On the Solution of Differential Equations Concerned with Problems of Applied Mechanics by Fourier Sine Transformation Method

By

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Synopsis

In the Fourier sine transformation method, representing the deflection curve of a beam or column by a certain series such as one each term of which satisfies the terminal conditions, this series is substituted in the differential equation concerned with the deflection curve and then multiplying this equation all through by sin $\frac{n\pi x}{l}$ and integrating from one end to the other of a beam or column the coefficients contained in the series are to be determined. This process is the same as that used in expressing any function in terms of Fourier trigonometrical series. By applying this method, however complicated a differential equation may be, it is easily solved. In this paper the author showed the applications of Fourier sine transformation method in studies of the various kinds of problems concerned with deflection, critical load and frequency of oscillation of bars and moreover the relation between this method and others was described.

Contents

I. ON THE RELATION AMONG ENERGY, GALERKIN'S AND FOURIER SINE TRANSFORMATION METHODS.

Taking, as an example, the case of a beam with variable cross section under the simultaneous action of axial tension and lateral loads, the differential equation of the deflection curve is

$$\frac{d^2}{dx^2}\left\{EI(x)\frac{d^2y}{dx^2}\right\}-N\frac{d^2y}{dx^2}-p(x)=0 \qquad (1)$$

in which E is the modulus of elasticity, $I(x)$ the moment of inertia of the cross section, N the axial tension and $p(x)$ the intensity of distributed lateral load.

One of the special methods of solution of this equation is as follows: First, express the deflection curve in the form of the series

$$y=\Sigma a_n\ \varphi_n(x) \qquad (2)$$

each term of which is to satisfy the terminal conditions of a beam. In such a case, whatever the lateral loads are, a single mathematical expression holds for the entire length of a beam and it is not necessary to discuss each one separately.

Substituting the value of y given by (2) in the left hand side of Eq. (1) and denoting this result for simplificatin, by $\varepsilon(x)$ one gets

$$\frac{d^2}{dx^2}\left\{EI(x)\frac{d^2\,\Sigma a_n\,\varphi_n(x)}{dx^2}\right\}-N\frac{d^2\,\Sigma\,a_n\,\varphi_n(x)}{dx^2}-p(x)=\varepsilon(x) \qquad (3)$$

Then, determining the coefficients $a_1,\ a_2,\ a_3,\ldots.$ in the series (2) by

$$\int_0^l \varepsilon(x)\,\varphi_n(x)\,dx=0 \quad (n=1,\ 2,\ 3,\ \ldots.), \qquad (4)$$

deflection curve can be determined. This process is called Galerkin's method. In this case, an approximate result can be obtained by taking for y an expression with several coefficients and then ajusting them so as to satisfy Eq. (4). By taking only the first term or two terms in the series of y, a satisfactory approximate result is obtained for any practical application.

Coefficients $a_1,\ a_2,\ldots\ldots$ are also determined by energy method as follows: In Eq. (1) $p(x)$ is the lateral load per unit length of beam. Considering the lateral load only, Eq. (1) becomes

$$\frac{d^2}{dx^2}\left\{EI(x)\frac{d^2y}{dx^2}\right\}=p(x). \qquad (5)$$

Thus, it can be recognized that $\frac{d^2}{dx^2}\left\{EI(x)\frac{d^2y}{dx^2}\right\}$ is the elastic force of the beam per unit length to be in equilibrium with $p(x)$. When the beam is subjected to an axial force only Eq. (1) becomes

$$\frac{d^2}{dx^2}\left\{EI\,(x)\,\frac{d^2y}{dx^2}\right\} = N\frac{d^2y}{dx^2}. \tag{6}$$

By comparing Eq. (6) with (5) it is obvious that $N\dfrac{d^2y}{dx^2}$ is the lateral force caused by axial force.

Therefore, three forces $\dfrac{d^2}{dx^2}\left\{EI\,(x)\,\dfrac{d^2y}{dx^2}\right\}dx$, $p\,(x)\,dx$ and $N\dfrac{d^2y}{dx^2}\,dx$ act in the differential length dx of the beam and they are in equilibrium. If the beam is given a very small displacement δy from the position of equilibrium, the total virtual work is to be zero. This follows from the principle of virtual displacements and therefore one gets

$$\int_0^l\left[\frac{d^2}{dx^2}\left\{EI\,(x)\,\frac{d^2y}{dx^2}\right\}\delta y - N\frac{d^2y}{dx^2}\delta y - p(x)\delta y\right]dx = 0. \tag{7}$$

Small displacements of the beam from the position of equilibrium can be obtained by slight variation of the coefficients a_1, a_2, a_3, ... If any coefficient a_n is given an increase δa_n, the term $(a_n + \delta a_n)\,\varphi_n(x)$ is obtained in series (2), instead of the term $a_n\,\varphi_n(x)$, the other terms remaining unchanged.

Thus the increase δa_n in the coefficient a_n represents an additional small deflection of the beam given by the curve $\delta a_n\ \varphi_n(x)$ superposed upon the original deflection curve and Eq. (7) becomes

$$\int_0^l\left[\frac{d^2}{dx^2}\left\{EI\,(x)\,\frac{dy^2}{dx^2}\right\} - N\frac{d^2y}{dx^2} - p\,(x)\right]\varphi_n(x)\,dx = 0 \quad (n = 1,\ 2,\ 3......) \tag{8}$$

With the value of y given by (2) coefficients a_1, a_2, a_3, ... can be determined and it is observed that Eq. (8) obtained by energy method coincides with Galerkin's formula (4).

In other problems such as the critical load and frequency of oscillation of bars with variable cross section, the results obtained by the application of Galerkin's method appear on the way of the calculation process in the energy method, and after all it is seen that Galerkin's method is more advantageous for the solution of the differential equation.

In the Fourier series transformation method, it is the same with the above two methods to express the deflection curve in the form of Eq. (2). Integrating Eq. (1) over the length of the beam after substituting the value of y and multiplying this equation all through by any function $\phi_n(x)$, it follows that

$$\int_0^l\left[\frac{d^2}{dx^2}\left\{EI\,(x)\,\frac{d^2\,\Sigma\,a_n\,\varphi_n(x)}{dx^2}\right\} - N\frac{d^2\,\Sigma\,a_n\,\varphi_n(x)}{dx^2} - p\,(x)\right]\phi_n(x)\,dx = 0. \tag{9}$$
$$(n = 1,\ 2,\ 3......)$$

from which coefficients a_1, a_2, a_3,......can be determined. This process is the same as that used in expressing any function in terms of Fourier trigonometrical series.

Galerkin's method is nothing but a special case in the Fourier series transformation method. In studying deflection, critical load and frequency of oscillation of a prismatical bar it is advantageous sometimes to use $\phi_n(x)$ in the form of $\sin \frac{n\pi x}{l}$. In such a case, this method is called Fourier sine transformation method.

The series of y which is to be assumed at first in the calculation of differential equations will be given in the following various kinds of examples.

II. APPLICATION OF FOURIER SINE TRANSFORMATION METHOD TO THE STUDY OF BENDING OF THE VARIOUS KINDS OF BEAMS.

1. Simple Beam with Uniform Cross Section under the Simultaneous Action of Axial Tension and Lateral Load.

As an easy example to illustrate the method of Fourier sine transformation the author takes a simple beam with uniform cross section under the simultaneous action of axial tension and lateral load (Fig. 1). Denoting by EI the flexural rigidity, by N the axial tension and by $p(x)$ the lateral distributing load, the differential equation of the deflection curve is

$$EI\frac{d^4y}{dx^4} - N\frac{d^2y}{dx^2} - p(x) = 0. \quad (10)$$

Fig. 1

The deflection curve in this case can be represented in the form of a sine series:

$$y = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l}. \quad (11)$$

Each term of the series satisfies the end conditions, since each term, together with its second derivative, becomes zero at the ends of beam. Thus the deflections of the beam and the bending moments at the ends are equal to zero.

The second and fourth derivatives of y with respect to x are, respectively,

$$\frac{d^2y}{dx^2} = -\sum_{n=1}^{\infty} a_n\left(\frac{n\pi}{l}\right)^2 \sin\frac{n\pi x}{l} \quad \text{and} \quad \frac{d^4y}{dx^4} = \sum_{n=1}^{\infty} a_n\left(\frac{n\pi}{l}\right)^4 \sin\frac{n\pi x}{l}.$$

Substitutiug these values in Eq. (10) and performing the Fourier sine transformation, the following equation is obtained:

$$\int_0^l \left\{EI\sum_{n=1}^{\infty} a_n\left(\frac{n\pi}{l}\right)^4 \sin\frac{n\pi x}{l} + N\sum_{n=1}^{\infty} a_n\left(\frac{n\pi}{l}\right)^2 \sin\frac{n\pi x}{l} - p(x)\right\} \sin\frac{n\pi x}{l}\,dx = 0. \quad (12)$$

Taking into account that

$$\int_0^l \sin\frac{m\pi x}{l}\sin\frac{n\pi x}{l}\,dx = 0\ (m \neq n) \text{ and } \int_0^l \sin^2\frac{n\pi x}{l}\,dx = \frac{l}{2},$$

an equation for determining the coefficient a_n is found as follows:

$$EIa_n\left(\frac{n\pi}{l}\right)^4\frac{l}{2} + Na_n\left(\frac{n\pi}{l}\right)^2\frac{l}{2} = \int p(x)\sin\frac{n\pi x}{l}\,dx.$$

Thus

$$a_n = \frac{2\,l^3\int p(x)\sin\frac{n\pi x}{l}\,dx}{EI\pi^4(n^4+n^2w^2)} \quad \text{in which} \quad w^2 = \frac{Nl^2}{EI\pi^2}. \tag{13}$$

Substituting such expressions for the cofficients in the series (11), one gets

$$y = \frac{2\,l^3}{EI\pi^4}\sum_{n=1}^{\infty}\frac{\int p(x)\sin\frac{n\pi x}{l}\,dx}{n^4+n^2w^2}\sin\frac{n\pi x}{l}. \tag{14}$$

When the load is uniformly spread at the rate of p per unit length run over from $x=u$ to $x=v$ (Fig. 2) one gets

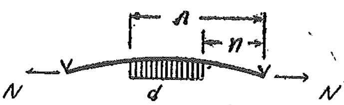

Fig. 2

$$\int_u^v p\sin\frac{n\pi x}{l}\,dx = \frac{pl}{n\pi}\left(\cos\frac{n\pi u}{l} - \cos\frac{n\pi v}{l}\right)$$

and accordingly

$$y = \frac{2pl^4}{EI\pi^5}\sum_{n=1}^{\infty}\frac{\left(\cos\frac{n\pi u}{l} - \cos\frac{n\pi v}{l}\right)}{n^5+n^3w^2}\sin\frac{n\pi x}{l}. \tag{15}$$

In the particular case of a uniform load applied from $x=0$ to $x=l$ (Fig. 3),

$$\cos\frac{n\pi u}{l} - \cos\frac{n\pi v}{l} = 1-(-1)^n$$

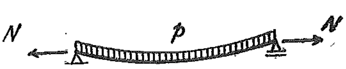

Fig. 3

and thus

$$y = \frac{4pl^4}{EI\pi^5}\sum_{n=1,3,5\cdots}^{\infty}\frac{1}{n^5+n^3w^2}\sin\frac{n\pi x}{l}. \tag{16}$$

The bending moment of the beam is obtained by differentiating the above equation. Thus

$$M = -EI\frac{d^2y}{dx^2} = \frac{4pl^2}{\pi^3}\sum_{n=1,3,5\cdots}^{\infty}\frac{1}{n^3+nw^2}\sin\frac{n\pi x}{l}. \tag{17}$$

In the case of a single concentrated load P applied at $x=u$ (Fig. 4), assuming that the uniform load of $\frac{P}{\xi}$ extends from $x=u$ to $x=u+\xi$ one gets

$$\int_u^{u+\xi}\frac{P}{\xi}\sin\frac{n\pi x}{l}\,dx = \frac{Pl}{\xi n\pi}\left(\cos\frac{n\pi u}{l} - \cos\frac{n\pi(u+\xi)}{l}\right).$$

If ξ is directly taken as zero, the right hand side of the above equation appears in the so-called indeterminate form 0/0. So ξ should be made to zero in both the differential coefficients of a numerator and a denominator of the above formula. Thus

$$\frac{Pl}{n\pi}\left[\frac{\sin\frac{n\pi(u+\xi)}{l}\cdot\frac{n\pi}{l}}{1}\right]_{\xi=0}=P\sin\frac{n\pi u}{l}.$$

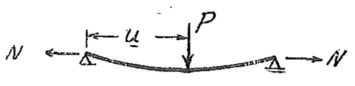

Fig. 4

The new result will be true for the load P concentrated at $x=u$. Then the deflection curve and the bending moment become, respectively,

$$y=\frac{2Pl^3}{EI\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4+n^2w^2}\sin\frac{n\pi u}{l}\sin\frac{n\pi x}{l}\qquad(x=0\sim u)\qquad(18)$$

and

$$M=\frac{2Pl}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2+w^2}\sin\frac{n\pi u}{l}\sin\frac{n\pi x}{l}\qquad(x=0\sim u).\qquad(19)$$

2. Simple Beam with Uniform Cross Section under the Simultaneous Action of Axial Compression and Lateral Load.

In this case, the differential equation of the deflection curve is

$$EI\frac{d^4y}{dx^4}+N\frac{d^2y}{dx^2}-p(x)=0\qquad(20)$$

If $-w^2$ is replaced for w^2 in the results one has just got the new results will be true for the simple beam under an axial compressive force and thus

(a) in the case of any distributed load:

$$y=\frac{2l^3}{EI\pi^4}\sum_{n=1}^{\infty}\frac{\int p(x)\sin\frac{n\pi x}{l}dx}{n^4-n^2w^2}\sin\frac{n\pi x}{l},\qquad(21)$$

(b) in the case of full uniform load:

$$y=\frac{4pl^4}{EI\pi^5}\sum_{n=1,3,5\cdots}^{\infty}\frac{1}{n^5-n^3w^2}\sin\frac{n\pi x}{l}\qquad(22)$$

$$M=\frac{4pl^2}{\pi^3}\sum_{n=1,3,5\cdots}^{\infty}\frac{1}{n^3-nw^2}\sin\frac{n\pi x}{l}.\qquad(23)$$

(c) in the case of single load concentrated at $x=u$:

$$y=\frac{2Pl^3}{EI\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4-n^2w^2}\sin\frac{n\pi u}{l}\sin\frac{n\pi x}{l}\qquad(x=0\sim u)\qquad(24)$$

$$M=\frac{2Pl}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2-w^2}\sin\frac{n\pi u}{l}\sin\frac{n\pi x}{l}\quad (x=0\sim u). \tag{25}$$

3. Simple Beam on an Elastic Bed (Fig. 5).

If a simple beam with hinged ends is placed on an elastic bed, the reaction of which, at each cross section of the beam, is proportional to the deflection at that cross section, the differential equation of the deflection curve is

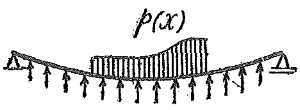

Fig. 5

$$EI\frac{d^4y}{dx^4}+\alpha y-p(x)=0 \tag{26}$$

where α is the modulus of foundation. α has the dimension of a force divided by the square of the length. It represents the magnitude of the reaction of the foundation per unit length of the beam if the deflection is equal to unity.

The general expression for the deflection curve of a beam with hinged ends can be represented by the series

$$y=\sum_{n=1}^{\infty}a_n\sin\frac{n\pi x}{l}. \tag{27}$$

Performing the Fourier sine transformation with the values of y given by Eq. (27), one gets

$$\int_0^l\left\{EI\sum a_n\left(\frac{n\pi}{l}\right)^4\sin\frac{n\pi x}{l}+\alpha\sum a_n\sin\frac{n\pi x}{l}-p(x)\right\}\sin\frac{n\pi x}{l}\,dx=0 \tag{28}$$

from which

$$EIa_n\left(\frac{n\pi}{l}\right)^4\frac{l}{2}+\alpha a_n\frac{l}{2}=\int p(x)\sin\frac{n\pi x}{l}\,dx.$$

Using the following notation:

$$w^2=\frac{\alpha l^4}{EI\pi^4}$$

the coefficient a_n and the deflection curve are respectively,

$$a_n=\frac{2l^3\int p(x)\sin\frac{n\pi x}{l}\,dx}{EI\pi^4(n^4+w^2)} \tag{29}$$

and

$$y=\frac{2l^3}{EI\pi^4}\sum_{n=1}^{\infty}\frac{\int p(x)\sin\frac{n\pi x}{l}\,dx}{n^4+w^2}\sin\frac{n\pi x}{l}. \tag{30}$$

In the case of partial uniform load applied from $x=u$ to $x=v$:

$$y = \frac{2pl^4}{EI\pi^5} \sum_{n=1}^{\infty} \frac{\left(\cos\frac{n\pi u}{l} - \cos\frac{n\pi v}{l}\right)}{n^5 + n\,w^2} \sin\frac{n\pi x}{l}. \tag{31}$$

In the ease of full uniform load:

$$y = \frac{4pl^4}{EI\pi^5} \sum_{n=1,3,5\cdots}^{\infty} \frac{1}{n^5 + n\,w^2} \sin\frac{n\pi x}{l}. \tag{32}$$

In the case of a single load concentrated at $x = u$:

$$y = \frac{2Pl^3}{EI\pi^4} \sum_{n=1}^{\infty} \frac{1}{n^4 + w^2} \sin\frac{n\pi u}{l} \sin\frac{n\pi x}{l} \quad (x = 0 \sim u) \tag{33}$$

4. Stiffening Girder of a Suspension Bridge.

In the ordinary calculation of the suspension bridge, the effect of deflection of a stiffening girder under load is not taken into account and it is assumed consequently that the upward pull of the hanger on the girder is constant. In the suspension bridge in which the span is long and the stiffening girder comparatively flexible, the effect of deflection under load must be taken into account. If taken into account, the upward pull of the hanger at any point on the girder would be

$$q(x) = \beta g - H_g (1 + \beta) \frac{d^2 y}{dx^2} \tag{34}$$

in which

$$\beta = \frac{H_g}{H},$$

$H_g =$ horizontal component of cable stress due to dead load,

$H =$ additional horizontal component of cable stress due to any cause, as live load or temperature change,

$g =$ dead load per unit length on cable including its own weight,

and $y =$ the deflection of the girder at the point in question, due to any given live load.

If the live load per unit length is denoted by $p(x)$, the stiffening girder is subjected to the simultaneous action of $p(x)$ and $-q(x)$ and consequently the differential equation of deflection curve of the stiffening girder becomes

$$EI \frac{d^4 y}{dx^4} = p(x) - q(x).$$

With the value of $q(x)$ given by (34) the above equation reduces to

$$EI \frac{d^4 y}{dx^4} - H_g (1+\beta) \frac{d^2 y}{dx^2} - p(x) + \beta g = 0 \tag{35}$$

in which EI is the flexural rigidity of the stiffening girder.

