}{1}}} - \frac{\sin\pi\lambda_{\frac{2}{1}}\xi_1}{\sin\pi\lambda_{\frac{2}{1}}}\right\}\sin\pi\eta_1;$$

$$w_2 = A\,w_2', \quad w_2' = \frac{\varphi_1}{\mu_2'\varphi_2}\left\{\cosh\pi\lambda_{\frac{1}{2}}\left(\frac{\cosh\pi\lambda_{\frac{1}{2}}\xi_2}{\cosh\pi\lambda_{\frac{1}{2}}} - \frac{\sinh\pi\lambda_{\frac{1}{2}}\xi_2}{\sinh\pi\lambda_{\frac{1}{2}}}\right) - \cosh\pi\lambda_{\frac{2}{2}}\left(\frac{\cosh\pi\lambda_{\frac{2}{2}}\xi_2}{\cosh\pi\lambda_{\frac{2}{2}}} - \frac{\sinh\pi\lambda_{\frac{2}{2}}\xi_2}{\sinh\pi\lambda_{\frac{2}{2}}}\right)\right\}\sin\pi\eta_2,$$

where

$$\left.\begin{matrix}\lambda_{\frac{1}{1}} \\ \lambda_{\frac{2}{1}}\end{matrix}\right\} = \frac{1}{2}\sqrt{2P_1 - 1 \mp 2\sqrt{P_1(P_1-1)}}, \qquad \left.\begin{matrix}\lambda_{\frac{1}{2}} \\ \lambda_{\frac{2}{2}}\end{matrix}\right\} = \frac{1}{2}\left(\sqrt{1-P_2} \pm i\sqrt{P_2}\right),$$

A = an indefinite constant.

Both terms inside the braces of the above second expression have the conjugate complex value of each other, then the contents in the braces can be replaced by twice the imaginary part of the first term. Therefore, we find as the expression of w_2'

$$w_2' = \frac{2\sqrt{P_1(P_1-1)}}{\mu_2'\sqrt{P_2(1-P_2)}}$$

$$\frac{\left\{\begin{aligned} &\cosh\pi\sqrt{1-P_2}\ \sinh\frac{\pi}{2}\sqrt{1-P_2}\,\xi_2\sin\frac{\pi}{2}\sqrt{P_2}\,\xi_2-\cos\pi\sqrt{P_2}\sinh\frac{\pi}{2}\sqrt{1-P_2}\,\xi_2\sin\frac{\pi}{2}\sqrt{P_2}\,\xi_2 \\ +\ &\sin\pi\sqrt{P_2}\sinh\frac{\pi}{2}\sqrt{1-P_2}\,\xi_2\cos\frac{\pi}{2}\sqrt{P_2}\,\xi_2-\sinh\pi\sqrt{1-P_2}\ \cosh\frac{\pi}{2}\sqrt{1-P_2}\,\xi_2\sin\frac{\pi}{2}\sqrt{P_2}\,\xi_2\end{aligned}\right\}}{\cosh\pi\sqrt{1-P_2}\ -\ \cos\pi\sqrt{P_2}}.$$

The numerical results by the above are given in Table 22 and Fig. 19 can be drawn.

Table 22.

$\xi_1,\ \xi_2$ / η	w_1' 0.25	w_1' 0.50	w_1' 0.75	w_1' 0.875	w_2' 0.05	w_2' 0.25	w_2' 0.50	w_2' 0.75
0.125	0.492 89	0.600 32	0.311 79	0.129 31	−0.032 00	−0.115 47	−0.134 04	−0.084 50
0.250	0.910 77	1.109 26	0.576 12	0.238 94	−0.059 12	−0.213 37	−0 247 67	−0.156 15
0.375	1.189 97	1.449 32	0.752 73	0.312 19	−0.077 25	−0.278 78	−0.323 60	−0.204 02
0.500	1.288 01	1.568 73	0.814 75	0.337 92	−0.083 62	−0.301 75	−0.350 26	−0.220 83

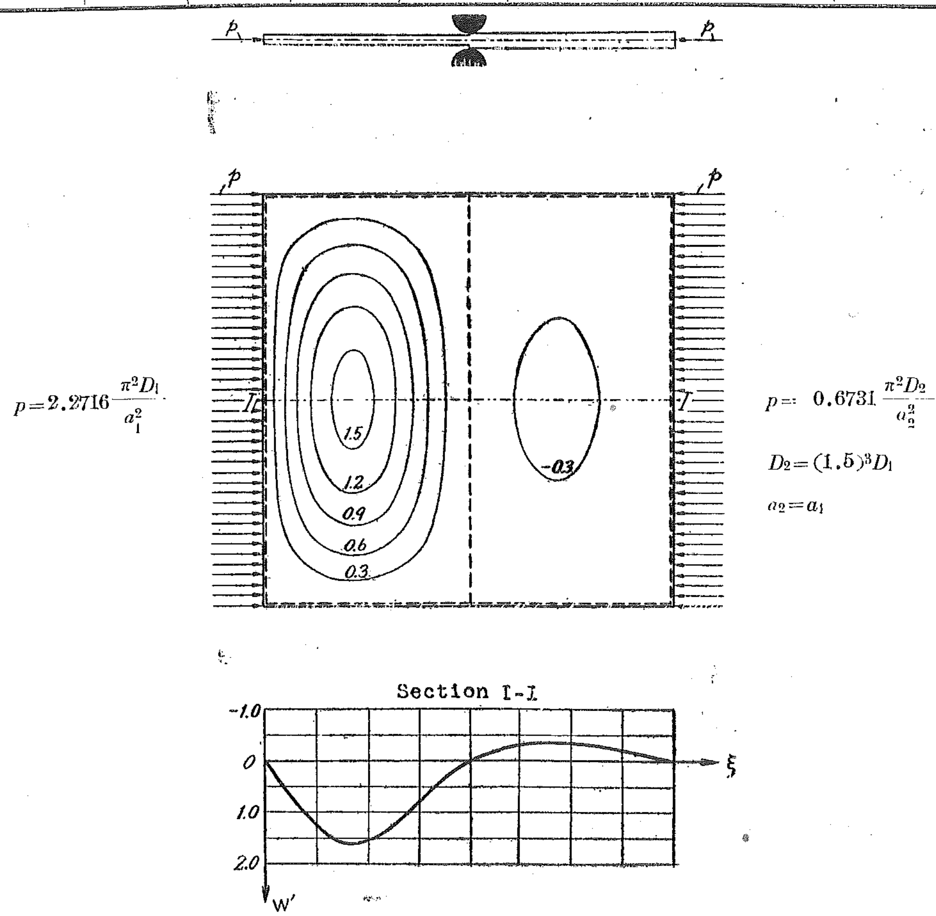


Fig. 19.

(2) *The case where both the end sides are free and only the load q_r is acting.*—Referring to (11.c), the conditions at End side-2 are

$$U_2 = -\frac{T_2 T_2' - \bar{S}_2 S_2''}{T_2'}, \qquad V_2 = 1 .$$

The conditional equation in the present case is written by (11.g) as

$$U_1 = U_2(F_1) - \mu_2' \frac{\varphi_2}{\varphi_1}(G_1) = 0 .$$

Then, referring to (11.f) and observing $\mu_2' = \dfrac{D_2}{D_1}\left(= 3.375\right)$, we obtain such a final equation as

$$\frac{T_2 T_2' - \bar{S}_2 S_2''}{D_2 \varphi_2 T_2'} + \frac{T_1 T_1' - \bar{S}_1 S_1''}{D_1 \varphi_1 T_1'} = 0 . \tag{16.3}$$

Let us investigate now about the limits within which the least root lies. Supposing the special case where the thicknesses are the same through both elementary plates, the least critical value in such a case, obviously, corresponds to reversely symmetric buckling with respect to the central supporting line, owing to the symmetric boundary conditions. Therefore, considering about one elementary plate, its deflection surface may be equivalent to what a single plate, which is specified as $\dfrac{a}{b} = \dfrac{1}{2}$ and has a free edge along one long side and is simply supported along the other three sides, conforms to at buckling due to the compressive forces acting perpendicularly to the short sides.[37] Or, we can consider that the deflection surface of the composite plate in this case coincides with what a single square plate, both the opposite end sides of which are free, conforms to at the higher mode of buckling with a central nodal line parallel to the free edges. Besides, it is already known that the critical value for such mode of buckling is 2.6722.[38]

In the next, denoting by q' the critical load for the case when h_1 is thickness through both elementary plates, and by q'' another one for the case when h_2 is so, we can write, by the foregoing considerations, as follows:

$$\frac{q' b^2}{D_1 \pi^2} = 2.6722, \qquad \frac{q'' b^2}{D_2 \pi^2} = 2.6722$$

Now, denoting by q_1 or q_2 the desired critical load in the present example, we can easily assume

37) For this case, we can find the numerical result $k=0.698$, in such a case as $\nu=0.25$, from Table 32 in S. Timoshenko's "Theory of Elastic Stability," 1936, p. 339. If we rewrite this result in advantageous form for comparison, we have $Q=2,792$.

38) See Table 2 (p. 8) in paper by C. Haraguchi, *loc. cit.*, p. 247.

$$q' < q_1 < q'', \qquad q' < q_2 < q''.$$[39]

Then.

$$Q_1 = \frac{q_1 b^2}{D_1 \pi^2} > \frac{q' b^2}{D_1 \pi^2} = 2.6722$$

$$Q_2 = \frac{q_2 b^2}{D_2 \pi^2} < \frac{q'' b^2}{D_2 \pi^2} = 2.6722$$

Now, observing $Q_1 = 2.25\, Q_2$,we have

$$2.672 < Q_1 < 6.012\,, \qquad 1.188 < Q_2 < 2.672 \tag{16.4}$$

These intervals are involved within the limits of (10.8) in the case (1) in § 10 on roughly speaking, and then (10.9) can be used also for expression of λ_r. At this time, the expressions of T_r, T_r', S_r, S_r'' and so on become of real form. And the least root of Eq. (16.3) can be obtained within the limits (16.4), *i, e.,*

$$\begin{cases} Q_1 = 3.9280 \\ Q_2 = 1.7458 \end{cases}$$

Using the functions of Case-2 in Table 1, the deflection surfaces corresponding to the above are expressed as follows:

$$w_1 = A\, w_1',$$

$$w_1' = \left\{(\underset{1}{\beta_2} \cosh\pi \underset{1}{\lambda_1}\, \xi_1 + \underset{1}{\beta_1} \cos\pi \underset{1}{\lambda_2}\, \xi_1) - 0.879\,458\, (\underset{1}{\gamma_2} \sinh\pi \underset{1}{\lambda_1}\, \xi_1 + \underset{1}{\gamma_1} \sin\pi \underset{1}{\lambda_2}\, \xi_1)\right\} \sin\pi\eta_1\,;$$

$$w_2 = A\, w_2',$$

$$w_2' = \left\{0.103\,879\, (\cosh\pi \underset{2}{\lambda_1}\, \xi_2 - \cos\pi \underset{2}{\lambda_2}\, \xi_2) - 0.120\,979 \sinh\pi \underset{2}{\lambda_1}\, \xi_2 \right.$$
$$\left. - 0.264\,753 \sin\pi \underset{2}{\lambda_2}\, \xi_2 \right\} \sin\pi\eta_2\,,$$

where A is an indefinite constant.

The numerical results by these are given in Table 23 and then Fig. 20 is obtained.

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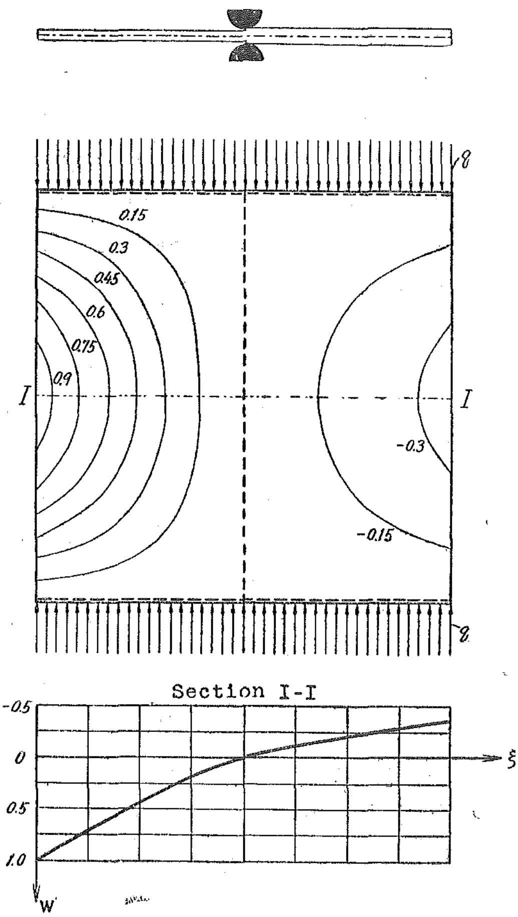
39) This is understood in the following manner. Let us suppose first the imaginary case where each elementary plate is separated from the other. Next, let these plates are submitted to the similar compressive forces at the same time. When the loads increase gradually and have reached to amount of q', the plate having the thickness h_1 begins to buckle but the other having the thickness h_2 will yet remain plane without buckling. Therefore, the composite plate composed by connecting both the plates cannot buckle because it results by the above considerations that the rigidity of the former plate be helped by that of the latter plate. From this, we can conclude $q' < q_1$. Next, when the loads have reached to q'', the latter plate buckles too. Considering that the former plate had been buckled previously, the composite plate in this time must be already buckled because the rigidity of the latter plate have been weakened by the formerly buckled plate. From this, we can conclude $q_2 < q''$ Observing $q_1 < q_2$ from (10.4), we can finally establish the inequalities in the paper.

Table 23.

η \ ξ_1, ξ_2	w_1' 0	0.25	0.50	0.75	1.00	w_2' 0.25	0.50	0.75	1.00
0.125	0.379 22	0.272 18	0.168 57	0.071 85	0.000 00	−0.043 43	−0.077 33	−0.106 49	−0.135 65
0.250	0.700 71	0.502 93	0.311 49	0.132 76	−0.000 01	−0.080 26	−0.142 88	−0.196 78	−0.250 66
0.375	0.915 52	0.657 11	0.406 98	0.173 46	−0.000 02	−0.104 86	−0.186 69	−0.257 11	−0.327 50
0.500	0.990 96	0.711 25	0.440 52	0.187 75	−0.000 02	−0.113 50	−0.202 07	−0.278 29	−0.354 48

$q_1 = 3.9280 \dfrac{\pi^2 D_1}{b^2}$



$q_2 = 1.7458 \dfrac{\pi^2 D_2}{b^2}$

$D_2 = (1.5)^3 D_1$

$q_2/q_1 = h_2/h_1 = 1.5.$

Fig. 20

§ 17. Plate Having a Hinged Joint and No Support along the Joint.

The recurrence formulas in this case are given by (15) and conditions at End side-2 by (15a), (15b) or (15c) and also the conditional equation by (15d), (15e) or (15f), corresponding to each of the various end side conditions.

(1) *The case where both the end sides are simply supported and only the load q is acting.*—From (15a), at End side-2

$$U_2 = -T_2' ; \qquad V_2 = 1 .$$

And from (15d), the conditional equation is

$$V_1 = 0 .$$

Then, from the recurrence formulas (**15**) and (**14**), we can write

$$V_1 = \mu_2'' \frac{T_2'}{S_1'} + \frac{\bar{\varphi}_2}{\bar{\varphi}_1} \frac{T_1'}{S_1'} = 0 . \tag{17.1}$$

Considering now $\mu_2'' = 3.375$, the above is transformed into

$$3.375\, T_2' + \frac{\bar{\varphi}_2}{\bar{\varphi}_1} T_1' = 0 . \tag{17.2}$$

In the next place, let us establish the limits within which the least root of Eq. (17.2) is searched. Supposing first the special case where the thicknesses of both elemenary plates are the same ones, since the joining line is there at the centre, the shearing stresses and the torsional moment become zero along this joining line, and moreover the bending moment zero by account of hinged joint. From this, it follows that each elementary plate buckles as a single plate having one free edge along the formerly joined edge and the other three sides simply supported. And at this time it is already known that the least critical value of the above described single plate is 2.6722.[40] Then, by using this for comparison, we finally obtain the limits (16.4) for the present example, in the same manner as in the previous example. These limits can again be modified more closely. Supposing the case where the joint in this case is replaced by a rigidly connected joint, the rigidity of the composite plate must be increased, and this supposed case becomes to be equivalent to the case (1) in § 10. Therefore we can obtain the following inequalities:

$$2.672 < Q_1 < 5.988, \qquad 1.188 < Q_2 < 2.661 \tag{17.3}$$

The expressions of λ_r become the same as in the case (2) in § 16. Searching the least root of Eq. (17.2) within the limits (17.3), we obtain

$$\begin{cases} Q_1 = 4.6157 \\ Q_2 = 2.0514 \end{cases}$$

The deflection surfaces in this case can be expressed, by using the functions of Case-2 in Table 1 for both, as follows:

$$w_1 = A\, w_1' .$$

$$w_1' = \left\{ \frac{\sinh \pi \lambda_1^1 \xi_1}{\beta_1^1 \sinh \pi \lambda_1^1} + \frac{\sin \pi \lambda_2^1 \xi_1}{\beta_2^1 \sin \pi \lambda_2^1} \right\} \sin \pi \eta_1 ;$$

40) See paper by C. Haraguchi, *loc. cit* p.247.

$$w_2 = A\, w_2',$$

$$w_2' = \frac{\bar{\varphi}_1}{\bar{\varphi}_2}\left\{\frac{\cosh\pi\lambda_{\substack{1\\2}}}{\beta_{\substack{1\\2}}}\left(\frac{\cosh\pi\lambda_{\substack{1\\2}}\xi_2}{\cosh\pi\lambda_{\substack{1\\2}}}-\frac{\sinh\pi\lambda_{\substack{1\\2}}\xi_2}{\sinh\pi\lambda_{\substack{1\\2}}}\right)+\frac{\cos\pi\lambda_{\substack{2\\2}}}{\beta_{\substack{2\\2}}}\left(\frac{\cos\pi\lambda_{\substack{2\\2}}\xi_2}{\cos\pi\lambda_{\substack{2\\2}}}-\frac{\sin\pi\lambda_{\substack{2\\2}}\xi_2}{\sin\pi\lambda_{\substack{2\\2}}}\right)\right\}\sin\pi\eta_2\,,$$

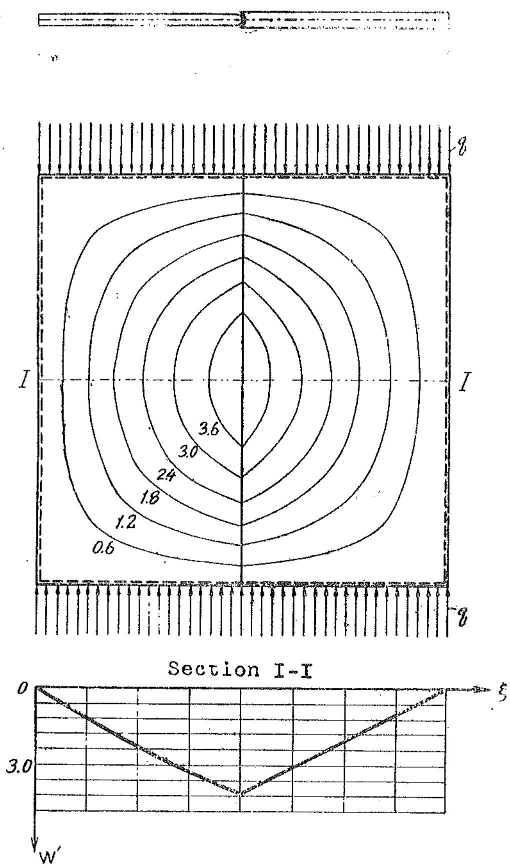
where A denotes an unknown constant, and $\bar{\varphi}_1=\dfrac{1}{\beta_{\substack{1\\1}}}+\dfrac{1}{\beta_{\substack{2\\1}}}$, $\bar{\varphi}_2=\dfrac{1}{\beta_{\substack{1\\2}}}+\dfrac{1}{\beta_{\substack{2\\2}}}$.

The numerical results are shown in Table 24 and then Fig. 21 is obtained.

Table 24

$\xi_1,\ \xi_2$ / η	w_1'				w_2'			
	0.25	0.50	0.75	1.00	0	0.25	0.50	0.75
0.125	0.484 62	0.918 77	1.278 03	1.594 21	1.594 21	1.183 43	0.792 51	0.399 02
0.250	0.895 48	1.697 70	2.361 53	2.945 76	2.945 76	2.186 73	1.464 40	0.737 31
0.375	1.170 00	2.218 14	3.085 47	3.848 81	3.848 80	2.857 09	1.913 32	0.963 34
0.500	1.266 40	2.400 90	3.339 69	4.165 92	4.165 91	3.092 49	2.070 96	1.042 72

$q_1 = 4.6157\,\dfrac{\pi^2 D_1}{b^2}$



$q_2 = 2.0514\,\dfrac{\pi^2 D_2}{b^2}$

$D_2 = (1.5)^3 D_1$

$q_2/q_1 = h_2/h_1 = 1.5$

Fig. 21.

(2) *The case where both end sides are simply supported and only the load* *p is acting*.—The conditions at the end sides and the conditional equation are expressed by the same form as in the previous example and then Eq. (17.1) or Eq. (17.2) can be used also in this case but the expressions of λ, become of different forms as a matter of course.

Now, for establishment of the limits within which the least root should be searched, it seems that the proper results for comparison have not been obtained. Then, let us observe first the case where a single plate, which is specified as $\frac{a}{b}=\frac{1}{2}$ and is free at one long side and simply supported at the other three sides, carries the compressive forces perpendicular to the long edges.

This case is equivalent to that in which either one of the elementary plates is omitted from the composite plate. Suppose that the second is omitted and the first remains only, then Eq. (17.2) becomes to be transformed into the following form:

$$T_1' = \frac{\gamma_1 + P_1\lambda_1}{\underset{1}{\beta_1}}\coth\pi\lambda_1 - \frac{\gamma_2 + P_1\lambda_2}{\underset{1}{\beta_2}}\coth\pi\lambda_2 = 0\,. \tag{17.4}$$

Deriving the least root of this equation is comparatively easy, *i.e.*,

$$P_1 = 0.5107$$

And, the deflection surface of the above is shown in Fig. 22. By the way, supposing

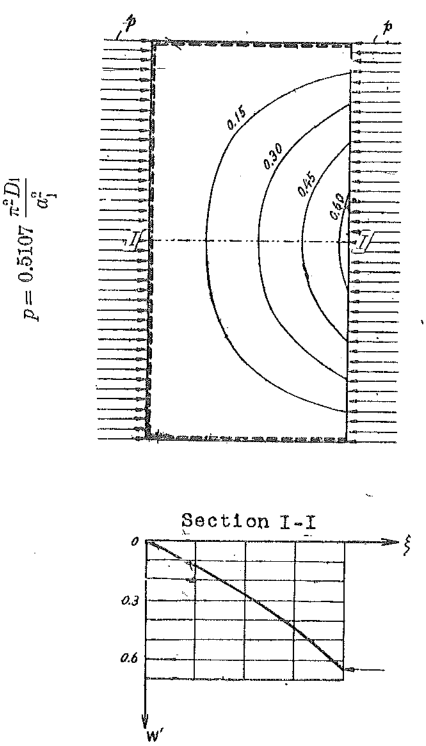


Fig. 22.

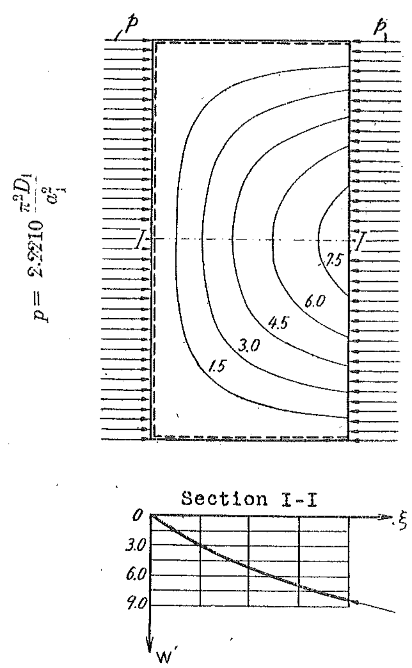


Fig. 22'.

the special case where the compressive force p at the free edge always acts in the middle plane of buckled plate, the critical value can be required by such an equation as is derived by omission of $P_1 \underset{1}{\lambda_1}$ and $P_1 \underset{1}{\lambda_2}$ from Eq. (17.4). Thus the following result is obtained:

$$P_1 = 2.2210$$

Then, we see that this is considerably larger then the previous one. The deflection surface, in this case, becomes such as Fig. 22′. The relations between the ratio of side lengths $\frac{a}{b}$ and the critical value P for both the above cases can be known from Fig. 23 in which the curve A corresponds to the former case and the curve B the latter and moreover that of a plate whose four sides are simply supported is plotted together for comparison.

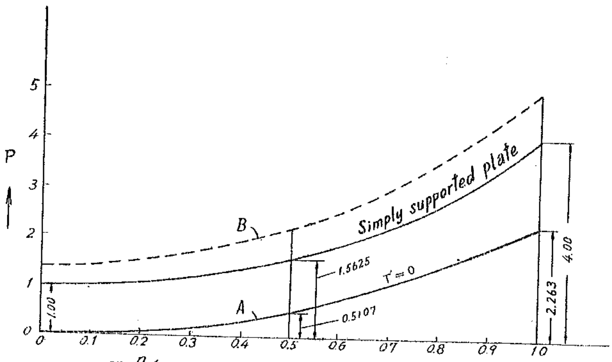


Fig. 23.

For references, requring the least critical value of the previously specified plate when the long edge opposite to the free edge is clamped, we have

$$P = 0.6565,$$

and its deflection surface is shown in Fig. 24. Let us investigate next the limits for the least critical value of the present composite plate by using the above obtained results. Supposing first the extreme case $h_2 = \infty$, the first elementary plate becomes a single plate which is simply supported at all sides and in this time $P_1 = 1.5625$ as already known. In the next place, supposing the other extreme case $h_2 = h_2$, the conditional equation (17.1) will coincide with Eq. (17.4) owing to the symmetic character of the boundary conditions, and then it follows that $P_1 = 0.5107$ as above seen. Accordingly, we can establish the following inequalities:

$$0.5107 < P_1 < 1.5625\,, \qquad 0.1513 < P_2 < 0.463 \qquad (17.5)$$

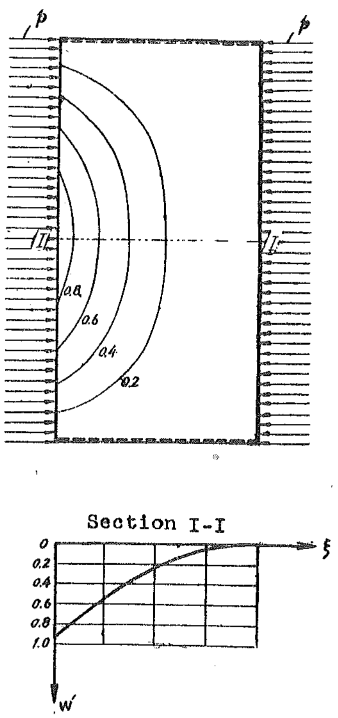


Fig. 24.

Now, in this case, both $\underset{1}{\lambda_1}$ and $\underset{1}{\lambda_2}$ become imaginary, and $\underset{2}{\lambda_1}$ and $\underset{2}{\lambda_2}$ become of the mutually conjugate complex values. Let us search the least root of Eq. (17.2) within the limit (17.5). Then we obtain

$$\begin{cases} P_1 = 1.1030 \\ P_2 = 0.3268 \end{cases}$$

The deflection surfaces are expressed in the following manner, by using the functions of Case-4 in Table 1 for w_1 and those of Case-1 for w_2 respectively:

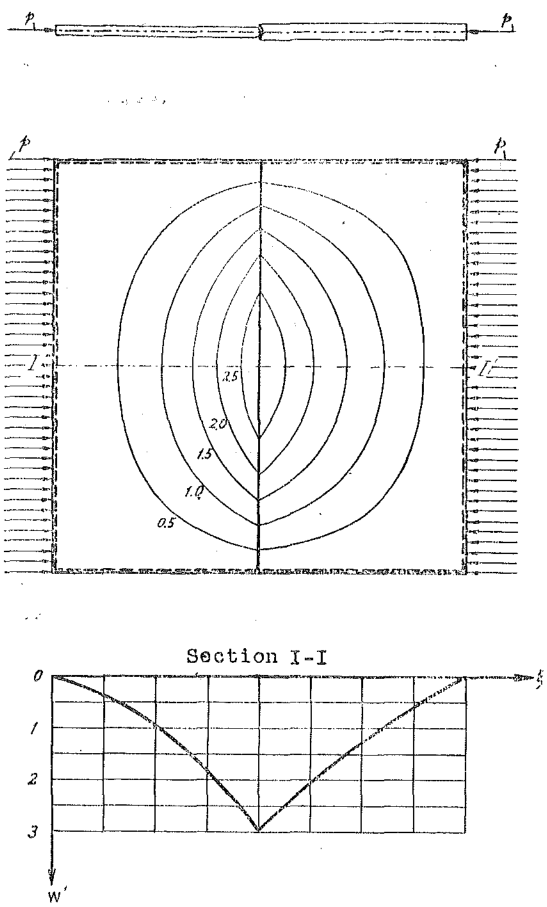
$$w_1 = A\,w_1',$$

$$w_1' = \left\{ \frac{\sin \pi \lambda_1 \xi_1}{\beta_1 \sin \pi \lambda_1} - \frac{\sin \pi \lambda_2 \xi_1}{\beta_2 \sin \pi \lambda_2} \right\} \sin \pi \eta_1 ;$$

$$w_2 = A\,w_2',$$

$$w_2' = \frac{4\sqrt{P_1(P_1-1)}}{(0.3P_1+0.1225)\sqrt{P_2(1-P_2)}} \left[(P_2-0.35) \sinh\frac{\pi}{2}\sqrt{1-P_2}\,\xi_2 \sin\frac{\pi}{2}\sqrt{P_2}\,\xi_2 + \sqrt{P_2(1-P_2)} \cosh\frac{\pi}{2}\sqrt{1-P_2}\,\xi_2 \cos\frac{\pi}{2}\sqrt{P_2}\,\xi_2 \right.$$

$p = 1.1030\,\dfrac{\pi^2 D_1}{a_1^2}$



$p = 0.3268\,\dfrac{\pi^2 D_2}{a_2^2}$

$D_2 = (1.5)^3 D_1$

$a_2 = a_1$

Fig. 25

$$-\frac{\left\{\begin{array}{l}(P_2-0.35)(\sinh\pi\sqrt{1-P_2}\cosh\frac{\pi}{2}\sqrt{1-P_2}\,\xi_2\sin\frac{\pi}{2}\sqrt{P_2}\,\xi_2-\sin\pi\sqrt{P_2}\sinh\frac{\pi}{2}\sqrt{1-P_2}\,\xi_2\cos\frac{\pi}{2}\sqrt{P_2}\,\xi_2)\\+\sqrt{P_2(1-P_2)}\,(\sinh\pi\sqrt{1-P_2}\sinh\frac{\pi}{2}\sqrt{1-P_2}\,\xi_2\cos\frac{\pi}{2}\sqrt{P_2}\,\xi_2+\sin\pi\sqrt{P_2}\cosh\frac{\pi}{2}\sqrt{1-P_2}\,\xi_2\sin\frac{\pi}{2}\sqrt{P_2}\,\xi_2)\end{array}\right\}}{\cosh\pi\sqrt{1-P_2}-\cos\pi\sqrt{P_2}}\Bigg]\sin\pi\eta_2,$$

in which A is an indefinite constant. The numerical results by the above are given in Table 25 and these illustrations are shown in Fig. 25.