The general expression for the deflection curve of the stiffening girder can be represented by the series

$$y = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l}. \tag{36}$$

With the value of y given by the above eqation, employing the Fourier sine transformation method, Eq. (35) reduced to

$$\int_0^l \left[EI \sum a_n \left(\frac{n\pi}{l}\right)^4 \sin \frac{n\pi x}{l} + H_g (1+\beta) \sum a_n \left(\frac{n\pi}{l}\right)^2 \sin \frac{n\pi x}{l} - p + \beta g \right] \sin \frac{n\pi x}{l}\, dx = 0.$$

Taking into account that

$$\int_0^l \sin \frac{m\pi x}{l} \sin \frac{n\pi x}{l}\, dx = 0 \,(m \neq n) \text{ and } \int_0^l \sin^2 \frac{n\pi x}{l}\, dx = \frac{l}{2},$$

the above equation becomes

$$EI a_n \left(\frac{n\pi}{l}\right)^4 \frac{l}{2} + H_g (1+\beta)\, a_n \left(\frac{n\pi}{l}\right)^2 \frac{l}{2} = \int (p - \beta g) \sin \frac{n\pi x}{l}\, dx. \tag{37}$$

Considering that $p(x)$ is uniformly spread from $x = u$ to $x = v$ and g over a total span, one gets

$$\int_u^v p \sin \frac{n\pi x}{l}\, dx = \frac{pl}{n\pi} \left(\cos \frac{n\pi u}{l} - \cos \frac{n\pi v}{l} \right)$$

and

$$\int_0^l \beta g \sin \frac{n\pi x}{l}\, dx = \frac{\beta g l}{n\pi} (1 - \cos n\pi).$$

Using the following notation,

$$w^2 = \frac{H_g l^2}{EI\pi^2} (1 + \beta) \tag{38}$$

the coefficient a_n and deflection curve are respectively

$$a_n = \frac{2l^4}{EI\pi^5} \frac{p \left(\cos \frac{n\pi u}{l} - \cos \frac{n\pi v}{l} \right) - \beta g (1 - \cos n\pi)}{n^5 + n^3 w^2} \tag{39}$$

and

$$y = \frac{2l^4}{EI\pi^5} \sum_{n=1}^{\infty} \frac{p \left(\cos \frac{n\pi u}{l} - \cos \frac{n\pi v}{l} \right) - \beta g (1 - \cos n\pi)}{n^5 + n^3 w^2} \sin \frac{n\pi x}{l}. \tag{40}$$

In the case of full uniform live load:

$$y = \frac{4l^4(p-\beta g)}{EI\pi^5} \sum_{n=1,3,5\cdots}^{\infty} \frac{1}{n^5+n^3 w^2} \sin\frac{n\pi x}{l}, \tag{41}$$

$$M = \frac{4l^2(p-\beta g)}{\pi^3} \sum_{n=1,3,5\cdots}^{\infty} \frac{1}{n^3+n w^2} \sin\frac{n\pi x}{l}. \tag{42}$$

In the case of a single load P concentrated at $x=u$:

$$y = \frac{2Pl^3}{EI\pi^4} \sum_{n=1}^{\infty} \frac{\sin\frac{n\pi u}{l} - 2\beta g(1-\cos n\pi)}{n^4+n^2 w^2} \sin\frac{n\pi x}{l}, \tag{43}$$

$$M = \frac{2Pl}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin\frac{n\pi u}{l} - 2\beta g(1-\cos n\pi)}{n^2+w^2} \sin\frac{n\pi x}{l}. \tag{44}$$

The same results have also been given by Timoshenko applying the energy method but the process of its calculation is more labourious in comparison with the above described one. In the energy method, the process of calculation is as follows:

The general expression for strain energy of bending is given by the equation

$$V = \frac{EI}{2}\int_0^l \left(\frac{d^2y}{dx^2}\right)^2 dx.$$

With the value of y given by Eq. (36), this expression reduces to

$$V = \frac{EI}{2}\int_0^l \left\{ \sum_{n=1}^{\infty} \left(\frac{n\pi}{l}\right)^4 a_n^2 \sin^2\frac{n\pi x}{l} + 2\left(\frac{\pi}{l}\right)^4 \sum_{m}^{except. n} a_n a_m n^2 m^2 \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l} \right\} dx.$$

Taking into account that

$$\int_0^l \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,dx = 0 \quad (n \neq m),$$

the above expression becomes

$$V = \frac{EI}{2}\int_0^l \sum_{n=1}^{\infty} \left(\frac{n\pi}{l}\right)^4 a_n^2 \sin^2\frac{n\pi x}{l}\,dx.$$

The change in strain enenrgy of the beam, due to the small increase δa_n in a_n, is

$$\delta V = EI\,\delta a_n \int_0^l a_n \left(\frac{n\pi}{l}\right)^4 \sin^2\frac{nnx}{l}\,dx = EI\,\delta a_n\, a_n \left(\frac{n\pi}{l}\right)^4 \frac{l}{2}. \tag{45}$$

Next, the change in deflection of the beam, due to the small increase δa_n in a_n, is

$$\delta y = \delta a_n \sin \frac{n\pi x}{l}$$

and accordingly the loads p and q produce the work

$$\delta W = \int p dx \, \delta y - \int q \, dx \, \delta y$$

$$= \int p \, \delta a_n \sin \frac{n\pi x}{l} dx - \int \left\{ \beta g - H_g (1+\beta) \frac{d^2 y}{dx^2} \right\} \delta a_n \sin \frac{n\pi x}{l} dx$$

$$= \delta a_n \int (p - \beta g) \sin \frac{n\pi x}{l} dx - H_g (1+\beta) \delta a_n \int_0^l \left\{ \sum_{n=1}^{\infty} a_n \left(\frac{n\pi}{l} \right)^2 \sin \frac{n\pi x}{l} \right\} \sin \frac{n\pi x}{l} dx$$

$$= \delta a_n \int (p - \beta g) \sin \frac{n\pi x}{l} dx - H_g (1+\beta) \delta a_n . a_n \left(\frac{n\pi}{l} \right)^2 \frac{l}{2} . \qquad (46)$$

Equating this to the work done δV, the same equation as (37) can be obtained.

In order to determine the value of β, equating the work done in the cable due to stress with the work done by vertical displacement of load, the following equation is obtained:

$$\frac{H_g L}{E_c A_c} \beta \left(1 + \frac{\beta}{2} \right) = \frac{16}{\pi} \frac{f}{l} \left(1 + \frac{\beta}{2} \right) \sum_{n=1,3,5\cdots}^{\infty} \frac{a_n}{n} + \frac{\pi^2}{4l} (1+\beta) \sum_{n=1}^{\infty} n^2 a_n^2 \qquad (47)$$

in which

$$L = l \left(1 + 8 \frac{f^2}{l^2} \right) + 2b \sec^2 \gamma,$$

l = length of span, f = centre sag of cable, b = length of one back stay cable and γ = inclination of back stay cable.

This equation can be solved by successive approximations. The method developed above for the case of a stiffening girder with uniform flexural rigidity can be extended to case of variable flexural rigidity.

5. Fixed Beam under a Lateral Load.

Taking the simplest case of the fixed beam with uniform cross section subjected to a lateral load only, the differential equation of the deflection is

$$EI \frac{d^4 y}{dx^4} - p(x) = 0. \qquad (48)$$

The general expression for y which makes y and $\frac{dy}{dx}$ zero at both ends is

$$y = \sum_{n=1}^{\infty} a_n \left\{ -\left(\frac{x^3}{l^3} - \frac{2x^2}{l^2} + \frac{x}{l} \right) - (-1)^n \left(\frac{x^3}{l^3} - \frac{x^2}{l^2} \right) + \frac{1}{n\pi} \sin \frac{n\pi x}{l} \right\}. \qquad (49)$$

In the case of a symmetrical loading, $\frac{dy}{dx} = 0$ at $x = \frac{l}{2}$, so that

$$y = \sum_{n=1,3,5\cdots}^{\infty} a_n \left(\frac{x^2}{l^2} - \frac{x}{l} + \frac{1}{n\pi} \sin \frac{n\pi x}{l} \right). \tag{50}$$

From Eq. (49) the fourth derivative of y with respect to x becomes

$$\frac{d^4 y}{dx^4} = \sum_{n=1}^{\infty} a_n \frac{n^3 \pi^3}{l^4} \sin \frac{n\pi x}{l}.$$

Proceeding as in the previous articles with the above value the following equation is obtained:

$$\int_0^l \left[EI \sum_{n=1}^{\infty} a_n \frac{n^3 \pi^3}{l^4} \sin \frac{n\pi x}{l} - p(x) \right] \sin \frac{n\pi x}{l} \, dx = 0. \tag{51}$$

Now since

$$\int_0^l \sin \frac{m\pi x}{l} \sin \frac{n\pi x}{l} \, dx = 0 \quad (m \neq n) \text{ and } \int_0^l \sin^2 \frac{n\pi x}{l} \, dx = \frac{l}{2}$$

it follows that

$$a_n = \frac{2l^3 \int p(x) \sin \frac{n\pi x}{l} \, dx}{EI n^3 \pi^3} \tag{52}$$

and accordingly

$$y = \frac{2l^3}{EI\pi^3} \sum_{n=1}^{\infty} \frac{\int p(x) \sin \frac{n\pi x}{l} \, dx}{n^3} \left\{ - \left(\frac{x^3}{l^3} - \frac{2x^2}{l^2} + \frac{x}{l} \right) - (-1)^n \left(\frac{x^3}{l^3} - \frac{x^2}{l^2} \right) + \frac{1}{n\pi} \sin \frac{n\pi x}{l} \right\} \tag{53}$$

In the case of full uniform load,

$$a_n = \frac{4pl^4}{EI n^4 \pi^4} \qquad \text{for } n = \text{odd}, \quad a_n = 0 \quad \text{for } n = \text{even},$$

and

$$- \left(\frac{x^3}{l^3} - \frac{2x^2}{l^2} + \frac{x}{l} \right) - (-1)^n \left(\frac{x^3}{l^3} - \frac{x^2}{l^2} \right) + \frac{1}{n\pi} \sin \frac{n\pi x}{l}$$

$$= \frac{x^2}{l^2} - \frac{x}{l} + \frac{1}{n\pi} \sin \frac{n\pi x}{l} \text{ for } n = \text{odd}.$$

Therefore

$$y = \frac{4pl^4}{EI\pi^4} \sum_{n=1,3,5\cdots}^{\infty} \frac{1}{n^4} \left(\frac{x^2}{l^2} - \frac{x}{l} + \frac{1}{n\pi} \sin \frac{n\pi x}{l} \right) \tag{54}$$

and accordingly

$$M = \frac{8pl^2}{\pi^4} \sum_{n=1,3,5\cdots}^{\infty} \frac{1}{n^4} \left(\frac{n\pi}{2} \sin \frac{n\pi x}{l} - 1 \right). \tag{55}$$

If Galerkin's method is used for this problem, the Galerkin formula becomes

$$\int_0^l \left\{ EI \sum_{n=1}^{\infty} a_n \frac{n^3\pi^3}{l^4} \sin\frac{n\pi x}{l} - p(x) \right\} \left\{ - \left(\frac{x^3}{l^3} - \frac{2x^2}{l^2} + \frac{x}{l} \right.\right.$$

$$\left.\left. - (-1)^n \left(\frac{x^3}{l^3} - \frac{x^2}{l^2} \right) + \frac{1}{n\pi} \sin\frac{n\pi x}{l} \right\} = 0. \tag{56}$$

Taking into account that

$$\int_0^l \sin\frac{m\pi x}{l} \sin\frac{n\pi x}{l}\, dx = 0 \quad (m \neq n),$$

the above equation reduces to

$$\int_0^l \left\{ EI \sum_{n=1}^{\infty} a_n \frac{n^3\pi^3}{l^4} \sin\frac{n\pi x}{l} \right\} \left\{ - \left(\frac{x^3}{l^3} - \frac{2x^2}{l^2} + \frac{x}{l} \right) - (-1)^n \left(\frac{x^3}{l^3} - \frac{x^2}{l^2} \right) \right\} dx$$

$$+ \int_0^l EI a_n \frac{n^2\pi^2}{l^4} \sin^2\frac{n\pi x}{l}\, dx = \int_0^l p(x) \left\{ - \left(\frac{x^3}{l^3} - \frac{2x^2}{l^2} + \frac{x}{l} \right) - (-1)^n \left(\frac{x^3}{l^3} - \frac{x^2}{l^2} \right) \right\} dx$$

$$+ \int_0^l p(x) \frac{1}{n\pi} \sin\frac{n\pi x}{l}\, dx. \tag{57}$$

Now putting 1, 2, 3……r in n of the above equation, r equations may be obtained. Next adding all these equations and then dividing with r one gets

$$\int_0^l \left\{ EI \sum_{n=1}^{\infty} a_n \frac{n^3\pi^3}{l^4} \sin\frac{n\pi x}{l} \right\} \left\{ - \left(\frac{x^3}{l^3} - \frac{2x^2}{l^2} + \frac{x}{l} \right) - (-1)^n \left(\frac{x^3}{l^3} - \frac{x^2}{l^2} \right) \right\} dx$$

$$+ \frac{1}{r} \int_0^l EI \sum_{n=1}^{r} a_n \frac{n^2\pi^2}{l^4} \sin^2\frac{n\pi x}{l}\, dx = \int_0^l p(x) \left\{ - \left(\frac{x^3}{l^3} - \frac{2x^2}{l^2} + \frac{x}{l} \right) - (-1)^n \left(\frac{x^3}{l^3} - \frac{x^2}{l^2} \right) \right\} dx$$

$$+ \frac{1}{r} \int_0^l p(x) \sum_{n=1}^{r} \frac{1}{n\pi} \sin\frac{n\pi x}{l}\, dx. \tag{58}$$

When r is taken to be infinite,

$$\frac{1}{r} \int_0^l EI \sum_{n=1}^{r} a_n \frac{n^2\pi^2}{l^4} \sin\frac{n\pi x}{l}\, dx = 0 \quad \text{and} \quad \frac{1}{r} \int p(x) \sum_{n=1}^{r} \frac{1}{n\pi} \sin\frac{n\pi x}{l}\, dx = 0,$$

since integrated values would have the finite value.

Thus the following relation may be obtained, from Eq. (58).

$$\int_0^l \left\{ EI \sum_{n=1}^{\infty} a_n \frac{n^3\pi^3}{l^4} \sin\frac{n\pi x}{l} \right\} \left\{ - \left(\frac{x^3}{l^3} - \frac{2x^2}{l^2} + \frac{x}{l} \right) - (-1)^n \left(\frac{x^3}{l^3} - \frac{x^2}{l^2} \right) \right\} dx$$

$$= \int_0^l p(x)\left\{-\left(\frac{x^3}{l^3}-\frac{2x^2}{l^2}+\frac{x}{l}\right)-(-1)^n\left(\frac{x^3}{l^3}-\frac{x^2}{l^2}\right)\right\}dx$$

and accordingly Eq. (56) becomes

$$\int_0^l EI a_n \frac{n^2\pi^2}{l^4}\sin^2\frac{n\pi x}{l}\,dx = \int_0^l p(x)\frac{1}{n\pi}\sin\frac{n\pi x}{l}\,dx$$

or

$$\int_0^l EI\, a_n \frac{n^3\pi^3}{l^4}\sin^2\frac{n\pi x}{l}\,dx = \int_0^l p(x)\sin\frac{n\pi x}{l}\,dx$$

which agrees with Eq. (51).
Therefore it may be seen that the Fourier sine transformation method is more advantageous than Galerkin's for the problems with regard to fixed beams.

6. Cantilever under a Lateral Load.

The differential equation of the deflection curve for a cantilever with uniform cross section subjected to a lateral load only is

$$EI\frac{d^4y}{dx^4} - p(x) = 0\,. \tag{59}$$

The expression for y which satisfies the terminal conditions is

$$y = \sum_{n=1}^{\infty} a_n\left\{\frac{1}{2}n^2\pi^2(-1)^n\left(\frac{x^3}{3l^3}-\frac{x^2}{l^2}\right)-\frac{x}{l}+\frac{1}{n\pi}\sin\frac{n\pi x}{l}\right\} \tag{60}$$

taking the origin of x at the fixed end.

Substituting the value of y given by Eq. (60) and using the Fourier sine transformation method, Eq. (59) reduces to

$$\int_0^l\left\{EI\sum_{n=1}^{\infty}a_n\frac{n^3\pi^3}{l^4}\sin\frac{n\pi x}{l} - p(x)\right\}\sin\frac{n\pi x}{l}\,dx = 0 \tag{61}$$

from which one gets, in the same way mentioned in the previous problems;

$$a_n = \frac{2l^3\int p(x)\sin\frac{n\pi x}{l}\,dx}{EIn^3\pi^3} \tag{62}$$

and

$$y = \frac{2l^3}{EI\pi^3}\sum_{n=1}^{\infty}\frac{\int p(x)\sin\frac{n\pi x}{l}\,dx}{n^3}\left\{\frac{1}{2}n^2\pi^2(-1)^n\left(\frac{x^3}{3l^3}-\frac{x^2}{l^2}\right)-\frac{x}{l}+\frac{1}{n\pi}\sin\frac{n\pi x}{l}\right\}. \tag{63}$$

In the special case of full uniform load

$$a_n = \frac{4pl^4}{EIn^4\pi^4} \quad \text{for } n = \text{odd}, \quad a_n = 0 \text{ for } n = \text{even}$$

and

$$y = \frac{4pl^4}{EI\pi^4} \sum_{n=1,3,5\cdots}^{\infty} \frac{1}{n^4} \left\{ \frac{1}{2} n^2\pi^2 (-1)^n \left(\frac{x^3}{3l^3} - \frac{x^2}{l^2} \right) - \frac{x}{l} + \frac{1}{n\pi} \sin \frac{n\pi x}{l} \right\}$$

or

$$y = \frac{4pl^4}{EI} \left[\sum_{n=1,3,5\cdots}^{\infty} \left\{ \frac{1}{2} \frac{1}{n^2\pi^2} \left(\frac{x^2}{l^2} - \frac{x^3}{3l^3} \right) - \frac{1}{n^4\pi^4} \frac{x}{l} + \frac{1}{n^5\pi^5} \sin \frac{n\pi x}{l} \right\} \right]. \quad (64)$$

7. Beam with One End Fixed and the Other Hinged.

In the case where one end of the beam is fixed and the other hinged, the expression for y which satisfies the terminal conditions is

$$y = \sum_{n=1}^{\infty} a_n \left\{ \frac{1}{2} \left(\frac{x^3}{l^3} - \frac{3x^2}{l^2} + \frac{2x}{l} \right) + \frac{1}{n\pi} \sin \frac{n\pi x}{l} \right\} \quad (65)$$

taking the origin of x at the fixed end.

The process of the solution of the problems concerned with this beam is the same as in the previous cases.

8. Fixed Beam with Uniform Cross Section under the Simultaneous Action of Axial and Lateral Loads.

Denoting by EI the flexural rigidity and by N the axial tension, the differential equation of the deflection curve is

$$EI \frac{d^4y}{dx^4} - N \frac{d^2y}{dx^2} - p = 0. \quad (66)$$

Considering the full uniform load, the deflection curve can be represented in the form of a series, as mentioned above in article 5;

$$y = \sum_{n=1,3,5\cdots}^{\infty} a_n \left(\frac{x^2}{l^2} - \frac{x}{l} + \frac{1}{n\pi} \sin \frac{n\pi x}{l} \right). \quad (67)$$

The second and fourth derivatives of y with respect to x are, respectively

$$\frac{d^2y}{dx^2} = \sum_{n=1,3,5\cdots}^{\infty} a_n \left(\frac{2}{l^2} - \frac{n\pi}{l^2} \sin \frac{n\pi x}{l} \right) \text{ and } \frac{d^4y}{dx^4} = \sum_{n=1,3,5\cdots}^{\infty} a_n \frac{n^3\pi^3}{l^4} \sin \frac{n\pi x}{l}.$$

It follows, on the substitution of these values into Eq. (66), that

$$EI \sum_{n=1,3,5\cdots}^{\infty} a_n \frac{n^3\pi^3}{l^4} \sin \frac{n\pi x}{l} - N \sum_{n=1,3,5\cdots}^{\infty} a_n \left(\frac{2}{l^2} - \frac{n\pi}{l^2} \sin \frac{n\pi x}{l} \right) - p(x) = 0 \quad (68)$$

Performing the operations of the Fourier transformation method one gets

$$EI \int_0^l a_n \frac{n^3\pi^3}{l^4} \sin^2 \frac{n\pi x}{l}\, dx - N \int_0^l \sum_{n=1,3,5\cdots}^{\infty} a_n \frac{2}{l^2} \sin \frac{n\pi x}{l} + N \int_0^l a_n \frac{n\pi}{l^2} \sin^2 \frac{n\pi x}{l}\, dx$$

$$= p\int_0^l \sin\frac{n\pi x}{l}\,dx\,. \tag{69}$$

Taking into account that

$$\int_0^l \sin^2\frac{n\pi x}{l}\,dx = \frac{l}{2} \quad \text{and} \quad \int_0^l \sin\frac{n\pi x}{l}\,dx = \frac{2l}{n\pi} \quad (n = \text{odd})$$

the above equation becomes

$$EIa_n\frac{n^3\pi^3}{2l^3} - \frac{4N}{ln\pi}\sum_{n=1,3,5\cdots}^{\infty} a_n + Na_n\frac{n\pi}{2l} = \frac{2pl}{n\pi}\,. \tag{70}$$

Using the notaion

$$w^2 = \frac{Nl^2}{EI\pi^2}$$

Eq. (70) reduces to

$$a_n - \frac{8l^2N}{EI\pi^4(n^4+n^2w^2)}\sum_{n=1,3,5\cdots}^{\infty} a_n = \frac{4l^4p}{EI\pi^4(n^4+n^2w^2)}\,. \tag{71}$$

Making r equations with the substitution of 1. 3. 5, ... into n and adding all these equations one gets

$$\sum_{n=1,3,5\cdots}^{r} a_n - \frac{8l^2N}{EI\pi^4}\sum_{n=1,3,5\cdots}^{r}\frac{1}{n^4+n^2w^2}\sum_{n=1,3,5\cdots}^{\infty} a_n = \frac{4l^4p}{EI\pi^4}\sum_{n=1,3,5\cdots}^{r}\frac{1}{n^4+n^2w^2}.$$

When r is taken as infinite, this equation becomes

$$\left(1 - \frac{8l^2N}{EI\pi^4}\sum_{n=1,3,5,\cdots}^{\infty}\frac{1}{n^4+n^2w^2}\right)\sum_{n=1,3,5\cdots}^{\infty} a_n = \frac{4l^4p}{EI\pi^4}\sum_{n=1,3,5\cdots}^{\infty}\frac{1}{n^4+n^2w^2}$$

from which

$$\sum_{n=1,3,5\cdots}^{\infty} a_n = \frac{\dfrac{4l^4p}{EI\pi^4}\displaystyle\sum_{n=1,3,5\cdots}^{\infty}\frac{1}{n^4+n^2w^2}}{1 - \dfrac{8l^2N}{EI\pi^4}\displaystyle\sum_{n=1,3,5\cdots}^{\infty}\frac{1}{n^4+n^2w^2}}\,. \tag{72}$$

Substituting this value for Σa_n into Eq. (71), the coefficient is determined as follows:

$$a_n = \frac{4pl^4}{EI\pi^4(n^4+n^2w^2)}\left[1 - \frac{8w^2}{\pi^2}\sum_{n=1,3,5\cdots}^{\infty}\frac{1}{n^4+n^2w^2}\right]^{-1}. \tag{73}$$

For simplification, using the following notation,

$$\alpha = \left[1 - \frac{8w^2}{\pi^2}\sum_{n=1,3,5\cdots}^{\infty}\frac{1}{n^4+n^2w^2}\right]^{-1}$$

the deflection curve of the beam and the bending moment are given by

$$y = \frac{4pl^4\alpha}{EI\pi^4} \sum_{n=1,3,5\cdots}^{\infty} \frac{1}{n^4+n^2w^2}\left(\frac{x^2}{l^2}-\frac{x}{l}+\frac{1}{n\pi}\sin\frac{n\pi x}{l}\right) \tag{74}$$

$$M= \frac{8pl^2\alpha}{\pi^4} \sum_{n=1,3,5\cdots}^{\infty} \frac{1}{n^4+n^2w^2}\left(\frac{n\pi}{2}\sin\frac{n\pi x}{l}-1\right). \tag{75}$$

In the case where a fixed beam is subjected to the simultaneous action of axial compression and full uniform lateral load, deflection curve and bending moment are obtained by replacing w^2 by $-w^2$ in the above results.

9. Fixed Beam on an Elastic Bed.

If a beam with fixed ends is placed on an elastic bed, the differential equation of the deflection curve is, like that described for a simple beam on an elastic bed,

$$EI\frac{d^4y}{dx^4}+\alpha y - p(x) = 0 \tag{76}$$

in which α is the modulus of foundation. Considering, for simplicity, the full uniform load, the expression of y which satisfies the terminal consitions is

$$y = \sum_{n=1,3,5\cdots}^{\infty} a_n\left(\frac{x^2}{l^2}-\frac{x}{l}+\frac{1}{n\pi}\sin\frac{n\pi x}{l}\right). \tag{77}$$

With this value of y Eq. (76) reduces to

$$EI\sum_{n=1,3,5\cdots}^{\infty} a_n\frac{n^3\pi^3}{l^4}\sin\frac{n\pi x}{l}+\alpha\sum_{n=1,3,5\cdots}^{\infty} a_n\left(\frac{x^2}{l^2}-\frac{x}{l}+\frac{1}{n\pi}\sin\frac{n\pi x}{l}\right)-p=0. \tag{78}$$

By the same proceeding as in the previous problem, the coefficient a_n becomes as follows:

$$a_n = \frac{4pl^4}{EI\pi^4(n^4+w^2)}\beta \tag{79}$$

in which, for simplification, the following notations are used:

$$w^2=\frac{\alpha l^4}{EI\pi^4},\quad \beta=\left[1-\frac{8\alpha l^4}{EI\pi^6}\sum_{n=1,3,5\cdots}^{\infty}\frac{1}{n^2(n^4+w^2)}\right]^{-1}.$$

Thus, the deflection curve and the bending moment are given by

$$y = \frac{4pl^4\beta}{EI\pi^4} \sum_{n=1,3,5\cdots}^{\infty} \frac{1}{n^4+w^2}\left(\frac{x^2}{l^2}-\frac{x}{l}+\frac{1}{n\pi}\sin\frac{n\pi x}{l}\right) \tag{80}$$

$$M= \frac{8pl^2\beta}{\pi^4} \sum_{n=1,3,5\cdots}^{\infty} \frac{1}{n^4+w^2}\left(\frac{n\pi}{2}\sin\frac{n\pi x}{l}-1\right). \tag{81}$$

In the problems regarding fixed beams with uniform cross section, it is considerably labourious to determine the coefficients of the expression of y. In such a case, it may be one of admirable methods to take the well-known normal function of the oscillation of a fixed beam which is represented by an expression of the type

$$y = \sum_{n=1}^{\infty} a_n \left\{ \cos\beta_n x - \cosh\beta_n x - \frac{\cos\beta_n l - \cosh\beta_n l}{\sin\beta_n l - \sinh\beta_n l} (\sin\beta_n x - \sinh\beta_n x) \right\} \tag{82}$$

in which β_n must be the n-th root of

$$\cos\beta l \cosh\beta l = 1 . \tag{83}$$

For simplification, denoting by φ_n the expression in the brackets Eq. (82) becomes

$$y = \sum_{n=1}^{\infty} a_n \varphi_n . \tag{84}$$

Substituting this value of y into Eq. (76), multiplying this equation all through by φ_n and then integrating from one end to the other of the beam one gets

$$\int_0^l \left[EI \sum_{n=1}^{\infty} \beta_n^4 a_n \varphi_n + \alpha \sum_{n=1}^{\infty} a_n \varphi_n - p(x) \right] \varphi_n \, dx = 0. \tag{85}$$

The φ_n -values constitute a complete orthonormal set, that is,

$$\int_0^l \varphi_n \varphi_m \, dx = 0 \qquad \text{for} \quad n \neq m.$$

Using this relation, the coefficient a_n is given by

$$a_n = \frac{\int_0^l p(x) \varphi_n \, dx}{(EI\beta_n^4 + \alpha) \int_0^l \varphi_n^2 \, dx} . \tag{86}$$

The integral of φ_n^2 can be obtained in every particular case by direct integration of the terms in φ_n^2. This process is labourious, but it is easier to get a general result to cover all cases. This result is given in the late Lord Rayleigh's "Theory of Sound" as follows:

$$\int_0^l \varphi_n^2 \, dx = \frac{l}{4\beta_n^4} \left\{ \varphi_n D^4 \varphi_n - 2D\varphi_n D^3 \varphi_n + (D^2 \varphi_n)^2 \right\}_{x=l} \tag{87}$$

in which D indicates, for shortness, differentiation with respect to x, that is, D^2 for $\frac{d^2}{dx^2}$, D^3 for $\frac{d^3}{dx^3}$ etc. and the form of this integral is independent of the terminal condition at $x = 0$.

For $D^4 \varphi_n$ may, of course, be substituted its value $\beta_n^4 \varphi_n$.

If the end $x = l$ is simply supported then $\varphi_n = 0$ and $D^2 \varphi_n = 0$ at that end, so that

$$\int_0^l \varphi_n^2\, dx = -\frac{l}{2\beta_n^4}(D\varphi_n D^3 \varphi_n)_{x=l}\,. \tag{88}$$

If the end $x = l$ is fixed then $\varphi_n = 0$ and $D\varphi_n = 0$, so that

$$\int_0^l \varphi_n^2\, dx = \frac{l}{4\beta_n^4}(D^2 \varphi_n)^2_{x=l}\,. \tag{89}$$

If the end $x = l$ is free then $D^2 \varphi_n = 0$ and $D^3 \varphi_n = 0$, so that

$$\int_0^l \varphi_n^2\, dx = \frac{1}{4}\, l\,(\varphi_n^2)_{x=l} \tag{90}$$

Thus, if the fixed beam on an elastic bed is taken,

$$\int_0^l \varphi_n^2\, dx = \quad l\,(\cos\beta_n l + \cosh\beta_n l)^2$$

and accordingly

$$a_n = \frac{\int_0^l p(x)\,\varphi_n\, dx}{(EI\,\beta_n^4 + \alpha)\, l\,(\cos\beta_n l + \cosh\beta_n l)^2}\,. \tag{91}$$

In the same way, the problems of a beam with both ends free on an elastic bed may be solved taking the well-known normal function of the oscillation of a beam with both ends free which is represented by the expression of the type

$$y = \sum_{n=1}^{\infty} a_n \left\{ \cos\beta_n x + \cosh\beta_n x - \frac{\cos\beta_n l - \cosh\beta_n l}{\sin\beta_n l - \sinh\beta_n l}(\sin\beta_n x + \sinh\beta_n x)\right\}$$

in which β_n must be the n-th root of

$$\cos\beta l \cosh\beta l = 1.$$

10. Beams with Variable Cross Sections.

In the case of beams with variable cross sections, the general expression of deflection curve to be assumed at the beginning is also quite the same as the beam with uniform cross section. But the coefficients a_1, a_2, a_3,⋯⋯in the expression can not be individually determined. In this case by putting 1, 2, 3,⋯⋯ into n of the multiplier $\sin\frac{n\pi x}{l}$ in the Fourier sine transformation method, the simultaneous equations must be made and then solving these equations for a the coefficients a_1, a_2, a_3,⋯⋯ should be determined.

This calculation is labourious but for practical purposes it is enough to take the two or three terms in the series of expression of deflection curve with coefficients a_1 and a_2 or a_1, a_2 and a_3. Very satisfactory results may be obtained, by solving the corresponding two or three equations. Taking only the first term of the series, accuracy of the result is sufficient for any practical cases, too.

Calculating process with respect to bending of the beams with variable cross sections is omitted in this paper since it is the same as the calculations of critical load and frequency of oscillation of a bar with variable cross section which will be related in the later articles.

III. CALCULATION OF INFINITE SERIES.

With regard to the bending of the various kind of beams with uniform cross section, the results obtained by the Fourier sine transformation method were represented by the infinite series. Since this series rapidly converges, the first several terms give the deflection with a high degree of accuracy. Taking only the first term or two terms, accuracy of the result is sufficient for many practical purposes. However, in the calculations of bending moment and shearing force, many more terms must be taken than in calculations of deflection since the series concerned with those is slow converging.

If one wishes to calculate the infinite series itself, this calculation may be performed, as follows, giving quite the same results as in the ordinary calculation of the differential equation.

1. Beams under a Lateral Load Only.

Take, as an example, the case in which the uniform load is applied over the entire length of the simply supported beam. The deflection curve in this case is represented in the form:

$$y = \frac{4pl^4}{EI\pi^5}\sum_{n=1,3,5\cdots}^{\infty}\frac{1}{n^5}\sin\frac{n\pi x}{l} \tag{93}$$

which is obtained from Eq. (16), putting $w=0$.

In the equation,

$$\sum_{n=1,3,5\cdots}^{\infty}\frac{1}{n^5}\sin\frac{n\pi x}{l} = \frac{1}{96}\left(\frac{x}{l}-\frac{2x^3}{l^3}+\frac{x^4}{l^4}\right)$$

which is recognized by expanding $\frac{1}{96}\left(\frac{x}{l}-\frac{2x^3}{l^3}+\frac{x^4}{l^4}\right)$ in a Fourier sine series.

Therefore it is seen that the deflection is represetned by

$$y = \frac{pl^4}{24EI}\left(\frac{x}{l}-\frac{2x^3}{l^3}+\frac{x^4}{l^4}\right). \tag{94}$$

Again, the deflection at the middle of the same beam is

$$y_{x=\frac{l}{2}} = \frac{4pl^4}{EI\pi^5} \sum_{n=1,3,5\cdots}^{\infty} \frac{1}{n^5} \sin \frac{n\pi}{2} \tag{95}$$

which is obtained from Eq. (93), taking $x = \dfrac{l}{2}$.

Using the relation

$$\sum_{n=1,3,5\cdots}^{\infty} \frac{1}{n^5\pi^5} \sin \frac{n\pi}{2} = \frac{5}{1536}, \tag{95}$$

it follows that

$$y = \frac{5pl^4}{384\,EI}. \tag{96}$$

In the same manner all the results represented in the form of the series, concerned with the bending of the various kinds of beam with uniform cross section having only a lateral load, can be reduced to the algebric functions using the following relations:

$$\left.\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{n\pi} \sin \frac{n\pi x}{l} = \frac{1}{2}\left(1-\frac{x}{l}\right)\\
&\sum \frac{1}{n^2\pi^2} \cos \frac{n\pi x}{l} = \frac{1}{6} - \frac{x}{2l} + \frac{x^2}{4l^2}\\
&\sum \frac{1}{n^3\pi^3} \sin \frac{n\pi x}{l} = \frac{1}{12}\left(\frac{2x}{l} - \frac{3x^2}{l^2} + \frac{x^3}{l^3}\right)\\
&\sum \frac{1}{n^4\pi^4} \cos \frac{n\pi x}{l} = \frac{1}{48}\left(\frac{8}{15} - \frac{4x^2}{l^2} + \frac{4x^3}{l^3} - \frac{x^4}{l^4}\right)\\
&\sum \frac{1}{n^5\pi^5} \sin \frac{n\pi x}{l} = \frac{1}{48}\left(\frac{8x}{15l} - \frac{4x^3}{3l^3} + \frac{x^4}{l^4} - \frac{x^5}{5l^5}\right)
\end{aligned}\right\} \tag{97}$$

$$\left.\begin{aligned}
&\sum_{n=1}^{\infty} \frac{(-1)^n}{n\pi} \sin \frac{n\pi x}{l} = -\frac{x}{2l}\\
&\sum \frac{(-1)^n}{n^2\pi^2} \cos \frac{n\pi x}{l} = -\frac{1}{12} + \frac{x^2}{4l^2}\\
&\sum \frac{(-1)^n}{n^3\pi^3} \sin \frac{n\pi x}{l} = -\frac{1}{12}\left(\frac{x}{l} - \frac{x^3}{l^3}\right)\\
&\sum \frac{(-1)^n}{n^4\pi^4} \cos \frac{n\pi x}{l} = -\frac{1}{48}\left(\frac{7}{15} - \frac{2x^2}{l^2} + \frac{x^4}{l^4}\right)\\
&\sum \frac{(-1)^n}{n^5\pi^5} \sin \frac{n\pi x}{l} = -\frac{1}{48}\left(\frac{7x}{15l} - \frac{2x^3}{3l^3} + \frac{x^5}{5l^5}\right)
\end{aligned}\right\} \tag{98}$$

$$\sum_{n=1,3,5\cdots}^{\infty} \frac{1}{n\pi} \sin \frac{n\pi x}{l} = \frac{1}{4}$$

$$\left.\begin{aligned}
&\sum_{\text{odd}} \frac{1}{n^2\pi^2} \cos\frac{n\pi x}{l} = \frac{1}{8}\left(1 - \frac{2x}{l}\right)\\
&\sum_{\text{odd}} \frac{1}{n^3x^3} \sin\frac{n\pi x}{l} = \frac{1}{8}\left(\frac{x}{l} - \frac{x^2}{l^2}\right)\\
&\sum_{\text{odd}} \frac{1}{n^4\pi^4} \cos\frac{n\pi x}{l} = \frac{1}{96}\left(1 - \frac{6x^2}{l^2} + \frac{4x^3}{l^3}\right)\\
&\sum_{\text{odd}} \frac{1}{n^5\pi^5} \sin\frac{n\pi x}{l} = \frac{1}{96}\left(\frac{x}{l} - \frac{2x^3}{l^3} + \frac{x^4}{l^4}\right)
\end{aligned}\right\} \tag{99}$$

$$\left.\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{n\pi} \sin\frac{n\pi}{2} = \frac{1}{4}, && \sum_{n=1}^{\infty} \frac{(-1)^n}{n\pi} \sin\frac{n\pi}{2} = -\frac{1}{4}\\
&\sum \frac{1}{n^2\pi^2} \cos\frac{n\pi}{2} = -\frac{1}{48}, && \sum \frac{(-1)^n}{n^2\pi^2} \cos\frac{n\pi}{2} = -\frac{1}{48}\\
&\sum \frac{1}{n^3\pi^3} \sin\frac{n\pi}{2} = \frac{1}{32}, && \sum \frac{(-1)^n}{n^3\pi^3} \sin\frac{n\pi}{2} = -\frac{1}{32}\\
&\sum \frac{1}{n^4\pi^4} \cos\frac{n\pi}{2} = -\frac{1}{16}\cdot\frac{7}{720}, && \sum \frac{(-1)^n}{n^4\pi^4} \cos\frac{n\pi}{2} = -\frac{1}{16}\cdot\frac{7}{720}\\
&\sum \frac{1}{n^5\pi^5} \sin\frac{n\pi}{2} = \frac{5}{1536}, && \sum \frac{(-1)^n}{n^5\pi^5} \sin\frac{n\pi}{2} = -\frac{5}{1536}\\
&\sum_{n=1}^{\infty} \frac{1}{n^2\pi^2} = \frac{1}{6}, && \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2\pi^2} = -\frac{1}{12}\\
&\sum \frac{1}{n^4\pi^4} = \frac{1}{90}, && \sum \frac{(-1)^n}{n^4\pi^4} = -\frac{7}{720}\\
&\sum_{\text{odd}} \frac{1}{n^2\pi^2} = \frac{1}{8}, && \sum_{\text{odd}} \frac{1}{n\pi} \sin\frac{n\pi}{2} = \frac{1}{4}\\
&\sum_{\text{odd}} \frac{1}{n^4\pi^4} = \frac{1}{96}, && \sum_{\text{odd}} \frac{1}{n^3\pi^3} \sin\frac{n\pi}{2} = \frac{1}{32}\\
& && \sum_{\text{odd}} \frac{1}{n^5\pi^5} \sin\frac{n\pi}{2} = \frac{5}{1536}
\end{aligned}\right\} \tag{100}$$

2. Simple Beam under the Simultaneous Action of Axial Tension and Lateral Load.

If $\cos w\theta$ is expanded in a Fourier cosine series, it follows that

$$\cos w\theta = \frac{2w \sin w\pi}{\pi}\left\{\frac{1}{2w^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos n\theta}{n^2 - w^2}\right\} \ (0 < w < n). \tag{101}$$

Next, putting iw in the place of w in the above equation and using the relations of $\sin iw = i \sinh w$ and $\cos iw = \cosh w$, it follows that

$$\cosh w\theta = \frac{2w \sinh w\pi}{\pi}\left\{\frac{1}{2w^2} - \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos n\theta}{n^2+w^2}\right\} \tag{102}$$

which may be also obtained by expanding $\cosh w\theta$ in a Fourier series.