Table 25.

η \ ξ_1, ξ_2	w_1' 0.25	0.50	0.75	1.00	w_2' 0	0.25	0.50	0.75
0.125	0.139 30	0.354 14	0.689 35	1.137 68	1.137 67	0.787 50	0.489 76	0.233 90
0.250	0.257 40	0.654 38	1.273 77	2.102 19	2.102 17	1.455 13	0.904 98	0.399 30
0.375	0.336 31	0.854 98	1 664 25	2.746 63	2.746 61	1.901 21	1.182 41	0.564 70
0.500	0.364 02	0.925 43	1.801 38	2.972 93	2.972 91	2.057 85	1.279 83	0.611 23

Furthermore, requiring the second root of Eq. (17.2), we have

$$\begin{cases} P_1 = 1.6030 \\ P_2 = 0.4749 \end{cases}$$

The deflection surfaces corresponding to the above can also be expressed by the same expressions as before. And, the numerical results and the graphical ones are Table 26 and Fig. 26 respectively.

Table 26.

η \ ξ_1, ξ_2	w_1' 0.25	0.50	0.75	1.00	w_2' 0	0.25	0.50	0.75
0.125	4.455 22	6.640 66	5.653 88	2.494 16	2.494 13	1.708 15	1.050 38	0.496 33
0.250	8.232 29	12.270 51	10.447 15	4.608 67	4.608 62	3.156 30	1.940 87	0.917 12
0.375	10.755 97	16.032 13	13.649 81	6.021 49	6.021 42	4.123 89	2.535 86	1.198 27
0.500	11.642 17	17.353 04	14.774 44	6.517 61	6.517 54	4.463 67	2.744 80	1.297 00

Next, let us investigate about the relation between the thickness ratio h_1/h_2 and the critical value P. Writing the conditional equation again,

$$\mu_2''\, T_2' + \frac{\bar{\varphi}_2}{\bar{\varphi}_1}\, T_1' = 0\,.$$

Considering that the following relations hold too[41]:

$$\frac{P_1}{P_2} = \frac{h_2^3}{h_1^3} = \mu_2''.$$

41) See (10.14) in [Case (2), § 10].

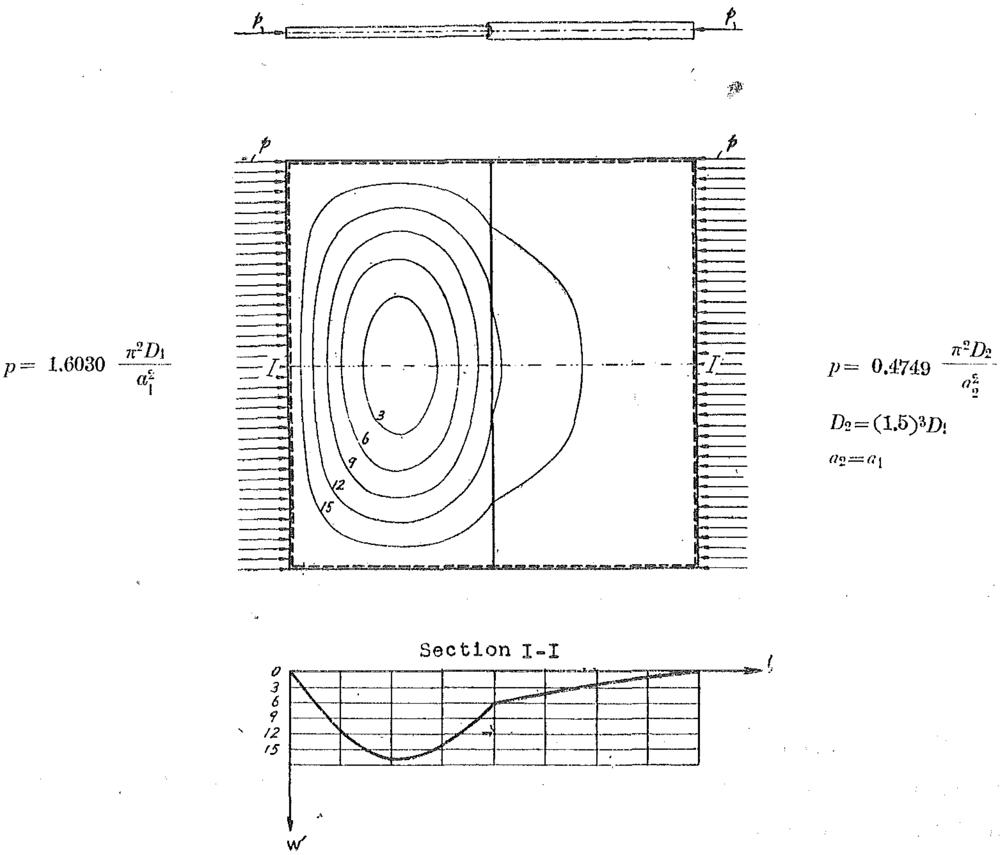


Fig. 26.

we can write down as follows:

$$\frac{P_1}{P_2} = -\frac{\dfrac{T_1'}{\bar{\varphi}_1}}{\dfrac{T_2'}{\bar{\varphi}_2}} = \mu_2'', \tag{17.6}$$

in which

$$T_r' = \frac{\underset{r}{\gamma_1} + P_r \underset{r}{\lambda_1}}{\underset{r}{\beta_1}} \coth \pi \underset{r}{\lambda_1} - \frac{\underset{r}{\gamma_2} + P_r \underset{r}{\lambda_2}}{\underset{r}{\beta_2}} \coth \pi \underset{r}{\lambda_2}, \qquad \bar{\varphi}_r = \frac{1}{\underset{r}{\beta_1}} - \frac{1}{\underset{r}{\beta_2}}.$$

Let us trace next the value of $T_r'/\bar{\varphi}_r$ as a function of P_r. Then we find that its sign changes at $P_r = 0.5107$ and its absolute value becomes infinite at $P_r = 1.5625$ [see Fig.27]. Therefore, in the first instance, plot the portion of $(T_r'/\bar{\varphi}_r)$-curve in the interval $0 < P_r < 0.5107$ at inverse position with respect to P_r-axis. Next, taking any point b on the portion of the curve in the interval $0.5107 < P_r < 1.5625$ and denoting by a the point at which the straight line connecting the origin and the point b intersects with the curve of $-T_r'/\bar{\varphi}_r$, the abscissas of the points a and

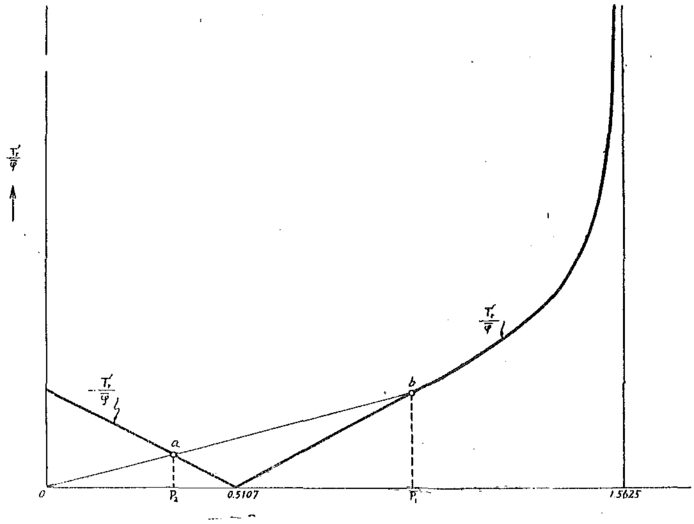


Fig. 27.

b obviously satisfy the first part of the relations (17.6). Hence, considering the above things and the second relation $h_2/h_1=(P_1/P_2)^{1/3}$, we can trace the value of P_r as a function of h_2/h_1 by means of the graphical method, and therefore Fig. 28 can be drawn finally.

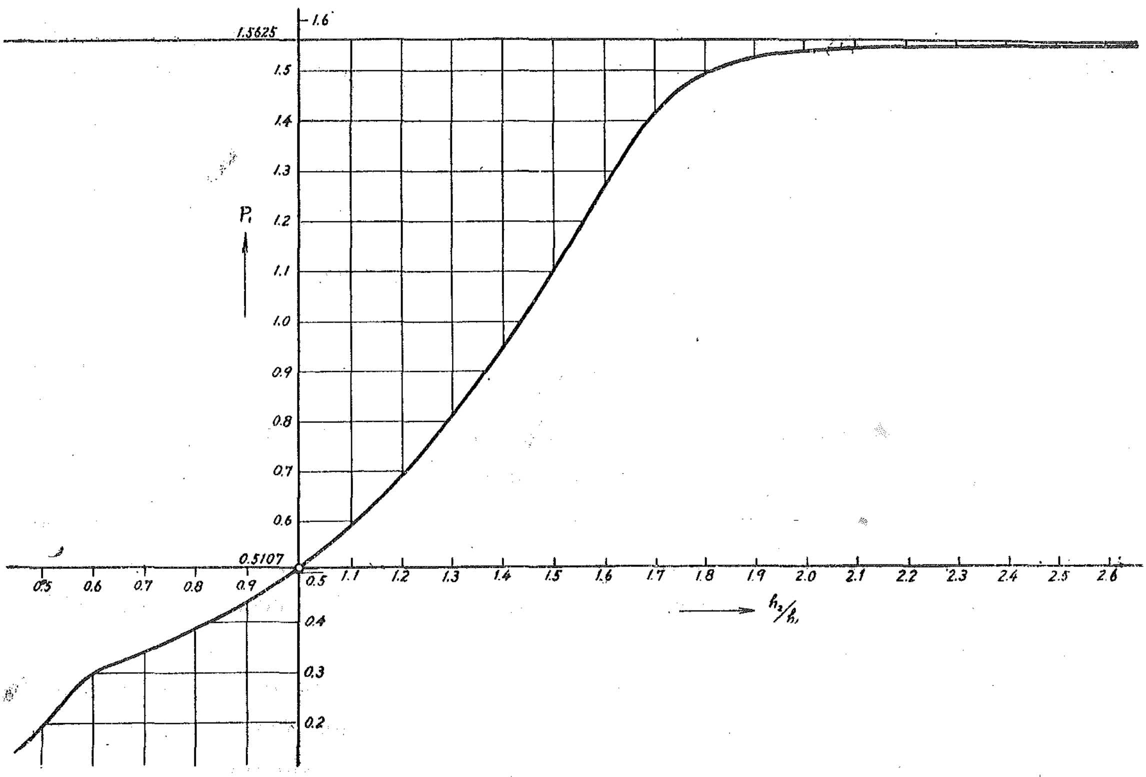


Fig. 28.

Yet, it is remarked that the values of P_1 for the case when $h_2/h_1<1$ are nothing but that of P_2 for the case above considered, *i. e.*, when $h_2/h_1>1$.

Table 27.–Summary of the numerical results

Joining conditions	End side conditions	Load condition	Least root	2-nd root
Rigid joint and rigid supporting	Both end sides are simply supported	Only p is acting	$P_1=2.2716$ $P_2=0.6731$	
〃	Both end sides are free.	Only q is acting	$Q_1=3.9280$ $Q_2=1.7458$	
Hinged joint and no supporting	Both end sides are simply supported	Only q is acting	$Q_1=4.6157$ $Q_2=2.0514$	
〃	〃	Only p is acting	$P_1=1.1030$ $P_2=0.3268$	$P_1=1.6030$ $P_2=0.4749$

§ 18. Plate Having an Elastically built Joint and No Support along the Joint.

The coefficient of restraint υ_{r-1}, concerning the elastically built joint between the $(r-1)$-th elementary plate and the r-th, denotes an angle which follows buckling at the joint. Then, denoting by M_{r-1} the bending moment per unit length along the joint line and by $\delta\theta_{r-1}$ the angle at the joint caused by the moment, υ_{r-1} is specified as follows:

$$\upsilon_{r-1}=\frac{\delta\theta_{r-1}}{M_{r-1}}.$$

And, observing $\varepsilon_r'=D_r\,\upsilon_{r-1}$ and $\kappa'=\varepsilon_r'\dfrac{\pi}{a_r}$, it follows that

$$\kappa_r'=\left(\frac{\delta\theta_{r-1}}{M_{r-1}}\right)\frac{D_r\pi}{a_r}, \tag{24}$$

from which, we find that κ_r' is a dimensionless quantity. Then let us use hereafter κ_r' as an expression of the restraint for convenience.

Now in the present case, the end side conditions and the conditional equation for buckling both can be expressed by those in [§ 6-a)] as explained in [§ 8-g)]. However the recurrence formulas (**9**), in this case, must be accompanied by (**17**) instead of (**8**).

As an example, let us investigate about the case where both end sides are simply supported. The conditions at the end edge coincide with those in § 10, Then, Eq. (10.2) and (10.3) can be applied as the conditional equation. Substituting (10.2) into (10.3), we have

$$\Delta_1 \equiv (\underset{1}{F_2}\,\underset{1}{H_1}-\underset{1}{F_1}\,\underset{1}{H_2})\sinh\pi\underset{2}{\lambda_1}\sinh\pi\underset{2}{\lambda_2}+(\underset{1}{F_2}\,\underset{1}{I_1'}-\underset{1}{F_1}\,\underset{1}{I_2'})\sinh\pi\underset{2}{\lambda_1}\cosh\pi\underset{2}{\lambda_2}$$
$$+(\underset{1}{G_2'}\,\underset{1}{H_1}-\underset{1}{G_1'}\,\underset{1}{H_2})\cosh\pi\underset{2}{\lambda_1}\sinh\pi\underset{2}{\lambda_2}+(\underset{1}{G_2'}\,\underset{1}{I_1'}-\underset{1}{G_1'}\,\underset{1}{I_2'})\cosh\pi\underset{2}{\lambda_1}\cosh\pi\underset{2}{\lambda_2}=0. \qquad \mathbf{(25)}$$

The expression included by the parenthesis of each term must be given by the joining conditions, *i. e.*, by (**17**) for this case. Indeed, by substituting and by some transformations, we obtain the following expressions:

$$\Delta_s + \kappa_2' \Delta_h = 0\,,^{42)} \qquad (18.1)$$

where

$$\left.\begin{aligned}
\Delta_s = {} & \mu_2'\,\varphi_1\,\varphi_2\,(\underset{1}{\lambda_1}\,\underset{1}{\lambda_2}\coth\pi\underset{1}{\lambda_1}\coth\pi\underset{1}{\lambda_2}+\mu_2^2\,\underset{2}{\lambda_1}\,\underset{2}{\lambda_2}\coth\pi\underset{2}{\lambda_1}\coth\pi\underset{2}{\lambda_2}\\
& +\Big\{(\underset{1}{\tau_2'}\,\underset{1}{\chi_1}\coth\pi\underset{1}{\lambda_1}-\underset{1}{\tau_1'}\,\underset{1}{\chi_2}\coth\pi\underset{1}{\lambda_2})\coth\pi\underset{2}{\lambda_1}\\
& -(\underset{1}{\tau_2}\,\underset{1}{\chi_1'}\coth\pi\underset{1}{\lambda_1}-\underset{1}{\tau_1}\,\underset{1}{\chi_2'}\coth\pi\underset{1}{\lambda_2})\coth\pi\underset{2}{\lambda_2}\Big\}\,,\\
\Delta_h = {} & \mu_2\,\underset{2}{\beta_1}\,\underset{2}{\beta_2}\,\varphi_1\,(\mu_2''\,T_2'+\frac{\bar{\varphi}_2}{\bar{\varphi}_1}\,T_1')\,,\\
\text{in which}\quad T_r' = {} & \frac{\underset{r}{\gamma_1}+P_r\underset{r}{\lambda_1}}{\underset{r}{\beta_1}}\coth\pi\underset{r}{\lambda_1}-\frac{\underset{r}{\gamma_2}+P_r\underset{r}{\lambda_2}}{\underset{r}{\beta_2}}\coth\pi\underset{r}{\lambda_2}\,,\quad \varphi_r=\underset{r}{\beta_1}-\underset{r}{\beta_2}\,,\\
\bar{\varphi}_r = {} & \frac{1}{\underset{r}{\beta_1}}-\frac{1}{\underset{r}{\beta_2}}\,.
\end{aligned}\right\} \qquad (18.2)$$

Next, supposing the extreme case where $\kappa_2'=0$, we have $\frac{\delta\theta_1}{M_1}=0$. This means the case of a rigidly connected joint and in this time, Eq. (18.1) becomes as follows:

$$\Delta_s = 0\,.$$

By some transformations, it can be proved that the above coincides with the conditional equation in § 10. Again, supposing the other extreme case where $\kappa_2'=\infty$, we have $\frac{\delta\theta_1}{M_1}=\infty$. From this, considering that the magnitude of $\delta\theta_1$ must not become infinite, it follows that $M_1 = 0$. This means the case of a hinged joint. Now by rewriting (18.1)

$$\frac{1}{\kappa_2'}\Delta_s + \Delta_h = 0\,.$$

Then, when $\kappa_2' = \infty$, the above becomes

$$\Delta_h = 0\,.$$

From this, the conditional equation (17.2) in § 17 is also yielded.

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42) In this equation, the factor $\frac{\sqrt{P(1-P)}}{4\,(\cosh\pi\sqrt{1-P}-\cos\pi\sqrt{P})}$ which the left hand side of the equation must be multiplied by is omitted for shortness because that can not become zero unless $P=0$ or $P=1$.

In the next place, let us investigate the special case where only the load p is acting and $h_1 = h_2$. At this time, it follows that

$$P_1 = P_2 = P, \qquad \underset{1}{\lambda_1} = \underset{2}{\lambda_1} = \lambda_1, \qquad \underset{1}{\lambda_2} = \underset{2}{\lambda_2} = \lambda_2, \qquad \varphi_1 = \varphi_2 = \varphi,$$

$$\mu_2 = \mu_2' = \mu_2'' = 1, \qquad T_1' = T_2' = T'.$$

Then the expressions (18.2) becomes simpler as follows:

$$\begin{aligned} \Delta_s &= 4\lambda_1 \lambda_2 \varphi^2 \coth\pi\lambda_1 \coth\pi\lambda_2, \\ \Delta_h &= 2\beta_1 \beta_2 \varphi T', \end{aligned} \tag{18.3}$$

in which the notation κ is used instead of κ_2' for simplicity. Now, in the special case $\kappa = 0$, it is concluded that $P = 1$,[43] since the composite plate becomes a single plate. And in the other special case $\kappa = \infty$, it follows that $P = 0.5107$ as understood from Fig. 28, since the composite plate becomes the one which is joined with a perfectly smooth hinge. Therefore, at any value of κ the following inequalities may be held:

$$0.5107 < P < 1. \tag{18.4}$$

At this time, λ_1 and λ_2 will become of the mutually conjugate complex values and accordingly the conditional equation is written as

$$\left.\begin{aligned} &\Delta = \Delta_s + \kappa\Delta_h = 0, \\ &\text{or} \\ &\kappa = -\frac{\Delta_s}{\Delta_h}, \\ &\text{where} \\ &\Delta_s = 2\sqrt{P(P-1)}\,\frac{\sinh^2\pi\sqrt{1-P} + \sin^2\pi\sqrt{P}}{\cosh\pi\sqrt{1-P} + \cos\pi\sqrt{P}}, \\ &\Delta_h = (0.5775 - P)\sqrt{P}\sinh\pi\sqrt{1-P} - (P - 0.1225)\sqrt{1-P}\sin\pi\sqrt{P}. \end{aligned}\right\} \tag{18.5}$$

Plotting the relation between the least value of P and the coefficient κ by the above, we obtain Fig. 29 which shows that the magnitude of P diminishes gradually according as the value of κ increases, namely, such a case approaches that with a smooth hinge.

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43) For a square plate having the side length b, such a result as $\frac{pb^2}{D\pi^2} = 4$ is already known. In the present case, observing $a_1 = a_2 = \frac{b}{2}$, we have $P = P_1 = \frac{pa_1^2}{D\pi^2} = \frac{pb^2}{D\pi^2}\left(\frac{a_1}{b}\right)^2 = 1$

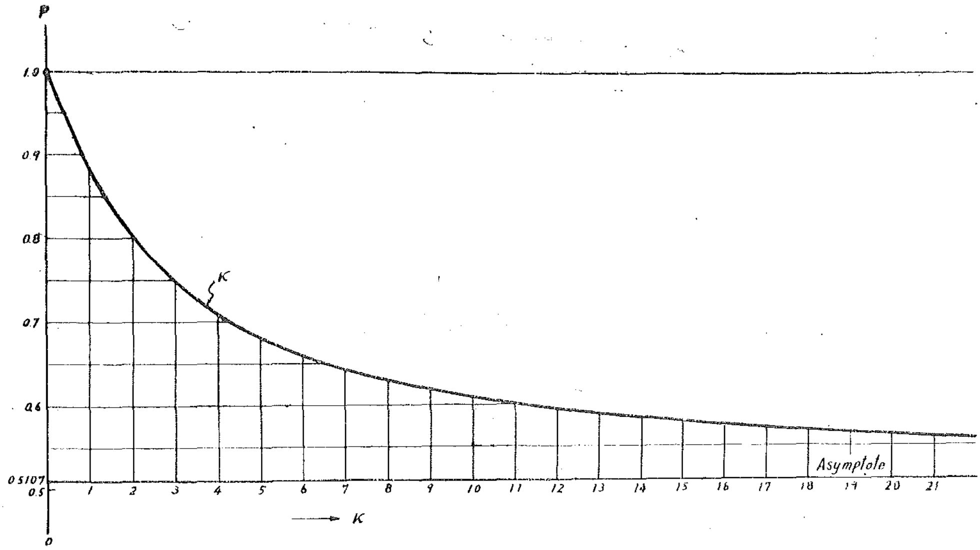

Fig. 29.

§ 19. Formulae for the Plates Supported by an Intermediate Elastic Beam.

In this section, some investigations are presented for the case where an elastically deformable beam is furnished to the joint of such a composite plate as considered before. And again, for brevity of investigations, let the consideration confine itself to the case of such a composite plate as is simply supported along both end edges.

(1) *The case with a rigidly connected joint* [Case c in Fig. 2].—Eq. (**25**) in the previous section can be used again as the conditional equation for this case. Substituting the joining conditions (**12**) for this case in the parentheses of each term of Eq. (**25**) respectively and arranging them, the following equation is finally obtained:

$$\Delta = \Delta_s + \delta\Delta_s = 0\,, \tag{19.1}$$

where

$$\left.\begin{aligned}
\Delta_s &= \text{that given in (18.2)},\\
\delta\Delta_s &= \mu_2'\,\varphi_2\,T_1\,(\bar{\mu}_2' - \mu_2\bar{\mu}_2\,\underset{2}{\lambda_1}\,\underset{2}{\lambda_2}\coth\pi\underset{2}{\lambda_1}\coth\pi\underset{2}{\lambda_2})\\
&\quad + \varphi_1\,T_2\,(\mu_2\bar{\mu}_2' - \bar{\mu}_2\,\underset{1}{\lambda_1}\,\underset{1}{\lambda_2}\coth\pi\underset{1}{\lambda_1}\coth\pi\underset{1}{\lambda_2})\\
&\quad - \bar{\mu}_2\,\bar{\mu}_2'\,T_1\,T_2\,,
\end{aligned}\right\} \tag{19.2}$$

in which

$$T_r = \underset{r}{\lambda_1}\coth\pi\underset{r}{\lambda_1} - \underset{r}{\lambda_2}\coth\pi\underset{r}{\lambda_2}\,.$$

In the above, we find that the term $\delta\Delta_s$ is the newly additional one due to the supporting beam. And yet, in the practical computations, it must be remarked that such considerations as are discussed in § 3 for continuity at joints are kept also in this case.

(2) *The case with a hinged joint* [Case f in Fig. 2].—Using the expressions (**14**), (**15**), (**15a**) and (**15d**) in § 7, the conditional equation is given in the following form:

$$V_1 = T_2' F_1' + G_1' = 0 ,$$

where

$$T_2' = -\frac{\gamma_{\substack{1\\2}} + P_2\lambda_{\substack{1\\2}}}{\beta_{\substack{1\\2}}} \coth\pi\lambda_{\substack{1\\2}} - \frac{\gamma_{\substack{2\\2}} + P_2\lambda_{\substack{2\\2}}}{\beta_{\substack{2\\2}}} \coth\pi\lambda_{\substack{2\\2}} .$$

Substituting the joining conditions (**16**) for this case in the above, some arrangings give

$$\Delta = \Delta_h + \delta\Delta_h = 0 , \tag{19.3}$$

where

$$\left.\begin{aligned} \Delta_h &= \text{that given in } (18.2), \\ \Delta\delta_h &= -\mu_2\bar{\mu}_2' \beta_{\substack{1\\2}} \beta_{\substack{2\\2}} \varphi_1 \bar{\varphi}_2 . \end{aligned}\right\} \tag{19.4}$$

In the above, we find that the term $\delta\Delta_h$ is the newly additional one due to the supporting beam.

(3) *The case with an elastically built joint* [Case i in Fig.2].—Eq. (**25**) can, again in this case, be applied because the end side conditions are not different. And then substituting the joining conditions (**20**) for this case in Eq. (**25**) and arranging them, we finally obtain after some transformations

$$\Delta = (\Delta_s + \delta\Delta_s) + \kappa(\Delta_h + \delta\Delta_h) + \delta\Delta = 0 , \tag{19.5}$$

where

$$\left.\begin{aligned} &\Delta_s,\ \delta\Delta_s = \text{those in } (19.2), \\ &\Delta_h,\ \delta\Delta_h = \text{those in } (19.4), \\ &\delta\Delta = -\kappa_1 \bar{\mu}_2(\beta_{\substack{1\\1}} \beta_{\substack{2\\1}} T_1' T_2 + \bar{\mu}_2' \varphi_1 T_2 + \mu_2'' \varphi_1 \varphi_2 \lambda_{\substack{1\\2}} \lambda_{\substack{2\\2}} \coth\pi\lambda_{\substack{1\\2}} \coth\pi\lambda_{\substack{2\\2}}) \\ &\qquad -\kappa_2' \bar{\mu}_2(\mu_2'' \beta_{\substack{1\\2}} \beta_{\substack{2\\2}} T_1 T_2' + \bar{\mu}_2' \varphi_2 T_1 + \varphi_1 \varphi_2 \lambda_{\substack{1\\1}} \lambda_{\substack{2\\1}} \coth\pi\lambda_{\substack{1\\1}} \coth\pi\lambda_{\substack{2\\1}}), \\ &\kappa = \frac{\bar{\mu}_2''}{\mu_2} \kappa_1 + \kappa_2' = \frac{\mu_2' - \kappa_2' \bar{\mu}_2}{\mu_2} \kappa_1 + \kappa_2' = \left(\frac{\mu_2'}{\mu_2} \kappa_1 + \kappa_2' \right) - \frac{\bar{\mu}_2}{\mu_2} \kappa_1 \kappa_2' . \end{aligned}\right\} \tag{19.6}$$

Now, letting $\kappa_1 = \kappa_2' = 0$, this case is reduced to the one with a rigidly connected joint, and at this time it follows that Eq. (19.5) coincides with Eq. (19.1). Again, if $\kappa_1 = \kappa_2' = \infty$, the case with a hinged joint is yielded, and at the same time it will be able to write $1/\kappa_1 = 1/\kappa_2'$. From this, considering

$$\frac{\Delta}{\kappa_1 \kappa_2'} = 0$$

instead of Eq. (19.5), Eq. (19.3) is consequently obtained.

Let us observe next that Eq. (19.5) becomes to coincide with the formula of the plate without a supporting beam by equating the coefficients $\bar{\mu}_2$, $\bar{\mu}_2'$ to zero, that is

$$\delta\varDelta_s = 0\,, \qquad \delta\varDelta_h = 0\,, \qquad \delta\varDelta = 0\,.$$

Now, considering that κ_1, κ_2' are expressed in the same form as (**24**) in the previous section, the following relations are held[44]

$$\kappa = \frac{\mu_2'}{\mu_2}\kappa_1 + \kappa_2' = \frac{\mu_2'}{\mu_2}\left(\frac{\delta\theta_1'}{M}\right)\frac{D_1\pi}{a_1} + \left(\frac{\delta\theta_2'}{M}\right)\frac{D_2\pi}{a_2} = \left(\frac{\delta\theta_1' + \delta\theta_2'}{M}\right)\frac{D_2\pi}{a_2}\,,$$

where $\delta\theta_1'$ = The angle yielded between the plane of the 1-st elementary plate and the horizontal axis of the cross section of the beam,

$\delta\theta_2'$ = The angle between the plane of the 2-nd elementary plate and that of the beam.

In the case without the supporting beam, denoting by $\delta\theta_1$ the angle between the plane of the 1-st elementary plate and that of the 2-nd, we can write as

$$\delta\theta_1 = \delta\theta_1' + \delta\theta_2'\,.$$

Then,

$$\kappa = \left(\frac{\delta\theta_1}{M}\right)\frac{D_2\pi}{a_2} = \kappa_2' \text{ of the case without such a beam.}$$

Consequently, Eq. (19.5) becomes

$$\varDelta = \varDelta_s + \kappa_2'\,\varDelta_h = 0\,.$$

The above is nothing else Eq. (18.1) in § 18.