Then, by using the relation of

$$(-1)^{n+1}\cos n\theta = -(-1)^n \cos n\theta = -\cos n(\pi-\theta),$$

Eq. (101) reduces to

$$\sum_{n=1}^{\infty}\frac{\cos n(\pi-\theta)}{n^2+w^2} = -\frac{1}{2w^2} + \frac{\pi \cosh w\theta}{2w \sinh w\pi}. \tag{103}$$

Integrating three times in succession, it follows that

$$\sum_{n=1}^{\infty}\frac{\sin n(\pi-\theta)}{n(n^2+w^2)} = +\frac{1}{2w^2}\theta - \frac{\pi \sinh w\theta}{2w^2 \sinh w\pi} \tag{104}$$

$$\sum_{n=1}^{\infty}\frac{\cos n(\pi-\theta)}{n^2(n^2+w^2)} = +\frac{1}{4w^2}\theta^2 - \frac{\pi^2}{12w^2} + \frac{1}{2w^4} - \frac{\pi \cosh w\theta}{2w^3 \sinh w\pi} \tag{105}$$

$$\sum_{n=1}^{\infty}\frac{\sin n(\pi-\theta)}{n^3(n^2+w^2)} = -\frac{1}{12w^2}\theta^3 + \left(\frac{\pi^2}{12w^2} - \frac{1}{2w^4}\right)\theta + \frac{\pi \sinh w\theta}{2w^4 \sinh w\pi} \tag{106}$$

Replacing $\pi-\theta$ and θ by $\frac{\pi x}{l}$ and $\pi\left(1-\frac{x}{l}\right)$ respectively in the above equations it follows finally that

$$\left.\begin{aligned}
&\sum_{n=1}^{\infty}\frac{1}{n^2+w^2}\cos\frac{n\pi x}{l} = -\frac{1}{2w^2} + \frac{\pi \cosh w\pi\left(1-\frac{x}{l}\right)}{2w \sinh w\pi}\\
&\sum_{n=1}^{\infty}\frac{1}{n(n^2+w^2)}\sin\frac{n\pi x}{l} = \frac{\pi}{2w^2}\left(1-\frac{x}{l}\right) - \frac{\pi \sinh w\pi\left(1-\frac{x}{l}\right)}{2w^2 \sinh w\pi}\\
&\sum_{n=1}^{\infty}\frac{1}{n^2(n^2+w^2)}\cos\frac{n\pi x}{l} = \frac{\pi^2}{4w^2}\left(1-\frac{x}{l}\right)^2 - \frac{\pi^2}{12w^2} + \frac{1}{2w^4} - \frac{\pi \cosh w\pi\left(1-\frac{x}{l}\right)}{2w^3 \sinh w\pi}\\
&\sum_{n=1}^{\infty}\frac{1}{n^3(n^2+w^2)}\sin\frac{n\pi x}{l} = -\frac{\pi^3}{12w^2}\left(1-\frac{x}{l}\right)^3 + \left(\frac{\pi^3}{12w^2} - \frac{\pi}{2w^4}\right)\left(1-\frac{x}{l}\right)\\
&\qquad\qquad + \frac{\pi \sinh w\pi\left(1-\frac{x}{l}\right)}{2w^4 \sinh w\pi}.
\end{aligned}\right\} \tag{107}$$

Replacing θ and $\pi-\theta$ by $\frac{\pi x}{l}$ and $\pi\left(1-\frac{x}{l}\right)$ respectively in equations (103)~(106) and using the relations of

$$\sin n\pi\left(1-\frac{x}{l}\right)=-(-1)^n\sin\frac{n\pi x}{l} \text{ and } \cos n\pi\left(1-\frac{x}{l}\right)=(-1)^n\cos\frac{n\pi x}{l}$$

it follows that

$$\left.\begin{aligned}
&\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+w^2}\cos\frac{n\pi x}{l}=-\frac{1}{2w^2}+\frac{\pi\cosh\frac{w\pi x}{l}}{2w\sinh w\pi}\\
&\sum_{n=1}^{\infty}\frac{(-1)^n}{n(n^2+w^2)}\sin\frac{n\pi x}{l}=-\frac{\pi}{2w^2}\frac{x}{l}+\frac{\pi\sinh\frac{w\pi x}{l}}{2w^2\sinh w\pi}\\
&\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2(n^2+w^2)}\cos\frac{n\pi x}{l}=\frac{\pi^2}{4w^2}\frac{x^2}{l^2}-\frac{\pi^2}{12w^2}+\frac{1}{2w^4}-\frac{\pi\cosh\frac{w\pi x}{l}}{2w^3\sinh w\pi}\\
&\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3(n^2+w^2)}\sin\frac{n\pi x}{l}=\frac{\pi^3}{12w^2}\frac{x^3}{l^3}-\left(\frac{\pi^3}{12w^2}-\frac{\pi}{2w^4}\right)\frac{x}{l}-\frac{\pi\sinh\frac{w\pi x}{l}}{2w^4\sinh w\pi}.
\end{aligned}\right\}\quad(108)$$

Next, subtracting the equations in (108) from those in (107) it follows that

$$\left.\begin{aligned}
&\sum_{n=1,3,5\cdots}^{\infty}\frac{1}{n^2+w^2}\cos\frac{n\pi x}{l}=\frac{\pi\sinh\left(\frac{w\pi}{2}-\frac{w\pi x}{l}\right)}{4w\cosh\frac{w\pi}{2}}.\\
&\sum_{odd}\frac{1}{n(n^2+w^2)}\sin\frac{n\pi x}{l}=\frac{\pi}{4w^2}-\frac{\pi\cosh\left(\frac{w\pi}{2}-\frac{w\pi x}{l}\right)}{4w^2\cosh\frac{w\pi}{2}}\\
&\sum_{odd}\frac{1}{n^2(n^2+w^2)}\cos\frac{n\pi x}{l}=\frac{\pi^2}{8w^2}\left(1-\frac{2x}{l}\right)-\frac{\pi\sinh\left(\frac{w\pi}{2}-\frac{w\pi x}{l}\right)}{4w^3\cosh\frac{w\pi}{2}}\\
&\sum_{odd}\frac{1}{n^3(n^2+w^2)}\sin\frac{n\pi x}{l}=\frac{\pi^3}{8w^2}\left(\frac{x}{l}-\frac{x^2}{l^2}\right)-\frac{\pi}{4w^4}+\frac{\pi\cosh\left(\frac{w\pi}{2}-\frac{w\pi x}{l}\right)}{4w^4\cosh\frac{w\pi}{2}}
\end{aligned}\right\}\quad(109)$$

The equations in (107) ~ (109) are useful for the calculation of the infinite series which appears in the results concerned with the beam under the simultaneous action of the axial tension and lateral load.

Take, as an example, the case in which a simply supported beam is subjected to the simultaneous action of axial tension and full uniform lateral load. In such a case the deflection curve is

$$y=\frac{4pl^4}{EI\pi^5}\sum_{n=2,3,5\cdots}^{\infty}\frac{1}{n^5+n^3w^2}\sin\frac{n\pi x}{l}.$$

Using the fourth relation in Eq. (109), this deflection curve reduces to

$$y=\frac{4pl^4}{EI}\left\{\frac{1}{8w^2\pi^2}\left(\frac{x}{l}-\frac{x^2}{l^2}\right)-\frac{1}{4w^4\pi^4}+\frac{\cosh\left(\frac{w\pi}{2}-\frac{w\pi x}{l}\right)}{4w^4\pi^4\cosh\frac{w\pi}{2}}\right\}. \tag{110}$$

Using the notaion $\mu^2=\frac{Nl^2}{4EI}$ in the place of $w^2=\frac{Nl^2}{EI\pi^2}$, the above equation takes the form

$$y=\frac{pl^4}{8EI\mu^2}\left\{\frac{x}{l}-\frac{x^2}{l^2}-\frac{1}{2\mu^2}+\frac{\cosh\left(\frac{2\mu x}{l}-\mu\right)}{2\mu^2\cosh\mu}\right\}$$

or

$$y=\frac{pl^2}{2}\left\{\frac{x}{l}-\frac{x^2}{l^2}-\frac{1}{2\mu^2}\right\}+\frac{pl^2\cosh\left(\frac{2\mu x}{l}-\mu\right)}{4\mu^2 N\cosh\mu}. \tag{110'}$$

In the same manner the bending moment becomes

$$M=\frac{4pl^2}{\pi^3}\sum_{n=1,3,5\cdots}^{\infty}\frac{1}{n^3+nw^2}\sin\frac{n\pi x}{l}=\frac{pl^2}{w^2\pi^2}\left\{1-\frac{\cosh\left(\frac{w\pi}{2}-\frac{w\pi x}{l}\right)}{\cosh\frac{w\pi}{2}}\right\} \tag{111}$$

or

$$M=\frac{pl^2}{4\mu^2}\left\{1-\frac{\cosh\left(\frac{2\mu x}{l}-\mu\right)}{\cosh\mu}\right\}. \tag{111'}$$

which may be also obtained by direct differentiation of Eq. (110').

In the case in which a beam is subjected to the simultaneous action of axial tension and a single concentrated load P at $x=u$, the deflection curve and bending moment are, respectively,

$$y=\frac{2Pl^3}{EI\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4+n^2w^2}\sin\frac{n\pi u}{l}\sin\frac{n\pi x}{l}\quad(x=0\sim u)$$

and

$$M=\frac{2Pl}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2+w^2}\sin\frac{n\pi u}{l}\sin\frac{n\pi x}{l}\quad(x=0\sim u).$$

Replacing $\pi - \theta$ and θ by $\pi\left(\frac{u}{l}-\frac{x}{l}\right)$ and $\pi\left(1-\frac{u}{l}+\frac{x}{l}\right)$, respectively, in Eq. (105) it follows that

$$\sum_{n=1}^{\infty}\frac{\cos n\pi\left(\frac{u}{l}-\frac{x}{l}\right)}{n^2(n^2+w^2)}=\frac{\pi^2}{4w^2}\left(1-\frac{u}{l}+\frac{x}{l}\right)^2-\frac{\pi^2}{12w^2}+\frac{1}{2w^4}-\frac{\pi\cosh w\pi\left(1-\frac{u}{l}+\frac{x}{l}\right)}{2w^3\sinh w\pi} \tag{112}$$

and also replacing $\pi-\theta$ and θ by $\pi\left(\frac{u}{l}+\frac{x}{l}\right)$ and $\pi\left(1-\frac{u}{l}-\frac{x}{l}\right)$, respectively, in Eq. (105) it follows that

$$\sum_{n=1}^{\infty}\frac{\cos n\pi\left(\frac{u}{l}+\frac{x}{l}\right)}{n^2(n^2+w^2)}=\frac{\pi^2}{4w^2}\left(1-\frac{u}{l}-\frac{x}{l}\right)^2-\frac{\pi^2}{12w^2}+\frac{1}{2w^4}-\frac{\pi\cosh w\pi\left(1-\frac{u}{l}-\frac{x}{l}\right)}{2w^3\sinh w\pi}. \tag{113}$$

From these two equations the following equation can be obtained:

$$\sum_{n=1}^{\infty}\frac{1}{n^2(n^2+w^2)}\left\{\cos n\pi\left(\frac{u}{l}-\frac{x}{l}\right)-\cos n\pi\left(\frac{u}{l}+\frac{x}{l}\right)\right\}$$

$$=\frac{\pi^2}{w^2}\left(1-\frac{u}{l}\right)\frac{x}{l}-\frac{\pi\sinh w\pi\left(1-\frac{u}{l}\right)\sinh\frac{w\pi x}{l}}{w^3\sinh w\pi}.$$

Using the relation of

$$2\sin\frac{n\pi u}{l}\sin\frac{n\pi x}{l}=\cos n\pi\left(\frac{u}{l}-\frac{x}{l}\right)-\cos n\pi\left(\frac{u}{l}+\frac{x}{l}\right)$$

the above equation takes the form

$$\sum_{n=1}^{\infty}\frac{1}{n^2(n^2+w^2)}\sin\frac{n\pi u}{l}\sin\frac{n\pi x}{l}=\frac{\pi^2}{2w^2}\left(1-\frac{u}{l}\right)\frac{x}{l}-\frac{\pi\sinh w\pi\left(1-\frac{u}{l}\right)\sinh\frac{w\pi x}{l}}{2w^3\sinh w\pi} \tag{114}$$

In the same manner the following equations are obtained:

$$\left.\begin{aligned}&\sum_{n=1}^{\infty}\frac{1}{n(n^2+w^2)}\sin\frac{n\pi u}{l}\cos\frac{n\pi x}{l}=\frac{\pi}{2w^2}\left(1-\frac{u}{l}\right)-\frac{\pi\sinh w\pi\left(1-\frac{u}{l}\right)\cosh\frac{w\pi x}{l}}{2w^2\sinh w\pi}\\&\sum_{n=1}^{\infty}\frac{1}{n^2+w^2}\sin\frac{n\pi u}{l}\sin\frac{n\pi x}{l}=\frac{\pi\sinh w\pi\left(1-\frac{u}{l}\right)\sinh\frac{w\pi x}{l}}{2w\sinh w\pi}.\end{aligned}\right\} \tag{115}$$

Using the above relations the expressions of the deflection and bending moment given by infinite series reduce, respectively, to

$$y = \frac{Pl^3}{EI}\left\{\frac{1}{w^2\pi^2}\left(1-\frac{u}{l}\right)\frac{x}{l} - \frac{\sinh w\pi\left(1-\frac{u}{l}\right)\sinh\frac{w\pi x}{l}}{\pi^4 w^3 \sinh w\pi}\right\} \tag{116}$$

or

$$y = \frac{Pl}{N}\left\{\left(1-\frac{u}{l}\right)\frac{x}{l} - \frac{\sinh 2\mu\left(1-\frac{u}{l}\right)\sinh\frac{2\mu x}{l}}{2\mu\sinh 2\mu}\right\}. \tag{116'}$$

$$(x = 0 \sim u)$$

and

$$M = \frac{Pl\sinh w\pi\left(1-\frac{u}{l}\right)\sinh\frac{w\pi x}{l}}{\pi w\sinh w\pi} \tag{117}$$

or

$$M = \frac{Pl\sinh 2\mu\left(1-\frac{u}{l}\right)\sinh\frac{2\mu x}{l}}{2\mu\sinh 2\mu}. \tag{117'}$$

In the special case in which a single load is placed on the middle of the span it follows that, given $u = \frac{l}{2}$,

$$y = \frac{Pl}{N}\left\{\frac{x}{2l} - \frac{\sinh\mu\sinh\frac{2\mu x}{l}}{2\mu\sinh 2\mu}\right\} = \frac{Pl}{2N}\left\{\frac{x}{l} - \frac{\sinh\frac{2\mu x}{l}}{2\mu\cosh\mu}\right\}$$

and

$$M = \frac{Pl\sinh\frac{2\mu x}{l}}{4\mu\cosh\mu}.$$

3. Simple Beam under the Simultaneous Action of Axial Compression and Lateral Load.

From Eq. (101) it follows that

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos n\theta}{n^2 - w^2} = -\frac{1}{2w^2} + \frac{\pi\cos w\theta}{2w\sin w\pi}. \tag{118}$$