As the final remarks, it must be described that the multiplier

$$\frac{\sinh\pi\underset{1}{\lambda_1}\,\sinh\pi\underset{1}{\lambda_2}\,\sinh\pi\underset{2}{\lambda_1}\,\sinh\pi\underset{2}{\lambda_2}}{\underset{1}{\lambda_1}\,\underset{1}{\lambda_2}\,\varphi_1^2}$$

has been omitted in (19.1) and (19.5), because the equation obtained by equating that to zero yields only the critical load for the case where each elementary plate is individually considered as a single plate simply supported along all the edges, and then can not yield the critical load of such a composite plate as considered now. On the other hand, the factors $(\mu_2\,\underset{2}{\beta_1}\,\underset{2}{\beta_2}\,\varphi_1\,S_1')$ have been multiplied in Eq. (19.3) for convenience of comparison.

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44) It can be written that $\frac{\mu_2'}{\mu_2} = \frac{D_2a_1}{D_1a_2}$, referring to the footnotes of Table 5.

CHAPTER IV

FORMULAE FOR CONTINUOUS PLATES.

The formulas which have been discussed in the preceding chapters are those which can be applied to the general cases where the various dimensions, such as thickness and length, of each elementary plate are arbitrary, and yet those are obviously applicable to the special cases where each elementary plate has common dimensions. Practically, such cases seem to be rather encountered and are of practical importance, Then it seems that the simplification of the treatments for these cases is especially desired.

Now, considering the case where the constants h_r, E_r, ν_r of each elementary plate are the same ones respectively, the composite plate of the case a) or b) [see Fig. 2] becomes to be equivalent to a single plate or that supported by the elastic beams.[45] On the other hand, let us call temporarily the composite plates of such cases as b), d), e), f), h), j) [see Fig. 2] *continuous plates* and show, in the following, that the comparatively simple treatments are possible for these cases by means of so-called "Differenzengleichungen," *i. e.*, the finite difference equations.

In the first instance, denote as follows:

$$a_1 = a_2 = \cdots\cdots\cdots\cdots = a_r = \cdots\cdots\cdots\cdots = a_k = a\,;$$
$$D_1 = D_2 = \cdots\cdots\cdots\cdots = D_r = \cdots\cdots\cdots\cdots = D_k = D\,,$$

Then $$\mu_r = \mu_r' = 1\,.$$

And in this case, we can put

$$P_1 = P_2 = \cdots\cdots\cdots\cdots = P_r = \cdots\cdots\cdots\cdots = P_k = P.$$

Again, considering that the following relations hold:

$$E_1 h_1 = E_2 h_2 = \cdots\cdots\cdots\cdots = E_r h_r = \cdots\cdots\cdots\cdots = E_k h_k,$$

and referring to the discussions in § 3, we can conclude that

$$q_1 = q_2 = \cdots\cdots\cdots\cdots = q_r = \cdots\cdots\cdots\cdots = q_k.$$

Accordingly, we can write

$$Q_1 = Q_2 = \cdots\cdots\cdots\cdots = Q_r = \cdots\cdots\cdots\cdots Q_k = Q.$$

Then, observing the foregoing relations, we see that the following denotations are

45) *Vid.* such as S. Timoshenko's "Theory of Elastic Stability," 1936, pp. 371~379.

possible for any suffix r:

$$\lambda_{\underset{r}{1}} = \lambda_1 , \qquad \lambda_{\underset{r}{2}} = \lambda_2 .$$

And, by

$$\nu_1 = \nu_2 = \cdots\cdots\cdots\cdots\cdots = \nu_r = \cdots\cdots\cdots\cdots\cdots = \nu_k ,$$

we can write again for any r

$$\left.\begin{aligned} \beta_{\underset{r}{1}} &= \beta_1, \\ \beta_{\underset{r}{2}} &= \beta_2, \end{aligned}\right\} \qquad \left.\begin{aligned} \gamma_{\underset{r}{1}} &= \gamma_1, \\ \gamma_{\underset{r}{2}} &= \gamma_2. \end{aligned}\right\}$$

Furthermore, it will be concluded that the following denotations are possible:

$$\begin{aligned}
&T_1 = T_2 = \cdots\cdots = T_r = \cdots\cdots = T_k = T = \lambda_1 \coth\pi\lambda_1 - \lambda_2 \coth\pi\lambda_2, \\
&T_1' = T_2' = \cdots\cdots = T_r' = \cdots\cdots = T_k' = T' = \frac{\gamma_1 + P\lambda_1}{\beta_1} \coth\pi\lambda_1 - \frac{\gamma_2 + P\lambda_2}{\beta_2} \coth\pi\lambda_2 , \\
&S_1 = S_2 = \cdots\cdots = S_r = \cdots\cdots = S_k = S = \lambda_1 \operatorname{cosech}\pi\lambda_1 - \lambda_2 \operatorname{cosech}\pi\lambda_2, \\
&\bar{S}_1 = \bar{S}_2 = \cdots\cdots = \bar{S}_r = \cdots\cdots = \bar{S}_k = \bar{S} = \frac{\lambda_1}{\beta_1} \operatorname{cosech}\pi\lambda_1 - \frac{\lambda_2}{\beta_2} \operatorname{cosech}\pi\lambda_2 , \\
&S_1'' = S_2'' = \cdots\cdots = S_r'' = \cdots\cdots = S_k'' = S'' = (\gamma_1 + P\lambda_1) \operatorname{cosech}\pi\lambda_1 - (\gamma_2 + P\lambda_2) \operatorname{cosech}\pi\lambda_2, \\
&S_1' = S_2' = \cdots\cdots = S_r' = \cdots\cdots = S_k' = S' = \frac{\gamma_1 + P\lambda_1}{\beta_1} \operatorname{cosech}\pi\lambda_1 - \frac{\gamma_2 + P\lambda_2}{\beta_2} \operatorname{cosech}\pi\lambda_2 , \\
&W_1 = W_2 = \cdots\cdots = W_r = \cdots\cdots = W_k = W = T^2 - S^2, \\
&W_1' = W_2' = \cdots\cdots = W_r' = \cdots\cdots = W_r' = W_k' = T'^2 - S'^2.
\end{aligned}$$

(A) FORMULAE AND NUMERICAL ILLUSTRATIONS FOR THE FUNDAMENTAL CONTINUOUS PLATES.

Particularly, in both the cases, *i. e.*, *b*) *The case with the rigidly connecting joints and the rigid supportings along the joints*, and *e*) *The case with the hinged joints and no supportings*, any coefficients of restraints concerning the joints do not enter in the considerations, and then the numerical investigations will be comparatively simple. So, considering both the above cases as the fundamental ones, we shall now go into detail about such cases.

§ 20. Plates Having Uniform Thickness and Simply Supported with Multiple Equal Spans.

i) **General solution.**—The joining conditions for the present case to which (11) and (10) [§ 6-b)] are applicable, for the continuous plate, become of the simpler form as follows:

$$\left.\begin{aligned} U_{r-1} - U_r F + V_r G &= 0 ; \\ V_{r-1} + U_r F' - V_r G' &= 0 , \end{aligned}\right\} \tag{20.1}$$

where

$$\left.\begin{aligned} F &= \frac{T}{S}, \qquad G = \frac{W}{S}, \\ F' &= \frac{1}{S}, \qquad G' = \frac{T}{S} = F. \end{aligned}\right\} \tag{20.2}$$

The above equations can be considered as the simultaneous finite difference equations of the first order. The general solution is obtained by letting as follows:

$$\left.\begin{aligned} U_r &= B\,\omega^r; \\ V_r &= B'\omega^r, \end{aligned}\right\} \tag{20.3}$$

where B and B' are the new unknown constants. Substituting these in Eqs. (20.1), we have

$$\left.\begin{aligned} B(\omega^{r-1} - \omega^r F) + B'\,\omega^r G &= 0; \\ B'(\omega^{r-1} - \omega^r G') + B\,\omega^r F' &= 0. \end{aligned}\right\} \tag{20.4}$$

Eliminating B and B' from the above, the following characteristic equation is obtained:

$$\frac{1}{\omega^2} - \frac{1}{\omega}(F + G') + F\,G' - G\,F' = 0.$$

This equation can be transformed again as follows by using the expressions (20.2):

$$\omega^2 - 2\omega\frac{T}{S} + 1 = 0.$$

Denoting the two roots of the above by ω_1 and ω_2 respectively, we find that they are expressed in the following forms:

$$\left.\begin{matrix}\omega_1 \\ \omega_2\end{matrix}\right\} = \frac{T}{S} \pm \sqrt{\left(\frac{T}{S}\right)^2 - 1}.$$

And again, letting

$$\left.\begin{aligned} \omega_1 &= \rho\,e^{i\alpha}; \\ \omega_2 &= \rho\,e^{-i\alpha}, \end{aligned}\right\} \tag{20.5}$$

we obtain by taking $\rho = 1$

$$\cos\alpha = \frac{\omega_1 + \omega_1}{1} = \frac{T}{S}, \tag{26}$$

because the relations $\omega_1\omega_2 = \rho^2 = 1$ can be held.

Next, considering that the following expressions can be written by (20.5):

$$\omega = \cos\alpha \pm i\sin\alpha,$$

we have from the first equation in (20.4)

$$B' = \pm iB \frac{S}{W} \sin\alpha .$$

Accordingly, using again B and B' as new unknown constants for saving of the notations, the general solutions of Eqs. (20.1) can be given in the following forms by referring to (20.3):

$$\left.\begin{aligned} U_r &= B\cos r\alpha + B'\sin r\alpha , \\ V_r &= -\frac{S}{W}\sin\alpha\,(B\sin r\alpha - B'\cos r\alpha), \end{aligned}\right\} \qquad (\mathbf{27})$$

where α is the parameter which is defined by (**26**).

ii) **The end side conditions.** (1) *The conditions at End side-k.*

1) The case of the simply supported edge.—The conditions at End side-k which is simply supported are given by the following expressions from (**11**.a) [§ 6]:

$$U_k = -T; \qquad V_k = 1 .$$

Then, substituting the general solution (**27**) in the above, we have

$$B\cos k\alpha + B'\sin k\alpha = -T;$$

$$B\sin k\alpha - B'\cos k\alpha = -\frac{W}{S\sin\alpha} .$$

From these; B and B' are expressed as follows:

$$\left.\begin{aligned} B &= -S\cos(k+1)\alpha , \\ B' &= -S\sin(k+1)\alpha . \end{aligned}\right\} \qquad (20.6)$$

2) The case of the clamped edge.—The conditions for this case are given as follows by (**11**.b):

$$U_k = -\frac{W}{T}; \qquad V_k = 1 .$$

Substituting (**27**) and solving with respect to B and B', we have

$$\left.\begin{aligned} B &= -\frac{W}{T}\frac{\sin(k+1)\alpha}{\sin\alpha}, \\ B' &= \frac{W}{T}\frac{\cos(k+1)\alpha}{\sin\alpha}. \end{aligned}\right\} \qquad (20.7)$$

3) The case of the free edge.—The conditions are given as follows by (**11**.c):

$$U_k = -\frac{TT' - \bar{S}S''}{T'}; \qquad V_k = 1 .$$

Substituting (**27**) and solving, we get

$$\left.\begin{aligned} B &= -S\left\{\cos(k+1)\alpha - \frac{\bar{S}S''}{ST'}\cos k\alpha\right\}, \\ B' &= -S\left\{\sin(k+1)\alpha - \frac{\bar{S}S''}{ST'}\sin k\alpha\right\}. \end{aligned}\right\} \qquad (20.8)$$

(2) *The conditions at End side-1.*

1) The case of the simply supported edge.—The condition at End side-1 which is simply supported is given by the following expression from (11.d) [§ 6];

$$V_1 = 0.$$

Substituting the general solution (27) in the above, we obtain

$$B\sin\alpha - B'\cos\alpha = 0. \tag{20.9}$$

2) The case of the clamped edge.—The condition is given as follows by (11.e):

$$U_1 = 0.$$

Substituting the general solution (27) in this, we get

$$B\cos\alpha + B'\sin\alpha = 0. \tag{20.10}$$

3) The case of the free edge.—The condition for this case is expressed by (11·f) in the following form:

$$U_1 = U_2(F_1) - V_2(G_1) = 0,$$

where

$$(F_1) = \frac{T'}{\bar{S}}, \qquad (G_1) = \frac{TT' - \bar{S}S''}{\bar{S}}.$$

Substituting (27) in the above, we obtain

$$B\left(\sin\alpha - \frac{\bar{S}S''}{ST'}\sin 2\alpha\right) - B'\left(\cos\alpha - \frac{\bar{S}S''}{ST'}\cos 2\alpha\right) = 0. \tag{20.11}$$

iii) The conditional equations for buckling and their numerical illustrations. 1) *The case where both the end sides are simply supported.*—The combination of (20.6) and (20.9) yields the conditional equation for this case, *i. e.*, by eliminating B and B' by means of substitution of (20.6) into (20.9)

$$S\left\{\sin(k+1)\alpha\cos\alpha - \cos(k+1)\alpha\sin\alpha\right\} = 0,$$

from which, both the following cases result:

$$S = 0, \qquad \text{or} \qquad \sin k\alpha = 0.$$

Here, since $S = 0$ yields $B = B' = 0$ owing to (20.6), U_r and V_r become to be identically zero. Then, returning to the general treatings in § 6–b), we can find that it corresponds to the case where the plate remains flat without buckling. Thus, it is trivial for the problems. Therefore putting $S \neq 0$, the following equation must be held:

$$\sin k\alpha = 0 .$$

From this, we have

$$\alpha = s\frac{\pi}{k} \qquad (s = 0, 1, 2, \cdots\cdots, 2k\text{-}1). \tag{28}$$

Hence, regarding the preceding (26) together, the conditional equation can finally be written in the following form:

$$\cos s\frac{\pi}{k} = \frac{T}{S} = \frac{\lambda_1 \coth\pi\lambda_1 - \lambda_2 \coth\pi\lambda_2}{\lambda_1 \operatorname{cosech}\pi\lambda_1 - \lambda_2 \operatorname{cosech}\pi\lambda_2} \tag{29}$$

$$(s=0, 1, 2, \cdots\cdots\cdots 2k-1).$$

Now, let us observe the extreme case when $b = \infty$.[46] At this time, it follows that

$$\lambda_1 = 0, \qquad \lambda_2 = i\frac{Z}{\pi} \qquad \text{where} \qquad Z = a\sqrt{\frac{p}{D}}\,.$$

And then, rewriting (29) as

$$\cos s\frac{\pi}{k} = \frac{\dfrac{\pi\lambda_1}{\sinh\pi\lambda_1}\cosh\pi\lambda_1 \sinh\pi\lambda_2 - \pi\lambda_2 \cosh\pi\lambda_2}{\dfrac{\pi\lambda_1}{\sinh\pi\lambda_1}\sinh\pi\lambda_2 - \pi\lambda_2},$$

and taking into account that $\lim\limits_{\pi\lambda_1 \to 0} \dfrac{\pi\lambda_1}{\sinh\pi\lambda_1} = 1$, the above equation is transformed into the following form by the limitation process $\lambda_1 \to 0$, $\pi\lambda_2 = iZ$:

$$\cos s\frac{\pi}{k} = \frac{\sin Z - Z\cos Z}{\sin Z - Z}\,. \tag{20.12}$$

This is nothing than the formula of a continuous column with multiple spans.[47] It is already known that the least critical load is determined when $Z = \pi$ by taking $s = k$ at the case where both ends are simply supported.[48]

Now, let us investigate numerically about such cases that a/b, the ratio of width of a plate to span, is 1/2. [In the numerical examples, it will always be assumed in the same manner hereafter.]

First, the case where only the load p is acting will be observed. By referring to the above mentioned result concerning the continuous column, we shall temporarily assume $s = k$ for Eq. (29). Then the left hand side of Eq. (29) becomes of

46) It is well known that in such an extreme case the formulas are reduced to those of the beam. See such as S. Iguchi, "Eigenwertprobleme," and A. Nadai, "Elastische Platten," s. 70.

47) Fr. Bleich und E. Melan. "Die Gewöhenlichen und Partiellen Differenzengleichungen der Baustatik," 1927, s. 216.

48) S. Ban, "Theory on the Stability of Rigid Frames by Slope Deflection Method," Journal of the Institute of Japanese Architects, Vol. 45, No. 551, pp. 9~11.

the magnitude -1 regardless of k, and on the other hand we can find that Eq.(29) is satisfied when $P=1.5625$ as shown in Fig.30[49] by plotting the curve expressing the relation between T/S and P.[50] The above facts means that the continuous plate

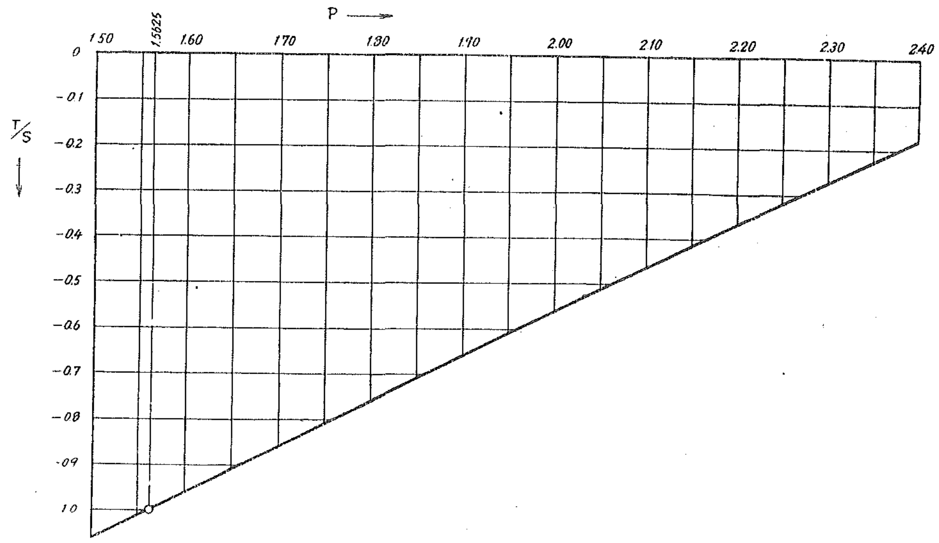

Fig. 30.

in the present case will buckle in the similar type as a single plate to be simply supported along its all edges, because the critical value of the former agrees with that of the latter. That is to say, we can understand the above facts by observing that the simplest distortion of such a continuous plate occurs when the portion in each span deflects in alternately opposite side just like the simplest distortion of a single plate.*

In the next place, let us consider the case where only the load q is acting. It is easy to suppose as such a previous mode of distortion may again correspond to the least critical value in this case. Hence, it can be justified that the portion of each span is considered as a single plate respectively. In the present case, since $b=2a$, when such a single plate buckles into two half-waves in y-direction, that is submitting to the least critical load, *viz.*, $Q=16$. [By the way, when such a plate buckles in a half-wave, $Q=25$.] Plotting T/S for $m=2$ depending on Q [Fig. 31],

49) Particularly, in the case when $k=1$, the considered plate becomes to be a single plate which is simply supported along all the edges. In this time, it follows that $P=1.5625$ [see the footnote 36)]. Therefore, it may be reasonable that the value of T/S is plotted as a function of P in the vicinity of this value.

50) Substituting directly $P=1.5625$ into Eq. (29), the right hand side of the equation becomes of indefinite form, *i.e.*, $\frac{T}{S}=-\frac{\infty}{\infty}$. But the graphical illustration shows that $\lim_{P\to 1.5625}\frac{T}{S}=-1$ holds.

* See the writer's paper concerning the experimental investigations in *Memoirs of the Faculty of Engineering*, Hokkaido University, Vol. 8, No. 2, 1949 (Japanese).

we can readily find that the above value of Q satisfies Eq. (**29**).

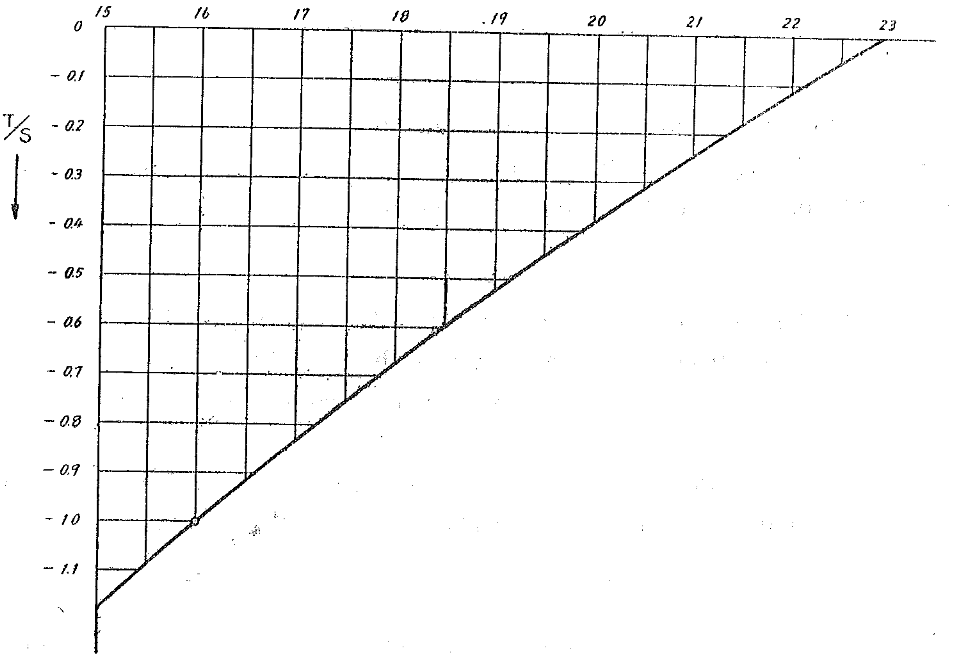

Fig. 31. $(m=2)$

2) *The case where one end side is clamped and the other is simply supported.*—Since which end side is considered to be clamped is out of the question about such a continuous plate, let us assume now that End side-1 is clamped.[51]

Eliminating B and B' from both (20.6) and (20.10), we have

$$S\left\{\cos(k+1)\alpha\cos\alpha+\sin(k+1)\alpha\sin\alpha\right\}=0\,.$$

But, observing that $S\neq 0$ as before, we find

$$\cos k\alpha=0\,.$$

From this

$$\alpha=\left(s+\frac{1}{2}\right)\frac{\pi}{k}\qquad(s=0,1,2,\cdots\cdots\cdots\cdots,2k-1). \tag{30}$$

Hence, taking the foregoing (**26**) into account together, the conditional equation for this case is written as follows:

$$\cos\left(s+\frac{1}{2}\right)\frac{\pi}{k}=\frac{T}{S}=\frac{\lambda_1\coth\pi\lambda_1-\lambda_2\coth\pi\lambda_2}{\lambda_1\operatorname{cosech}\pi\lambda_1-\lambda_2\operatorname{cosech}\pi\lambda_2} \tag{31}$$
$$(s=0,1,2,\cdots\cdots\cdots\cdots,2k-1).$$

51) Considering that End side-2 is clamped, we must eliminate B and B' from (20.7) and (20.9), but the final result will become the same provided that $W/T\neq 0$ is taken instead of $S\neq 0$.

At the case when $k=1$, *viz.*, that of the single plate, the following equation is immediately reduced from the above:

$$\lambda_1 \coth\pi\lambda_1 - \lambda_2 \coth\pi\lambda_2 = 0\,. \tag{20.13}$$

This is the formula already known.[52]

For the numerical example, let us consider first the case where only the load p is acting. Provided that $k=1$, the conditional equation (**31**) is reduced to Eq. (20.13) as described before, and its least root is obtained as follows:

$$P = 2.5966.$$

Let us suppose next the extreme case when $k=\infty$. In this time, it seems that the effect by the clamped edge tends to vanish as Fr.Bleich has indicated with respect to a continuous column.[53] Thus, it may be assumed that $P=1.5625$ in such an extreme case.

By the above considerations, we can establish the following limits within which the least critical value of P must be searched for such a k as is larger than 2:

$$1.5625 < P < 2.5966$$

Now, the curve T/S depending upon P represents a negative monotonic increasing function within the above interval as shown in Fig. 30, then we can understand that the minimum value which is possible for the left side of Eq. (**31**) gives the least root P for this case. But, since s must be taken on an integral number such as 0, 1, 2,........, $2k-1$, we have to put $s=k-1$ so that $\left(s+\dfrac{1}{2}\right)\dfrac{\pi}{k}$ becomes the nearest value to π. Then Eq. (**31**) is transformed as follows:

$$\cos\left(1-\frac{1}{2k}\right)\pi = \frac{T}{S}\,. \tag{31$'$}$$

Table 28.*

k	$\left(1-\dfrac{1}{2k}\right)\pi$	$\cos\left(1-\dfrac{1}{2k}\right)\pi$	P
2	$3/4\ \pi$	$-0.707\,11$	1.853
3	$5/6\ \pi$	$-0.866\,03$	1.695
4	$7/8\ \pi$	$-0.923\,88$	1.637
5	$9/10\ \pi$	$-0.951\,06$	1.611
6	$11/12\ \pi$	$-0.965\,93$	1.597
∞	π	$-\ 1$	1.5625

Calculating by this the value of T/S for any k, we can find the value of P by observing the point at which the curve of T/S just indicate such a value [see Fig. 30]. The practical examples for the various k are shown in Table 28. Here, taking $k=\infty$ in (**31**$'$), we obtain $T/S=-1$ and then $P=1.5625$ by means of Fig. 30.

52) See S. Iguchi, "Die Eigenwertprobleme," Memoirs of the Faculty of Eng., H.I.U., Vol. 4, No. 4, s. 41.

53) See paper by Fr. Bleich, *loc. cit.*, p. 275.

* See the writer's paper, *loc. cit.*, p. 276.

This fact agrees with the initial assumption.

By the above conclusion, it can be seen that the effect by the clamped edge diminishes rapidly according as the number of the spans k increases.

In the next place, let us obsearve the case where only the load q is acting. We can consider that the buckling with two half-waves in y-direction corresponds to the least critical load as in the preceding example. Then, using the curve of T/S for $m=2$ [Fig. 31] and requiring the least critical value of Q for the various k, the numerical results shown in Table 29 are found. Again in this case, we can see that the effect by the clamped edge diminishes rapidly according as the number of spans increases.

Table 29.

k	$\cos\left(1-\frac{1}{2k}\right)\pi$	Q $[m=2]$
2	The same as in Table 28.	17.77
3		16.80
4		16.45
5		16.28
6		16.20
∞		16

3) *The case where both the end sides are clamped.*—The conditional equation is represented by the combination of (20.7) and (20.10). Eliminating B and B' as before, we have

$$-\frac{W}{T}\left\{\sin(k+1)\alpha\cos\alpha-\cos(k+1)\alpha\sin\alpha\right\}=0.$$

From this

$$\frac{W}{T}=0, \qquad \text{or} \qquad \sin k\alpha=0.$$

If we take $\frac{W}{T}=0$, $B=B'=0$ results by (20.7) and then it becomes that the plate remains plane. So, putting $\frac{W}{T}\neq 0$, we have

$$\sin k\alpha=0.$$

Then

$$\alpha=s\frac{\pi}{k} \qquad (s=0,1,2,\cdots\cdots\cdots,2k\text{-}1).$$

Accordingly we find that the conditional equation in this case has the same form as Eq. (**29**).

Now, considering such an extreme case as $b=\infty$, we obtain again the formula (20.12) for a continuous column, and it is already known that its least critical load is obtained by puttiug $s=k\text{-}1$.[54)]

Let us observe next the case when $k=1$, *viz.*, that of a single plate. Letting $s=0$, the conditional equation (**29**) yields

$$\lambda_1\tanh\frac{\pi}{2}\lambda_1-\lambda_2\tanh\frac{\pi}{2}\lambda_2=0. \qquad (20.14)$$

54) See paper by Fr. Bleich *loc. cit.*, p. 275

This is nothing but the already known formula.[55]

Next, at the case when $k=2$, putting $s=1$ in Eq. (**29**) gives

$$\lambda_1 \coth\pi\lambda_1 - \lambda_2 \coth\pi\lambda_2 = 0. \tag{20.15}$$

This coincides with Eq. (20.13), or is nothing else the conditional equation for the single plate which is clamped at one end side and simply supported at the other. Now, let us place the two configurations of distortion of such a plate in a row and in the inversely symmetric form by joining together the simply supported edge of one and that of the other. Then, we shall obtain the simplest configuration of the buckled continuous plate considered now. Accordingly, we can understand that the preceding conclusion is reasonable. Referring to the above special cases, it can be supposed that the least critical value will generally be obtained by putting $s=k-1$ in Eq. (**29**).

For the numerical example, let us consider first the case where only the load p is acting. Suppose $k=2$, we obtain $P=2.5966$ by (20.15) as previously known. And next, for $k=\infty$, we can suppose $P=1.5625$ as in the foregoing example. Therefore, for such a k as is larger than two, the following limits can be established:

$$1.5625 < P < 2.5966$$

The right member of Eq. (**29**), *viz.*, T/S is a monotonic increasing function of negative value as previously shown in Fig. 30. And, it has been explained in the preceding example that the least root of the conditional equation can be obtained by putting $s=k$, but if so, Eq. (**29**) becomes free of k and it follows that the continuous plate in the present example remains plane by the similar reason as Fr. Bleich had indicated with respect to a continuous column. Hence, let $s=k-1$, then Eq. (**29**) becomes

$$\cos\left(1-\frac{1}{k}\right)\pi = \frac{T}{S}. \tag{32}$$

The least roots of the above for the various k can be obtained by observing the curve of T/S in Fig. 30 as explained before. Yet, the following remarks must be added: Putting now $k=2k'$ in the above equation, Eq. (**32**) becomes to coincide with Eq.(**31**′), and therefore we may conclude that the least critical load of the continuous plate clamped at one end, simply supported at the other and having k' spans is the same as that of the continuous plate clamped at both ends and having $2k'$ spans. Observing that the configuration of the latter can be made by placing two ones of the former in a row and in the inversely symmetric form with the meeting of the simply supported end edges of such two, we can understand that the foregoing conclusion is reasonable. Again, letting $k=\infty$, Eq. (**32**) becomes

55) See paper by S. Iguchi, *loc. cit.*, p. 278.

$$\frac{T}{S} = -1$$

From this, we get $P=1.5625$, then the increase of number of the spans may cause diminution of the effects of the clamped end edge. Indeed, by using Fig. 30, the least roots of (**32**) are found as shown in Table. 30.

In the next place, in the case where only the load q is acting, the least values of Q will be obtained for $m=2$ as readily understood. Then by using Fig. 31 the numerical results in Table 31 are obtained. Also in this case, the increase of k can be seen to cause diminution of the effects of the clamped edge.

Table 30.*

k	$\left(1-\frac{1}{k}\right)\pi$	$\cos\left(1-\frac{1}{k}\right)\pi$	P
2	$1/2\ \pi$	0	2.5966
3	$2/3\ \pi$	$-0.500\,00$	2.062
4	$3/4\ \pi$	$-0.707\,11$	1.853
5	$4/5\ \pi$	$-0.809\,02$	1.752
6	$5/6\ \pi$	$-0.866\,03$	1.695
∞	π	-1	1.5625

Table 31.

k	$\cos\left(1-\frac{1}{k}\right)\pi$	$Q\ [m=2]$
2	The same as in Table. 30	22.96
3		19.15
4		17.77
5		17.14
6		16.80
∞		16

4) *The case where one end side is free and the other simply supported.*— The conditional equations for this case are represented by the combination of (20.6) and (20.11), or of (20.8) and (20.9); both give the same result. Eliminating B and B' from these, we obtain after some transformations

$$\sin k\alpha - \frac{\bar{S}\,S''}{ST'} \sin(k-1)\alpha = 0\,.$$

Then, we have

and from (**26**)

$$\left.\begin{aligned} \frac{\sin k\alpha}{\sin(k-1)\alpha} &= \frac{\bar{S}\,S''}{ST'}, \\ \cos\alpha = \frac{T}{S} &= \frac{\lambda_1 \coth\pi\lambda_1 - \lambda_2\coth\pi\lambda_2}{\lambda_1 \operatorname{cosech}\pi\lambda_1 - \lambda_2 \operatorname{cosech}\pi\lambda_2}. \end{aligned}\right\} \qquad (\mathbf{33})$$

The above can be considered as the conditional equations having a parameter α. Letting $k=1$, the denominator of the left member of the first equation becomes zero, and therefore we must equate that of the right member to zero. Accordingly by referring to the second equation, we can finally consider as follows:

$$T' = \frac{\gamma_1 + P\lambda_1}{\beta_1}\coth\pi\lambda_1 - \frac{\gamma_2 + P\lambda_2}{\beta_2}\coth\pi\lambda_2 = 0 \qquad (20.16)$$

* See the writer's paper, *loc. cit.*, p. 276.

This must be the formula for a single plate.

Particularly, in the case where only the load q exists but p is absent, we can write as

$$\underset{r}{\gamma_1} = -\underset{r}{\lambda_1}\underset{r}{\beta_2}, \qquad \underset{r}{\gamma_2} = -\underset{r}{\lambda_2}\underset{r}{\beta_1},$$

then the preceding formula is transformed into

$$\frac{\lambda_1}{\beta_1^2}\coth\pi\lambda_1 - \frac{\lambda_2}{\beta_2^2}\coth\pi\lambda_2 = 0 \tag{20.17}$$

or

$$\lambda_2\beta_1^2\tanh\pi\lambda_1 = \lambda_1\beta_2^2\tanh\pi\lambda_2 .$$

This coincides with the formula as already known.[56]

For the numerical examples, let us consider first the case when only the load p is acting. In the case when $k=1$, the continuous plate becomes a single plate which has no supportings along a long side edge and is simply supported along the other three. Then, at this time, we may obtain $P=0.5107$ as before described. And in order to require the least root P at the special case when $k=\infty$, let us transform the left member of the first equation of **(33)** as follows:

$$\frac{\sin(k-1)\alpha}{\sin k\alpha} = \cos\alpha - \cot k\alpha \sin\alpha .$$

Next, putting as $\alpha = i\alpha'$ and considering $\lim\limits_{k\to\infty}\coth k\alpha' = 1$, we have

$$\begin{aligned}\lim_{k\to\infty}\frac{\sin(k-1)\alpha}{\sin k\alpha} &= \lim_{k\to\infty}(\cosh\alpha' - \coth k\alpha' \sinh\alpha') \\ &= \cosh\alpha' - \sinh\alpha' = \frac{T}{S} \pm\sqrt{\left(\frac{T}{S}\right)^2 - 1},\end{aligned}$$

where

$$\cos\alpha = \cosh\alpha' = \frac{T}{S}.$$

Hence, when $k=\infty$, we find that P can be obtained by the following formula:

$$\frac{ST'}{SS''} = \frac{T}{S} \pm\sqrt{\left(\frac{T}{S}\right)^2 - 1}. \tag{34}$$

The above formula gives $P=0.5967$ as the least root. Accordingly, we can establish the following limits for any k:

$$0.5107 < P < 0.5967$$

In the next place, calculating the values of $\frac{T}{S}$ depending on P within the

56) See S. Timoshenko's book referred often before; p. 339, Eq. (i), *loc. cit.* P.156

above limits, we can plot the curve of the left member of the first equation of (33) in the following manner; first, calculate the values of $\cos\alpha$ and $\sin\alpha$ by the next expressions:

$$\cos\alpha = \frac{T}{S}; \qquad \sin\alpha = \sqrt{1-\left(\frac{T}{S}\right)^2},$$

then by using such a recurrence formula as

$$\sin n\alpha = 2\cos\alpha \sin(n-1)\alpha - \sin(n-2)\alpha, \tag{20.18}$$

we can compute the value of the left member of Eq. (33) for a certain k. Thus, the curves of both sides of the first of Eq. (33) can be plotted depending upon P. Accordingly, we can graphically find the desired value of P for such a k by observing the point at which both curves intersect with each other. These actual results are given in Table 32. This table shows that the value of P tends rapidly to a certain limit according to the increase of number of the spans, but the effect of the free end edge does not vanish.

Table 32.

k	1	2	3	4	∞
P	0.5107	0.591	0.596	0.596	0.5967

In the next place, let us consider the case where only the load q is acting. Particularly, in the case when $k=2$, the continuous plate may be considered to be composed of the two single plates :– one having a free edge along the long side and simply supported at the remaining three, and the other simply supported along all the sides. Since each the plate as above supposed has the side ratio $\frac{a}{b}=\frac{1}{2}$, when the former one buckles in a half-wave in y-direction, $Q=2.6722$ and when in two half-waves, $Q=5.610$; and on the other hand, when the latter one buckles in a half-wave in y-direction, $Q=25$, and when in two half-waves, $Q=16$. From these, we can suppose that, when the continuous plate considered now buckles in a half-wave in y-direction, the least critical value Q must lie within the following limits:

$$2.672 < Q < 25$$

On the other hand, when that plate buckles in two half-waves, Q' [temporarily denoted as such for distinguishing from the previous Q] can be supposed as follows:

$$5.610 < Q' < 16$$

But, it is impossible to judge which case gives the smaller value of Q by comparing both of the above cases of the limits. Then we must investigate about these cases respectively.

Table 33.

k	1	2	3	4	∞
Q	2.6722	3.805	3.871	3.874	3.874
Q'	5.61	6.06	6.05	6.05	6.05

Proceeding as explained before with respect to each the case [Fig. 32], the actual operations give the numerical results in Table 33.

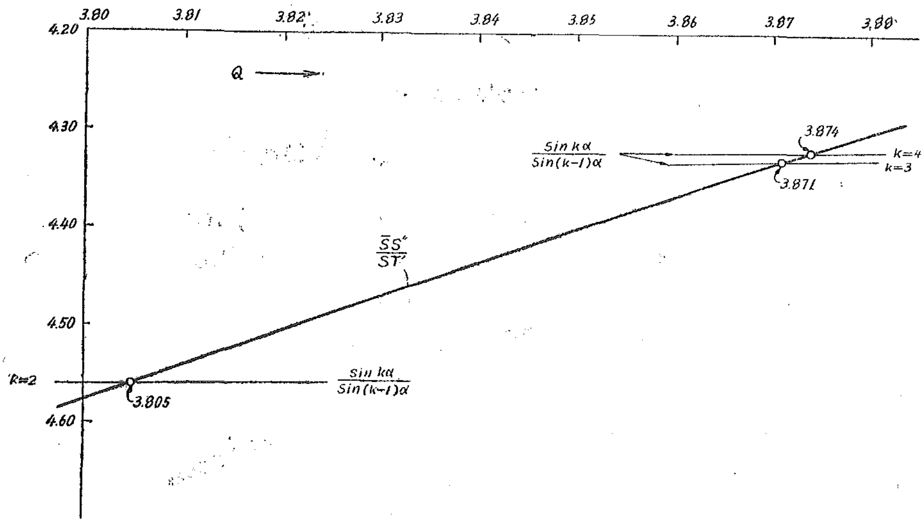

Fig. 32.

From these results, we find that the least critical value in this case corresponds to the buckling in a half-wave, and moreover we can see that the effect of free edge remains without vanishing in spite of rapid convergence of Q when k increases.

For references, let us require the deflection surface of the special case when $k=2$, $m=1$ and accordingly $Q=3.805$. That is

$$w_1 = A\, w_1',$$

$$w_1' = \left[\frac{\cosh\pi\lambda_1}{\beta_1}\left(\frac{\cosh\pi\lambda_1\,\xi_1}{\cosh\pi\lambda_1}-\frac{\sinh\pi\lambda_1\,\xi_1}{\sinh\pi\lambda_1}\right)+\frac{\cos\pi\lambda_2}{\beta_2}\left(\frac{\cos\pi\lambda_2\xi_1}{\cos\pi\lambda_2}-\frac{\sin\pi\lambda_2\xi_1}{\sin\pi\lambda_2}\right)\right.$$
$$\left.+\frac{\bar{S}}{2T}\left(\frac{\sinh\pi\lambda_1\xi_1}{\sinh\pi\lambda_1}-\frac{\sin\pi\lambda_2\xi_1}{\sin\pi\lambda_2}\right)\right]\sin\pi\eta_1\,;$$

$$w_2 = A\, w_2',$$

$$w_2' = \frac{\bar{S}}{2T}\left[\cosh\pi\lambda_1\left(\frac{\cosh\pi\lambda_1\xi_2}{\cosh\pi\lambda_1}-\frac{\sinh\pi\lambda_1\xi_2}{\sinh\pi\lambda_1}\right)-\cos\pi\lambda_2\left(\frac{\cos\pi\lambda_2\xi_2}{\cos\pi\lambda_2}-\frac{\sin\pi\lambda_2\xi_2}{\sin\pi\lambda_2}\right)\right]\sin\pi\eta_2\,.$$

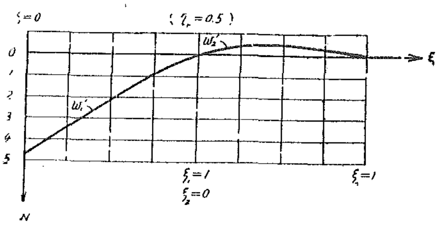

Fig. 33.

Table 34.

$\xi_1,\ \xi_2$	0	0.25	0.50	0.75	1
w_1'	4.707 57	3.396 49	2.124 04	0.924 23	0
w_2'	0	−0.419 99	−0.461 33	−0.282 17	0

By using these, requiring the deflection curve along the line $\eta_r=0.5$, the numerical and corresponding graphical results shown in Table 34 and in Fig. 33 respectively are obtained.

5) *The case where one end side is free and the other clamped.*—In this case, the conditional equations can be obtained by the combination of (20.7) and (20.11) or of (20.8) and (20.10). As a matter of course, each of the combinations gives the same result. Eliminating B and B' in the similar manner as before,

$$\cos k\alpha - \frac{\bar{S}S''}{ST'}\cos(k-1)\alpha = 0 .$$

Then

and from (**26**)

$$\left.\begin{aligned} \frac{\cos k\alpha}{\cos(k-1)\alpha} &= \frac{\bar{S}S''}{ST'}, \\ \cos\alpha &= \frac{T}{S} . \end{aligned}\right\} \tag{35}$$

These are the conditional equations having a parameter α. Particularly, when $k=1$, the left member of the first equation becomes $\cos\alpha$. Then, substituting the second equation into this, we obtain

$$T\,T' = \bar{S}\,S''. \tag{20.19}$$

Supposing the special case when the load q is acting and the load p is absent, since the following relations are obtainable:

$$\gamma_{1_r} = -\lambda_{1_r}\beta_{2_r}, \qquad \gamma_{2_r} = -\lambda_{2_r}\beta_{1_r},$$

then the foregoing (20.19) is readily transformed into the following expression:

$$-2\beta_1\beta_2 + (\beta_1^2+\beta_2^2)\cosh\pi\lambda_1\cosh\pi\lambda_2 = \frac{\lambda_1^2\beta_2^2+\lambda_2^2\beta_1^2}{\lambda_1\lambda_2}\sinh\pi\lambda_1\sinh\pi\lambda_2 .$$

This is the formula as already known.[57]

For the numerical example, let us observe first the case where only the load p is acting. When $k=1$, the continuous plate is nothing than the single plate freely suspended at one long side, clamped at the opposite side and simply supported at the remaining two sides. And its critical value is known as already obtained in the preceding numerical example, *i. e.*,

$$P = 0.6565$$

Next, let us consider the extreme case when $k=\infty$. Noting $\lim\limits_{k\to\infty}\tanh k\alpha' = 1$,

57) See S. Timoshenko's book previously referred to; p. 341, Eq. (*m*).

we have

$$\lim_{k\to\infty}\frac{\cos(k-1)\alpha}{\cos\alpha}=\lim_{k\to\infty}(\cosh\alpha'-\tanh k\alpha'\sinh\alpha')$$

$$=\cosh\alpha'-\sinh\alpha'=\frac{T}{S}\pm\sqrt{\left(\frac{T}{S}\right)^2-1}\,.$$

Therefore, we find that the critical value can be obtained again by the foregoing (**34**). That is

$$P=0.5967$$

Accordingly, for any k, we can put

$$0.5967<P<0.6565$$

Then, within the above limits, plotting $\frac{ST'}{SS''}$ -curve and $\frac{\cos(k-1)\alpha}{\cos k\alpha}$ -curve in the same manner as before, the intersecting points of them give the desired value P corresponding to a certain k. The actual results are given in Table 35. Again in this case, the convergency of P is very rapid but the effect of the free edge does not vanish.

Table 35.

k	1	2	3	4	∞
P	0.6565	0.601	0.597	0.596	0.5967

In the next place, let us consider the case where only the load q is acting. First, when $k=2$, we can suppose as follows by comparing with the preceding example [Table 33]:

$$Q>3.805$$

And we can, also in this case, suppose without much error that the least critical vaues of Q are obtained by letting $m=1$ and the convergency of Q corresponding to the increase of k is very rapid as already observed in the preceding example.

Table 36.

k	1	2	3	4	∞
$[m=1]Q$	5.344	3.946	3.878	3.874	3.874

The numerical results by the actual operations are given in Table 36. Now, comparing Table 32 and Table 33 with Table 35 and Table 36 respectively, we can find that in the former case the critical values constitute a monotonic increasing sequence and in the latter case those constitute a monotonic decreasing one and moreover both have the common limiting value.

6) *The case where both the end sides are free.*—The conditional equations in this case must be obtained from the combination of (20.8) and (20.11). That is to say, substituting (20.8) in (20.11), the following equation is obtained by some

transformations:

$$\left(\frac{\bar{S}S''}{ST'}\right)^2 \sin(k-2)\alpha - 2\left(\frac{\bar{S}S''}{ST'}\right)\sin(k-1)\alpha + \sin k\alpha = 0\,. \tag{20.20}$$

From this,

$$\left\{\frac{\bar{S}S''}{ST'} - \frac{\sin(k-1)\alpha + \sin\alpha}{\sin(k-2)\alpha}\right\}\left\{\frac{\bar{S}S''}{ST'} - \frac{\sin(k-1)\alpha - \sin\alpha}{\sin(k-2)\alpha}\right\} = 0\,.$$

Then

and from (**26**)

$$\left.\begin{aligned} \frac{\sin(k-1)\alpha \pm \sin\alpha}{\sin(k-2)\alpha} &= \frac{\bar{S}S''}{ST'}, \\ \cos\alpha &= \frac{T}{S}. \end{aligned}\right\} \tag{36}$$

These are the conditional equations having a parameter α. In this case, we can not deduce the formula for a single plate from the above by putting $k=1$, by the reason why every edge, parallel to y-direction, of such a single plate is deflectable notwithstanding the condition of no deflection at the intermediate supports are fundamentally used for the reduction of the formulas. When $k=2$, taking the upper sign in the left member of the first equation in (**36**), we can again obtain Eq. (20.17) in the same manner as in the special case $k=1$ of the formula (**33**). The above conclusion can be understood by observing that the two configurations of distortion of the single plate which is free at one long side and simply supported at the remaining three can correspond to the first mode of buckling of the continuous plate considered now provided that those two are placed in a row and in inversely symmetric form with the meeting of their long edges simply supported [see Fig. 34 (a)]. Next, taking the lower sign in the first equation of (**36**), the left hand side becomes of indefinite form, then the limitation process may be taken as follows: that is

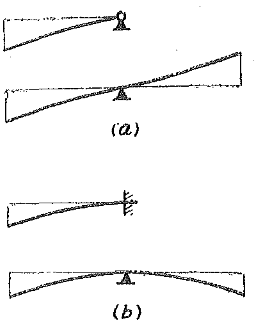

Fig. 34.

$$\lim_{\varepsilon\to 0}\frac{\sin(1+\varepsilon)\alpha - \sin\alpha}{\sin\varepsilon\alpha} = \lim_{\varepsilon\to 0}\frac{\dfrac{\partial}{\partial\varepsilon}\left\{\sin(1+\varepsilon)-\sin\alpha\right\}}{\dfrac{\partial}{\partial\varepsilon}\sin\varepsilon\alpha} = \cos\alpha.$$

Observing this, we can readily obtain Eq. (20.19). This result can be understood by considering that the two configurations of distortion of the single plate which is free at one long side and clamped at the opposite side can correspond to the second mode of buckling of the continuous plate provided that those two are placed in a row and in symmetric form with the meeting of their clamped edges [see Fig. 34 (b)].

In general speaking, considering the configurations of distortion of the continuous plate with an even number of spans, we can suppose two types at the central support, *i. e.*, one is the inversely symmetric type as in Fig. 34 (*a*) and the other the symmetric type as in Fig. 34 (*b*). Then, considering about the portion of half the numbers of spans in one side of the central support, we can suppose that, in the former case, the continuous plate is equivalent to that having half the numbers of spans which is free at one end side and simply supported at the other, and, in the latter case, the continuous plate to another one which is free at one end side and clamped at the other. These facts can be proved as follows: letting now that $2k'$ denotes the total number of spans, and putting $k=2k'$ in the left member of the first equation in (**36**), we have

$$\frac{\sin(k-1)\alpha\pm\sin\alpha}{\sin(k-2)\alpha}=\frac{\sin(2k'-1)\alpha\pm\sin\alpha}{\sin 2(k'-1)\alpha}=\frac{\sin k'\alpha\cos(k'-1)\alpha+\cos k'\alpha\sin(k'-1)\alpha\pm\sin\alpha}{2\sin(k'-1)\alpha\cos(k'-1)\alpha}.$$

Noting
$$\cos k'\alpha\sin(k'-1)\alpha+\sin\alpha=\sin k'\alpha\cos(k'-1)\alpha\,;$$
$$\sin k'\alpha\cos(k'-1)\alpha-\sin\alpha=\cos k'\alpha\sin(k'-1)\alpha\,,$$

the following relations are obtained:

$$\frac{\sin(k-1)\alpha\pm\sin\alpha}{\sin(k-2)\alpha}=\begin{cases}\dfrac{\sin k'\alpha}{\sin(k'-1)\alpha} & \text{for the upper sign,}\\[2ex] \dfrac{\cos k'\alpha}{\cos(k'-1)\alpha} & \text{for the lower sign.}\end{cases}$$

Thus, it was proved that the formulas (**36**), for the upper sign, becomes to coincide with the formulas (**33**) and, for the lower sign, with the formulas (**35**).

Besides, it can be proved that the above facts are generally held regardless of whether k is even or not. To do this, let us investigate that the deflections at both end sides must have an equal amount in the same or opposite side according to a symmetric buckling or an inversely symmetric buckling with respect to the center line of the continuous plate perpendicular to x-axis, because the circumstances of the present continuous plate are symmetrical about such a center line.

Referring to the expression (**3**) in § 4, the deflection in the first span is expressed in the following form:

$$w_1=\left\{\underset{1}{K_m}\cosh\pi\lambda_1\xi_1+\underset{1}{L_m}\sinh\pi\lambda_1\xi_1+\underset{1}{M_m}\cosh\pi\lambda_2\xi_1+\underset{1}{N_n}\sinh\pi\lambda_2\xi_1\right\}\sin\pi\eta_1\,.$$

Particularly, at End side-1, by putting $\xi_1=0$ in the above

$$\Big[w_1\Big]_{\xi_1=0}=\left\{\underset{1}{K_m}+\underset{1}{M_m}\right\}\sin\pi\eta_1\,.$$

Yet, by referring to (6.19) in § 6, we can write the above expression as follows:

$$\Big[w_1\Big]_{\xi_1=0}=-B_{\substack{m\\1}}(\beta_1-\beta_2)\sin\pi\eta_1. \tag{20.21}$$

In the same manner, the deflection at End side-k is expressed as follows by putting $\xi_k=1$:

$$\Big[w_k\Big]_{\xi_k=1}=\Big\{K_{\substack{m\\k}}\cosh\pi\lambda_1+L_{\substack{m\\k}}\sinh\pi\lambda_1+M_{\substack{m\\k}}\cosh\pi\lambda_2+N_{\substack{m\\k}}\sinh\pi\lambda_2\Big\}\sin\pi\eta_k.$$

Now, the integration constants in the above can be replaced with $B_{\substack{m\\k}}$ by using (6.11) and (6.17) in § 6. The actual substitution gives

$$\Big[w_k\Big]_{\xi_k=1}=B_{\substack{m\\k}}\bar{\varphi}\frac{(\gamma_1+P\lambda_1)\operatorname{cosech}\pi\lambda_1-(\gamma_2+P\lambda_2)\operatorname{cosech}\pi\lambda_2}{\dfrac{\gamma_1+P\lambda_1}{\beta_1}\coth\pi\lambda_1-\dfrac{\gamma_2+P\lambda_2}{\beta_2}\coth\pi\lambda_2}\sin\pi\eta_k=B_{\substack{m\\k}}\bar{\varphi}\frac{S''}{T'}\sin\pi\eta_k. \tag{20.22}$$

On the other hand, from the first and the third expressions in (6.20), we obtain

$$L_{\substack{m\\1}}=-B_{\substack{m\\1}}\beta_2\coth\pi\lambda_1+B_{\substack{m\\2}}\operatorname{cosech}\pi\lambda_1,$$

$$N_{\substack{m\\1}}=\ B_{\substack{m\\1}}\beta_1\coth\pi\lambda_2-B_{\substack{m\\2}}\operatorname{cosech}\pi\lambda_2.$$

Then, the second expression in (6.18) is transformed in the following form:

$$B_{\substack{m\\1}}\beta_1\beta_2T'-B_{\substack{m\\2}}S''=0$$

or

$$\frac{B_{\substack{m\\1}}}{B_{\substack{m\\2}}}=\frac{S''}{\beta_1\beta_2T'}$$

Referring to this and (20.21), and putting $\eta_1=\eta_k$, (20.22) becomes

$$\Big[w_k\Big]_{\xi_k=1}=-\frac{B_{\substack{m\\k}}}{B_{\substack{m\\2}}}B_{\substack{m\\1}}(\beta_1-\beta_2)\sin\pi\eta_1=\frac{B_{\substack{m\\k}}}{B_{\substack{m\\2}}}\Big[w_1\Big]_{\xi_1=0}. \tag{20.23}$$

Again, representing the coefficient in the right member of the above with the second expression of (6.16) and substituting (27) in it, we have

$$\frac{B_{\substack{m\\k}}}{B_{\substack{m\\2}}}=\frac{V_k}{V_2}=\frac{B\sin k\alpha-B'\cos k\alpha}{B\sin 2\alpha-B'\cos 2\alpha}.$$

Yet, substituting (20.8) in B and B'

$$\frac{B_{\substack{m\\k}}}{B_{\substack{m\\2}}}=\frac{\sin\alpha}{\sin(k-1)\alpha-\dfrac{\bar{S}S''}{ST'}\sin(k-2)\alpha}.$$

Now, $\dfrac{\bar{S}S''}{ST'}$ in the above can be expressed with trigonometric functions of α by using the first equation in (**36**), then we finally obtain

$$\frac{B_{\frac{m}{k}}}{B_{\frac{m}{2}}} = \mp 1 .$$

The double signs in the above correspond to those in the left member of the first equation in (**36**), *i. e.*, that upper sign corresponds to this upper one and that lower to this lower. Then, accordingly (20.23) can be reduced into the following forms:

$$\Big[w_k \Big]_{\xi_k=1} = \mp \Big[w_1 \Big]_{\xi_1=0} .$$

Namely, it has been proved.

In the next place; let us consider the numerical examples. By the foregoing considerations, it can be understood that the numerical results in preceding cases of boundary conditions, *viz.*, in 4). 5), is again applicable to the present case, when the number of spans is even. But, when it is odd, the new calculations must be necessary.

Now, in the extreme case when $k=\infty$, by putting $\alpha=i\alpha'$, we find that the left member of the first equation of (**36**) can be transformed into the same form as before, *viz.*,

$$\lim_{k\to\infty} \frac{\sin(k-2)\alpha}{\sin(k-1)\alpha \pm \sin\alpha} = \cosh\alpha' - \sinh\alpha' = \frac{T}{S} \pm \sqrt{\left(\frac{T}{S}\right)^2 - 1} .$$

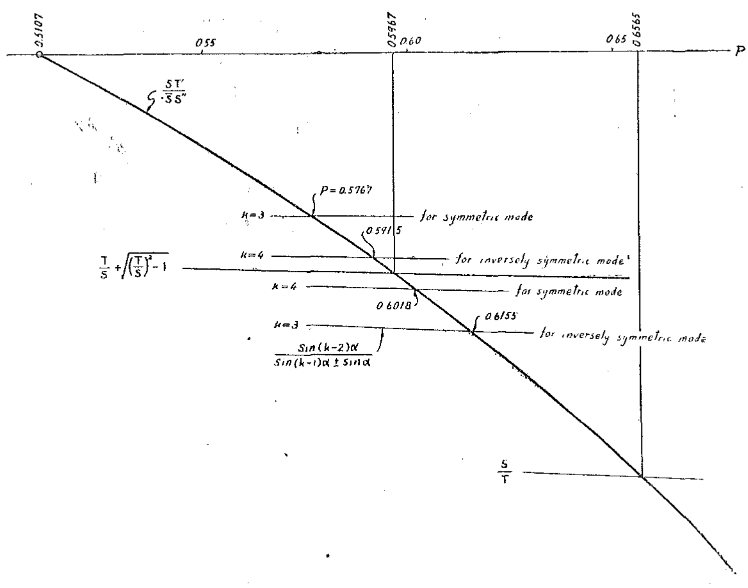

Fig. 35.

Then, in such an extreme case, the same results as already known must be obtained. For instance, when only the load p is acting, we have $P=0.5967$. Proceeding in the same manner as before, the actual treatings give the numerical results in Table 37 [see Fig. 35].

Table 37.–Least critical values of P in the case when both the end sides are free.

k	2	3	4	5	6	∞
Inversely Symmetric buckling	0.5107	0.615_5	0.591_5	0.598_2	0.596_3	0.5967
Symmetric buckling	0.6565	0.576_7	0.601_8	0.595_5	0.597_0	

Next, when only the load q is acting, the results in Table 38 are obtained.

Table 38.–Least critical values of Q in the case when both the end sides are free.

k	2	3	4	5	6	∞
Inversely Symmetric buckling	2.673	4.190	3.805	3.890	3.871	3.874
Symmetric buckling	5.344	3.580	3.946	3.855	3.878	

Observing the above Tables, we can conclude that, when the number of spans is even, the least critical loads correspond to the inversely symmetric mode of buckling and, when the number of spans is odd, the least ones to the symmetric mode respectively, and moreover, the sequences of these critical values have rapid convergency but the effects of the free end edges do not tend to vanish. Furthermore, it seems that we can say as follows: when a continuous plate with any spans more than three has a free end edge, its critical value may practically be determined by the formula (34). For references, some modes of distortion in section of x-direction are illustrated in Fig. 36.

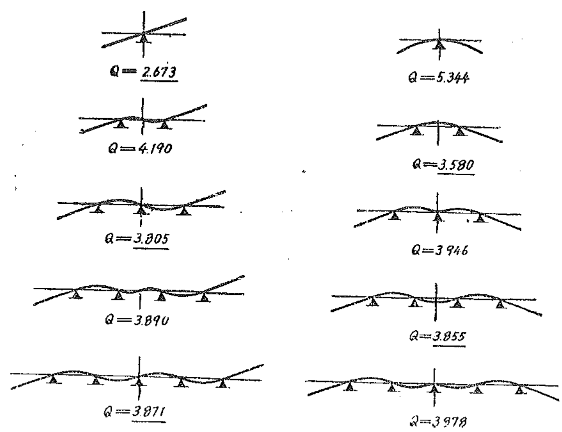

Fig. 36.