Proceeding as in the previous article, the following equations are reduced from the above equation:

$$\left.\begin{aligned}
\sum_{n=1}^{\infty}\frac{\cos n(\pi-\theta)}{n^2-w^2} &= \frac{1}{2w^2} - \frac{\pi\cos w\theta}{2w\sin w\pi}\\
\sum_{n=1}^{\infty}\frac{\sin n(\pi-\theta)}{n(n^2-w^2)} &= -\frac{1}{2w^2}\theta + \frac{\pi\sin w\theta}{2w^2\sin w\pi}
\end{aligned}\right\} \tag{119}$$

$$\left.\begin{aligned}
\sum_{n=1}^{\infty}\frac{\cos n(\pi-\theta)}{n^2(n^2-w^2)} &= -\frac{1}{2w^2}\theta^2+\frac{\pi^2}{12w^2}+\frac{1}{2w^4}-\frac{\pi\cos w\theta}{2w^3\sin w\pi}\\
\sum_{n=1}^{\infty}\frac{\sin n(\pi-\theta)}{n^3(n^2-w^2)} &= \frac{1}{12w^2}\theta^3-\left(\frac{\pi^2}{12w^2}+\frac{1}{2w^4}\right)\theta+\frac{\pi\sin w\theta}{2w^4\sin w\pi}
\end{aligned}\right\}$$

$$\left.\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{n^2-w^2}\cos\frac{n\pi x}{l} &= \frac{1}{2w^2}-\frac{\pi\cos w\pi\left(1-\frac{x}{l}\right)}{2w\sin w\pi}\\
\sum_{n=1}^{\infty}\frac{1}{n(n^2-w^2)}\sin\frac{n\pi x}{l} &= -\frac{\pi}{2w^2}\left(1-\frac{x}{l}\right)+\frac{\pi\sin w\pi\left(1-\frac{x}{l}\right)}{2w^2\sin w\pi}\\
\sum_{n=1}^{\infty}\frac{1}{n^2(n^2-w^2)}\cos\frac{n\pi x}{l} &= -\frac{\pi^2}{4w^2}\left(1-\frac{x}{l}\right)^2+\frac{\pi^2}{12w^2}-\frac{1}{2w^4}-\frac{\pi\cos w\pi\left(1-\frac{x}{l}\right)}{2w^3\sin w\pi}\\
\sum_{n=1}^{\infty}\frac{1}{n^3(n^2-w^2)}\sin\frac{n\pi x}{l} &= \frac{\pi^3}{12w^2}\left(1-\frac{x}{l}\right)^3-\left(\frac{\pi^3}{12w^2}+\frac{\pi}{2w^4}\right)\left(1-\frac{x}{l}\right)\\
&\quad+\frac{\pi\sin w\pi\left(1-\frac{x}{l}\right)}{2w^4\sin w\pi}
\end{aligned}\right\} \tag{120}$$

$$\left.\begin{aligned}
\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2-w^2}\cos\frac{n\pi x}{l} &= \frac{1}{2w^2}-\frac{\pi\cos\frac{w\pi x}{l}}{2w\sin w\pi}\\
\sum_{n=1}^{\infty}\frac{(-1)^n}{n(n^2-w^2)}\sin\frac{n\pi x}{l} &= -\frac{1}{2w^2}\frac{x}{l}-\frac{\pi\sin\frac{w\pi x}{l}}{2w^2\sin w\pi}\\
\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2(n^2-w^2)}\cos\frac{n\pi x}{l} &= -\frac{\pi^2}{4w^2}\frac{x^2}{l^2}+\frac{\pi^2}{12w^2}+\frac{1}{2w^4}-\frac{\pi\cos\frac{w\pi x}{l}}{2w^3\sin w\pi}\\
\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3(n^2-w^2)}\sin\frac{n\pi x}{l} &= -\frac{\pi^3}{12w^2}\frac{x^3}{l^3}+\left(\frac{\pi^3}{12w^2}+\frac{\pi}{2w^4}\right)\frac{x}{l}-\frac{\pi\sin\frac{w\pi x}{l}}{2w^4\sin w\pi}
\end{aligned}\right\} \tag{121}$$

$$\left.\begin{aligned}
\sum_{n=1,3,5\cdots}^{\infty}\frac{1}{n^2-w^2}\cos\frac{n\pi x}{l} &= \frac{\pi\sin\left(\frac{w\pi}{2}-\frac{w\pi x}{l}\right)}{4w\cos\frac{w\pi}{2}}\\
\sum_{odd}\frac{1}{n(n^2-w^2)}\sin\frac{n\pi x}{l} &= -\frac{\pi}{4w^2}+\frac{\pi\cos\left(\frac{w\pi}{2}-\frac{w\pi x}{l}\right)}{4w^2\cos\frac{w\pi}{2}}
\end{aligned}\right\} \tag{122}$$

$$\left.\begin{aligned}
&\sum_{ad}\frac{1}{n^2(n^2-w^2)}\cos\frac{n\pi x}{l}=-\frac{\pi^2}{8w^2}\left(1-\frac{2x}{l}\right)+\frac{\pi\sin\left(\frac{w\pi}{2}-\frac{w\pi x}{l}\right)}{4w^3\cos\frac{w\pi}{2}}\\
&\sum_{ad}\frac{1}{n^3(n^2-w^2)}\sin\frac{n\pi x}{l}=-\frac{\pi^3}{8w^2}\left(\frac{x}{l}-\frac{x^2}{l^2}\right)-\frac{\pi}{4w^4}+\frac{\pi\cos\left(\frac{w\pi}{2}-\frac{w\pi x}{l}\right)}{4w^4\cos\frac{w\pi}{2}}
\end{aligned}\right\}$$

$$\left.\begin{aligned}
&\sum_{n=1}^{\infty}\frac{1}{n^2-w^2}\sin\frac{n\pi u}{l}\sin\frac{n\pi x}{l}=\frac{\pi\sin w\pi\left(1-\frac{u}{l}\right)\sin\frac{w\pi x}{l}}{2w\sin w\pi}\\
&\sum_{n=1}^{\infty}\frac{1}{n(n^2-w^2)}\sin\frac{n\pi u}{l}\cos\frac{n\pi x}{l}=-\frac{\pi}{2w^2}\left(1-\frac{u}{l}\right)+\frac{\pi\sin w\pi\left(1-\frac{u}{l}\right)\cos\frac{w\pi x}{l}}{2w^2\sin w\pi}\\
&\sum_{n=1}^{\infty}\frac{1}{n^2(n^2-w^2)}\sin\frac{n\pi u}{l}\sin\frac{n\pi x}{l}=-\frac{\pi^2}{2w^2}\left(1-\frac{u}{l}\right)\frac{x}{l}\\
&\qquad+\frac{\pi\sin w\pi\left(1-\frac{u}{l}\right)\sin\frac{w\pi x}{l}}{2w^3\sin w\pi}.
\end{aligned}\right\}\quad(123)$$

Using the above equations, the deflection curve and the bending moment in the case in which a beam is subjected to the simultaneous action of axial compression and full uniform load become as follows:

$$\begin{aligned}
y&=\frac{4pl^4}{EI\pi^5}\sum_{n=1,3,5\cdots}^{\infty}\frac{1}{n^5-n^3w^2}\sin\frac{n\pi x}{l}\\
&=\frac{4pl^4}{EI}\left\{-\frac{1}{8w^2\pi^2}\left(\frac{x}{l}-\frac{x^2}{l^2}\right)-\frac{1}{4w^4\pi^4}+\frac{\cos\left(\frac{w\pi}{2}-\frac{w\pi x}{l}\right)}{4w^4\pi^4\cos\frac{w\pi}{2}}\right\}
\end{aligned}\quad(124)$$

or taking the notation of

$$\mu^2=\frac{l^2N}{4EI},$$

Eq. (124) takes the form

$$y=\frac{pl^2}{2}\left\{\frac{x^2}{l^2}-\frac{x}{l}-\frac{1}{2\mu^2}\right\}+\frac{pl^2\cos\left(\frac{2\mu x}{l}-\mu\right)}{4\mu^2N\cos\mu}\quad(124')$$

$$M=\frac{4pl^2}{\pi^3}\sum_{n=1,3,5\cdots}^{\infty}\frac{1}{n^3-nw^2}\sin\frac{n\pi x}{l}$$

$$= \frac{pl^2}{w^2\pi^2}\left\{-1+\frac{\cos\left(\frac{w\pi}{2}-\frac{w\pi x}{l}\right)}{\cos\frac{w\pi}{2}}\right\}, \tag{125}$$

or

$$M = \frac{pl^2}{4\mu^2}\left\{\frac{\cos\left(\mu-\frac{2\mu x}{l}\right)}{\cos\mu}-1\right\}. \tag{125'}$$

In the case in which a beam is subjected to the simultaneous action of axial compression and a single load at $x = u$, it follows that

$$y = \frac{2Pl^3}{EI\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4-n^2w^2}\sin\frac{n\pi u}{l}\sin\frac{n\pi x}{l}$$

$$= \frac{Pl^3}{EI}\left\{-\frac{1}{w^2\pi^2}\left(1-\frac{u}{l}\right)\frac{x}{l}+\frac{\sin w\pi\left(1-\frac{u}{l}\right)\sin\frac{w\pi x}{l}}{w^3\pi^3\sin w\pi}\right\} \tag{126}$$

or

$$y = \frac{pl}{N}\left\{-\left(1-\frac{u}{l}\right)\frac{x}{l}+\frac{\sin 2\mu\left(1-\frac{u}{l}\right)\sin\frac{2\mu x}{l}}{2\mu\sin 2\mu}\right\} \tag{126'}$$

$$(x = 0 \sim u)$$

and

$$M = \frac{2Pl}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2-w^2}\sin\frac{n\pi u}{l}\sin\frac{n\pi x}{l}$$

$$= \frac{Pl\sin w\pi\left(1-\frac{u}{l}\right)\sin\frac{w\pi x}{l}}{w\pi\sin w\pi} \tag{127}$$

or

$$M = \frac{Pl\sin 2\mu\left(1-\frac{u}{l}\right)\sin\frac{2\mu x}{l}}{2\mu\sin 2\mu}. \tag{127'}$$

In the special case in which a single load is placed in the middle of the span, it follows that

$$y = \frac{Pl}{2N}\left\{-\frac{x}{l}+\frac{\sin\frac{2\mu x}{l}}{2\mu\cos\mu}\right\} \tag{128}$$

and

$$M = \frac{Pl\sin\frac{2\mu x}{l}}{4\mu\cos\mu}. \tag{129}$$

These results may also be directly obtained by replacing w by iw in the results obtained in the case in which a beam is subjected to axial tension.

4. Fixed Beam under the Simultaneous Action of Axial and Lateral Loads. Putting $x = 0$ in the third formula in Eq. (109), it follows that

$$\sum_{n=1,3,5\cdots}^{\infty} \frac{1}{n^2(n^2+w^2)} = \frac{\pi^2}{8w^2} - \frac{\pi \sinh \frac{w\pi}{2}}{4w^3 \cosh \frac{w\pi}{2}}.$$

Using the notation $\mu^2 = \frac{Nl^2}{4EI}$ in the place of $w^2 = \frac{Nl^2}{EI\pi^2}$ the above formula reduces to

$$\sum_{n=1,3,5\cdots}^{\infty} \frac{1}{\pi^4 n^2(n^2+w^2)} = \frac{1}{32\mu^2}\left(1-\frac{\tanh \mu}{\mu}\right).$$

Consequently, α in the formulae of (74) and (75) reduces to

$$\alpha = \left[1 - \frac{8w^2}{\pi^2} \sum_{n=1,3,5\cdots}^{\infty} \frac{1}{n^4+n^2w^2}\right]^{-1} = \left[1 - 32\,\mu^2 \sum_{n=1,3,5\cdots}^{\infty} \frac{1}{\pi^4 n^2(n^2+w^2)}\right]^{-1}$$

$$= \frac{\mu}{\tanh \mu}.$$

Substituting the relation of $w = \frac{2}{\pi}\,\mu$ into the forth formula of Eq. (109), it follows that

$$\sum_{n=1,3,5\ldots}^{\infty} \frac{1}{\pi^5 n^3(n^2+w^2)} \sin \frac{n\pi x}{l} = \frac{1}{32\,\mu^2}\left\{\frac{x}{l} - \frac{x^2}{l^2} - \frac{1}{2\,\mu^2} + \frac{\cosh\left(\frac{2\mu x}{l}-\mu\right)}{2\mu^2 \cosh \mu}\right\}.$$

Therefore the deflection curve in the case in which a fixed beam is subjected to the simultaneous action of axial tension and full lateral uniform load becomes as follows:

$$y = \frac{4pl^4\alpha}{EI\pi^4} \sum_{n=1,3,5\cdots}^{\infty} \frac{1}{n^4+n^2w^2}\left(\frac{x^2}{l^2} - \frac{x}{l} + \frac{1}{n\pi}\sin\frac{n\pi x}{l}\right)$$

$$= \frac{4pl^4}{32\mu^2 EI}\left\{\left(\frac{x}{l}-\frac{x^2}{l^2}\right) - \frac{1}{2\mu \tanh \mu} + \frac{\cosh\left(\frac{2\mu x}{l}-\mu\right)}{2\mu \cosh \mu}\right\}$$

$$= \frac{4pl^4}{32\mu^2 EI}\left\{\left(\frac{x}{l}-\frac{x^2}{l^2}\right) + \frac{\cosh\left(\frac{2\mu x}{l}-\mu\right) - \cosh \mu}{2\mu \cosh \mu \tanh \mu}\right\}$$

$$= \frac{4pl^4}{32\mu^2 EI}\left\{\left(\frac{x}{l}-\frac{x^2}{l^2}\right) + \frac{\sinh \frac{\mu x}{l} \sinh\left(\frac{\mu x}{l}-\mu\right)}{\mu \sinh \mu}\right\} \qquad (130)$$

or

$$y = \frac{pl^2}{2N}\left\{\left(\frac{x}{l}-\frac{x^2}{l^2}\right) - \frac{\sinh\frac{\mu x}{l}\sinh\left(\mu-\frac{\mu x}{l}\right)}{\mu\sinh\mu}\right\} \tag{130'}$$

In the same manner, the bending moment becomes

$$M = \frac{pl^2}{4\mu^2}\left\{1-\frac{\mu\cosh\left(\mu-\frac{2\mu x}{l}\right)}{\sinh\mu}\right\} \tag{131}$$

which may also be directly obtained by direct differentiation of Eq. (130).

In the case in which a fixed beam is subjected to the simultaneous action of axial compression and full lateral uniform load, the deflection curve and the bending moment are obtained by replacing μ by $i\mu$ in the above results, so that

$$y = \frac{4pl^4}{32\mu^2 EI}\left\{\left(\frac{x^2}{l^2}-\frac{x}{l}\right) + \frac{\sin\frac{\mu x}{l}\sin\left(\frac{\mu x}{l}-\mu\right)}{\mu\sin\mu}\right\} \tag{132}$$

or

$$y = \frac{pl^2}{2N}\left\{\left(\frac{x^2}{l^2}-\frac{x}{l}\right) - \frac{\sin\frac{\mu x}{l}\sin\left(\mu-\frac{\mu x}{l}\right)}{\mu\sin\mu}\right\} \tag{132'}$$

and

$$M = \frac{pl^2}{4\mu^2}\left\{\frac{\mu\cos\left(\mu-\frac{2\mu x}{l}\right)}{\sin\mu}-1\right\}. \tag{133}$$

IV. APPLICATION OF FOURIER SINE TRANSFORMATION METHOD TO THE STUDY OF BUCKLING OF BARS WITH VARIABLE CROSS SECTION.

1. Unsymmetrical Bars with Variable Cross Section.

Denoting by $I(x)$ the moment of inertia of the cross section at x and by P the axial compressive force, the differential equation of the buckling curve is

$$EI(x)\frac{d^2y}{dx^2} + Py = 0. \tag{134}$$

Taking the case of a bar with simply supported ends, the deflection curve in this case can be represented in the form of a sine series:

$$y = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l}. \tag{135}$$

Each term of the series satisfies the end conditions, since each term, together with its second derivative, becomes zero at the ends of the bar.

The second derivative of y with respect to x, from Eq. (135), is

$$\frac{d^2 y}{dx^2} = -\sum_{n=1}^{\infty} a_n \left(\frac{n\pi}{l}\right)^2 \sin \frac{n\pi x}{l}. \tag{136}$$

Substituting (135) and (136) into Eq. (134), it follows that

$$-I(x) \sum_{n=1}^{\infty} a_n \left(\frac{n\pi}{l}\right)^2 \sin \frac{n\pi x}{l} + \frac{P}{E} \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} = 0.$$

Multipling this by $\sin \frac{n\pi x}{l}$ and integrating from one end to the other of the bar it follows, taking the ends at $x = 0$ and $x = l$, that

$$\int_0^l \left\{ I(x) \sum_{n=1}^{\infty} a_n \left(\frac{n\pi}{l}\right)^2 \sin \frac{n\pi x}{l} - \frac{P}{E} \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} \right\} \sin \frac{n\pi x}{l}\, dx = 0. \tag{137}$$

Taking into account that

$$\int_0^l \sin \frac{m\pi x}{l} \sin \frac{n\pi x}{l}\, dx = 0 \quad (m \neq n) \text{ and } \int_0^l \sin^2 \frac{n\pi x}{l}\, dx = \frac{l}{2}$$

the above equation reduces to

$$\int_0^l I(x) \left\{ a_n \left(\frac{n\pi}{l}\right)^2 \sin^2 \frac{n\pi x}{l} + \sum_{m=1}^{\text{except } n} a_m \left(\frac{m\pi}{l}\right)^2 \sin \frac{m\pi x}{l} \sin \frac{n\pi x}{l} \right\} dx - a_n \frac{l}{2} \frac{P}{E} = 0 \tag{138}$$

Eq. (138) is extended over all values of n and m in a system of homogeneous linear equations in $a_1, a_2, a_3, \cdots\cdots$ as follows:

$$\left.\begin{array}{l} r_{11}\, a_1 + r_{12}\, a_2 + r_{13}\, a_3 + \cdots\cdots \\ r_{21}\, a_1 + r_{22}\, a_2 + r_{23}\, a_2 + \cdots\cdots \\ r_{31}\, a_1 + r_{32}\, a_2 + r_{33}\, a_2 + \cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{array}\right\} \tag{139}$$

Buckling of the bar becomes possible when a system of equations gives for coefficients a_n a solution different from zero, i.e., when the determinant of a system (139) becomes equal to zero. Thus it follows that

$$\begin{vmatrix} r_{11} & r_{12} & r_{13} & \cdots & \cdots \\ r_{21} & r_{22} & r_{23} & \cdots & \cdots \\ r_{31} & r_{32} & r_{23} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0 \tag{140}$$

from which the critical load P is determined.

Now, for an example, take a case in which the moment of inertia of the bar varies as follows:

$$I(x) = a + bx + cx^2 + dx^3 + ex^4 . \tag{141}$$

Substituting this into Eq. (138) and using the formulae:

$$\int_0^l \sin^2 \frac{n\pi x}{l} dx = \frac{l}{2} ; \qquad \int_0^l x \sin^2 \frac{n\pi x}{l} dx = \frac{l^2}{4} ;$$

$$\int_0^l x^2 \sin^2 \frac{n\pi x}{l} dx = \frac{l^3}{2}\left(\frac{1}{3} - \frac{1}{2n^2\pi^2}\right) ; \int_0^l x^3 \sin^2 \frac{n\pi x}{l} dx = \frac{l^4}{8}\left(1 - \frac{3}{n^2\pi^2}\right) ;$$

$$\int_0^l x^4 \sin^2 \frac{n\pi x}{l} dx = \frac{l^5}{2}\left(\frac{1}{5} - \frac{1}{n^2\pi^2} + \frac{3}{2n^4\pi^4}\right) ; \int_0^l \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l} dx = 0 ;$$

$$\int_0^l x \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l} dx = \frac{-4nml^2}{(n^2-m^2)^2\pi^2} \text{, when } n+m \text{ is an odd number;}$$

$$\int_0^l x \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l} dx = 0, \text{ when } n+m \text{ is an even number;}$$

$$\int_0^l x^2 \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l} dx = \frac{4nml^3}{(n^2-m^2)^2\pi^2} (-1)^{m+n} ;$$

$$\int_0^l x^3 \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l} dx = \frac{-6nml^4}{(n^2-m^2)^2\pi^2}\left\{1 - \frac{8(n^2+m^2)}{(n^2-m^2)^2\pi^2}\right\},$$

when $n + m$ is an odd number;

$$\int_0^l x^3 \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l} dx = \frac{6nml^4}{(n^2-m^2)^2\pi^2}, \text{ when } n+m \text{ is an even number;}$$

$$\int_0^l x^4 \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l} dx = \frac{8nml^5}{(n^2-m^2)^2\pi^2}\left\{1 - \frac{12(n^2+m^2)}{(n^2-m^2)^2\pi^2}\right\}(-1)^{m+n} ;$$

a system of homogeneous linear equations in $a_1, a_2, \cdots\cdots$ is finally obtained of the following type:

$$a_n\left\{n^2\pi^2\left[\frac{a}{2l} + \frac{b}{4} + \frac{cl}{2}\left(\frac{1}{3} - \frac{1}{2n^2\pi^2}\right) + \frac{dl^2}{8}\left(1 - \frac{3}{n^2\pi^2}\right) + \frac{el^3}{2}\left(\frac{1}{5} - \frac{1}{n^2\pi^2} + \frac{3}{2n^4\pi^4}\right)\right] - \frac{lP}{2E}\right\}$$

$$+ \sum_m^{m+n:\,even} a_m \frac{nm^3}{(n^2-m^2)^2}\left\{4cl + 6dl^2 + 8el^3\left[1 - \frac{12(n^2+m^2)}{\pi^2(n^2-m^2)^2}\right]\right\} - \sum_m^{m+n\cdot\,odd} a_m \frac{nm^3}{(n^2-m^2)^2}\left\{4b\right.$$

$$\left. + 4cl + 6dl^2\left[1 - \frac{8(n^2+m^2)}{\pi^2(n^2-m^2)^2}\right] + 8el^3\left[1 - \frac{12(n^2+m^2)}{\pi^2(n^2-m^2)^2}\right]\right\} = 0. \tag{142}$$

Taking the values of $n = 1$, $m = 2, 3, 4 \cdots$; $n = 2$, $m = 1, 3, 4 \cdots$; $n = 3$, $m = 1, 2, 4 \cdots$; $\cdots\cdots$, the above equation reduces to a group of equations as follows:

$$a_1\left\{\pi^2\left[\frac{a}{2l}+\frac{b}{4}+\frac{cl}{2}\left(\frac{1}{3}-\frac{1}{2\pi^2}\right)+\frac{dl^2}{8}\left(1-\frac{3}{\pi^2}\right)+\frac{el^3}{2}\left(\frac{1}{5}-\frac{1}{\pi^2}+\frac{3}{2\pi^4}\right)\right]-\frac{Pl}{2E}\right\}$$

$$-\frac{16}{9}a_2\left\{2b+2cl+3dl^2\left(1-\frac{40}{9\pi^2}\right)+4el^3\left(1-\frac{60}{9\pi^2}\right)\right\}+\frac{27}{32}a_3\left\{2cl\right.$$

$$\left.+3dl^2+4el^3\left(1-\frac{15}{8\pi^2}\right)\right\}-\cdots\cdots=0$$

$$\frac{a_1}{9}\left\{4b+4cl+6dl^2\left(1-\frac{40}{9\pi^2}\right)+8el^3\left(1-\frac{60}{9\pi^2}\right)\right\}-a_2\left\{2\pi^2\left[\frac{a}{2l}+\frac{b}{4}\right.\right.$$

$$\left.\left.+\frac{cl}{2}\left(\frac{1}{3}-\frac{1}{8\pi^2}\right)+\frac{dl^2}{8}\left(1-\frac{3}{4\pi^2}\right)+\frac{el^3}{2}\left(\frac{1}{5}-\frac{1}{4\pi^2}+\frac{3}{32\pi^4}\right)\right]-\frac{Pl}{4E}\right\}$$

$$+\frac{27}{25}a_3\left\{4b+4cl+6dl^2\left(1-\frac{104}{25\pi^2}\right)+8el^3\left(1-\frac{156}{25\pi^2}\right)\right\}-\cdots\cdots=0$$

$$\frac{a_1}{64}\left\{4cl+6dl^2+5el^3\left(1-\frac{15}{8\pi^2}\right)\right\}-\frac{8}{25}a_2\left\{4b+4cl+6dl^2\left(1-\frac{104}{25\pi^2}\right)\right.$$

$$\left.+8el^3\left(1-\frac{156}{25\pi^2}\right)\right\}+a_3\left\{3\pi^2\left[\frac{a}{2l}+\frac{b}{4}+\frac{cl}{2}\left(\frac{1}{3}-\frac{1}{18\pi^2}\right)+\frac{dl^2}{8}\left(1-\frac{3}{9\pi^2}\right)\right.\right.$$

$$\left.\left.+\frac{el^3}{2}\left(\frac{1}{5}-\frac{1}{2\pi^2}+\frac{1}{54\pi^4}\right)\right]-\frac{Pl}{6E}\right\}-\cdots\cdots=0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \qquad (142')$$

General equation (138) can also be obtained by applying the energy method as follows: By equating the internal work to the external work, axial force is expressed by

$$P=E\,\frac{\int_0^l I(x)\left(\frac{d^2y}{dx^2}\right)^2dx}{\int_0^l\left(\frac{dy}{dx}\right)^2dx}. \qquad (143)$$

Substituting in this expression the series (135) for y, the critical load is determined by finding such relations between the coefficients $a_1,\ a_2,\ a_3\cdots\cdots$ as to make expression (143) a minimum. The condition for the critical load to be at its minimum is

$$\frac{\partial P}{\partial a_n}=0$$

or, for simplification, denoting by N the numerator and by D the denominator in the right hand side of Eq. (143), it follows that

$$\frac{\partial}{\partial a_n}\left(\frac{N}{D}\right)=\frac{1}{D}\frac{\partial}{\partial a_n}(N)-\frac{N}{D^2}\frac{\partial}{\partial a_n}(D)=0.$$

Using the relation

$$P = E\,\frac{N}{D} \quad \text{or} \quad \frac{N}{D} = \frac{P}{E},$$

the above equation reduces to

$$\frac{\partial}{\partial a_n}(N) - \frac{P}{E}\,\frac{\partial}{\partial a_n}(D) = 0$$

or

$$\frac{\partial}{\partial a_n}\left(N - \frac{P}{E}\,D\right) = 0$$

from which one gets

$$\frac{\partial}{\partial a_n}\left\{\int_0^l I(x)\left(\frac{d^2y}{dx^2}\right)^2 dx - \frac{P}{E}\int_0^l\left(\frac{dy}{dx}\right)^2 dx\right\} = 0. \tag{144}$$

The first and second derivatives of y with respect to x, from Eq. (135), are, respectively,

$$\frac{dy}{dx} = \sum_{n=1}^{\infty} a_n\left(\frac{n\pi}{l}\right)\cos\frac{n\pi x}{l} \quad \text{and} \quad \frac{d^2y}{dx^2} = -\sum_{n=1}^{\infty} a_n\left(\frac{n\pi}{l}\right)^2 \sin\frac{n\pi x}{l}.$$

Hence

$$\left(\frac{dy}{dx}\right)^2 = \sum_n a_n^2\left(\frac{n\pi}{l}\right)^2\cos^2\frac{n\pi x}{l} + 2\sum_m^{\text{except } n} a_n\, a_m\, nm\left(\frac{\pi}{l}\right)^2\cos\frac{n\pi x}{l}\cos\frac{m\pi x}{l}$$

$$\left(\frac{d^2y}{dx^2}\right)^2 = \sum_n a_n^2\left(\frac{n\pi}{l}\right)^4\sin^2\frac{n\pi x}{l} + 2\sum_m^{\text{except } n} a_n\, a_m\, n^2 m^2\left(\frac{\pi}{l}\right)^4\sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}$$

Substituting these into Eq. (144) for $\left(\frac{d^2y}{dx^2}\right)^2$ and $\left(\frac{dy}{dx}\right)^2$ it follows that

$$\int_0^l I(x)\left\{2a_n\left(\frac{n\pi}{l}\right)^4\sin^2\frac{n\pi x}{l} + 2\sum_m^{\text{except } n} a_m\, n^2 m^2\left(\frac{\pi}{l}\right)^4\sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\right\} dx$$

$$-\frac{P}{E}\int_0^l 2a_n\left(\frac{n\pi}{l}\right)^2\cos^2\frac{n\pi x}{l} + 2\sum_m^{\text{except } n} a_m\, n\, m\left(\frac{\pi}{l}\right)^4\cos\frac{n\pi x}{l}\cos\frac{m\pi x}{l}\Big\} dx = 0.$$

Taking into account that

$$\int_0^l \cos\frac{n\pi x}{l}\cos\frac{m\pi x}{l}\,dx = 0 \quad (n \neq m), \quad \int_0^l \cos^2\frac{n\pi x}{l}\,dx = \frac{l}{2}$$

one finally gets

$$\int_0^l I(x)\left\{a_n\left(\frac{n\pi}{l}\right)^2\sin^2\frac{n\pi x}{l} + \sum_m^{\text{except } n} a_m\left(\frac{m\pi}{l}\right)^2\sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\right\} dx - a_n\,\frac{Pl}{2E} = 0$$

which agrees with Eq. (138). In the present problem, however, the process of calculation by the energy method is more labourious than by Fourier sine transformation method.

For examples of the application of Eq. (142), take a bar whose moment of inertia of the cross section varies as a certain power of the distance from the lower end (Fig. 6) so that the moment of inertia of any cross section at x is

$$I(x) = I_c \left(\frac{\lambda + x}{\lambda + \frac{l}{2}} \right)^n \tag{145}$$

where I_c is the moment of inertia at the middle of the bar.
By taking various values for n, various shapes of the column are obtained. Assuming that $n = 1$ in Eq. (145), one gets the case of a column in the form of a plate of constant thickness (Fig. 6. I) and of varying width h.

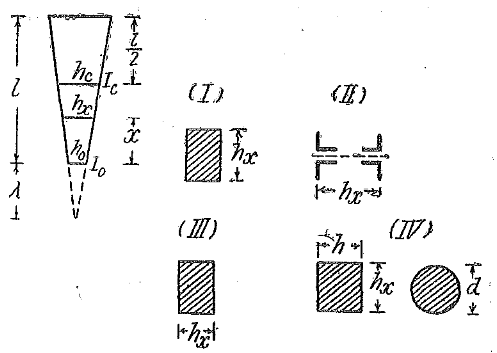

Fig. 6

The assumption $n = 2$ represents, with sufficient accuracy, the case of a built-up column consisting of four angles connected by lacing bars (Fig. 6. II). In this case the cross sectional area of the column remains constant and the moment of inertia is approximately proportional to the square of the distance of the centroids of the angles from the axes of symmetry of the cross section.

By taking $n = 3$ one gets the case of a column in the form of a plate of constant width and of varying thickness h (Fig. 6.III).

By taking $n = 4$ one gets such cases as a solid truncated cone or a pyramid (Fig. 6. IV).

Then the differential equation of the curve, in general, can be solved by means of Bessel's functions and in the particular case of $n = 2$, the solution can be obtained in a simple manner. In this paper, however, the author shows the solution by means of Fourier sine transformation method using the result given by Eq. (142).

The coefficients in $I(x) = a + bx + cx^2 + dx^3 + ex^4$, corresponding to each case, become as follows:

I. In the case of $n = 1$;

$$a = I_c \frac{\lambda}{\left(\lambda + \frac{l}{2}\right)}, \quad b = I_c \frac{1}{\left(\lambda + \frac{l}{2}\right)}, \qquad c = d = e = 0.$$

II. In the case of $n = 2$;

$$a = I_c \frac{\lambda^2}{\left(\lambda + \frac{l}{2}\right)^2}, \quad b = I_c \frac{2\lambda}{\left(\lambda + \frac{l}{2}\right)^2}, \quad c = I_c \frac{1}{\left(\lambda + \frac{l}{2}\right)^2}, \quad d = e = 0.$$

III. In the case of $n = 3$;

$$a = I_c \frac{\lambda^3}{\left(\lambda + \frac{l}{2}\right)^3}, \quad b = I_c \frac{3\lambda^2}{\left(\lambda + \frac{l}{2}\right)^3}, \quad c = I_c \frac{3\lambda}{\left(\lambda + \frac{l}{2}\right)^3}, \quad d = I_c \frac{1}{\left(\lambda + \frac{l}{2}\right)^3}, \quad e = 0.$$

IV. In the case of $n = 4$;

$$a = I_c \frac{\lambda^4}{\left(\lambda + \frac{l}{2}\right)^4}, \quad b = I_c \frac{4\lambda^3}{\left(\lambda + \frac{l}{2}\right)^4}, \quad c = I_c \frac{6\lambda^2}{\left(\lambda + \frac{l}{2}\right)^4}, \quad d = I_c \frac{4\lambda}{\left(\lambda + \frac{l}{2}\right)^4},$$

$$e = I_c \frac{1}{\left(\lambda + \frac{l}{2}\right)^4}.$$

Substituting these into Eq. (142) and taking the two terms in expression (135) with cofficients a_1 and a_2, the equations, from which the critical loads corresponding to each are to be approximately determined, are obtained as follows:

I.
$$(\alpha - k)(4\alpha - k) - \left(\frac{32}{9}\right)^2 = 0 \tag{146}$$

in which

$$\alpha = \pi^2 \left(\frac{\lambda}{l} + \frac{1}{2}\right), \qquad k = \frac{Pl\left(\lambda + \frac{l}{2}\right)}{EI_c}.$$

II.
$$\left(\alpha - \frac{1}{2} - k\right)\left(4\alpha - \frac{1}{2} - k\right) - \left(\frac{16}{9}\right)^2 \beta^2 = 0$$

in which

$$\alpha = \pi^2 \left(\frac{\lambda^2}{l^2} + \frac{\lambda}{l} + \frac{1}{3}\right), \quad \beta = 2\left(2\frac{\lambda}{l} + 1\right), \quad k = \frac{P\left(\lambda + \frac{l}{2}\right)^2}{EI_c}.$$

III.
$$(\alpha - \beta - k)(4\alpha - \beta - k) - \left(\frac{16}{9}\right)^2 \delta^2 = 0 \tag{148}$$

in which

$$\alpha = \pi^2 \left(\frac{\lambda^3}{l^3} + \frac{3}{2}\frac{\lambda^2}{l^2} + \frac{\lambda}{l} + \frac{1}{4}\right), \quad \beta = \frac{3}{2}\frac{\lambda}{l} + \frac{3}{4},$$

$$\delta = 6\frac{\lambda^2}{l^2} + 6\frac{\lambda}{l} + 3\left(1 - \frac{40}{9\pi^2}\right), \quad k = \frac{P\left(\lambda + \frac{l}{2}\right)^3}{lEI_c}.$$

$$\text{IV.} \qquad (\alpha - \beta + \gamma - k)\left(4\alpha - \beta + \frac{1}{4}\gamma - k\right) - \left(\frac{16}{9}\right)^2 \delta^2 = 0 \tag{149}$$

in which

$$\alpha = \pi^2\left(\frac{\lambda^4}{l^4} + 2\frac{\lambda^3}{l^3} + 2\frac{\lambda^2}{l^2} + \frac{\lambda}{l} + \frac{1}{5}\right), \quad \beta = 3\frac{\lambda^2}{l^2} + 3\frac{\lambda}{l} + 1, \ \gamma = \frac{3}{2\pi^2},$$

$$\delta = 8\frac{\lambda^3}{l^3} + 12\frac{\lambda^2}{l^2} + 12\frac{\lambda}{l}\left(1 - \frac{40}{9\pi^2}\right) + 4\left(1 - \frac{60}{9\pi^2}\right), \quad k = \frac{P\left(\lambda + \frac{l}{2}\right)^4}{l^2 EI_c}.$$

Solving the above equations, the smaller root gives the critical load in each case. This value can be represented in general by the formula

$$P_{cr} = C\frac{\pi^2 EI_c}{l^2} \tag{150}$$

in which $\frac{\pi^2 EI_c}{l^2}$ is the critical load of a bar with uniform cross section having the value of I_c. Several values of the factor C are given below in Table 1, in which I_0 is the moment of inertia of the bottom cross section. When the ratio I_0/I_c approaches zero, the values of C should be determined taking the three or more terms in expression (135) and therefore the values of C in such cases were calculated taking the three or more terms.

Table 1. Critical Loads for Unsymmetrical Bars with Variable Cross Section. coeff: $\frac{\pi^2 EI_c}{l^2}$

n \ I_0/I_c	0.01	0.02	0.04	0.06	0.08	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	0.754	0.764	0.780	0.794	0.807	0.819	0.868	0.904	0.932	0.954	0.971	0.985	0.994	0.998
2	0.456	0.519	0.595	0.646	0.684	0.715	0.820	0.881	0.922	0.951	0.971	0.985	0.994	0.998
3	0.336	0.419	0.519	0.585	0.634	0.675	0.802	0.874	0.920	0.950	0.971	0.985	0.994	0.998
4	0.284	0.373	0.482	0.555	0.610	0.653	0.792	0.869	0.918	0.950	0.971	0.985	0.994	0.998

2. Symmetrical Bars with Variable Cross Section.

In the case in which the variation of the moment of inertia of the cross section is continuous throughout the whole length of a bar, the formulae obtained in the previous article can be used. When the variation, however, is discontinuous at the middle of a bar, the integration in Eq. (138) should be performed subdiving it into two parts.

Thus, taking in account that

$$\int_0^{\frac{l}{2}} \sin\frac{m\pi x}{l}\sin\frac{n\pi x}{l}\,dx = 0 \quad (m \neq n), \quad \int_0^{\frac{l}{2}} \sin^2\frac{n\pi x}{2}\,dx = \frac{l}{4}$$

the formula corresponding to Eq. (138) becomes

$$\int_0^{\frac{l}{2}} I(x)\left\{a_n\left(\frac{n\pi}{l}\right)^2\sin^2\frac{n\pi x}{l}+\sum_m^{except.n} a_m\left(\frac{m\pi}{l}\right)^2\sin\frac{m\pi x}{l}\sin\frac{n\pi x}{l}\right\}dx - a_n\frac{Pl}{4E}=0\,.$$

In the case of a symmetrical bar, considering a symmetrical shape of the buckled curve, all the values of n and m are to be taken odd only.