§. 21. Plates Composed of a Number of the Same Elementary Plates with Hinged Joints.

i) **General solution.**—For the case of the continuous plate, the joint conditions (14), (15) [§ 7-e)] are rewritten as follows:

$$\left.\begin{aligned} U_{r-1}-U_r F+V_r G&=0\,;\\ V_{r-1}+U_r F'-V_r G'&=0\,, \end{aligned}\right\} \tag{21.1}$$

where

$$\left.\begin{aligned} F&=\frac{T'}{S'}, \qquad & G&=\frac{W'}{S'},\\ F'&=\frac{1}{S'}, \qquad & G'&=\frac{T'}{S'}=F. \end{aligned}\right\} \tag{21.2}$$

The above are of the same form as (20.1). And it is only the point different from (20.1) that (21.2) must be used instead of (20.2) as their coefficients. Therefore, the treatment by the method of "Differenzengleichungen" is the same as before, and in this case the general solutions can be expressed as follows by referring to **(27)**:

$$\left.\begin{aligned} U_r&=B\cos r\alpha+B'\sin r\alpha,\\ V_r&=-\frac{S'}{W'}\sin\alpha\,(B\sin r\alpha-B'\cos r\alpha), \end{aligned}\right\} \tag{37}$$

where, α is the parameter defined by the following equation as in **(26)**:

$$\cos\alpha=\frac{T'}{S'}. \tag{38}$$

ii) **End side conditions.** (1) *The conditions at End side-k.*

1) The case of the simply supported edge.—From **(15. a)** [§7-e)]

$$U_k=-T'\,; \qquad V_k=1\,.$$

Substituting **(37)** in the above, the following expressions of the same form as (20.6) are obtained:

$$\left.\begin{aligned} B&=-S'\cos(k+1)\alpha\,,\\ B'&=-S'\sin(k+1)\alpha\,. \end{aligned}\right\} \tag{21.3}$$

2) The case of the clamped edge.—From **(15.b)**

$$U_k=-\frac{TT'-\bar{S}S''}{T}\,; \qquad V_k=1\,.$$

Substituting **(37)**, the following are obtained:

$$\left.\begin{aligned} B&=-S'\left\{\cos(k+1)\alpha-\frac{\bar{S}S''}{S'T}\cos k\alpha\right\},\\ B'&=-S'\left\{\sin(k+1)\alpha-\frac{\bar{S}S''}{S'T}\sin k\alpha\right\}. \end{aligned}\right\} \tag{21.4}$$

3) The case of the free edge.—From **(15.c)**

$$U_k = -\frac{W'}{T'}\,; \qquad V_k = 1\,.$$

Substitute (**37**), then we have

$$\left.\begin{aligned} B &= -\frac{W'}{T'}\frac{\sin(k+1)\alpha}{\sin\alpha}\,, \\ B' &= \frac{W'}{T'}\frac{\cos(k+1)\alpha}{\sin\alpha}\,. \end{aligned}\right\} \tag{21.5}$$

(2) *The conditions at End side-1.*

1) The case of the simply supported edge.—From (**15.d**)

$$V_1 = 0\,.$$

Substituting the general solutions (**37**) in the above, we have

$$B\sin\alpha - B'\cos\alpha = 0\,. \tag{21.6}$$

2) The case of the clamped edge.—From (**15.f**)

$$U_1 = U_2(F_1) - V_2(G_1) = 0\,,$$

where

$$(F_1) = \frac{T}{S''}\,, \qquad (G_1) = \frac{TT' - \bar{S}S''}{S''}\,.$$

Substituting (**37**), the following expression is obtained:

$$B\left(\sin\alpha - \frac{\bar{S}S''}{S'T}\sin 2\alpha\right) - B'\left(\cos\alpha - \frac{\bar{S}S''}{S'T}\cos 2\alpha\right) = 0\,. \tag{21.7}$$

3) The case of the free edge.—From (**15.g**)

$$U_1 = 0\,.$$

Substitute (**37**) in this, then we obtain

$$B\cos\alpha + B'\sin\alpha = 0\,. \tag{21.8}$$

iii) The conditional equation.—As understood by observing the foregoing expressions, the general solutions and the end side conditions in this case are of the same forms as in § 20. Then we can consider that the conditional equations reduced by the combinations of these expressions must become of the same forms as before, but yet it must be remarked that, in the present case, S, T, W, $\frac{\bar{S}S''}{ST'}$ in the formulas of § 20 are replaced with S', T'. W', $\frac{\bar{S}S''}{S'T}$ respectively.

1) *The case where both the end sides are simply supported.*—The conditional equation for this case is expressed by the combination of (21.3) and (21.6), and that

is analogous to the combination of (20.6) and (20.9) in § 20. Then, the conditional equation can be expressed in the same form as Eq. (29); *i.e.*,

$$\cos s\frac{\pi}{k}=\frac{T'}{S'}=\frac{\dfrac{\gamma_1+P\lambda_1}{\beta_1}\coth\pi\lambda_1-\dfrac{\gamma_2+P\lambda_2}{\beta_2}\coth\pi\lambda_2}{\dfrac{\gamma_1+P\lambda_1}{\beta_1}\operatorname{cosech}\pi\lambda_1-\dfrac{\gamma_2+P\lambda_2}{\beta_2}\operatorname{cosech}\pi\lambda_2}. \tag{39}$$

$$(s=0,\ 1,\ 2,\cdots\cdots\cdots,2k-1)$$

For the numerical examples, les us consider first the case when only the load p is acting. Now, supposing the special case when $k=2$, we can acknowledge $P=0.5107$ provided that we note that each elementary plate buckles in such a configuration as in Fig. 22 due to the symmetrical circumstances. Thus, plotting the value of T'/S' as the function of P in the neighborhood of this value, we obtain Fig. 37 which shows that T'/S' is a monotonic increasing function of P. From this,

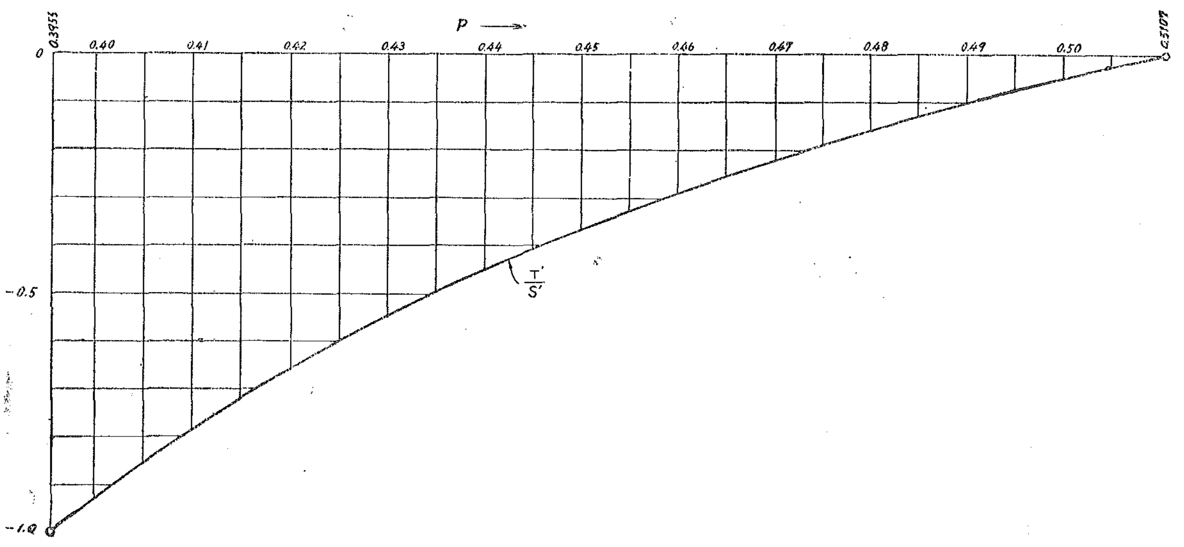

Fig. 37.

we must take $s=k$ to minimize $\cos s\dfrac{\pi}{b}$, then it follows that $\cos s\dfrac{\pi}{k}=\cos\pi=-1$. So, it becomes that the formula gives $P=0.3955$ regardless of k. But, as shown later, this result is nothing but that of the case where both end sides are free. And accordingly it can be proved that the plate must remain plane under such a value of P, and it is unsuitable for the problem. Then, putting $s=k-1$ to require the next small value of P, Eq. (39) is transformed into

$$\cos\left(1-\frac{1}{k}\right)\pi=\frac{T'}{S'} \tag{39'}$$

In the above equation, calculating the left hand side for a certain k, we can obtain

P for such a k by means of observing Fig. 37. In Table 39, there are given some numerical results of P. Particularly at the extreme case when $k=\infty$, Eq. (**39′**) becomes

$$-1=\frac{T'}{S'}$$

Then we get $P=0.3955$ from Fig. 37. This shows that the effects of the end side vanish in accordance with the case where both end sides are free.

Table 39.

k	$\left(1-\frac{1}{k}\right)\pi$	$\cos\left(1-\frac{1}{k}\right)\pi$	P
2	$1/2\ \pi$	0	0.5107
3	$2/3\ \pi$	$-0.500\ 75$	0.434
4	$3/4\ \pi$	$-0.707\ 11$	0.415
5	$4/5\ \pi$	$-0.809\ 02$	0.408
6	$5/6\ \pi$	$-0.865\ 60$	0.404
7	$6/7\ \pi$	$-0.900\ 97$	0.401
8	$7/8\ \pi$	$-0.923\ 88$	0.400
∞	π	-1	0.3955

For references, let us investigate next the shape function, *i. e.*, the deflection surface. Taking $m=1$ as a rule for the least critical load, the deflection surface of r-th elementary plate can be expressed as follows by (**3**) in §4:

$$w_r=\left\{K_r\cosh\pi\lambda_1\xi_r+L_r\sinh\pi\lambda_1\xi_r+M_r\cosh\pi\lambda_2\xi_r+N_r\sinh\pi\lambda_2\xi_r\right\}\sin\pi\eta_r.$$

Using (7.1), (7.2) and the expressions of L_{r-1} and N_{r-1} to represent by A_r and B_r every constant in the above expression and substituting (6.16), *viz.*, the following expressions:

$$\left.\begin{aligned}A_r&=A\,U_r\,,\\ B_r&=A\,V_r\,,\end{aligned}\right\}$$

we can obtain as the expression of the deflection surface of the r-th elementary plate

$$\begin{aligned}w_r=A\Bigg[V_r\left\{\frac{\cosh\pi\lambda_1}{\beta_1}\left(\frac{\cosh\pi\lambda_1\xi_r}{\cosh\pi\lambda_1}-\frac{\sinh\pi\lambda_1\xi_r}{\sinh\pi\lambda_1}\right)-\frac{\cosh\pi\lambda_2}{\beta_2}\left(\frac{\cosh\pi\lambda_2\xi_r}{\cosh\pi\lambda_2}-\frac{\sinh\pi\lambda_2\xi_r}{\sinh\pi\lambda_2}\right)\right\}\\ +V_{r+1}\left(\frac{\sinh\pi\lambda_1\xi_r}{\beta_1\sinh\pi\lambda_1}-\frac{\sinh\pi\lambda_2\xi_r}{\beta_2\sinh\pi\lambda_2}\right)\Bigg]\sin\pi\eta_r.\end{aligned}\tag{21.9}$$

In the above expression, V_r is of the form in (**37**), *i. e.*,

$$V_r=-\frac{S'}{W'}\sin\alpha\left\{B\sin r\alpha-B'\cos r\alpha\right\}.$$

Since B, B' are given by (21.3) when End side-k is simply supported, by substituting these and considering $\cos\alpha=T'/S'$, $W'=T'^2-S'^2$, the above expression can be transformed into

$$V_r=\frac{\sin\,(k+1-r)\,\alpha}{\sin\alpha}.$$

Particularly, when both the end sides are simply supported as considered now, from (39′)

$$\alpha=\left(1-\frac{1}{k}\right)\pi.$$

Then V_r is finally written as follows:

$$V_r=\frac{\sin(k+1-r)\left(1-\frac{1}{k}\right)\pi}{\sin\left(1-\frac{1}{k}\right)\pi}=(-1)^r\frac{\sin\left(k+\frac{r-1}{k}\right)\pi}{\sin\frac{\pi}{k}}.$$

From this, the following expression can hold again:

$$V_{r+1}=-(-1)^r\frac{\sin\left(k+\frac{r}{k}\right)\pi}{\sin\frac{\pi}{k}}.$$

Accordingly, the expression (21.9) is transformed as follows:

$$w_r=(-1)^r\frac{A}{\sin\frac{\pi}{k}}\left[\sin\left(k+\frac{r-1}{k}\right)\pi\left\{\frac{\cosh\pi\lambda_1}{\beta_1}\left(\frac{\cosh\pi\lambda_1\xi_r}{\cosh\pi\lambda_1}-\frac{\sinh\pi\lambda_1\xi_r}{\sinh\pi\lambda_1}\right)\right.\right.$$

$$\left.-\frac{\cosh\pi\lambda_2}{\beta_2}\left(\frac{\cosh\pi\lambda_2\xi_r}{\cosh\pi\lambda_2}-\frac{\sinh\pi\lambda_2\xi_r}{\sinh\pi\lambda_2}\right)\right\}$$

$$\left.-\sin\left(k+\frac{r}{k}\right)\pi\left\{\frac{\sinh\pi\lambda_1\xi_r}{\beta_1\sinh\pi\lambda_1}-\frac{\sinh\pi\lambda_2\xi_r}{\beta_2\sinh\pi\lambda_2}\right\}\right]\sin\pi\eta_r.$$

Next, substituting the given guantities such as $\nu=0.3$, $a/b=1/2$ in this, we get

$$w_r=\frac{C}{(0.1225+0.3P)\sin\frac{\pi}{k}}w_r',$$

$$w_r'=(-1)^r\left[\sin\left(k+\frac{r-1}{k}\right)\pi\left\{\left(\begin{array}{c}(0.35-P)\sinh\gamma\xi_r\sin\beta\xi_r\\ -\sqrt{P(1-P)}\cosh\gamma\xi_r\cos\beta\xi_r\end{array}\right)\right.\right.$$

$$\left.-\frac{\left[\begin{array}{l}\{(0.35-P)\sinh2\gamma-\sqrt{P(1-P)}\sin2\beta\}\cosh\gamma\xi_r\sin\beta\xi_r\\ -\{(0.35-P)\sin2\beta+\sqrt{P(1-P)}\sinh2\gamma\}\sinh\gamma\xi_r\cos\beta\xi_r\end{array}\right]}{\cosh2\gamma-\cos2\beta}\right\}$$

$$-\sin\left(k+\frac{r}{k}\right)\pi$$

$$\times \frac{2\left[\begin{array}{l}\{(0.35-P)\sinh\gamma\cos\beta-\sqrt{P(1-P)}\cosh\gamma\sin\beta\}\cosh\gamma\xi_r\sin\beta\xi_r \\ -\{(0.35-P)\cosh\gamma\sin\beta+\sqrt{P(1-P)}\sinh\gamma\cos\beta\}\sinh\gamma\xi_r\cos\beta\xi_r\end{array}\right]}{\cosh 2\gamma-\cos 2\beta}\sin\pi\eta_r, \quad (21.10)$$

where

$$\gamma=\frac{\pi}{2}\sqrt{1-P}, \qquad \beta=\frac{\pi}{2}\sqrt{P},$$

$$C= \text{an indefinite constant } [=4iA].$$

As an example, computing the above with respect to the case when $k=5$, we obtain Table 40 and Fig. 38 in which the values along the line $\eta_r=0.5$ are shown. From these, we find that such a plate as considered now buckles in zigag form and it agrees with our common idea.

Table 40.-Values of w_r' along $\eta_r=0.5$

r \ ξ_r	0	0.125	0.250	0.500	0.750	0.875	1
1	0	0.028 804	0.058 408	0.123 002	0.198 898	0.242 087	0.288 878
2	0.288 879	0.195 484	0.104 393	−0.076 015	−0.263 411	−0.362 896	−0.467 413
3	−0.467 415	−0.345 102	−0.227 319	−0.000 006	0.227 311	0.345 093	0.467 413
4	0.467 415	0.362 903	0.263 417	0.076 024	−0.104 386	−0.195 477	−0.288 878
5	−0.288 879	−0.242 089	−0.198 899	−0.123 005	−0.058 412	0.028 807	0

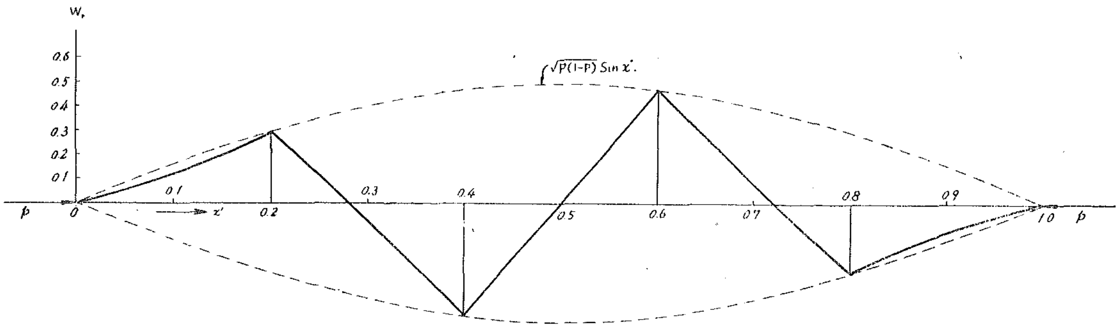

Fig. 38.

Particularly, put $\xi_r=0$ in (21.10), then the deflection at every joint can be obtained, that is

$$\left[w_r'\right]_{\xi_r=0}=(-1)^{r+k-1}\sqrt{P(1-P)}\sin\frac{r-1}{k}\pi\ \sin\pi\eta_r.$$

Then, replacing $(r-1)/k$ by x' which is specified as $x'=0$ at End side-1 and $x'=1$ at End side-k, we find that the deflection surface w' of the continuous plate is circumscribed by the following two surfaces along every joint line:

$$w'=\pm\sqrt{P(1-P)}\sin x'\pi\ \sin\pi\eta_r.$$

In the next place, let us consider the case when only the load q is acting. First, for the special case when $k=2$, we have known $Q=2.6722$[58] because each elementary plate buckles as a single plate which has a free long edge and the other three edges simply supported. Next, for the extreme case when $k=\infty$, we shall find $Q=0.9288$ by observing Fig. 39 because the conditional equation becomes $T'/S'=1$ and that corresponds to the symmetric buckling of a single plate having both the free end sides as clarified later on [in the case 4) where both end sides are free]. At the present case, we may understand the above fact by supposing that the effects of the end sides vanish when $k=\infty$. Then, when $k>2$, the following limits may be established:

$$0.928 < Q < 2.672$$

Plotting the value of T'/S' as the function of Q within these limits, we find that T'/S' is a monotonic decreasing function of Q. Therefore, the least value of Q corresponds to the maximum of $\cos(s\cdot\pi/k)$ within such limits, and then we must take $s=0$ to make $\cos(s\cdot\pi/k)$ as close as possible to unit. But if so, the formula (39) becomes free of k and of no use because it results that the continuous plate remains plane. Accordingly, taking $s=1$, the value of $\cos(s\cdot\pi/k)$ can be determined for any k, then by referring to T'/S'-curve in Fig. 39, we obtain the results in Table 41.

Table 41.

k	$\cos\cdot\pi/k$	Q
2	0	2.6722
3	0.5	1.650
4	0.707 11	1.325
5	0.809 02	1.178
6	0.866 03	1.101
∞	1	0.9288

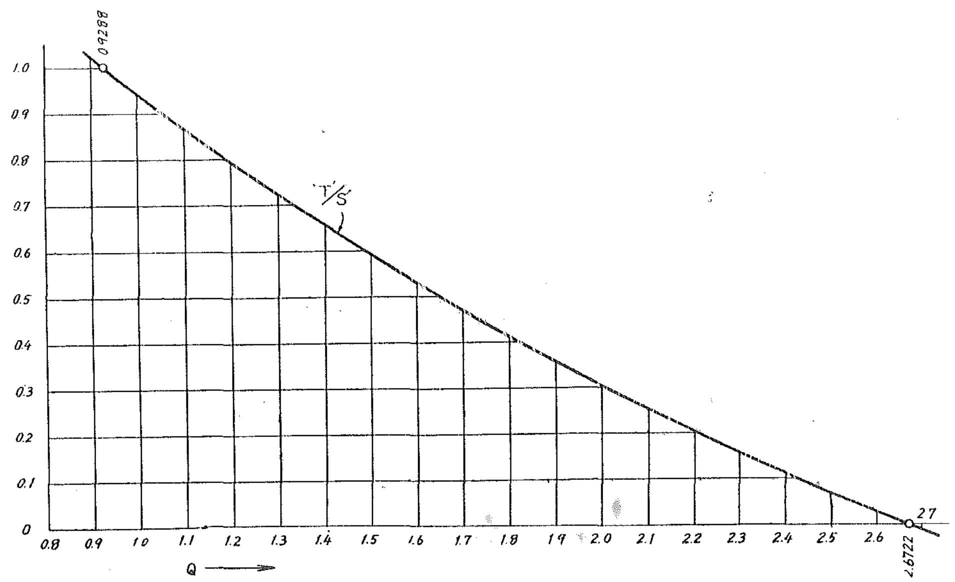

Fig. 39.

58) See the foregoing footnote 35).

We can finally conclude by these results that the sequence of Q rapidly converges to the value of the case where both end sides are free according to increasing of k. And yet, at the case $k=1$, the plate considered now coincides with a single plate simply supported along all sides and the least critical load is obtained by taking $m=2$ because the side ratio a/b is $1/2$. From this we must obtain $Q=16$ as already known. In fact, it can be proved that the formula (**39**) also yields such a result.

2) *The case where one end side is clamped and the other simply supported.* —The conditional equations for this case are given by the combination of (21.4) and (21.6), or of (21.3) and (21.7). Each combination is respectively similar to that of (20.8) and (20.9), or (20.6) and (20.11) in § 20 and therefore we can readily write down the conditional equations for the present case as follows by referring to the foregoing formulas (**33**):

$$\left.\begin{aligned} \frac{\sin k\alpha}{\sin(k-1)\alpha}&=\frac{\bar{S}S''}{S'T}, \\ \cos\alpha&=\frac{T'}{S'}. \end{aligned}\right\} \qquad (\mathbf{40})$$

These are the formulas having a parameter α. For a single plate, taking $k=1$, the above yields

$$T=\lambda_1 \coth\pi\lambda_1-\lambda_2 \coth\pi\lambda_2=0.$$

This is nothing than (20.13).

For the numerical examples, let us take first the case where only the load p is acting. Considering the special case when $k=1$, we can obtain $P=2.5966$ from the foregoing (20.13). And next, consider the extreme case when $k=\infty$, then, as in the previous examples, we can suppose that the present case becomes to coincide with the case where both end sides are free and accordingly $P=0.3955$. Again, considering the case $k=2$, such a continuous plate is nothing but that which is constructed by joining each free edge of the two elementary plates with a hinge :- one is the single plate having a long free edge, the opposite clamped edge and the remaining two simply supported, and the other is the single plate obtained by replacing the long clamped edge of the former with a simply supported one, and we have known that $P=0.6565$ for the former and $P=0.5107$ for the latter [see Fig. 22 and Fig. 24]. Thus, for such a continuous plate as $k=2$, the following limits for P are established:

$$0.5107 < P < 0.6565$$

Then, when $2 < k < \infty$, we can suppose

$$0.3955 < P < 0.6565$$

Now, it becomes that $T'/S'<1$ within the above limits. So, the conditional equation for the extreme case $k=\infty$ can not be required in such a form as the preceding (**34**). Since the formerly obtained result $P=0.3955$ for the case $k=\infty$ satisfies

$$\cos\alpha=\frac{T'}{S'}=-1,$$

then, at this time it may be concluded that $\alpha=n\pi$ in which n is odd. Next, considering

$$\lim_{\substack{\alpha\to n\pi \\ (n=1,\ 3,\ 5,\cdots\cdots)}}\frac{\sin k\alpha}{\sin(k-1)\alpha}=-1-\frac{1}{k-1}, \tag{21.11}$$

and plotting $\frac{\sin k\alpha}{\sin(k-1)\alpha}$ as a function of P, such a curve must pass through the point whose coordinates are expressed by $P=0.3955$ and (21.11). Then the ordinate of such a curve must be -1 when $k=\infty$. Again $\frac{\sin k\alpha}{\sin(k-1)\alpha}$ tends to zero, when

$$\alpha=m\frac{\pi}{k} \qquad (m=0,\ 1,\ 2,\ 3,\cdots\cdots\cdots),$$

and to $\pm\infty$, when

$$\alpha=m\frac{\pi}{k-1} \qquad (m=0,\ 1,\ 2,\ 3,\cdots\cdots\cdots).$$

From these, we see that the larger k is, the larger the frequency of oscillation of $\frac{\sin k\alpha}{\sin(k-1)\alpha}$-curve between $\pm\infty$ becomes, under the small variation of α, *viz.*, that of P.[59] Accordingly, it is concluded that at the extreme case when $k=\infty$ such a curve must pass through the point of the ordinate -1 at $P=0.3955$ and tends to coincide with the straight line passing through the point of such an ordinate and perpendicular to P-axis. On the other hand, $\frac{SS''}{S'T}$-curve is almost parallel to P-axis as shown in Fig. 40, then it must intersect with the former. Thus, it results that $P=0.3955$ when $k=\infty$, and it agrees with the initial conjecture. Besides, since $\frac{\sin k\alpha}{\sin(k-1)\alpha}$-curve is an oscillatory one, then such a curve intersects with $\frac{\bar{S}S''}{S'T}$-curve many times and the intersecting points give the critical loads of higher order. And, according as $k\to\infty$, the wave length of the above curve tends to zero. Then, at the limit $k=\infty$ we may consider that each of the critical loads of higher order becomes to be repeated at $P=0.3955$. The actual procedures are illustrated in Fig. 40 and from this

59) α is related to P by means of $\cos\alpha=\frac{T'}{S'}$.

we find the numerical results in Table 42. The convergency of this sequence is comparatively slow.

Table 42.

k	1	2	3	4	5	6	7	∞
P	2.5966	0.5915	0.459	0.424	0.412	0.406	0.403	0.3955

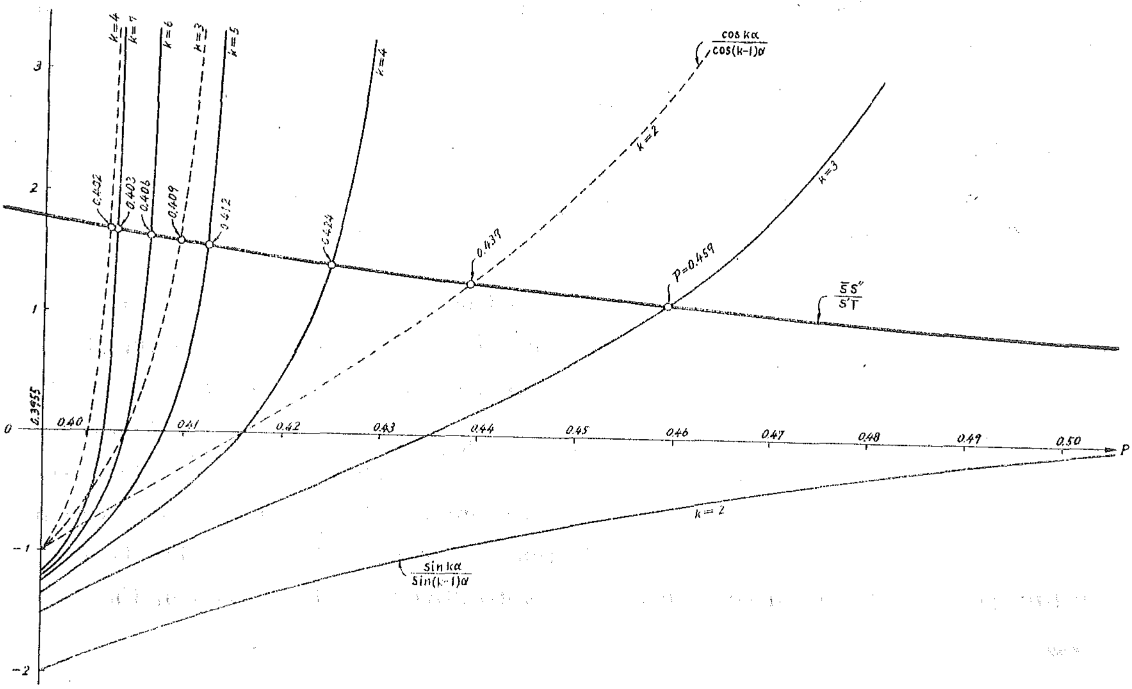

Fig. 40.

In the second place, let us investigate the case where only the load q is acting. At the case $k=1$, we obtain again (20.13) for a single plate. Next, at the case $k=2$, it is considered as before that the continuous plate is composed of the two single plates:—one having a long clamped edge and the opposite free edge, and the other having a long simply supported edge and the opposite free edge, by connecting the free edges of both. And the critical values of such single plates are already known such as $Q=5.344$ for the former and $Q=2.673$ for the latter.[60] Then, we can establish the following limits for the least critical value of the continuous plate considered now:

$$2.673 < Q < 5.344$$

Using the formulas (40), the actual operation gives $Q=3.805$ within the above limits. For references, let us require the deflection surface of this case. That is

60) See Table 33 and Table 36.

$$w_1' = \left\{(\cosh\pi\lambda_1\xi_1 - \cos\pi\lambda_2\xi_1) - 1.865\,550\ (\lambda_2\sinh\pi\lambda_1\xi_1 - \lambda_1\sin\pi\lambda_2\xi_1)\right\}\sin\pi\eta_1;$$

$$w_2' = 2.356\,785\left\{(\beta_2\cosh\pi\lambda_1\xi_2 + \beta_1\cos\pi\lambda_2\xi_2)\right.$$

$$\left. - (\beta_2\coth\pi\lambda_1\sinh\pi\lambda_1\xi_2 + \beta_1\cot\pi\lambda_2\sin\pi\lambda_2\xi_2)\right\}\sin\pi\eta_2\,.$$

Calculating the values of w_1' and w_2' along the line $\eta_\nu = 0.5$ by the above, we obtain Table 43 and Fig. 41.

Table 43.

$\xi_1,\ \xi_2$	0	0.125	0.250	0.500	0.750	1.000
w_1'	0	0.067 95	0.246 18	0.811 86	1.520 70	2.298 56
w_2'	2.298 55	—	1.796 17	1.262 36	0.656 29	0

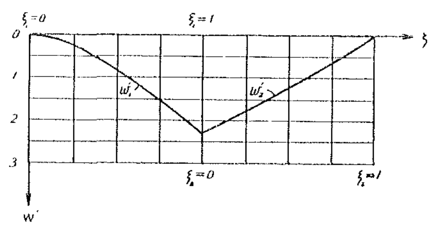

Fig. 41.

Next, let us consider the extreme case $k = \infty$. In this time it may be supposed that the effects of the supported end edge vanish and $Q = 0.9288$ is obtained. Plotting $\dfrac{SS''}{S'T}$-curve and $\dfrac{\sin k\alpha}{\sin(k-1)\alpha}$-curve in the interval $0.928 < Q < 3.805$, we can readily find the critical values of Q by the intersecting points of those curves, when k is comparatively small, as seen in Fig. 42.[61]

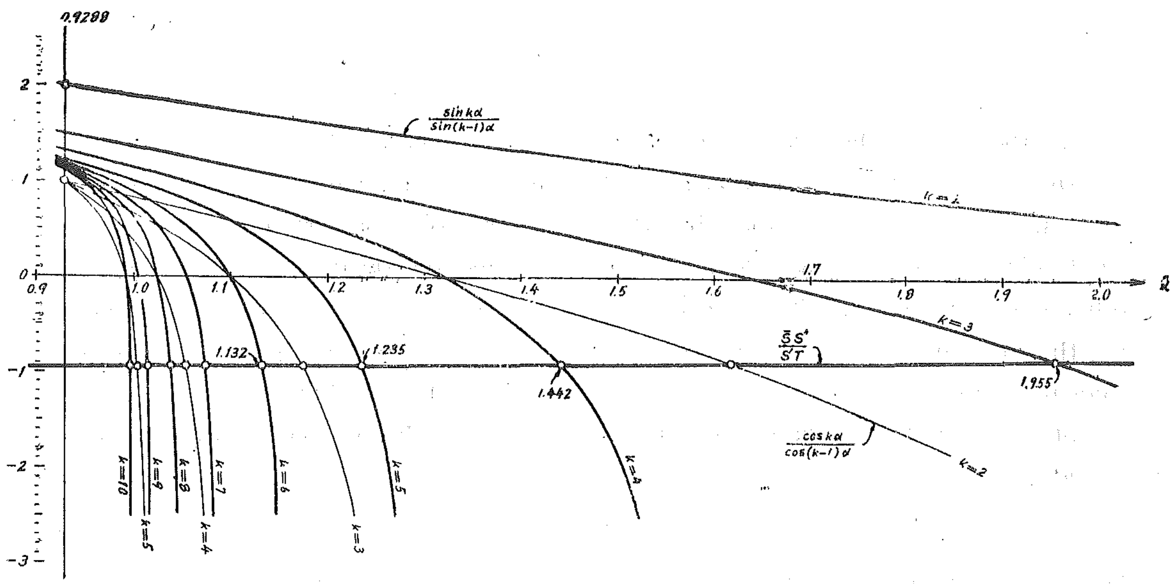

Fig. 42.

61) It is shown with heavy lines in the figure.

Considering the relation

$$\lim_{\substack{\alpha\to n\pi \\ (n=0,\,2,\,4,\cdots\cdots)}} \frac{\sin k\alpha}{\sin(k-1)\alpha}=1+\frac{1}{k-1}, \tag{21.12}$$

the right member of the above should represent the ordinate of $\dfrac{\sin k\alpha}{\sin(k-1)\alpha}$-curve at $Q=0.9288$: in this time $\alpha=n\pi$ in which n is even, because the relation T'/S' $(=\cos\alpha)=1$ is satisfied by such a value of Q. However, as observed before, $\dfrac{\sin k\alpha}{\sin(k-1)\alpha}$-curve is an oscillatory one between $\pm\infty$. Then in the same manner as before, we can know that the larger k is, the larger the frequency of oscillation of such a curve between $\pm\infty$ becomes. Then at the extreme case $k=\infty$, the curve becomes to coincide with the straight line passing through the point whose coordinates are given by $Q=0.9288$ and (21.12), and perpendicular to Q-axis, and accordingly, for such an extreme case it results that the critical value of Q is 0.9288. Then the initial supposition has been verified.

The actual operations give the numerical results in Table 44.

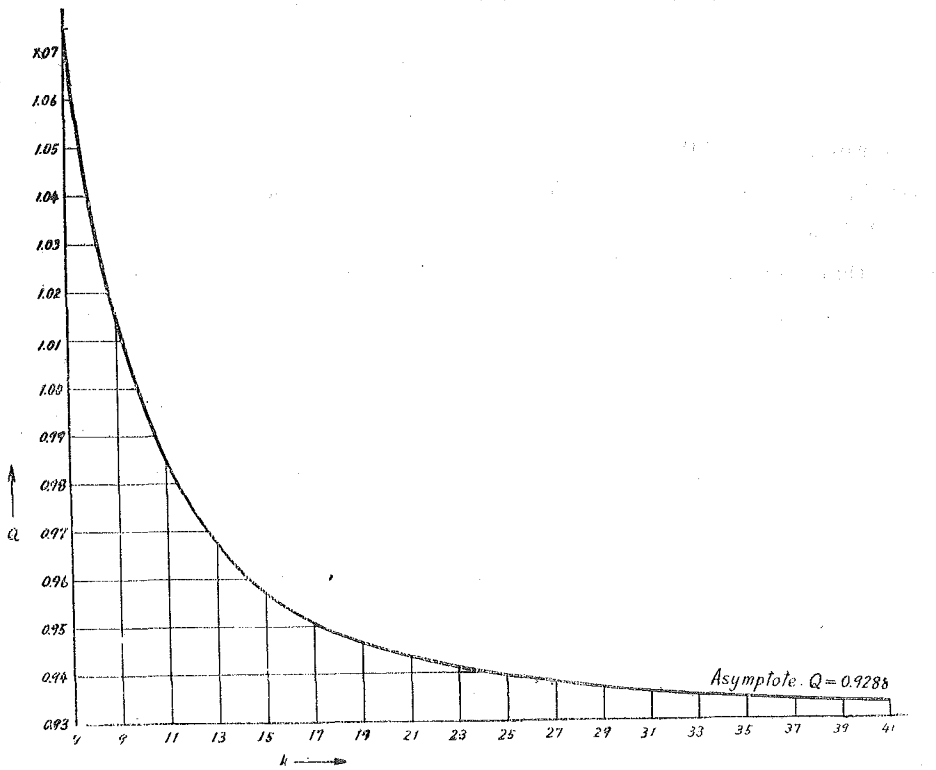

Fig. 43.

Table 44.

k	1	2	3	4	5	6	7	8	9	10	11	12	∞
Q	$[m=2]$ 22.96	3.805	1.955	1.442	1.235	1.132	1.075	1.038	1.014	0.997	0.985	0.975	0.9288

Observing Table 44, we find that the convergency of the sequence of Q, in this case, is comparatively slow. Fig. 43 shows the state of this fact.

3) *The case where one end side is free and the other simply supported.*— For convenience, let us discuss this case previously, though the order of the explanations is somewhat different from that in the previous section. The conditional equation for this case is given by the combination of (21.3) and (21.8), or of (21.5) and (21.6). Since each combination is respectively similar to that of (20.6) and (20·10), or of (20.7) and (20.9) in § 20, the following equation of such a form as (**31**) can be written:

$$\cos\left(s+\frac{1}{2}\right)\frac{\pi}{k}=\frac{T'}{S'} \tag{41}$$

$$(s=0,\ 1,\ 2,\cdots\cdots\cdots,2k-1).$$

Taking $k=1$ in the above, we can obtain for a single plate

$$T'=\frac{\gamma_1+P\lambda_1}{\beta_1}\coth\pi\lambda_1-\frac{\gamma_2+P\lambda_2}{\beta_2}\coth\pi\lambda_2=0.$$

This is nothing but (20.16) obtained before.

For the numerical examples, let us consider the case where only the load p is acting. When $k=1$, obviously $P=0.5107$ as found before; and next, it can be supposed that, when $k=\infty$, P will be 0.3955 because of vanishing of the effect caused by the simply supported end edge. Then for $1<k<\infty$, we can establish

$$0.3955<P<0.5107$$

As we can understand from Fig. 37, T'/S' is a negative monotonic increasing function of P within the above limits. Then the minimum value which is possible for the left hand side of Eq. (**41**) gives the least root P for this case. To do this, we have to put $s=k-1$ so that $\left(s+\frac{1}{2}\right)\frac{\pi}{k}$ becomes the nearest value to π. Hence, (**41**) becomes

$$\cos\left(1-\frac{1}{2k}\right)\pi=\frac{T'}{S'}. \tag{41'}$$

Then, referring to Fig 37, the values of P for various values of k can be obtained as shown in Table 45. Taking $k=\infty$ in the formula (**41**′), we find $P=0.3955$ by observing Fig. 37 because (**41**′), in this time, is transformed into $T'/S'=-1$.

Table 45.

k	$\cos\left(1-\frac{1}{2k}\right)\pi$	P
1	0	0.5107
2	−0.707 11	0.415
3	−0.865 60	0.404
4	−0.923 88	0.400
5	−0.951 06	0.398
6	−0.965 93	0.397
∞	−1	0.3955

Accordingly, it can be seen that the initial supposition is proper.

In the second place, let us consider the case when only the load q is acting. It is already known that $Q=2.6722$ when $k=1$.[62] Next, for the extreme case when $k=\infty$, we can suppose as $Q=0.9288$ by the reason why the effect of simple supporting at one end side can be neglected as often described. Then for $1<k<\infty$, the following limits are obtained:

$$0.928<Q<2.672$$

Within the above limits, T'/S' is a monotonic decreasing function of Q as shown in Fig. 39. Since the left member of (**41**) can not become larger than unit, the least root Q must be determined by making $\left(s+\frac{1}{2}\right)\frac{\pi}{k}$ as close as possible to zero. From this, $s=0$ must be taken. Therefore, we finally obtain

$$\cos\frac{\pi}{2k}=\frac{T'}{S'}\,. \qquad (\mathbf{41}'')$$

Table 46

k	$\cos\frac{\pi}{2k}$	Q
1	0	2.6722
2	0.707 11	1.325
3	0.866 03	1.101
4	0.923 88	1.023
5	0.951 06	0.990
6	0.965 93	0.972
∞	1	0.9288

Reading the values of Q corresponding to various values of k from Fig. 39, the results shown in Table 46 are obtained. Now, since we obtain $Q=0.9288$ by $T'/S'=1$ provided that $k=\infty$, it can be understood that the initial supposition is proper.

This table shows that the effect of simply supporting at one end side diminishes rapidly according to the increase in k.

4) *The case where both the end sides are free.*—The conditional equation for this case is expressed by the combination of (21.5) and (21.8). This combination is similar to that of (20.7) and (20.10) in § 20. Then we can see that the following relation results:

$$\sin k\alpha=0\,,$$

from which

$$\alpha=s\frac{\pi}{k} \qquad (s=0,\,1,\,2,\cdots\cdots\cdots\cdots,2k-1).$$

Therefore the conditional equation becomes of the same form as (**39**), but the fact that the desired least root in this case must be required by taking s different from the one in the foregoing case is similar to the relation between both the cases ,*viz.*,

62) See Table 33 in § 20.

"1) *The case where both the end sides are simply supported*" and "3) *The case where both the end sides are clamped*" in § 20. Writing Eq. (39) again,

$$\cos s\frac{\pi}{k}=\frac{T'}{S'}=\frac{\dfrac{\gamma_1+P\lambda_1}{\beta_1}\coth\pi\lambda_1-\dfrac{\gamma_2+P\lambda_2}{\beta_2}\coth\pi\lambda_2}{\dfrac{\gamma_1+P\lambda_1}{\beta_1}\operatorname{cosech}\pi\lambda_1-\dfrac{\gamma_2+P\lambda_2}{\beta_2}\operatorname{cosech}\pi\lambda_2}$$
$$(s=0,\ 1,\ 2,\cdots\cdots\cdots,2k-1).$$

Particularly, putting $k=1$ for a single plate, we have

$$\frac{T'}{S'}=\mp 1 \qquad \begin{cases}\text{upper sign corresponding to odd } s\,,\\ \text{lower sign corresponding to even } s\,.\end{cases} \tag{21.13}$$

Then, we get

$$\frac{\gamma_1+P\lambda_1}{\beta_1}(\coth\pi\lambda_1\pm\operatorname{cosech}\pi\lambda_1)-\frac{\gamma_2+P\lambda_2}{\beta_2}(\coth\pi\lambda_2\pm\operatorname{cosech}\pi\lambda_2)=0\,,$$

and accordingly, we can write

$$\left.\begin{aligned}&\frac{\gamma_1+P\lambda_1}{\beta_1}\coth\frac{\pi}{2}\lambda_1-\frac{\gamma_2+P\lambda_2}{\beta_2}\coth\frac{\pi}{2}\lambda_2=0 \qquad \text{for the upper sign,}\\ &\frac{\gamma_1+P\lambda_1}{\beta_1}\tanh\frac{\pi}{2}\lambda_1-\frac{\gamma_2+P\lambda_2}{\beta_2}\tanh\frac{\pi}{2}\lambda_2=0 \qquad \text{for the lower sign.}\end{aligned}\right\} \tag{21.14}$$

If there is no load of p, we have

$$\gamma_1=-\lambda_1\beta_2, \qquad \gamma_2=-\lambda_2\beta_1,$$

then (21.14) are transformed as follows:

$$\left.\begin{aligned}&\lambda_1\beta_2^2\coth\frac{\pi}{2}\lambda_1-\lambda_2\beta_1^2\coth\frac{\pi}{2}\lambda_2=0\,,\\ &\lambda_1\beta_2^2\tanh\frac{\pi}{2}\lambda_1-\lambda_2\beta_1^2\tanh\frac{\pi}{2}\lambda_2=0\,.\end{aligned}\right\} \tag{21.15}$$

The first equation of the above corresponds to the inversely symmetric buckling with respect to the line $\xi=1/2$ and the second equation to the symmetric one. These are the already known formulas.

For the numerical examples, let us consider first the case where only the load p is acting. For the special case $k=1$, the first equation of (21.15) gives $P=0.3955$ and the second gives $P=1.3123$ as the least root respectively. Now, let us investigate the deflection surfaces in these special cases for references. In such a case, the initially obtained relation becomes as follows because of $k=1$:

$$\alpha=s\pi \qquad (s=0,\ 1,\ 2,\cdots\cdots\cdots,2k-1).$$

Next, substituting (21.5) in (37), we obtain

$$V_r = \frac{\cos(k+1-r)\alpha}{\cos\alpha},$$

and accordingly we can write again

$$V_{r+1} = \frac{\cos(k-r)\alpha}{\cos\alpha}.$$

Then, putting $k=r=1$ in the above, we get for the case when s is odd

$$V_r = 1 \qquad \text{and} \qquad V_{r+1} = -1,$$

and for the case when s is even

$$V_r = 1 \qquad \text{and} \qquad V_{r+1} = 1.$$

So, the expression (21.9) about the deflection surface, in this case, becomes as follows:

$$w_r = A\left[\left\{\frac{\cosh\pi\lambda_1}{\beta_1}\left(\frac{\cosh\pi\lambda_1\xi}{\cosh\pi\lambda_1} - \frac{\sinh\pi\lambda_1\xi}{\sinh\pi\lambda_1}\right) - \frac{\cosh\pi\lambda_2}{\beta_2}\left(\frac{\cosh\pi\lambda_2\xi}{\cosh\pi\lambda_2} - \frac{\sinh\pi\lambda_2\xi}{\sinh\pi\lambda_2}\right)\right\} \mp \left(\frac{\sinh\pi\lambda_1\xi}{\beta_1\sinh\pi\lambda_1} - \frac{\sinh\pi\lambda_2\xi}{\beta_2\sinh\pi\lambda_2}\right)\right]\sin\pi\eta. \tag{21.16}$$

In the above expression, the upper of the double signs corresponds to the inversely symmetric buckling and the lower to the symmetric buckling. Yet, provided that y-axis is moved to the position $x=a/2$ by putting $\xi-(1/2)=\xi'$, we can readily prove that (21.16) becomes the odd function of ξ' for the first case, and the even function for the second case. The numerical results for $\eta=1/2$ are given in Fig. 44.[63]

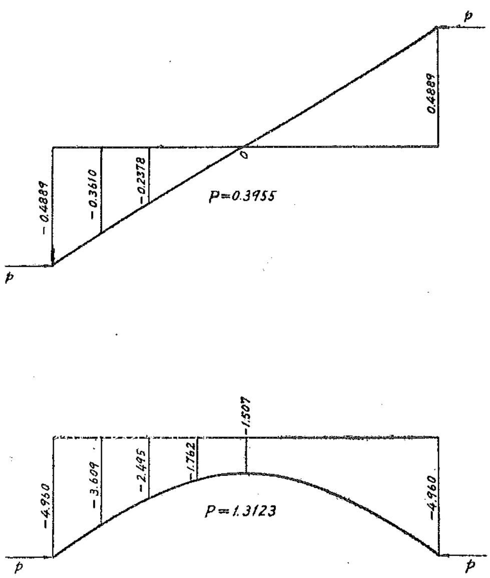

Fig. 44.

63) These numerical results are the values obtained by omitting the common factor from the ones by (21.16) for convenience of the calculation.

Now, as initially seen, the conditional equation for the general case is given as follows:

$$\cos s\frac{\pi}{k}=\frac{T'}{S'}.$$

Then, observing Fig. 37, it is obvious that the least root must be obtained by taking $s=k$. And in this time, the above equation becomes to coincide with the first of (21.13). This conclusion may be understood by considering the fact that the configuration of distortion of the present continuous plate can be constructed with some of the configuration shown in the upper part of Fig. 44 by placing them in a row and in zigzag form, and the reason why the conditional equation becomes free of k may be admitted. From this, it results that for any k the least root is always given by

$$P=0.3955$$

Let us consider next the case where only the load q is acting. For the special case $k=1$, (21.15) can be used again. Then, for the symmetric buckling, the second equation of (21.15) gives

$$Q=0.9288$$ [64]

Observing that T'/S'-curve represents a monotonic decreasing function of Q as shown in Fig. 39, we see that the same result as above is obtained by taking $s=0$ for any k and also Eq. (**39**) becomes to be free of k. These conclusions can be understood by the reason why each elementary plate will buckle in the same manner as a single plate having both free end sides owing to the symmetrical circumstances.

5) *The case where one end side is free and the other clamped.*—In this case, the conditional equations must be constructed by the combination of (21.5) and (21.7), or of (21.4) and (21.8), which is similar to that of (20.7) and (20.11), or of (20.8) and (20.10) in § 20 respectively. Hence, we obtain the following equations of the same form as (**35**):

$$\left.\begin{aligned}\frac{\cos k\alpha}{\cos (k-1)\alpha}&=\frac{\bar{S}S''}{S'T'},\\ \cos\alpha&=\frac{T'}{S'}.\end{aligned}\right\}\qquad(\mathbf{42})$$

These are the conditional equations for the present case which have a parameter α. Considering the particular case when $k=1$, we can readily prove that the above become to coincide with (20.19).

64) For a square plate, there is given a numerical result: $Q=0.9524$ in the paper by C. Haraguchi, Jour. of Civil Eng. Soc., Vol. 29, No. 12, p. 6. Then, the now desired root must be obtained within the neighbourhood of this value.

For the numerical examples, let us observe first the case where only the load p is acting. When $k=1$, the following result has already been obtained [see Fig. 24]:

$$P = 0.6565$$

And when $k=\infty$, we can readily suppose as in the foregoing examples

$$P = 0.3955$$

And in this time, it follows that $\cos\alpha = T'/S' = -1$. Therefore we can see that $\alpha = n\pi$ in which n is odd, *i. e.*, $\alpha \to n\pi$ according as $P \to 0.3955$. Considering now

$$\lim_{\substack{\alpha\to n\pi \\ (n=1,3,5,\cdots\cdots)}} \frac{\cos k\alpha}{\cos(k-1)\alpha} = -1, \tag{21.17}$$

since $\dfrac{\cos k\alpha}{\cos(k-1)\alpha}$ -curve is an oscillatory one depending upon P, by such a consideration as in the numerical examples about the preceding formulas (**40**), it will be seen that such a curve must coincide with the line $P=0.3955$ at the extreme case when $k=\infty$. So, its intersecting point with $\dfrac{\bar{S}S'}{S'T}$ -curve will be recognized to give $P=0.3955$. The actual procedures are illustrated by the broken line in Fig. 40 and from these the numerical results are tabulated in Table 47.

Table 47.

k	1	2	3	4	∞
P	0.6565	0.439	0.409	0.402	0.3955

Seeing this table, we find that the convergency of the sequence, in this case, comparatively slow.

In the next place. let us consider the case where only the load q is acting. It is already known in the preceding example that, for the least value, Q is 5.344 when $k=1$. Next, we can suppose that $Q=0.9288$ owing to disappearance of the effect of the clamped edge when $k=\infty$. At this time, referring to that $T'/S'=1$ holds, the second equation of (**42**) becomes

$$\cos\alpha = 1,$$

then it must be taken that $\alpha = n\pi$ in which n is even. Accordingly, $\dfrac{\cos k\alpha}{\cos(k-1)\alpha} = 1$. But, on the other hand, we can know that $\dfrac{\cos k\alpha}{\cos(k-1)\alpha}$-curve becomes to coincide with a straight line perpendicular to Q-axis when $k=\infty$ as plotted by the light lines in Fig. 42. Then it can be concluded that $Q=0.9288$ when $k=\infty$, and thus the numerical results in Table 48 are obtained.

Table 48.

k	1	2	3	4	5	6	7	∞
Q	5.344	1.618	1.175	1.053	1.004	0.979	0.965	0.9288

6) *The case where both the end sides are clamped.*—The conditional equations for this case are given by the combination of (21.4) and (21.7). Since this combination is similar to that of (20.8) and (20.11) in § 20, then the conditional equations can be written as follows:

$$\left.\begin{aligned} \frac{\sin(k-1)\alpha \pm \sin\alpha}{\sin(k-2)\alpha} &= \frac{SS''}{S'T}, \\ \cos\alpha &= \frac{T'}{S'}. \end{aligned}\right\} \tag{43}$$

The reason why the formula for a single plate can not be deduced from the above by putting $k=1$ can be understood in the same manner as about the foregoing formulas (**36**).

First, in the case $k=2$, we can readily obtain by taking the upper sign in the first equation of (**43**)

$$\lambda_1 \coth\pi\lambda_1 - \lambda_2 \coth\pi\lambda_2 = 0\,.$$

This is nothing but (20.15). This result may be understood by supposing that the configuration of the buckled continuous plate can be constructed by connecting those of such two plates as has one clamped end side and the opposite simply supported end side in the manner shown in Fig 45 (a). Second place, taking the lower sign in the first equation of (**43**) and making such a limitation process as done with respect to (**36**), we can obtain (20.19). This result may be understood also by observing the illustrating figure of Fig. 45 (b). In both the above cases, the latter can be supposed to give the least critical value because of simplicity of distortion.

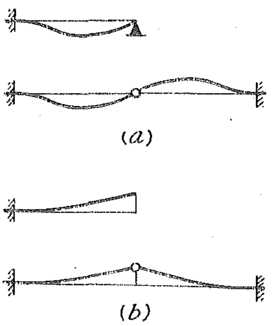

Fig. 45.

And, taking $k=2k'$, it follows that

$$\frac{\sin(k-1)\alpha \pm \sin\alpha}{\sin(k-2)\alpha} = \begin{cases} \dfrac{\sin k'\alpha}{\sin(k'-1)\alpha} & \text{for the upper sign,} \\[2ex] \dfrac{\operatorname{osc} k'\alpha}{\cos(k'-1)\alpha} & \text{for the lower sign.} \end{cases}$$

Then, if k is an even number, we can conclude as follows: taking the upper sign, (**43**) becomes to coincide with (**40**), and, taking the lower sign, (**43**) coincide with (**42**). Thus, the former case corresponds to the inversely symmetric buckling and the latter one to the symmetric buckling in the similar way as in the examples concerning (**36**). In general, it may be proved that the upper sign corresponds to the inversely symmetric buckling and the lower one to the symmetric buckling.

For the numerial examples, let us begin with the case where only the load p is acting. By the above considerations, if the number of the elementary plates is even, the numerical results in Table 42 and Table 47 can be used again for this case. And, if the number of the elementary plate is odd, the actual procedures to acquire the critical values must be added. These actual results are shown in Table 49.

Table. 49.–Least critical values of P in the case when both the end sides are clamped.

k	2	3	4	5	6	∞
Inversely Symmetric buckling	2.5966	0.4975	0.5915	0.418_2	0.459	0.3955
Symmetric buckling	0.6565	0.8107	0.439_0	0.502_3	0.409_3	

Observing this table, we find that the least critical values correspond to the inversely symmetric buckling when k is odd and to the symmetric when k is even. And moreover, their configurations in section of x-direction can be supposed as illustrated in Fig. 46.

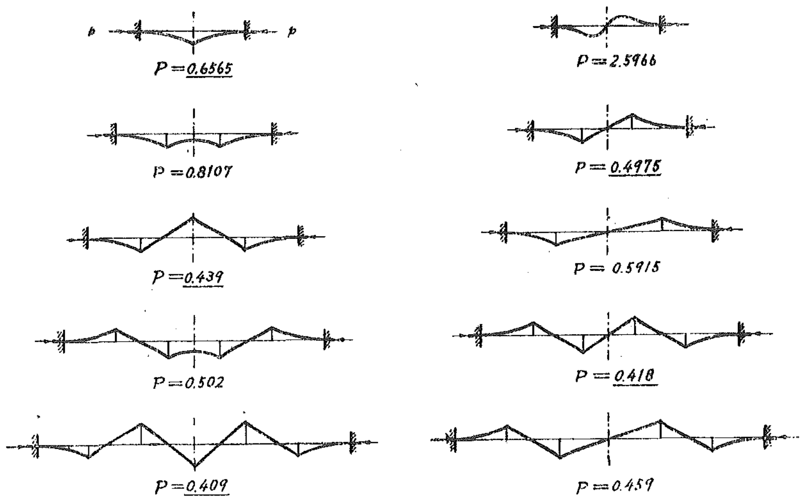

Fig. 46.

Next, let us consider the case where only the load q is acting. Proceeding as before, we obtain the following numerical results.

Table 50.-Least critical values of Q in the case when both the end sides are clamped.

k	2	3	4	5	6	7	8	9	10	∞
Inversely Symmetric buckling	$[m=2]$ 22.96	6.280	3.805	2.550	1.955	1.635	1.442	1.320	1.235	0.9288
Symmetric buckling	5.344	2.455	1.618	1.315	1.175	1.098	1.053	1.025	1.004	

From this, we find that the least critical values always correspond to the symmetric buckling.

(B) FORMULAE FOR THE CONTINUOUS PLATES FURNISHED WITH VARIOUS RESTRAINTS ALONG THE CONNECTING JOINTS.

The method of solving by means of the finite difference equations is applicable to the following cases having coefficients of restraint concerning joint, when these coefficients of every joint are the same ones. And it can be supposed that such procedures of solving are similar as in the preceding sections. Henceforth, the formulas for such cases will be enumerated without the numerical examples. For advantage of explanations, let us give the additional notations as follows:

$B_1 = B_2 = \cdots\cdots = B_r = \cdots\cdots = B_k = B =$ Flexural rigidity of a supporting beam,
$C_1 = C_2 = \cdots\cdots = C_r = \cdots\cdots = C_k = C =$ Torsional rigidity of a supporting beam,
$Q_1' = Q_2' = \cdots\cdots = Q_r' = \cdots\cdots = Q_k' = Q' =$ Axial thrust in a supporting beam,

$$\bar{\mu}_1 = \bar{\mu}_2 = \cdots\cdots = \bar{\mu}_r = \cdots\cdots = \bar{\mu}_k = \bar{\mu},$$

$$\bar{\mu}_1' = \bar{\mu}_2' = \cdots\cdots = \bar{\mu}_r' = \cdots\cdots = \bar{\mu}_k' = \bar{\mu}',$$

$$\bar{\mu}_1'' = \bar{\mu}_2'' = \cdots\cdots = \bar{\mu}_r'' = \cdots\cdots = \bar{\mu}_k'' = \bar{\mu}'',$$

$$\kappa_1 = \kappa_2 = \cdots\cdots = \kappa_r = \cdots\cdots \kappa_{r-1} = \kappa_2' = \kappa_3' = \cdots\cdots = \kappa_r' = \cdots\cdots = \kappa_k' = \kappa.$$

§ 22. Plates Having Rigid Joints and Elastically Supported along the Joints in 2-nd Way:-Restricted by Twisting Moment and Prevented from Deflecting.

i) **General solution.**—In this case, the joining conditions given by (**13**) and (**11**) [§ 6-d)] are simplified as follows:

$$\left.\begin{aligned} U_{r-1} - U_r F + V_r G &= 0\,; \\ V_{r-1} + U_r F' - V_r G' &= 0\,, \end{aligned}\right\} \tag{22.1}$$

where

$$F = \frac{T}{S} - \frac{\bar{\mu}}{\varphi}\frac{W}{S}\,, \qquad G = \frac{W}{S}\,,$$

$$\left.\begin{aligned} F' &= \frac{1}{S} - \frac{\bar{\mu}}{\varphi}\frac{T}{S}, \qquad G' = \frac{T}{S}, \\ \bar{\mu} &= -\frac{C}{D}\left(\frac{\pi}{a}\right)\left(m\frac{a}{b}\right)^2. \end{aligned}\right\} \tag{22.2}$$

The solution of these equations can be obtained in the same manner as before, since the form of (22.1) is similar to that of (20.1).