If the variation of the moment of inertia is

$$I(x)=a+bx+cx^2+dx^3+ex^4 \quad \left(x=0\sim\frac{l}{2}\right), \qquad (152)$$

using the formulae in which n and m are obd numbers:

$$\int_0^{\frac{l}{2}} x\sin^2\frac{n\pi x}{l}dx=\frac{l^2}{4}\left(\frac{1}{4}+\frac{1}{n^2\pi^2}\right);\quad \int_0^{\frac{l}{2}} x^2\sin^2\frac{n\pi x}{l}=\frac{l^3}{8}\left(\frac{1}{6}+\frac{1}{n^2\pi^2}\right);$$

$$\int_0^{\frac{l}{2}} x^3\sin^2\frac{n\pi x}{l}dx=\frac{l^4}{8}\left(\frac{1}{16}+\frac{3}{4n^2\pi^2}-\frac{3}{n^4\pi^4}\right);$$

$$\int_0^{\frac{l}{2}} x^4\sin^2\frac{n\pi x}{l}dx=\frac{l^5}{8}\left(\frac{1}{40}+\frac{1}{2n^2\pi^2}-\frac{3}{n^4\pi^4}\right);$$

$$\int_0^{\frac{l}{2}} x\sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}dx=\frac{l^2}{(n+m)^2\pi^2}\quad\text{when } \frac{n+m}{2} \text{ is an odd number;}$$

$$\int_0^{\frac{l}{2}} x\sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}dx=\frac{-l^2}{(n-m)^2\pi^2}\quad\text{when } \frac{n+m}{2} \text{ is an even number;}$$

$$\int_0^{\frac{l}{2}} x^2\sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}dx=-\frac{(n^2+m^2)\,l^3}{(n^2-m^2)^2\pi^2}(-1)^{\frac{n+m}{2}}$$

$$\int_0^{\frac{l}{2}} x^3\sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}dx=3l^4\left\{\frac{n^2+m^2}{4(n^2-m^2)^2\pi^2}-\frac{2}{(n+m)^4\pi^4}\right\}$$

$$\text{when } \frac{n+m}{2} \text{ is an odd number;}$$

$$\int_0^{\frac{l}{2}} x^3\sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}dx=-3l^4\left\{\frac{n^2+m^2}{4(n^2-m^2)^2\pi^2}-\frac{2}{(n-m)^4\pi^4}\right\}$$

$$\text{when } \frac{n+m}{2} \text{ is an even number;}$$

$$\int_0^{\frac{l}{2}} x^4\sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}dx=-\frac{2l^5}{(n^2-m^2)^2\pi^2}\left\{\frac{n^2+m^2}{4}-\frac{6(n^4+6n^2m^2+m^4)}{(n^2-m^2)^2\pi^2}\right\}(-1)^{\frac{n+m}{2}}$$

the formula corresponding to Eq. (142) is finally obtained of the following type:

$$a_n\left\{n^2\pi^2\left[\frac{a}{4l}+\frac{b}{4}\left(\frac{1}{4}+\frac{1}{n^2\pi^2}\right)+\frac{cl}{8}\left(\frac{1}{6}+\frac{1}{n^2\pi^2}\right)+\frac{dl^2}{8}\left(\frac{1}{16}+\frac{3}{4n^2\pi^2}-\frac{3}{n^4\pi^4}\right)\right.\right.$$

$$+\frac{el^3}{8}\left(\frac{1}{40}+\frac{1}{2n^2\pi^2}-\frac{3}{n^4\pi^4}\right)\Big]-\frac{Pl}{4E}\Big\}+\sum_m^{\frac{m+n}{2}:odd} a_m\frac{m^2}{(n+m)^2}\Big\{b+\frac{cl\,(n^2+m^2)}{(n-m)^2}$$

$$+3dl^2\left[\frac{n^2+m^2}{4\,(n-m)^2}-\frac{2}{(n+m)^2\,\pi^2}\right]+\frac{2\,el^3}{(n-m)^2}\left[\frac{n^2+m^2}{4}-\frac{6\,(n^4+6n^2m^2+m^4)}{(n^2-m^2)^2\,\pi^2}\right]\Big\}$$

$$-\sum_m^{\frac{m+n}{2}:even} a_m\frac{m^2}{(n-m)^2}\Big\{b+\frac{cl\,(n^2+m^2)}{(n+m)^2}+3dl^2\left[\frac{n^2+m^2}{4\,(n+m)^2}-\frac{2}{(n-m)^2\,\pi^2}\right]$$

$$+\frac{2el^3}{(n+m)^2}\left[\frac{n^2+m^2}{4}-\frac{6\,(n^4+6n^2m^2+m^4)}{(n^2-m^2)^2\,\pi^2}\right]\Big\}=0$$

$$(n,\ m=odd). \qquad (153)$$

Taking the values of $n=1$, $m=3, 5, 7\cdots$; $n=3$, $m=1, 5, 7\cdots$; $n=5, m=1, 3, 7\cdots$; $\cdots$,
the above equation reduces to a group of equations as follows:

$$a_1\Big\{\pi^2\Big[\frac{a}{l}+b\left(\frac{1}{4}+\frac{1}{\pi^2}\right)+\frac{cl}{2}\left(\frac{1}{6}+\frac{1}{\pi^2}\right)+\frac{dl^2}{2}\left(\frac{1}{16}+\frac{3}{4\pi^2}-\frac{3}{\pi^4}\right)$$

$$+\frac{el^3}{2}\left(\frac{1}{40}+\frac{1}{2\pi^2}-\frac{3}{\pi^4}\right)\Big]-\frac{Pl}{E}\Big\}-9a_3\Big\{b+\frac{5}{8}cl+\frac{3}{4}dl^2\left(\frac{5}{8}-\frac{2}{\pi^2}\right)$$

$$+\frac{el^3}{8}\left(\frac{5}{2}-\frac{51}{4\pi^2}\right)\Big\}+\frac{25}{9}a_5\Big\{b+\frac{13}{8}cl+\frac{3}{4}dl^2\left(\frac{13}{8}-\frac{2}{9\pi^2}\right)$$

$$+\frac{el^3}{8}\left(\frac{13}{2}-\frac{97}{12\pi^2}\right)\Big\}-\cdots=0$$

$$a_1\Big\{b+\frac{5}{8}cl+\frac{3}{4}dl^2\left(\frac{5}{8}-\frac{2}{\pi^2}\right)+\frac{el^3}{8}\left(\frac{5}{2}-\frac{51}{4\pi^2}\right)\Big\}-a_3\Big\{9\pi^2\Big[\frac{a}{l}+b\left(\frac{1}{4}+\frac{1}{9\pi^2}\right)$$

$$+\frac{cl}{2}\left(\frac{1}{6}+\frac{1}{9\pi^2}\right)+\frac{dl^2}{2}\left(\frac{1}{16}+\frac{1}{12\pi^2}-\frac{1}{27\pi^4}\right)+\frac{el^3}{2}\left(\frac{1}{40}+\frac{1}{18\pi^2}-\frac{1}{27\pi^4}\right)\Big]-\frac{Pl}{E}\Big\}$$

$$+25\,a_5\Big\{b+\frac{17}{32}cl+\frac{3}{4}dl^2\left(\frac{17}{32}-\frac{2}{\pi^2}\right)+\frac{el^3}{32}\left(\frac{17}{2}-\frac{771}{16\pi^2}\right)\Big\}-\cdots=0$$

$$\frac{a_1}{9}\Big\{b+\frac{13}{8}cl+\frac{3}{4}dl^2\left(\frac{13}{8}-\frac{2}{9\pi^2}\right)+\frac{el^3}{8}\left(\frac{13}{2}-\frac{97}{12\pi^2}\right)\Big\}-9a_3\Big\{b+\frac{17}{32}cl$$

$$+\frac{3}{4}dl^2\left(\frac{17}{32}-\frac{2}{\pi^2}\right)+\frac{el^3}{32}\left(\frac{17}{2}-\frac{771}{16\pi^2}\right)\Big\}+a_5\Big\{25\pi^2\Big[\frac{a}{l}+b\left(\frac{1}{4}+\frac{1}{25\pi^2}\right)$$

$$+\frac{cl}{2}\left(\frac{1}{16}+\frac{1}{25\pi^2}\right)+\frac{dl^2}{2}\left(\frac{1}{16}+\frac{3}{100\pi^2}-\frac{3}{625\pi^4}\right)+\frac{el^3}{2}\left(\frac{1}{40}+\frac{1}{50\pi^2}-\frac{3}{625\pi^4}\right)\Big]-\frac{Pl}{E}\Big\}$$

$$-\cdots=0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \qquad (153')$$

Now, for example, take a bar whose moment of inertia of the cross section varies as a certain power of the distance from the lower end (Fig. 7) so that the moment of inertia of any cross section at x is

$$I(x) = I_c \left(\frac{\lambda + x}{\lambda + \frac{l}{2}} \right)^n, \quad \left(x = 0 \sim \frac{l}{2} \right) \qquad (154)$$

in which I_c is the moment of inertia at the middle of the bar.

In the same manner as in an unsymmetrical bar, taking two terms in expression (135) with a_1 and a_3, the equations, from which the critical loads corresponding to the cases of $n = 1 \sim 4$ are to be approximately determined, are obtained as follows:

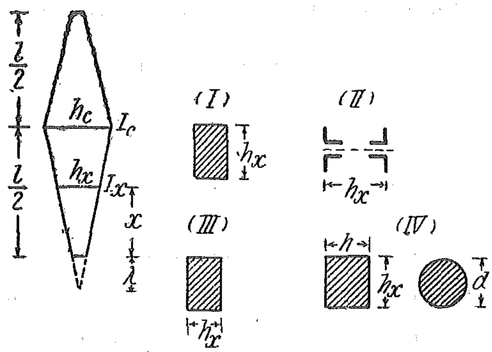

Fig. 7

I. $(\alpha+1-k)(9\alpha+1-k)-9=0 \qquad (155)$

in which

$$\alpha = \pi^2 \left(\frac{\lambda}{l} + \frac{1}{4} \right), \quad k = \frac{Pl\left(\lambda + \frac{l}{2}\right)}{EI_c}.$$

II. $$(\alpha + \beta - k)(9\alpha + \beta - k) - 9\left(\beta + \frac{1}{8}\right)^2 = 0 \qquad (156)$$

in which

$$\alpha = \pi^2 \left(\frac{\lambda^2}{l^2} + \frac{\lambda}{2l} + \frac{1}{12} \right), \quad \beta = \frac{2\lambda}{l} + \frac{1}{2}, \quad k = \frac{P\left(\lambda + \frac{l}{2}\right)^2}{EI_c}.$$

III. $$(\alpha + \beta - \gamma - k)\left(9\alpha + \beta - \frac{\gamma}{9} - k\right) - 9\delta^2 = 0 \qquad (157)$$

in which

$$\alpha = \pi^2 \left(\frac{\lambda^3}{l^3} + \frac{3}{4}\frac{\lambda^2}{l^2} + \frac{1}{4}\frac{\lambda}{l} + \frac{1}{32} \right), \quad \beta = \frac{3\lambda^2}{l^2} + \frac{3}{2}\frac{\lambda}{l} + \frac{3}{8},$$

$$\gamma = \frac{3}{2\pi^2}, \quad \delta = 3\frac{\lambda^2}{l^2} + \frac{15}{8}\frac{\lambda}{l} + \frac{3}{4}\left(\frac{5}{8} - \frac{2}{\pi^2} \right), \quad k = \frac{P\left(\lambda + \frac{l}{2}\right)^3}{lEI_c}.$$

IV. $$(\alpha + \beta - \gamma - k)\left(9\alpha + \beta - \frac{\gamma}{9} - k\right) - 9\delta^2 = 0 \qquad (158)$$

in which

$$\alpha = \pi^2 \left(\frac{\lambda^4}{l^4} + \frac{\lambda^3}{l^3} + \frac{\lambda^2}{2l^2} + \frac{\lambda}{8l} + \frac{1}{80} \right), \quad \beta = \frac{4\lambda^3}{l^3} + \frac{3\lambda^2}{l^2} + \frac{3\lambda}{2l} + \frac{1}{4},$$

$$\gamma = \frac{3}{2\pi^3}\left(4\frac{\lambda}{l}+1\right), \quad \delta = \frac{4\lambda^3}{l^3} + \frac{15}{4}\frac{\lambda^2}{l^2} + 3\frac{\lambda}{l}\left(\frac{5}{8}-\frac{2}{\pi^2}\right) + \frac{1}{8}\left(\frac{5}{2}-\frac{51}{4\pi^2}\right),$$

$$k = \frac{P\left(\lambda+\frac{l}{2}\right)^4}{l^2 EI_c}.$$

The values of the factor C in this case become as shown in Table 2. The values of C in the case in which the ratio I_0/I_c approaches zero were also calculated taking the three or more terms in the series of y.

Table 2. Critical Loads for Symmetrical Bars with Variable Cross Section. Coeff.: $\frac{\pi^2 EI_c}{l^2}$

n \ I_0/I_c	0.01	0.02	0.04	0.06	0.08	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	0.595	0.604	0.618	0.632	0.645	0.656	0.711	0.757	0.798	0.836	0.872	0.906	0.938	0.970
2	0.350	0.394	0.450	0.489	0.520	0.547	0.645	0.715	0.771	0.820	0.863	0.902	0.937	0.970
3	0.258	0.313	0.386	0.436	0.473	0.508	0.622	0.709	0.762	0.814	0.861	0.899	0.935	0.970
4	0.218	0.280	0.356	0.410	0.452	0.486	0.610	0.694	0.758	0.811	0.858	0.898	0.935	0.970

V. APPLICATION OF FOURIER SINE TRANSFORMATION METHOD TO THE STUDY OF STABILITY OF THE UPPER CHORD OF A LOW-TRUSS BRIDGE.

As a special example among the buckling problems, take the upper chord of a low-truss bridge. In a low-truss bridge, there is no bracing in the upper horizontal plane (Fig. 8) and the upper chord is in the condition of a compressed bar, the lateral buckling of which is resisted by the elastic reactions of the vertical and diagonal members. At the supports there are usually frames of considerable rigidity so that the ends of the chord may be considered as immovable in a lateral direction. Thus the upper chord may be considered as a bar with hinged ends compressed by forces distributed along its length and elastically supported at intermediate points.

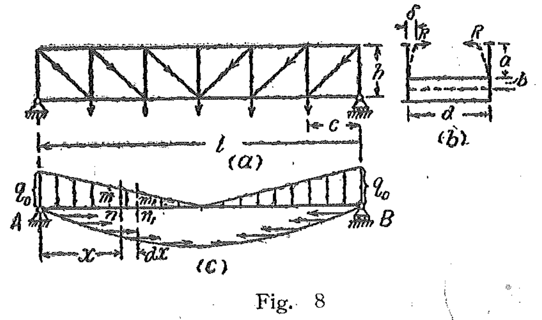

Fig. 8

A general method of solving problems of this kind is of the same sort as used in the case of continuous beams on elastic supports. However, the amount of work necessary to obtain the critical value of the compressive force increases rapidly with the number of elastic supports. The stability of the compressed chord can be increased by increasing the rigidity of the lateral supports.

If the proportions of the compressed chord and verticals of the bridge are such that the half-wave length of the buckled chord is large in comparison with one panel length of the bridge, a great simplification of the problem can be obtained by replacing the elastic supports by an equivalent elastic foundation and replacing the concentrated compressive forces, applied at the joints, by a continuously distributed load.

Assuming that the bridge is uniformly loaded, the compressive forces transmitted to the chord by the diagonals are proportional to the distances from the middle of the span, and the equivalent compressive load distribution is shown by the shaded area in Fig.8.

In calculating the modulus β of the elastic foundation, equivalent to the elastic resistance of the verticals, it is necessary to establish the relation between the force R, applied at the top of a vertical and the deflection that would be produced if the upper chord were removed. If only bending of the vertical is taken into account, then

$$\delta = \frac{Ra^3}{3EI_1},$$

where I_1 is the moment of inertia of one vertical. Taking into account the bending of the floor beam, and using notations indicated in the figure, it follows that

$$\delta = \frac{Ra^3}{3EI_1} + \frac{R(a+b)^2 d}{2EI_2},$$

where I_2 is the moment of inertia of the cross section of the floor beam. The force necessary to produce the deflection δ equal to unity is then,

$$R_0 = \frac{1}{\dfrac{a^3}{3EI_1} + \dfrac{(a+b)^2 d}{2EI_2}}. \tag{159}$$

and the modulus of the equivalent elastic foundation is

$$\beta = \frac{R_0}{c} \tag{160}$$

where c is the distance between verticals.

In this manner the problem of the stability of the compressed chord of the bridge is reduced to one of buckling of a bar with hinged ends, supported laterally by a continuous elastic medium and axially loaded by a continuous load, the intensity of which is proportional to the distance from the middle.

This problem was solved by Timoshenko by using the energy method. The same result can be obtained more easily by applying the Fourier sine transformation method.

With regard to the differental equation of the deflection curve of the buckled bar; it is noted that the intensity of the distributed compressive load at any cross section, distance x from the left support, is

$$q = q_0\left(1 - \frac{2x}{l}\right), \tag{161}$$

in which q_0 is the intensity of load at the ends. For a truss with parallel chords and a large number of panels, it can be concluded from elementary statics that the maximum intensity of the axial load can be assumed as follows:

$$q_0 = \frac{Q}{2h}, \tag{162}$$

in which Q is the total load on one truss and h the depth of the truss.

Accordingly, the total compressive force at any cross section, distance x from the left support, becomes

$$P = \frac{1}{2}(q + q_0)\,x = q_0\left(x - \frac{x^2}{l}\right),$$

from which the shearing force due to the compressive force is

$$S = q_0\left(x - \frac{x^2}{l}\right)\frac{dy}{dx}. \tag{163}$$

If the shearing force due to only compressive force is taken into considertion, then

$$EI\frac{d^3y}{dx^3} = -\,q_0\left(x - \frac{x^2}{l}\right)\frac{dy}{dx}$$

Differentiating this with respect to x, it follows that

$$EI\frac{d^4y}{dx^4} + q_0\left(1 - \frac{2x}{l}\right)\frac{dy}{dx} + q_0\left(x - \frac{x^2}{l}\right)\frac{d^2y}{dx^2} = 0.$$

Adding the effect of the elastic medium, the differential equation of the deflection curve of the buckled bar is finally obtained as follow:

$$EI\frac{d^4y}{dx^4} + q_0\left(x - \frac{x^2}{l}\right)\frac{d^2y}{dx^2} + q_0\left(1 - \frac{2x}{l}\right)\frac{dy}{dx} + \beta y = 0. \tag{164}$$

This equation can be also obtained by considering the equilibrium of a differential length of the buckled bar. In the above equation, the cross section of the upper chord is assumed as constant along its length and the modulus of elastic medium is also considered as constant. The differential equation of the deflection curve of the buckled bar is no longer a simple equation with constant cofficients. In its solution the Fourier sine transformation method can also be used to advantage.