Now putting

$$\left.\begin{aligned} U_r &= B\,\omega^r\,; \\ V_r &= B'\omega^r\,, \end{aligned}\right\} \tag{22.3}$$

and substituting these in (22.1), we have

$$\left.\begin{aligned} B\,(\omega^{r-1} - \omega^r F\,) + B'\,\omega^r G &= 0\,; \\ B'(\omega^{r-1} - \omega^r G') + B\,\omega^r F' &= 0\,. \end{aligned}\right\} \tag{22.4}$$

Eliminating B and B' from the above, we obtain as the characteristic equation

$$\frac{1}{\omega^2} - \frac{1}{\omega}(F + G') + F\,G' - G\,F' = 0\,.$$

Considering (22.2), the above equation is rewritten as follows:

$$\omega^2 - 2\omega\left\{\frac{T}{S} - \frac{\bar{\mu}}{2\varphi}\frac{W}{S}\right\} + 1 = 0.$$

Then, by solving this, we get

$$\left.\begin{matrix}\omega_1\\ \\ \omega_2\end{matrix}\right\} = \left(\frac{T}{S} - \frac{\bar{\mu}}{2\varphi}\frac{W}{S}\right) \pm \sqrt{\left(\frac{T}{S} - \frac{\bar{\mu}}{2\varphi}\frac{W}{S}\right)^2 - 1}\,.$$

For the purpose of representing by trigonometric functions, putting as follows:

$$\left.\begin{aligned} \omega_1 &= \rho\,e^{i\alpha}\,, \\ \omega_2 &= \rho\,e^{-i\alpha}\,, \end{aligned}\right\}$$

and taking $\rho = 1$ because of $\omega_1\,\omega_2 = \rho^2 = 1$, the following relation can be written:

$$\cos\alpha = \frac{\omega_1 + \omega_2}{2} = \frac{T}{S} - \frac{\bar{\mu}}{2\varphi}\frac{W}{S}\,. \tag{44}$$

Noting moreover $\omega = \cos\alpha \pm i\sin\alpha$, (22.4) gives the following relation:

$$B' = B\left(\pm i\frac{S}{W}\sin\alpha - \frac{\bar{\mu}}{2\varphi}\right).$$

Then, using B, B' again as new unknown constants for saving of the notations, the general solution is finally written in the following form:

$$\left.\begin{aligned} U_r &= B\cos r\alpha + B'\sin r\alpha,\\ V_r &= -\frac{S}{W}\left\{B\left(\sin\alpha\sin r\alpha + \frac{\bar{\mu}}{2\varphi}\,\frac{W}{S}\cos r\alpha\right) - B'\left(\sin\alpha\cos r\alpha - \frac{\bar{\mu}}{2\varphi}\,\frac{W}{S}\sin r\alpha\right)\right\}. \end{aligned}\right\} \quad (\mathbf{45})$$

where α is a parameter defined by (**44**).

ii) **The end side conditions.** (1) *The conditions at End side-k.*—These for the present case are given by the expressions of the same form as in the case [§6–**b**)]:

1) The case of the simply supported edge.—From (**11**.a)

$$U_k = -T; \qquad V_k = 1.$$

Substituting (**45**) in the above, we have

$$B\cos k\alpha + B'\sin k\alpha = -T;$$

$$B\left(\frac{S}{W}\sin\alpha\sin k\alpha + \frac{\bar{\mu}}{2\varphi}\cos k\alpha\right) - B'\left(\frac{S}{W}\sin\alpha\cos k\alpha - \frac{\bar{\mu}}{2\varphi}\sin k\alpha\right) = -1.$$

Solving the above two with respect to B and B',

$$\left.\begin{aligned} B &= \frac{S}{\sin\alpha}\left\{\sin k\alpha - \frac{T}{S}\sin(k+1)\alpha\right\},\\ B' &= -\frac{S}{\sin\alpha}\left\{\cos k\alpha - \frac{T}{S}\cos(k+1)\alpha\right\}. \end{aligned}\right\} \quad (22.5)$$

2) The case of the clamped edge.—From (**11**.b)

$$U_k = -\frac{W}{T}; \qquad V_k = 1.$$

Substituting (**45**) in the above, and solving them as before, we get

$$\left.\begin{aligned} B &= -\frac{W}{T}\,\frac{\sin(k+1)\alpha}{\sin\alpha},\\ B' &= \frac{W}{T}\,\frac{\cos(k+1)\alpha}{\sin\alpha}. \end{aligned}\right\} \quad (22.6)$$

3) The case of the free edge.—From (**11**.c)

$$U_k = -\frac{TT' - \bar{S}S''}{T'}; \qquad V_k = 1.$$

Substituting (**45**) in the above, and solving them as before, we get

$$\left.\begin{aligned} B &= \frac{S}{\sin\alpha}\left[\left\{\sin k\alpha - \frac{T}{S}\sin(k+1)\alpha\right\} + \frac{\bar{S}S''}{ST'}\left\{\sin(k+1)\alpha - \frac{T}{S}\sin k\alpha\right\}\right],\\ B' &= -\frac{S}{\sin\alpha}\left[\left\{\cos k\alpha - \frac{T}{S}\cos(k+1)\alpha\right\} + \frac{\bar{S}S''}{ST'}\left\{\cos(k+1)\alpha - \frac{T}{S}\cos k\alpha\right\}\right]. \end{aligned}\right\} \quad (22.7)$$

(2) *The conditions at End side-1.*—These for this case become of the same form as in the forgoing [§ 6–**b**)] except for the case where both end sides are free:

1) The case of the simply supported edge.—From (**11.d**)

$$V_1 = 0 .$$

Substituting (**45**) in this

$$B\left(\sin^2\alpha + \frac{\bar{\mu}}{2\varphi}\frac{W}{S}\cos\alpha\right) - B'\left(\cos\alpha - \frac{\bar{\mu}}{2\varphi}\frac{W}{S}\right)\sin\alpha = 0 . \tag{22.8}$$

2) The case of the clamped edge.—From (**11.e**)

$$U_1 = 0 .$$

Substituting (**45**),

$$B\cos\alpha + B'\sin\alpha = 0. \tag{22.9}$$

3) The case of the free edge.—From (**13′**)

$$\left.\begin{aligned} & U_1 = U_2\left\{(F_1) - \frac{\bar{\mu}}{\varphi}(G_1)\right\} - V_2(G_1) = 0 , \\ \text{where}\quad & (F_1) = \frac{T'}{S} , \qquad (G_1) = \frac{TT' - \bar{S}S''}{\bar{S}} . \end{aligned}\right\}$$

Substituting (**45**),

$$B\left\{\frac{W}{S^2}\cos 2\alpha + \left(\frac{T}{S} - \frac{\bar{S}S''}{ST'}\right)\left(\sin\alpha\sin 2\alpha - \frac{\bar{\mu}}{2\varphi}\frac{W}{S}\cos 2\alpha\right)\right\}$$
$$+ B'\left\{\frac{W}{S^2}\sin 2\alpha - \left(\frac{T}{S} - \frac{\bar{S}S''}{ST'}\right)\left(\sin\alpha\cos 2\alpha + \frac{\bar{\mu}}{2\varphi}\frac{W}{S}\sin 2\alpha\right)\right\} = 0. \tag{22.10}$$

iii) The conditional equations.—The various combinations of the preceding conditions at End side-k and at End side-1 can represent every case tabulated in Table 4.

1) *The case where both the end sides are simply supported.*—Eliminating B and B' from (22.5) and (22.8), and considering by (**44**)

$$\frac{\mu}{2\varphi}\frac{W}{S} = \frac{T}{S} - \cos\alpha,$$

we obtain after some arrangements

$$\left(\frac{T}{S}\right)^2 \sin k\alpha - 2\left(\frac{T}{S}\right)\sin(k-1)\alpha + \sin(k-2)\alpha = 0.$$

From this

$$\left\{\frac{T}{S} - \frac{\sin(k-1)\alpha + \sin\alpha}{\sin k\alpha}\right\}\left\{\frac{T}{S} - \frac{\sin(k-1)\alpha - \sin\alpha}{\sin k\alpha}\right\} = 0.$$

Then,

and by (**44**)

$$\left.\begin{aligned} \frac{\sin(k-1)\alpha \pm \sin\alpha}{\sin k\alpha} &= \frac{T}{S}, \\ \cos\alpha &= \frac{T}{S} - \frac{\bar{\mu}}{2\varphi}\frac{W}{S}. \end{aligned}\right\} \tag{46}$$

These are the conditional equations having a parameter α.

Now, provided that effects of twisting moments at the joints vanish, the above formulas must be proved to coincide with those of § 20. To do this, putting $\bar{\mu}=0$ in (**46**), it follows that.

$$\frac{\sin(k-1)\alpha \pm \sin\alpha}{\sin k\alpha} = \frac{T}{S} = \cos\alpha .$$

From this

$$\sin\alpha\,(\cos k\alpha \mp 1) = 0 .$$

Then

$$\sin\alpha\,(\cos^2 k\alpha - 1) = \sin^2\alpha \sin^2 k\alpha = 0 .$$

Hence

$$\sin\alpha \sin k\alpha = 0 . \tag{22.11}$$

Accordingly,

$$\alpha = s\frac{\pi}{k} \qquad (s=0, 1, 2, \cdots\cdots, 2k-1) .$$

This is nothing than (**28**). Thus we can see that the formula (**29**) should be arrived at

2) *The case where one end side is clamped and the other simply supported.*— The conditional equations for this case can be obtained by the combination of (22.6) and (22.8) or of (22.5) and (22.9). Naturally, both of these combinations give the same result. Eliminating B and B', we have

and from (**44**)

$$\left.\begin{aligned} \frac{\sin(k-1)\alpha}{\sin k\alpha} &= \frac{T}{S}, \\ \cos\alpha &= \frac{T}{S} - \frac{\bar{\mu}}{2\varphi}\frac{W}{S}. \end{aligned}\right\} \tag{47}$$

These are the conditional equations having a parameter α. Now, in these, putting $\bar{\mu}=0$,

$$\frac{\sin(k-1)\alpha}{\sin k\alpha} = \frac{T}{S} = \cos\alpha .$$

Then

$$\sin\alpha \cos k\alpha = 0 .$$

However, the equation $\sin\alpha = 0$ is involved by (22.11) and accordingly it can not be adopted in this case. Hence

$$\cos k\alpha = 0\,.$$

From this

$$\alpha=\left(s+\frac{1}{2}\right)\frac{\pi}{k} \qquad (s=0,1,2,\cdots\cdots,2k\text{-}1).$$

This is nothing but (**30**) and therefore the foregoing (**31**) can be understood to be arrived at,

3) *The case where both end sides are clamped.*—The combination of (22.6) and (22.9) gives the formula for this case. Eliminating B and B' from both of them, we get

$$\sin k\alpha = 0\,.$$

Then

$$\alpha = s\frac{\pi}{k} \qquad (s=0,1,2,\cdots\cdots,2k\text{-}1)\,.$$

Accordingly, the conditional equation is

$$\cos s\frac{\pi}{k}=\frac{T}{S}-\frac{\bar{\mu}}{2\varphi}\frac{W}{S}\,. \qquad (s=0,1,2,\cdots\cdots,2k\text{-}1) \tag{48}$$

Putting now $\bar{\mu}=0$ in this, we can prove that the above becomes to coincide with Eq. (**29**).

4) *The case where one end side is free and the other simply supported.*–In this case, the conditional equations are given by the combination of (22.7) and (22.8) or of (22.5) and (22.10). Then eliminating B and B' from such combinations, we can obtain the following conditional equations with a parameter α:

$$\left.\begin{aligned} &\frac{\sin(k-1)\alpha-\dfrac{T}{S}\sin k\alpha}{\sin(k-2)\alpha-\dfrac{T}{S}\sin(k-1)\alpha}=\frac{\left(\dfrac{\bar{S}S''}{ST'}\right)\dfrac{T}{S}-1}{\left(\dfrac{\bar{S}S''}{ST'}\right)-\dfrac{T}{S}}\,,\\ &\text{and from } (\mathbf{44})\\ &\cos\alpha=\frac{T}{S}-\frac{\bar{\mu}}{2\varphi}\frac{W}{S}\,. \end{aligned}\right\} \tag{49}$$

Putting now $\bar{\mu}=0$, it can readily be proved that the formulas (**33**) in § 20 be arrived at.

5) *The case where one end side is free and the other clamped.*—The combination of (22.7) and (22.9) or of (22.6) and (22.10) is used and each of them gives

the same result. The elimination as before gives

$$\left.\begin{aligned} \frac{\sin k\alpha}{\sin(k-1)\alpha} &= \frac{\left(\dfrac{\bar{S}S''}{ST'}\right)\dfrac{T}{S}-1}{\left(\dfrac{\bar{S}S''}{ST'}\right)-\dfrac{T}{S}}, \\ \text{and from } (\mathbf{44}) \qquad & \\ \cos\alpha &= \frac{T}{S}-\frac{\mu}{2\varphi}\frac{W}{S}, \end{aligned}\right\} \tag{50}$$

in which α is a parameter. Now putting $\bar{\mu}=0$, the formulas (**35**) in § 20 can readily be arrived at.

6) *The case where both end sides are free.*—The combination of (22.7) and (22.10) is applicable to this case. The elimination of B and B' and some arrangements give the following equation by referring to (**44**):

$$\left(\frac{T}{S}-\frac{\bar{S}S''}{ST'}\right)^2\left(\frac{T^2}{S^2}-2\frac{T}{S}\frac{\sin(k-1)\alpha}{\sin(k-2)\alpha}+\frac{\sin k\alpha}{\sin(k-2)\alpha}\right)-2\left(\frac{T}{S}-\frac{\bar{S}S''}{ST'}\right)\left(\frac{W}{S^2}\right)$$
$$\times\left(\frac{T}{S}-\frac{\sin(k-1)\alpha}{\sin(k-2)\alpha}\right)+\left(\frac{W}{S^2}\right)^2=0.$$

From this,

$$\left\{\left(\frac{T}{S}-\frac{\bar{S}S''}{ST'}\right)\left(\frac{T}{S}-\frac{\sin(k-1)\alpha+\sin\alpha}{\sin(k-2)\alpha}\right)-\frac{W}{S^2}\right\}$$
$$\times\left\{\left(\frac{T}{S}-\frac{\bar{S}S''}{ST'}\right)\left(\frac{T}{S}-\frac{\sin(k-1)\alpha-\sin\alpha}{\sin(k-2)\alpha}\right)-\frac{W}{S^2}\right\}=0.$$

Then

$$\left.\begin{aligned} \frac{\sin(k-1)\alpha\pm\sin\alpha}{\sin(k-2)\alpha} &= \frac{\left(\dfrac{\bar{S}S''}{ST'}\right)\dfrac{T}{S}-1}{\left(\dfrac{\bar{S}S''}{ST'}\right)-\dfrac{T}{S}}, \\ \text{and } (\mathbf{44}): \qquad & \\ \cos\alpha &= \frac{T}{S}-\frac{\bar{\mu}}{2\varphi}\frac{W}{S}. \end{aligned}\right\} \tag{51}$$

in which α is a parameter. Now put $\bar{\mu}=0$ in the above, then we obtain

$$\left(\frac{\bar{S}S''}{ST'}\right)\left\{\cos\alpha-\frac{\sin(k-1)\alpha+\sin\alpha}{\sin(k-2)\alpha}\right\}+\left\{\frac{\sin(k-1)\alpha+\sin\alpha}{\sin(k-2)\alpha}\cos\alpha-1\right\}=0,$$

or

$$\left(\frac{SS''}{ST'}\right)\left\{\cos\alpha-\frac{\sin(k-1)\alpha-\sin\alpha}{\sin(k-2)\alpha}\right\}+\left\{\frac{\sin(k-1)\alpha-\sin\alpha}{\sin(k-2)\alpha}\cos\alpha-1\right\}=0.$$

Multiplying them mutually, we can readily obtain (20.20) and then the formulas (36) can be reduced.

§ 23. Plates Having Hinged Joints and Elastically Supported along the Joints in 1-st Way:-Supported by an Elastic Beam.

i) **General solution.**—The recurrence formula in § 7-f), also in this case, is reduced in such simultaneous finite difference equations as (21.1). But the coefficients must be replaced by the following expressions reduced from (16) instead of (21.2).

$$\left.\begin{aligned} F&=\frac{T'}{S'}, & G&=\frac{W'}{S'}-\bar{\mu}'\bar{\varphi}\frac{T'}{S'},\\ F'&=\frac{1}{S'}, & G'&=\frac{T'}{S'}-\bar{\mu}'\bar{\varphi}\frac{1}{S'},\end{aligned}\right\} \tag{52}$$

where

$$\bar{\mu}'=\left\{\frac{B}{D}\left(\frac{m\pi}{b}\right)^2-\frac{Q'}{D}\right\}\left(\frac{a}{\pi}\right)\left(\frac{ma}{b}\right)^2.$$

Proceeding as before, the general solution for this case is obtained as follows:

$$\left.\begin{aligned} U_r&=B\cos r\alpha+B'\sin r\alpha,\\ V_r&=\frac{1}{S'\left\{\sin^2\alpha+\left(\dfrac{\bar{\mu}'\bar{\varphi}}{2S'}\right)^2\right\}}\left[\begin{aligned}&B\left\{\sin\alpha\sin r\alpha-\left(\frac{\bar{\mu}'\bar{\varphi}}{2S'}\right)\cos r\alpha\right\}\\ &+B'\left\{\sin\alpha\cos r\alpha+\left(\frac{\bar{\mu}'\bar{\varphi}}{2S'}\right)\sin r\alpha\right\}\end{aligned}\right].\end{aligned}\right\} \tag{53}$$

In the above, the parameter α is defined by

$$\cos\alpha=\frac{T'}{S'}-\frac{\bar{\mu}'\bar{\varphi}}{2S'}. \tag{54}$$

ii) **The end side conditions.** (1) *The conditions at End side-k.*—These are represented in the same forms as in § 7-e) by referring to § 7-f). Then, substituting (53) in them, and solving with respect to B and B', the following expressions are written for each case:

1) The case of the simply supported edge.—From (15.a)

$$U_k=-T'; \qquad V_k=1.$$

Then

$$\left.\begin{aligned} B &= \frac{S'}{\sin\alpha}\left\{\sin k\alpha - \frac{T'}{S'}\sin(k+1)\alpha\right\}, \\ B' &= -\frac{S'}{\sin\alpha}\left\{\cos k\alpha - \frac{T'}{S'}\cos(k+1)\alpha\right\}. \end{aligned}\right\} \tag{23.1}$$

2) The case of the clamped edge.—From (**15.b**)

$$U_k = -\frac{TT' - \bar{S}S''}{T}; \qquad V_k = 1.$$

Then

$$\left.\begin{aligned} B &= \frac{S'}{\sin\alpha}\left[\left\{\sin k\alpha - \frac{T'}{S'}\sin(k+1)\alpha\right\} - \frac{\bar{S}S''}{S'T}\left\{\sin(k-1)\alpha - \frac{T'}{S'}\sin k\alpha\right\}\right], \\ B' &= -\frac{S'}{\sin\alpha}\left[\left\{\cos k\alpha - \frac{T'}{S'}\cos(k+1)\alpha\right\} - \frac{\bar{S}S''}{S'T}\left\{\cos(k-1)\alpha - \frac{T'}{S'}\cos k\alpha\right\}\right]. \end{aligned}\right\} \tag{23.2}$$

3) The case of the free edge.—From (**15.c**)

$$U_k = -\frac{W'}{T'}; \qquad V_k = 1.$$

Then

$$\left.\begin{aligned} B &= \frac{S'}{\sin\alpha}\left[\left\{\sin k\alpha - \frac{T'}{S'}\sin(k+1)\alpha\right\} - \frac{S'}{T'}\left\{\sin(k-1)\alpha - \frac{T'}{S'}\sin k\alpha\right\}\right], \\ B' &= -\frac{S'}{\sin\alpha}\left[\left\{\cos k\alpha - \frac{T'}{S'}\cos(k+1)\alpha\right\} - \frac{S'}{T'}\left\{\cos(k-1)\alpha - \frac{T'}{S'}\cos k\alpha\right\}\right]. \end{aligned}\right\} \tag{23.3}$$

(2) *The conditions at End side-1.*—Referring to the explanation in §7-f), these conditions can be represented in the same forms as in §7-e) except concerning the case of the clamped edge. Then substituting (**53**) in them, we obtain the following equations respectively for each case:

1) The case of the simply supported edge.—From (**15.d**), *viz.*, $V_1 = 0$,

$$B\left(\frac{T'}{S'}\cos\alpha - 1\right) + B'\left(\frac{T'}{S'}\sin\alpha\right) = 0. \tag{23.4}$$

2) The case of the clamped edge.—The condition for this case can be expressed by Eq. (**16′**) in §7, *i.e.*,

$$U_1 = U_2(F_1) - V_2\{(G_1) - \bar{\mu}'\varphi(F_1)\} = 0,$$

where

$$(F_1) = \frac{T}{S''}, \qquad (G_1) = \frac{TT' - \bar{S}S''}{S''}.$$

Substituting (**53**) in the above.

$$B\left\{\left(\frac{T'}{S'}\cos\alpha - 1\right) - \frac{\bar{S}S''}{S'T}\left(\frac{T'}{S'}\cos 2\alpha - \cos\alpha\right)\right\} + B'\left\{\frac{T'}{S'}\sin\alpha - \frac{\bar{S}S''}{S'T}\left(\frac{T'}{S'}\sin 2\alpha - \sin\alpha\right)\right\} = 0. \tag{23.5}$$

3) The case of the free edge.—From (**15**.g), *viz.*, $U_1=0$

$$B\cos\alpha+B'\sin\alpha=0\,. \tag{23.6}$$

iii) The conditional equations. — Various combinations of the foregoing conditions can represent each case tabulated in Table 4.

1) *The case where both end sides are simply supported.*—The conditional equation is given by combination of (23.1) and (23.4). By the elimination of B and B', we have

$$\sin k\alpha=0\,.$$

Then

$$\alpha=s\frac{\pi}{k}\qquad (s=0,1,2,\cdots\cdots\cdots,2k-1)\,.$$

Accordingly, referring to (**54**), we finally get

$$\cos s\frac{\pi}{k}=\frac{T'}{S'}-\frac{\bar{\mu}'\bar{\varphi}}{2S'} \tag{55}$$

$$(s=0,1,2,\cdots\cdots\cdots,2k-1).$$

Now, putting $\bar{\mu}'=0$, the above equation coincides with Eq. (**39**) in § 21. That is to say, this fact is a matter of course by the reason why $\bar{\mu}'=0$ means the vanishing of the beams.

2) *The case where one end side is clamped and the other simply supported.*—The conditional equations are given by the combination of (23.2) and (23.4) or of (23.1) and (23.5). Obviously both combinations give the same result. By eliminating B and B' as before

$$\left.\begin{aligned}\frac{\sin k\alpha}{\sin(k-1)\alpha}&=\frac{\bar{S}S''}{S'T}\,,\\ \text{and (54), i.e.,}\quad \cos\alpha&=\frac{T'}{S'}-\frac{\bar{\mu}'\bar{\varphi}}{2S'}\,.\end{aligned}\right\} \tag{56}$$

These are the conditional equations having a parameter α. Now, putting $\bar{\mu}'=0$, we see that the above coincide with (**40**).

3) *The case where both end sides are clamped.*—The combination of (23.2) and (23.5) is used. By the elimination of B and B'

$$\left.\begin{aligned}\frac{\sin(k-1)\alpha\pm\sin\alpha}{\sin(k-2)\alpha}&=\frac{\bar{S}S''}{S'T}\,,\\ \text{and (54), i.e.,}\quad \cos\alpha&=\frac{T'}{S'}-\frac{\bar{\mu}'\bar{\varphi}}{2S'}\,,\end{aligned}\right\} \tag{57}$$

in which α is a parameter. Now, by $\bar{\mu}'=0$, we see that the above coincide with (**43**).

4) *The case where one end side is free and the other simply supported.*—The combination of (23.3) and (23.4) or of (23.1) and (23.6) is used and each of the combinations should give the same result. Eliminating B and B' as before, the following equations having a parameter α are obtained:

and (**54**), *i.e.*,
$$\left.\begin{aligned} &\frac{\sin(k-1)\alpha}{\sin k\alpha}=\frac{T'}{S'}\,, \\ &\cos\alpha=\frac{T'}{S'}-\frac{\bar{\mu}'\bar{\varphi}}{2S'}\,. \end{aligned}\right\} \tag{58}$$

Putting now $\mu'=0$ in the above, we have

$$\cos k\alpha=0\,.$$

Then it results that (**41**) can be arrived at.

5) *The case where one end side is free and the other clamped.*—The combination of (23.2) and (23.6) or of (23.3) and (23.5) is used and each of them should give the same result. Eliminating B and B', the following equations having a parameter α are obtained:

and (**54**), *i.e.*,
$$\left.\begin{aligned} &\frac{\sin(k-1)\alpha-\dfrac{T'}{S'}\sin k\alpha}{\sin(k-2)\alpha-\dfrac{T'}{S'}\sin(k-1)\alpha}=\frac{\bar{S}S''}{S'T}\,, \\ &\cos\alpha=\frac{T'}{S'}-\frac{\bar{\mu}'\bar{\varphi}}{2S'}\,. \end{aligned}\right\} \tag{59}$$

Now, to put $\mu'=0$ in the above gives the formulas (**42**).

6) *The case where both end sides are free.*—The combination of (23.3) and (23.6) gives the conditional equations as follows:

and (**54**), *i.e.*,
$$\left.\begin{aligned} &\frac{\sin(k-2)\alpha}{\sin(k-1)\alpha\pm\sin\alpha}=\frac{T'}{S'}\,, \\ &\cos\alpha=\frac{T'}{S'}-\frac{\bar{\mu}'\bar{\varphi}}{2S'}\,, \end{aligned}\right\} \tag{60}$$

in which α is a parameter. Now putting $\mu'=0$ and multiplying both cases of the first equation mutually, we can readily obtain the following relation by taking account of the second equation:

$$\sin k\alpha=0\,.$$

Then we can arrive at the formula (**39**) [§ 21-**iii**)-4)].

§ 24. Plates Having Elastically Built Joints and Simply Supported along the Joints.

i) General solution.—Referring to the explanation in § 8-h), the conditional equations have been given by (**11**); therefore such equations for this case are expressed in the same form as (20.1) by replacing the coefficients (20.2) by following (**61**):

$$\left.\begin{aligned} &F=\frac{T}{S}, \quad G=\frac{W}{S}+\kappa\varphi\frac{T}{S}, \\ &F'=\frac{1}{S}, \quad G'=\frac{T}{S}+\kappa\varphi\frac{1}{S}, \\ \text{where} \qquad &\kappa=\varepsilon\frac{\pi}{a}. \end{aligned}\right\} \tag{61}$$

Comparing the above with (**52**) in the previous section § 23, we see that the above formulas are obtained provided that T', S', W', $-\bar{\mu}'\varphi$ in (**52**) are replaced by T, S, W, $\kappa\varphi$ respectively. And moreover, considering that the joining conditions in both cases are expressed in the same form, we can say that the conditional equations are obtained by appling the above mentioned substitution for (**53**) and (**54**). That is

$$\left.\begin{aligned} &U_r=B\cos r\alpha+B'\sin r\alpha, \\ &V_r=\frac{1}{S\left\{\sin^2\alpha+\left(\frac{\kappa\varphi}{2S}\right)^2\right\}}\left[\begin{aligned} &B\left\{\sin\alpha\sin r\alpha+\left(\frac{\kappa\varphi}{2S}\right)\cos r\alpha\right\} \\ &+B'\left\{\sin\alpha\cos r\alpha-\left(\frac{\kappa\varphi}{2S}\right)\sin r\alpha\right\}\end{aligned}\right], \end{aligned}\right\} \tag{62}$$

where

$$\cos\alpha=\frac{T}{S}+\frac{\kappa\varphi}{2S}. \tag{63}$$

ii) The end side conditions.—Referring to § 8-h), the expressions in § 6-b) are considered to be applicable to this case except to the case where End side-1 is free.

(1) *The conditions at End side-k.*

1) The case of the simply supported edge.—The end side conditions are expressed as $U_k=-T$, $V_k=1$ by (**11**.a). These are the expressions to be obtained by the foregoing replacements from those in the paragraph [§ 23-**ii**)-(1)-1)].[65] Thus, B and B', in this case, can be written as follows by applying such replacements to (23.1):

65) This notation means the paragraph "§ 23 Plates Having Hinged Joints and Elastically Supported along the Joint in 1-st Way, —ii) The end side conditions,—(1) The conditions at End side-k,—1) The case of the simply supported edge." Henceforth such notations have such a meaning as above.

$$\left.\begin{aligned} B &= \frac{S}{\sin\alpha}\left\{\sin k\alpha - \frac{T}{S}\sin(k+1)\alpha\right\}, \\ B' &= -\frac{S}{\sin\alpha}\left\{\cos k\alpha - \frac{T}{S}\cos(k+1)\alpha\right\}. \end{aligned}\right\} \tag{24.1}$$

2) The case of the clamped edge.—The end side conditions are expressed as $U_k = -W/T$, $V_k = 1$ by (**11**.**b**). These are the expressions to be obtained by the foregoing replacements from those in the paragraph [§ 23-**ii**)-(1)-3)]. Then, B and B', in this case, can be written as follows by applying such replacements to (23.3):

$$\left.\begin{aligned} B &= \frac{S}{\sin\alpha}\left[\left\{\sin k\alpha - \frac{T}{S}\sin(k+1)\alpha\right\} - \frac{S}{T}\left\{\sin(k-1)\alpha - \frac{T}{S}\sin k\alpha\right\}\right], \\ B' &= -\frac{S}{\sin\alpha}\left[\left\{\cos k\alpha - \frac{T}{S}\cos(k+1)\alpha\right\} - \frac{T}{S}\left\{\cos(k-1)\alpha - \frac{T}{S}\cos k\alpha\right\}\right]. \end{aligned}\right\} \tag{24.2}$$

3) The case of the free edge.—The end side conditions are expressed as $U_k = -(TT' - \bar{S}S'')/T'$, $V_k = 1$ by (**11**.**c**). These are the expressions to be obtained by the foregoing replacements from those in the paragraph [§ 23-**ii**)-(1)-2)]. Then B and B', in this case, are written as follows by applying such replacements to (23.2):[66]

$$\left.\begin{aligned} B &= \frac{S}{\sin\alpha}\left[\left\{\sin k\alpha - \frac{T}{S}\sin(k+1)\alpha\right\} - \frac{\bar{S}S''}{ST'}\left\{\sin(k-1)\alpha - \frac{T}{S}\sin k\alpha\right\}\right], \\ B' &= -\frac{S}{\sin\alpha}\left[\left\{\cos k\alpha - \frac{T}{S}\cos(k+1)\alpha\right\} - \frac{\bar{S}S''}{ST'}\left\{\cos(k-1)\alpha - \frac{T}{S}\cos k\alpha\right\}\right]. \end{aligned}\right\} \tag{24.3}$$

(2) *The conditions at End side-1.*

1) The case of the simply supported edge.—From (**11**.**d**), $V_1 = 0$. This is of the same form as that in the paragraph [§ 23-**ii**)-(2)-1)]. Then, by the preceding replacements in (23.4), we have

$$B\left(\frac{T}{S}\cos\alpha - 1\right) + B'\left(\frac{T}{S}\sin\alpha\right) = 0. \tag{24.4}$$

2) The case of the clamped edge.—From (**11**.**e**), $U_1 = 0$. This is of the same form as that in [§23-**ii**)-(2)-3)]. Then the foregoing (23.6) can hold in this case. That is

$$B\cos\alpha + B'\sin\alpha = 0. \tag{24.5}$$

3) The case of the free edge.—The condition is written as follows by referring to (**18′**) [§ 8-**h**)]:

$$U_1 = U_2(F_1) - V_2\{(G_1) + \kappa\varphi(F_1)\} = 0,$$

66) In addition to the foregoing replacements, T' and S' must be replaced by T and S respectively.

where

$$(F_1)=\frac{T'}{\bar{S}}, \qquad (G_1)=\frac{TT'-\bar{S}S''}{\bar{S}}.$$

This is the expression which is obtained from that in [§ 23-**ii**)-(2)-2)] by the foregoing replacements.[67] Then, applying such replacements to (23.5), the condition is written as follows:

$$B\left\{\left(\frac{T}{S}\cos\alpha-1\right)-\frac{\bar{S}S''}{ST'}\left(\frac{T}{S}\cos 2\alpha-\cos\alpha\right)\right\}+B'\left\{\frac{T}{S}\sin\alpha-\frac{\bar{S}S''}{ST'}\left(\frac{T}{S}\sin 2\alpha-\sin\alpha\right)\right\}=0. \quad (24.6)$$

iii) The conditional equations. 1) *The case of the simply supported edge.*—The combination of (24.1) and (24.4) gives the conditional equation for this case. Now, the expressions of (24.1) and (24.4) can be obtained by the foregoing replacements from (23.1) and (23.4). Then, the combination of them, *i. e.*, Eq. (**55**) must be transformed into the conditional equation for the present case by such replacements. That is

$$\cos s\frac{\pi}{k}=\frac{T}{S}+\frac{\kappa\varphi}{2S} \qquad (\mathbf{64})$$
$$(s=0, 1, 2, \cdots\cdots\cdots, 2k-1).$$

Now, putting $\kappa=0$, the above expression coincides with the formula (**29**). This result can readily be understood by observing that $\kappa=0$ means the rigidly connected joint.[68]

2) *The case where one end side is clamped and the other simply supported.*—The combination of (24.2) and (24.4) or of (24.1) and (24.5) will give the conditional equations. Now, their expressions are nothing but those obtained by the replacements as described before from (23.3), (23.4), (23.1) and (23.6). Then the combination of them, *i. e.*, Eqs. (**58**) is transformed into the formulas for this case by such replacements. That is

$$\left.\begin{aligned} \frac{\sin(k-1)\alpha}{\sin k\alpha}&=\frac{T}{S}, \\ \cos\alpha&=\frac{T}{S}+\frac{\kappa\varphi}{2S}. \end{aligned}\right\} \qquad (\mathbf{65})$$

Putting now $\kappa=0$ in the above, we have

$$\cos k\alpha=0.$$

Accordingly, the formula (**31**) in § 20 is arrived at

67) In addition to such replacements, the interchange between S'' and $\bar{S}$ must be taken.
68) See § 18.

3) *The case where both end sides are clamped.*—The combination of (24.2) and (24.5) is used. Now, both of them are obtained by such replacements as before from (23.3) and (23.6). Then, applying such replacements to the formulas (**60**) derived from (23.3) and (23.6), we get

$$\left.\begin{aligned} \frac{\sin(k-2)\alpha}{\sin(k-1)\pm\sin\alpha}&=\frac{T}{S},\\ \cos\alpha&=\frac{T}{S}+\frac{\kappa\varphi}{2S}, \end{aligned}\right\} \tag{66}$$

Now, putting $\kappa=0$, the above equations give the following relation as in [§ 23-**iii**) -6)]:

$$\sin k\alpha=0.$$

Then, we see that the formula (**29**) in [§ 20-**iii**)-3)] can be arrived at,

4) *The case where one end side is free and the other simply supported.*—The combination of (24.3) and (24.4) or of (24.1) and (24.6) are used. Now, their expressions are obtained by the foregoing replacements form (23.2), (23.4), (23.1) and (23.5). Then the combination of them, *i. e.*, Eqs. (**56**) must be transformed into the conditional equations for the present case by such replacements. That is

$$\left.\begin{aligned} \frac{\sin k\alpha}{\sin(k-1)\alpha}&=\frac{\bar{S}S''}{ST'},\\ \cos\alpha&=\frac{T}{S}+\frac{\kappa\varphi}{2S}. \end{aligned}\right\} \tag{67}$$

Putting now $\kappa=0$, the formulas (**33**) in § 20 is arrived at.

5) *The case where one end side is free and the other clamped.*—The combination of (24.3) and (24.5) or of (24.2) and (24.6) are used. Now, their expressions are obtained by the foregoing replacements from (23.2), (23.6), (23.3) and (23.5). Then, the combination of them, *i. e.* Eqs. (**59**) must be transformed into the conditional equations for the present case by such replacements. That is

$$\left.\begin{aligned} \frac{\sin(k-1)\alpha-\dfrac{T}{S}\sin k\alpha}{\sin(k-2)\alpha-\dfrac{T}{S}\sin(k-1)\alpha}&=\frac{\bar{S}S''}{ST'},\\ \cos\alpha&=\frac{T}{S}+\frac{\kappa\varphi}{2S}. \end{aligned}\right\} \tag{68}$$

Putting now $\kappa=0$, the formulas (**35**) in § 20 can readily be arrived at.

6) *The case where both end sides are free.*—The combination of (24.3) and (24.6) is used. Since, both expressions are obtained by the replacements from (23.2)

and (23.5), then such replacements in Eqs. (57) must give the conditional equations for this case as follows:

$$\left.\begin{aligned} \frac{\sin(k-1)\pm\sin\alpha}{\sin(k-2)\alpha} &= \frac{SS''}{ST'}, \\ \cos\alpha &= \frac{T}{S}+\frac{\kappa\varphi}{2S}. \end{aligned}\right\} \qquad (69)$$

Put $\kappa=0$ in the above, then the formulas (36) in § 20 must be arrived at.

§ 25. Plates Having Elastically Built Joints and Elastically Supported along the Joints in 2-nd Way.

i) **General solution.**—Referring to the explanation in § 8–j), (11) can be used in this case. Therefore, (20.1) becomes to be applicable to the present case. But the coefficients in (20.1) must be represented by the following expressions obtained from (21) instead of (20.2):

$$\left.\begin{aligned} F &= \bar{\mu}''\frac{T}{S}-\frac{\bar{\mu}}{\varphi}\frac{W}{S}, \qquad G=\bar{\mu}''\frac{W}{S}+\kappa(1+\bar{\mu}'')\varphi\frac{T}{S}, \\ F' &= \bar{\mu}''\frac{1}{S}-\frac{\bar{\mu}}{\varphi}\frac{T}{S}, \qquad G'=\bar{\mu}''\frac{T}{S}+\kappa(1+\bar{\mu}'')\varphi\frac{1}{S}, \end{aligned}\right\} \qquad (70)$$

where

$$\bar{\mu}''=1-\kappa\bar{\mu}.$$

Substituting (20.3) in the expressions of the joining condition as before, we obtain the following characteristic equation:

$$\omega^2-2\omega\left[\mu''\frac{T}{S}-\frac{1}{2}\left\{\frac{\bar{\mu}}{\varphi}\frac{W}{S}-\kappa(1+\bar{\mu}'')\frac{\varphi}{S}\right\}\right]+1=0.$$

From this, the two roots are obtained. Representing these roots in the forms of trigonometric functions as in the previous sections, we can obtain the following expressions with the unknown constants B and B':

$$\left.\begin{aligned} U_r &= B\cos r\alpha+B'\sin r\alpha, \\ V_r &= -\frac{1}{G}\left\{B(\sin\alpha\sin r\alpha+R\cos r\alpha)-B'(\sin\alpha\cos r\alpha-R\sin r\alpha)\right\}, \end{aligned}\right\} \qquad (71)$$

where, for simplification, $\frac{1}{2}\left\{\frac{\bar{\mu}W}{\varphi S}+\kappa(1+\bar{\mu}'')\frac{\varphi}{S}\right\}$ is denoted by R.

In the above, α is the parameter defined as follows:

$$\cos\alpha=\bar{\mu}''\frac{T}{S}-\frac{1}{2}\left\{\frac{\bar{\mu}W}{\varphi S}-\kappa(1+\bar{\mu}'')\frac{\varphi}{S}\right\}. \tag{72}$$

ii) The end side conditions. (1) *The conditions at End side-k.*—As explained in § 8-j), the expressions in § 6-**b**) can be applied also. Since the operation processes are the same as before, then only the obtained results will be written below.

1) The case of the simply supported edge.

$$\left.\begin{aligned} B &= (TR-G)\frac{\sin k\alpha}{\sin\alpha}-T\cos k\alpha, \\ B' &= -(TR-G)\frac{\cos k\alpha}{\sin\alpha}-T\sin k\alpha. \end{aligned}\right\} \tag{25.1}$$

2) The case of the clamped edge.

$$\left.\begin{aligned} B &= \left(\frac{W}{T}R-G\right)\frac{\sin k\alpha}{\sin\alpha}-\frac{W}{T}\cos k\alpha, \\ B' &= -\left(\frac{W}{T}R-G\right)\frac{\cos k\alpha}{\sin\alpha}-\frac{W}{T}\sin k\alpha. \end{aligned}\right\} \tag{25.2}$$

3) The case of the free edge.

$$\left.\begin{aligned} B &= \left(\frac{TT'-\bar{S}S''}{T'}R-G\right)\frac{\sin k\alpha}{\sin\alpha}-\frac{TT'-\bar{S}S''}{T'}\cos k\alpha, \\ B' &= -\left(\frac{TT'-\bar{S}S''}{T'}R-G\right)\frac{\cos k\alpha}{\sin\alpha}-\frac{TT'-\bar{S}S''}{T'}\sin k\alpha. \end{aligned}\right\} \tag{25.3}$$

(2) *The conditions at End side-1.*

1) The case of the simply supported edge.—From (**11**.d), $V_1=0$. Then, substituting the general solution (**71**) in this, we have

$$B(\sin^2\alpha+R\cos\alpha)-B'(\cos\alpha\sin\alpha-R\sin\alpha)=0\,. \tag{25.4}$$

2) The case of the clamped edge.—From (**11**.e), $U_1=0$. Substitute (**71**) in this, then we have

$$B\cos\alpha+B'\sin\alpha=0\,. \tag{25.5}$$

3) The case of the free edge.—In this case, the following expression is obtained by (**21'**) in § 8-j):

$$U_1=U_2\left\{\bar{\mu}''(F_1)-\frac{\bar{\mu}}{\varphi}(G_1)\right\}-V_2\left\{\bar{\mu}''(G_1)+\kappa(1+\bar{\mu}'')\varphi(F_1)\right\}=0\,,$$

where

$$(F_1)=\frac{T'}{\bar{S}},\qquad (G_1)=\frac{TT'-\bar{S}S''}{\bar{S}}\,.$$

Substitute (71) in the above, then

$$B\left[\left\{\frac{\bar{\mu}''}{S}-\frac{\bar{\mu}}{\varphi}\left(\frac{T}{S}-\frac{\bar{S}S''}{ST'}\right)\right\}G\cos 2\alpha+\left\{\bar{\mu}''\left(\frac{T}{S}-\frac{\bar{S}S''}{ST'}\right)+\kappa(1+\bar{\mu}'')\frac{\varphi}{S}\right\}(\sin\alpha\sin 2\alpha+R\cos 2\alpha)\right]$$

$$+B'\left[\left\{\frac{\bar{\mu}''}{S}-\frac{\bar{\mu}}{\varphi}\left(\frac{T}{S}-\frac{\bar{S}S''}{ST'}\right)\right\}G\sin 2\alpha-\left\{\bar{\mu}''\left(\frac{T}{S}-\frac{\bar{S}S''}{ST'}\right)+\kappa(1+\bar{\mu}'')\frac{\varphi}{S}\right\}\right.$$

$$\left.\times(\sin\alpha\cos 2\alpha-R\sin 2\alpha)\right]=0. \qquad (25.6)$$

iii) The conditional equations.—The various combinations of the preceding conditions can give the conditional equation, for each of the end side conditions, which has the parameter α defined by (72).

1) *The case where both end sides are simply supported.*—Eliminating B and B' from (25.1) and (25.4),

$$(TR-G)R\frac{\sin(k-1)\alpha}{\sin\alpha}-G\cos(k-1)\alpha+T\sin(k-1)\alpha\sin\alpha=0. \qquad (73)$$

2) *The case where one end side is clamped and the other simply supported.*—Eliminating B and B' from (25.2) and (25.4) or from (25.1) and (25.5),

$$\bar{\mu}''\frac{\sin(k-1)\alpha}{\sin k\alpha}-\frac{T}{S}=0. \qquad (74)$$

3) *The case where both end sides are clamped.*—Eliminating B and B' from (25.2) and (25.5),

$$\kappa(1+\bar{\mu}'')\frac{\varphi}{S}\frac{\sin(k-1)\alpha}{\sin k\alpha}-1+\left(\frac{T}{S}\right)^2=0. \qquad (75)$$

4) *The case where one end side is free and the other simply supported.*—Eliminating B and B' from (25.3) and (25.4) or from (25.1) and (25.6),

$$\left(\frac{T}{S}-\frac{\bar{S}S''}{TS'}\right)\left(R^2+\sin^2\alpha\right)\frac{\sin(k-1)\alpha}{\sin\alpha}-\frac{G}{S}\left\{\cos(k-1)\alpha+R\frac{\sin(k-1)\alpha}{\sin\alpha}\right\}=0. \qquad (76)$$

5) *The case where one end side is free and the other clamped.*—Eliminating B and B' from (25.3) and (25.5) or from (25.2) and, (25.6),

$$\cot(k-1)\alpha\sin\alpha-R+\frac{G}{S\left(\dfrac{T}{S}-\dfrac{\bar{S}S''}{ST'}\right)}=0. \qquad (77)$$

6) *The case where both end sides are free.*—Eliminating B and B' from (25.3) and (25.6),

$$\left\{\frac{\bar{\mu}''}{S}-\frac{\bar{\mu}}{\varphi}\left(\frac{T}{S}-\frac{\bar{S}S''}{ST'}\right)\right\}G\left[\left(\frac{T}{S}-\frac{\bar{S}S''}{TS'}\right)\left\{R\frac{\sin(k-2)\alpha}{\sin\alpha}-\cos(k-2)\alpha\right\}-\frac{G}{S}\frac{\sin(k-2)\alpha}{\sin\alpha}\right]$$

$$+\left\{\bar{\mu}''\left(\frac{T}{S}-\frac{\bar{S}S''}{ST'}\right)+\kappa(1+\bar{\mu}'')\frac{\varphi}{S}\right\}\left[\left(\frac{T}{S}-\frac{\bar{S}S''}{ST'}\right)\left\{R^2\frac{\sin(k-2)\alpha}{\sin\alpha}+\sin\alpha\sin(k-2)\alpha\right\}\right.$$
$$\left.-\frac{G}{S}\left\{R\frac{\sin(k-2)\alpha}{\sin\alpha}+\cos(k-2)\alpha\right\}\right]=0. \quad (78)$$

To each of the above equations, the following procedures are applicable. First, calculate the value of $\cos\alpha$ by (72), then $\sin k\alpha$ and $\cos k\alpha$ for any k can be computed by using the following recurrence formulas:

$$\sin n\alpha = 2\cos\alpha\sin(n-1)\alpha - \sin(n-2)\alpha;$$
$$\cos n\alpha = 2\cos\alpha\cos(n-1)\alpha - \cos(n-2)\alpha.$$

Accordingly, the left members of the conditional equations are computed and then their roots can be determined by the graphical method as before.

Now, we can suppose that the continuous plates in this case should become those in § 20 provided that $\kappa=0$ and $C=0$. Indeed, in this time, since we have

$$\bar{\mu}=0,\quad \bar{\mu}''=1,\quad \cos\alpha=\frac{T}{S},\quad G=\frac{W}{S},\quad R=0,$$

then the preceding formulas can be proved to coincide with those in § 20.

Next, we can suppose that the present plates should become those in § 22 provided that $\kappa=0$. Since we have, in this time,

$$\bar{\mu}''=1,\qquad \cos\alpha=\frac{T}{S}-\frac{\bar{\mu}W}{2\varphi S},\quad G=\frac{W}{S},\quad R=\frac{\bar{\mu}W}{2\varphi S},$$

the preceding formulas can be proved to coincide with those in § 22.

In the third place, we can suppose that the present plates should become those in § 24 provided that $\kappa=\kappa'/2$, $C=0$. In this time, since we get

$$\bar{\mu}''=1,\quad \kappa(1+\bar{\mu}'')=\kappa',\quad \cos\alpha=\frac{T}{S}+\kappa\frac{\varphi}{2S},$$
$$G=\frac{W}{S}+\kappa\varphi\frac{T}{S},\qquad R=\kappa\frac{\varphi}{2S},$$

the preceding formulas can readily be proved to coincide with those in § 24.

APPENDIX I.

Some Extensions of Application of the Formulae.

Considering that every one of the composite plates discussed in this paper always is simply supported along the two opposite sides perpendicular to the joining lines, their solutions can be extended to be applicable to such cases as explained below.

Let the composite plate with the width b, which has any joint and end side conditions, buckle in such a mode as $m-1$ nodal lines occur there in x-direction. And denote by P_s and Q_s the critical values corresponding to such mode of buckling. Now, placing these l configurations in a row in y-direction (alternately in the inverse situation whan m is odd and, on the other hand, always in the same situation when m is even), the deflection surface of s-th plate in y-direction can be expressed as follows by referring to (3) in § 4:[69]

$$\underset{s}{w_r} = (-1)^{ms} X_r \sin m\pi \underset{s}{\eta_r}, \tag{79}$$

where X_r is a function of only ξ_r and now is free of s.

Observing that each of thus arranged deflection surfaces can smoothly connect with the neighbouring ones, it can be proved that the whole configuration constructed in such a way as above mentioned coincides with the deflection surfaces of the composite plates with the width lb which are buckled in the following states:

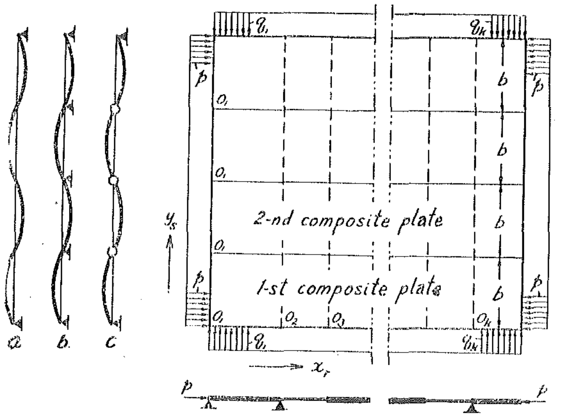

Fig. 47.

(1) The buckling having lm half-waves in y-direction. [Fig. 47- a]

(2) The case where the plate simply supported along the $l+1$ rigidly supporting lines arranged with equal intervals in y-direction buckles, having m half-waves in each interval. [Fig. 47-b]

(3) The case where the plate furnished with the $l-1$ smoothly hinged joints arranged with equal intervals in y-direction buckles, having m half-waves in each interval. [Fig. 46-c]

69) The situations of the coordinate axes are indicated in Fig. 47, where the deflection surface for $m=1$ and $l=4$ is drawn.

Then the critical value P_s or Q_s of the initally considered plate becomes to be applicable to the above various cases. This fact is readily proved as follows. Every one of the above three cases is analogous to each other in the mechanical point of view. That is to say, considering about the joining conditions between the $(s-1)$-th composite plate and the s-th, these can be represented in the following common expressions—omitting the suffix r for simplicity in below—:

At such a joint

1) The deflections are zero; *i. e.*,

$$\left| w_{s-1} \right|_{\eta_{s-1}=1} = 0; \qquad \left| w_s \right|_{\eta_s=0} = 0. \tag{a}$$

2) The slope displacements continue smoothly to each other; *i. e.*, at $\xi_{r_{s-1}} = \xi_{r_s}$

$$\left| \frac{\partial w_{s-1}}{\partial y_{s-1}} \right|_{\eta_{s-1}=1} = \left| \frac{\partial w_s}{\partial y_s} \right|_{\eta_s=0}. \tag{b}$$

3) The bending moments are zero; *i. e.*,

$$\left| \frac{\partial^2 w_{s-1}}{\partial y_{s-1}^2} + \nu \frac{\partial^2 w_{s-1}}{\partial x_{s-1}^2} \right|_{\eta_{s-1}=1} = 0; \qquad \left| \frac{\partial^2 w_s}{\partial y_s^2} + \nu \frac{\partial^2 w_s}{\partial x_s^2} \right|_{\eta_s=0} = 0, \tag{c}$$

however, since the conditions (a) always hold along the joint, we obtain

$$\left| \frac{\partial^2 w_{s-1}}{\partial x_{s-1}^2} \right|_{\eta_{s-1}=1} = 0; \qquad \left| \frac{\partial^2 w_s}{\partial x_s^2} \right|_{\eta_s=0} = 0.$$

Then, the present expressions are transformed as follows:

$$\left| \frac{\partial^2 w_{s-1}}{\partial y_{s-1}^2} \right|_{\eta_{s-1}=1} = 0; \qquad \left| \frac{\partial^2 w_s}{\partial y_s^2} \right|_{\eta_s=0} = 0. \tag{c'}$$

4) The lateral forces transmitted from both the neighbouring plates must be in equilibrium state; *i. e.*, at $\xi_{s-1} = \xi_s$

$$\left| \frac{\partial^3 w_{s-1}}{\partial y_{s-1}^3} + (2-\nu) \frac{\partial^3 w_{s-1}}{\partial y_{s-1} \partial x_{s-1}^2} \right|_{\eta_{s-1}=1} = \left| \frac{\partial^3 w_s}{\partial y_s^3} + (2-\nu) \frac{\partial^3 w_s}{\partial y_s \partial x_s^2} \right|_{\eta_s=0}. \tag{d}$$

However, provided that the condition (b) always hold along the joint, it follows that

$$\left| \frac{\partial^2}{\partial x_{s-1}^2} \left(\frac{\partial w_{s-1}}{\partial y_{s-1}} \right) \right|_{\eta_{s-1}=1} = \left| \frac{\partial^2}{\partial x_s^2} \left(\frac{\partial w_s}{\partial y_s} \right) \right|_{\eta_s=0}.$$

Then, the present relation can be written as follows:

$$\left|\frac{\partial^3 w_{s-1}}{\partial y_{s-1}^3}\right|_{\eta_{s-1}=1} = \left|\frac{\partial^3 w_s}{\partial y_s^3}\right|_{\eta_s=0}. \tag{d$'$}$$

Now, we see that the preceding cases (1) and (2) are represented by the above expressions (a) and (b) and the case (3) repesented by the above (c′) and (d′)

Next, the deflection surface of the $(s-1)$-th plate can be written as follows by referring to (**79**):

$$w_{s-1} = (-1)^{m(s-1)}\ X\sin m\pi\eta_{s-1} \tag{79.a}$$

In the similar way, that of the s-th as follows by (**79**):

$$w_s = (-1)^{ms}\ X\sin m\pi\eta_s. \tag{79.b}$$

By these, we can readily see that the preceding conditions (a) and (c′) hold from the beginning. In the second place, by (**79**.a), we get

$$\left|\frac{\partial w_{s-1}}{\partial y_{s-1}}\right|_{\eta_{s-1}=1} = (-1)^{ms}\frac{m\pi}{b}X; \quad \left|\frac{\partial^3 w_{s-1}}{\partial y_{s-1}^3}\right|_{\eta_{s-1}=1} = (-1)^{ms+1}\left(\frac{m\pi}{b}\right)^3 X,$$

and by (**79**.b)

$$\left|\frac{\partial w_s}{\partial y_s}\right|_{\eta_s=0} = (-1)^{ms}\frac{m\pi}{b}X; \quad \left|\frac{\partial^3 w_s}{\partial y_s^3}\right|_{\eta_s=0} = (-1)^{ms+1}\left(\frac{m\pi}{b}\right)^3 X.$$

Accordingly, it can be seen that at $\xi_{\underset{s-1}{r}} = \xi_{\underset{s}{r}}$ the preceding (b) and (d′) hold also.

Hence, it has been proved that the preceding conditions (a), (b), (c′) and (d′) are satisfied from the beginning. Therefore, we can understand that the critical value P_s or Q_s obtained by handling the s-th plate as a composite plate discussed previously in this paper must be applicable to such cases (1), (2) and (3) as before stated.

Finally, it must be remarked that the above considerations are sufficiently fit to the case (2), but the case (3) is not practical by the reason why the deflection at the joints generally occurs, so that, while the condition (c) can hold, the remaining conditions can not hold. Yet, the case (1) is nothing but the higher mode of buckling obtained by substituting $b=lb$, $m=lm$ in the formulas as discussed before in this paper. And, in the application to the case-(2) [see Fig. 47-b], practically we need not distinguish whether an end side parallel to x-axis is simply supported or clamped provided that the number of composite plates in y-direction is sufficiently large, because the effect of such an end side will become negligible.

APPENDIX II.

Considerations about a Hinged Joint having Deflection under a Compressive Force p.

Let us consider about the equilibrium condition of the lateral pressure along such a hinged joint without supportings as accompanied by some slope difference. First, assume the cross section perpendicular to the joint line between the $(r-1)$-th elementary plate and the r-th as shown in the upper part of Fig. 48-a. Next, imagining to separate the neighbouring plates from each other at the joint and applying the two pairs of equivalent opposite forces without disturbance of the equilibrium state as shown in the lower part of Fig. 48-a, the magnitude of the horizontal forces must be equal to the end thrust p and the vartical ones or the lateral pressures denoted by s are the statically indeterminate ones.

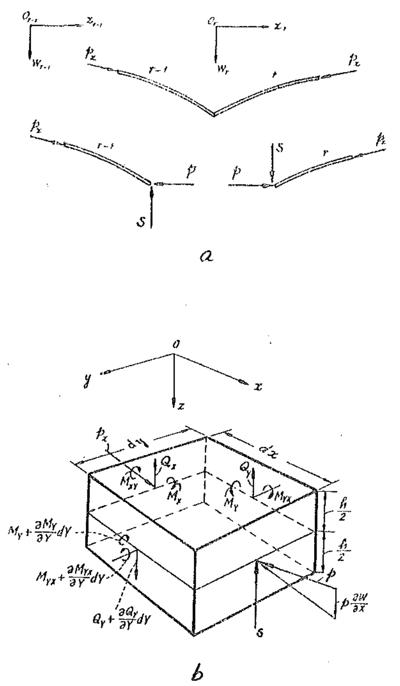

Fig. 48.

Next place, let us consider the equilibrium of an element cut out from the $(r-1)$-th plate along the joining edge as shown in Fig 48-b. The stresses acting on the sides of the element can be denoted in the usual manner[70] as in Fig. 48-b where the direct stress q_y acting in y-direction is omitted because of no effects. Resolving the horizontal force p in the component acting in the plane of the deformed plate and in the vertical component $p(\partial w/\partial x)$, the former component can be omitted from the present considerations.

Now considering the equilibrium of the stresses in z-direction, we have

$$Q_x dy - \frac{\partial Q_y}{\partial y} dy\, dx - p\frac{\partial w}{\partial x} dy + s dy = 0.$$

Then

$$Q_x - \frac{\partial Q_y}{\partial y} dx - p\frac{\partial w}{\partial x} + s = 0. \tag{a}$$

70) See, for instance, S. Timoshenko's books.

By the equilibrium of the moments around x-direction, we obtain

$$M_{xy}dy + \frac{\partial M_y}{\partial y} dy\, dx - Q_y dx\, dy - \frac{\partial Q_y}{\partial y} \frac{dy^2}{2} dx = 0 .$$

From this

$$M_{xy} + \frac{\partial M_y}{\partial y} dx - Q_y\, dx - \frac{\partial Q_y}{\partial y} dx \frac{dy}{2} = 0 .$$

By partial differentiation with respect to y

$$\frac{\partial M_{xy}}{\partial y} + \frac{\partial^2 M_y}{\partial y^2} dx - \frac{\partial Q_y}{\partial y} dx - \frac{\partial^2 Q_y}{\partial y^2} \frac{dy}{2} dx = 0 . \tag{b}$$

Hence, eliminating the term $\frac{\partial Q_y}{\partial y} dx$ from (a) and (b), we get

$$Q_x - \frac{\partial M_{xy}}{\partial y} - \frac{\partial^2 M_y}{\partial y^2} dx + \frac{\partial^2 Q_y}{\partial y^2} \frac{dy}{2} dx - p \frac{\partial w}{\partial x} + s = 0 .$$

Again, omitting the smaller terms of higher order,

$$s = - Q_x + \frac{\partial M_{xy}}{\partial y} + p \frac{\partial w}{\partial x} . \tag{c}$$

Provided that Q_x, M_{xy} in the right member of the above are rewritten in the usual forms as the function of deflection and moreover thus obtained expressions are furnished with the suffix $r-1$, we finally obtain the following expression:

$$\begin{aligned} s &= D_{r-1} \frac{\partial}{\partial x_{r-1}} \left(\frac{\partial^2 w_{r-1}}{\partial x_{r-1}^2} + \frac{\partial^2 w_{r-1}}{\partial y_{r-1}^2} \right) + D_{r-1}(1-\nu_{r-1}) \frac{\partial^3 w_{r-1}}{\partial x_{r-1} \partial y_{r-1}^2} + p \frac{\partial w_{r-1}}{\partial x_{r-1}} \\ &= D_{r-1} \left[\frac{\partial^3 w_{r-1}}{\partial x_{r-1}^3} + (2-\nu_{r-1}) \frac{\partial^3 w_{r-1}}{\partial x_{r-1} \partial y_{r-1}^2} \right] + p \frac{\partial w_{r-1}}{\partial x_{r-1}} . \end{aligned} \tag{d}$$

In the next place, considering about the joining edge of the r-th plate in the opposite side in the same manner as before, the directions of p and s should become to be oppsite from those of the $(r-1)$-th plate. Finally, by changing the signs of s, Q_x, M_{xy}, $p(\partial w/\partial x)$ in (c), the expression for this case must be obatined. But, such an expression as (c) remains identical as a whole notwithstanding this fact. Then, the expression concerning the r-th plate is written as follows:

$$s = D_r \left[\frac{\partial^3 w_r}{\partial x_r^3} + (2-\nu_r) \frac{\partial^3 w_r}{\partial x_r \partial y_r^2} \right] + p \frac{\partial w_r}{\partial x_r} . \tag{e}$$

Thus, observing (d) and (e), we obtain

$$D_{r-1}\left[\frac{\partial^3 w_{r-1}}{\partial x_{r-1}^3} + (2-\nu_{r-1}) \frac{\partial^3 w_{r-1}}{\partial x_{r-1} \partial y_{r-1}^2} \right] + p \frac{\partial w_{r-1}}{\partial x_{r-1}} = D_r \left[\frac{\partial^3 w_r}{\partial x_r^3} + (2-\nu_r) \frac{\partial^3 w_r}{\partial x_r \partial y_r^2} \right] + p \frac{\partial w_r}{\partial x_r} .$$

This is the desired relation concerning the lateral pressure along the joining edge.

For the free end edge, the condition can be obtained by putting $s=0$ in (c) because this case is equivalent to such a case as the plate in one side of the joint vanishes. That is,

$$\left.\begin{aligned}&\left[\frac{\partial^3 w_1}{\partial x_1^3}+(2-\nu_1)\frac{\partial^3 w_1}{\partial x_1 \partial y_1^2}\right]+\frac{p}{D_1}\frac{\partial w_1}{\partial x_1}=0 \qquad \text{at } x_1=0 \quad \text{or } \xi_1=0\,; \\ &\left[\frac{\partial^3 w_k}{\partial x_k^3}+(2-\nu_k)\frac{\partial^3 w_k}{\partial x_k \partial y_k^2}\right]+\frac{p}{D_k}\frac{\partial w_k}{\partial x_k}=0 \qquad \text{at } x_k=a_k \quad \text{or } \xi_k=1\,.\end{aligned}\right\}$$