The deflection curve of the buckled bar in the case of hinged ends can be represented by the series

$$y = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l}. \tag{165}$$

Substituting this into Eq. (164) it follows that

$$EI \sum_{n=1}^{\infty} \left(\frac{nx}{l}\right)^4 a_n \sin \frac{n\pi x}{l} - q_0\left(x - \frac{x^2}{l}\right) \sum_{n=1}^{\infty} \left(\frac{n\pi}{l}\right)^2 a_n \sin \frac{n\pi x}{l}$$
$$+ q_0 \left(1 - \frac{2x}{l}\right) \sum_{n=1}^{\infty} \left(\frac{n\pi}{l}\right) a_n \cos \frac{n\pi x}{l} + \beta \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} = 0$$

Multipling by $\sin \frac{n\pi x}{l}$ and integrating from one end to the other of the bar, one gets

$$\int_0^l \left\{ EI \sum_{n=1}^{\infty} \left(\frac{n\pi}{l}\right)^4 a_n \sin \frac{n\pi x}{l} - q_0\left(x - \frac{x^2}{l}\right) \sum_{n=1}^{\infty} \left(\frac{n\pi}{l}\right)^2 a_n \sin \frac{n\pi x}{l}\right.$$
$$\left. + q_0 \left(1 - \frac{2x}{l}\right) \sum_{n=1}^{\infty} \left(\frac{n\pi}{l}\right) a_n \cos \frac{n\pi x}{l} + \beta \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l}\right\} \sin \frac{n\pi x}{l}\, dx = 0. \tag{166}$$

Using the formulae:

$$\int_0^l \sin \frac{m\pi x}{l} \sin \frac{n\pi x}{l}\, dx = 0 \quad (m \neq n); \quad \int_0^l \sin^2 \frac{n\pi x}{l}\, dx = \frac{l}{2};$$

$$\int_0^l x \sin \frac{m\pi x}{l} \sin \frac{n\pi x}{l}\, dx = \left\{(-1)^{m+n} - 1\right\} \frac{2nml^2}{(n^2-m^2)^2 \pi^2};$$

$$\int_0^l x \sin^2 \frac{n\pi x}{l}\, dx = \frac{l^2}{4}; \quad \int_0^l x^2 \sin^2 \frac{n\pi x}{l}\, dx = \frac{l^3}{2}\left(\frac{1}{3} - \frac{1}{2n^2\pi^2}\right);$$

$$\int_0^l x^2 \sin \frac{m\pi x}{l} \sin \frac{n\pi x}{l}\, dx = (-1)^{m+n} \frac{4nml^3}{(n^2-m^2)^2 \pi^2}:$$

$$\int_0^l \cos \frac{n\pi x}{l} \sin \frac{n\pi x}{l}\, dx = 0; \quad \int_0^l \cos \frac{m\pi x}{l} \sin \frac{n\pi x}{l}\, dx = \left\{(-1)^{m+n} - 1\right\} \frac{nl}{(m^2-n^2)\pi};$$

$$\int_0^l x \cos \frac{n\pi x}{l} \sin \frac{n\pi x}{l}\, dx = -\frac{1}{2} \frac{l^2}{2n\pi};$$

$$\int_0^l x \cos \frac{m\pi x}{l} \sin \frac{n\pi x}{l}\, dx = (-1)^{m+n} \frac{nl^2}{(m^2-n^2)\pi};$$

a system of homogeneous linear equations in a_1, a_2, a_3, $\cdots$ of the following type is finally obtained:

$$a_n\left[(n^4+r)\pi^2-2\alpha\left(\frac{n^2\pi^2}{3}-1\right)\right]+16\alpha\sum_{m}^{m+n:\text{even}}\frac{mn\,(m^2+n^2)}{(m^2-n^2)^2}=0 \tag{167}$$

in which, for simplification, the following notations are used;

$$\alpha=\frac{q_0 l}{4}\div\frac{\pi^2 EI}{l^2},\quad r=\frac{\beta l^4}{\pi^4 EI}. \tag{168}$$

The summation in the second term of the above epuation is extended over all values of m different from n such that $m+n$ is an even number. Thus, the series represented by Eq. (167) can be subdivided into two groups, one containing the coefficients a_m with values of m taken odd and the second with all values of m taken even.

The equations of the first group are.

$$\left.\begin{aligned}
&\left[(1+r)\pi^2-2\alpha\left(\frac{\pi^2}{3}-1\right)\right]a_1+\alpha\left(\frac{15}{2}a_3+\frac{65}{18}a_5+\frac{175}{72}a_7+\cdots\right)=0\\
&\frac{15}{2}\alpha\,a_1+\left[(3^4+r)\pi^2-2\alpha(3\pi^2-1)\right]a_3+\alpha\left(\frac{255}{8}a_5+\frac{609}{50}a_7+\cdots\right)=0\\
&\frac{65}{18}\alpha\,a_1+\frac{255}{8}\alpha\,a_3+\left[(5^4+r)\pi^2-2\alpha\left(\frac{25}{3}\pi^2-1\right)\right]a_5+\alpha\left(\frac{1295}{18}a_7+\cdots\right)=0\\
&\frac{175}{72}\alpha\,a_1+\frac{609}{50}\alpha\,a_3+\frac{1295}{18}\alpha\,a_5+\left[(7^4+r)\pi^2-2\alpha\left(\frac{49}{3}\pi^2-1\right)\right]a_7+\cdots=0\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\right\} \tag{169}$$

The equations of the second group are:

$$\left.\begin{aligned}
&\left[(2^4+r)\pi^2-2\alpha\left(\frac{4}{3}\pi^2-1\right)\right]a_2+\alpha\left(\frac{160}{9}a_4+\frac{15}{2}a_6+\cdots\right)=0\\
&\frac{160}{9}\alpha\,a_2+\left[(4^4+r)\pi^2-2\alpha\left(\frac{16}{3}\pi^2-1\right)\right]a_4+\alpha\left(\frac{1248}{25}a_6+\cdots\right)=0\\
&\frac{15}{2}\alpha\,a_2+\frac{1248}{25}\alpha\,a_4+\left[(6^4+r)\pi^2-2\alpha\left(\frac{36}{3}\pi^2-1\right)\right]a_6+\cdots=0\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\right\} \tag{170}$$

Buckling of the chord becomes possible when one of the above two systems of equations give for coefficients a_m a solution different from zero, i. e., when the determinant of system (169) or of system (170) becomes equal to zero. Thus, one gets the determinant equations, from which the critical load is given, as follows:

$$\begin{vmatrix} A_1 & \frac{15}{2}\alpha & \frac{65}{18}\alpha & \frac{175}{72}\alpha & \cdots \\ \frac{15}{2}\alpha & A_3 & \frac{255}{8}\alpha & \frac{609}{50}\alpha & \cdots \\ \frac{65}{18}\alpha & \frac{255}{8}\alpha & A_5 & \frac{1295}{18}\alpha & \cdots \\ \frac{175}{72}\alpha & \frac{609}{50}\alpha & \frac{1295}{18}\alpha & A_7 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0 \qquad (171)$$

in which

$$A_n = \left[(n^4 + \gamma)\pi^2 - 2\alpha\left(\frac{n^2}{3}\pi^2 - 1\right)\right]$$

and

$$\begin{vmatrix} A_2 & \frac{160}{9}\alpha & \frac{15}{2}\alpha & \frac{1088}{222}\alpha & \cdots \\ \frac{160}{9}\alpha & A_4 & \frac{1248}{25}\alpha & \frac{160}{9}\alpha & \cdots \\ \frac{15}{2}\alpha & \frac{1248}{25}\alpha & A_6 & \frac{4800}{49}\alpha & \cdots \\ \frac{1088}{222}\alpha & \frac{160}{9}\alpha & \frac{4800}{49}\alpha & A_8 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0 \qquad (172)$$

in which

$$A_n = \left[(n_4 + \gamma)\pi^2 - 2\alpha\left(\frac{n^2}{3}\pi^2 - 1\right)\right].$$

Eq. (171) corresponds to a symmetrical shape of the buckled bar; Eq. (172) corresponds to an unsymmetrical shape of the buckled bar.

In the case in which the rigidity of the elastic medium is very small, the deflection curve of the buckled bar has only one half-wave and is symmetrical with respect to the middle, Eq. (171) should be used. The first approximation is obtained by taking only the first term in the series (165) and putting $a_3 = a_5 \cdots = 0$. Then, Eq. (171) becomes

$$(1 + \gamma)\pi^2 - 2\alpha\left(\frac{\pi^2}{3} - 1\right) = 0 \qquad (173)$$

from which

$$\alpha = \frac{\pi^2(1+\gamma)}{2\left(\frac{1}{3}\pi^2 - 1\right)}.$$

Using notations (168), it finally follows that

$$\left(\frac{q_0 l}{4}\right)_c = \frac{\pi^2(1+\gamma)}{2\left(\frac{1}{3}\pi^2 - 1\right)} \frac{\pi^2 EI}{l^2} \tag{174}$$

To get a better approximate result for the critical compressive force, the two terms in expression (165) with coefficients a_1 and a_3 are taken. The corresponding equation, from (171), is

$$\left[\pi^2 - 2\alpha\left(\frac{\pi^2}{3} - 1\right)\right]\left[81\pi^2 - 2\alpha(3\pi^2 - 1)\right] - \left(\frac{15}{2}\right)^2 \alpha^2 = 0. \tag{175}$$

Sovling this equation for α, the critical load may be obtained.

Where a greater restraint ($\gamma > 3$) is supplied by the vertical members of the truss, the buckled form of the chord may have two half-waves and an inflection point occurs at the middle of the bar. To calculate the critical load in such a case, Eq (172) should be used. With a further increase of γ, the buckled bar has three half-waves, and Eq. (171) should be again used in calculating the critical value of the compressive load.

In all these cases the critical load can be represented by the equation

$$\left(\frac{q_0 l}{4}\right)_{cr} = C\frac{\pi^2 EI}{l^2} \tag{176}$$

in which the coefficient C depends on the rigidity of the elastic medium. Several values of C calculated by Timoshenko are given in the following table:

Table 3. Values of C.

$\frac{\beta l^4}{16EI}$	0	5	10	15	22.8	56.5	100	162.8	200	300	500	1000
C	2.06	3.63	5.10	6.37	7.58	9.51	11.9	14.9	16.5	19.8	24.0	33.0

The method developed above for the case of a bar of uniform cross section supported by an elastic medium of a uniform rigidity along the length of the bar can be extended to include cases of chords of variable cross section and cases where the rigidities of the elastic supports vary along the length.

VI. APPLICATION OF FOURIER SINE TRANSFORMATION METHOD TO THE STUDY OF TRANSVERSE OSCILLATION OF BARS WITH VARIABLE CROSS SECTION.

I. Unsymmetrical Bars with Variable Cross Section.

The differential equation of the maximum deflection curve in the transverse oscillation of bar with variable cross section is representented by

$$\frac{d^2}{dx^2}\left\{EI(x)\frac{d^2y}{dx^2}\right\} - \frac{q(x)}{g}4n_0^2\pi^2 y = 0 \tag{177}$$

in which the following notations are used:

$q(x) =$ dead load per unit length of the bar including live load which oscillates with the bar;

$g =$ gravity acceleration;

$n_0 =$ frequency of free oscillation.

Assuming that $q(x)$ is constant along the length of the bar and using the notation of

$$k = \frac{4n_0^2\pi^2 q}{Eg} \tag{178}$$

the above equation becomes

$$\frac{d^2}{dx^2}\left\{I(x)\frac{d^2y}{dx^2}\right\} - ky = 0. \tag{179}$$

Integrating this with respect to x, it follows that

$$I(x)\frac{d^2y}{dx^2} = k\int_0^x y(\xi)\int_\xi^x dx\,d\xi + C_1x + C_2$$

$$= k\int_0^x (x-\xi)\,y(\xi)\,d\xi + C_1x + C_2$$

Taking a bar with hinged ends, the terminal conditons are $\frac{d^2y}{dx^2} = 0$ at both ends. Therefore

$$C_2 = 0 \quad \text{and} \quad C_1 = -\frac{k}{l}\int_0^l (l-\xi)\,y(\xi)\,d\xi\,.$$

Thus the differential equation (179) reduces to

$$I(x)\frac{d^2y}{dx^2} = k\int_0^x (x-\xi)\,y(\xi)\,d\xi - \frac{k}{l}x\int_0^l (l-\xi)\,y(\xi)\,d\xi\,. \tag{180}$$

The deflection curve in this case can be represented in the form of a sine series

$$y = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} \tag{181}$$

Each term of the series satisfies the end conditions, since each term, together with its second derivative, becoms zero at each end of the bar. Substituting this value into the differential equation (180), it follows that

$$I(x)\sum_{n=1}^{\infty} a_n \left(\frac{n\pi}{l}\right)^2 \sin \frac{n\pi x}{l} + k\int_0^x (x-\xi)\sum_{n=1}^{\infty} a_n \sin \frac{n\pi\xi}{l}\, d\xi$$

$$-\frac{k}{l}x\int_0^l (l-\xi)\sum_{n=1}^{\infty} a_n \sin \frac{n\pi\xi}{l}\, d\xi = 0\,. \tag{182}$$

Calculating the integrations in the second and third terms in the above equation it follows that

$$\int_0^x (x-\xi)\sin\frac{n\pi\xi}{l}\, d\xi = \frac{l}{n\pi}x - \left(\frac{l}{n\pi}\right)^2 \sin\frac{n\pi x}{l}\,,$$

$$\frac{x}{l}\int_0^l (l-\xi)\sin\frac{n\pi\xi}{l}\, d\xi = \frac{l}{n\pi}x\,.$$

Thus, Eq. (182) reduces to

$$I(x)\sum_{n=1}^{\infty} a_n \left(\frac{n\pi}{l}\right)^2 \sin\frac{n\pi x}{l} - k\sum_{n=1}^{\infty} a_n \left(\frac{l}{n\pi}\right)^2 \sin\frac{n\pi x}{l} = 0. \tag{183}$$

Multiplying by $\sin\frac{n\pi x}{l}$ and integrating from one end to the other of the bar, a system of equations in a_1, $a_2,\cdots$ of the following type is obtained:

$$\int_0^l I(x)\left\{a_n\left(\frac{n\pi}{l}\right)\sin^2\frac{n\pi x}{l} + \sum_{m}^{\text{except}\,n} a_m n^2 m^2 \left(\frac{\pi}{l}\right)^4 \sin\frac{m\pi x}{l}\sin\frac{n\pi x}{l}\right\}dx$$

$$-\frac{kl}{2}a_n = 0 \qquad (n = 1,\ 2,\ 3\cdots) \tag{184}$$

Oscillation of a bar becomes possible when the above system of equations gives for coefficients a_1, a_2, a_3, $\cdots$ a solution different from zero, i. e., when the determinant of system (184) becomes equal to zero. From this determinant equation the frequency of free oscillation is determined.

The same result can be also obtained by the energy method. Equating the internal work to the external work, frequency of free oscillation is expressed by the well-known formula

$$n_0 = \frac{gE}{4\pi^2} \frac{\int_0^l I(x)\left(\frac{d^2y}{dx^2}\right)^2 dx}{\int_0^l qy^2\,dx}. \tag{185}$$

Substituting in this expression the value of y given by (181), the frequency is determined by finding such relation between the coefficients a_1, a_2, a_3,$\cdots$ as to make expression (185) a minimum. In this method, however, it is more labourious to arrive at the result given by Eq. (184).

Now, take a bar whose moment of inertia of the cross section varies as follows:

$$I(x) = a + bx + cx^2 + dx^3 + ex^4. \tag{186}$$

Substituting this into Eq. (184) and calculating the integration, a system of homogeneous linear equations in a_1, a_2, $a_3\cdots$ of the following type is obtained:

$$\begin{aligned} a_n \Bigg\{ & n^4\pi^4\left[\frac{a}{2l^3}+\frac{b}{4l^2}+\frac{c}{2l}\left(\frac{1}{3}-\frac{1}{2n^2\pi^2}\right)+\frac{d}{8}\left(1-\frac{3}{n^2\pi^2}\right)+\frac{el}{2}\left(\frac{1}{5}-\frac{1}{n^2\pi^2}\right.\right. \\ & \left.\left.+\frac{3}{2n^4\pi^4}\right)\right]-\frac{kl}{2}+\sum_m^{m+n:\,even} a_m\frac{n^3m^3\pi^2}{(n^2-m^2)^2}\left\{\frac{4c}{l}+6d+8el\left[1-\frac{12(n^2+m^2)}{\pi^2(n^2-m^2)^2}\right]\right\} \\ & -\sum_m^{m+n:\,odd} a_m\frac{n^3m^3\pi^2}{(n^2-m^2)^2}\left\{\frac{4b}{l^2}+\frac{4c}{l}+6d\left[1-\frac{8(n^2+m^2)}{\pi^2(n^2-m^2)^2}\right]+8el\left[1-\frac{12(n^2+m^2)}{\pi^2(n^2-m^2)^2}\right]\right\}=0. \end{aligned} \tag{187}$$

From this equation one can also derive the equations corresponding to a bar whose moment of inertia of the cross section varies as a certain power of the distance from the lower end so that the moment of inertia of any cross section at x is, as in the buckling problem,

$$I(x) = I_c\left(\frac{\lambda + x}{\lambda + \frac{l}{2}}\right)^n, \qquad (n = 1,\ 2,\ 3,\ 4). \tag{188}$$

Taking two terms in expression (181), the equations, from which the frequencies are to be approximatately determined, are obtained as follows:

I. In the case of $n = 1$;

$$(\alpha - K)\left(4\alpha - \frac{K}{4}\right) - \left(\frac{16}{9}\right)^2 \times 4 = 0 \tag{189}$$

in which

$$\alpha = \pi^2\left(\frac{\lambda}{l}+\frac{1}{2}\right), \qquad K = \frac{kl^3\left(\lambda + \frac{l}{2}\right)}{\pi^2 I_c}$$

II. In the case of $n = 2$;

$$\left(\alpha-\frac{1}{2}-K\right)\left(4\alpha-\frac{1}{2}-\frac{K}{4}\right)-\left(\frac{16}{9}\right)^2\beta^2=0 \qquad (190)$$

in which

$$\alpha=\pi^2\left(\frac{\lambda^2}{l^2}+\frac{\lambda}{l}+\frac{1}{3}\right),\quad \beta=2\left(2\frac{l}{\lambda}+1\right),\quad K=\frac{kl^2\left(\lambda+\frac{l}{2}\right)^2}{\pi^2 I_c}.$$

III. In the case of $n=3$;

$$(\alpha-\beta-K)\left(4\alpha-\beta-\frac{K}{4}\right)-\left(\frac{16}{9}\right)^2\delta^2=0 \qquad (191)$$

in which

$$\alpha=\pi^2\left(\frac{\pi^3}{l^3}+\frac{3}{2}\frac{\lambda^2}{l^2}+\frac{\lambda}{l}+\frac{1}{4}\right),\quad \beta=\frac{3}{2}\frac{\lambda}{l}+\frac{3}{4}$$

$$\delta=6\frac{\lambda^2}{l^2}+6\frac{\lambda}{l}+3\left(1-\frac{40}{9\pi^2}\right),\quad K=\frac{kl\left(\lambda+\frac{l}{2}\right)^3}{\pi^2 I_c}.$$

IV. In the case of $n=4$;

$$(\alpha-\beta+\gamma-K)\left(4\alpha-\beta+\frac{1}{4}\gamma-\frac{K}{4}\right)-\left(\frac{16}{9}\right)^2\delta^2=0 \qquad (192)$$

in which

$$\alpha=\pi^2\left(\frac{\lambda^4}{l^4}+2\frac{\lambda^3}{l^3}+2\frac{\lambda^2}{l^2}+\frac{\lambda}{l}+\frac{1}{5}\right),\quad \beta=3\frac{\lambda^2}{l^2}+3\frac{\lambda}{l}+1,\quad \gamma=\frac{3}{2\pi^2},$$

$$\delta=8\frac{\lambda^3}{l^3}+12\frac{\lambda^2}{l^2}+12\frac{\lambda}{l}\left(1-\frac{40}{9\pi^2}\right)+4\left(1-\frac{60}{9\pi^2}\right),\quad K=\frac{k\left(\lambda+\frac{l}{2}\right)^4}{\pi^2 I_c}$$

Solving the above equations, the smaller root gives the frequency of the fundamental oscillation in each case.

2. Symmetrical Bars with Variable Cross Section.

Proceeding as in the previous article, an equation corresponding to Eq. (184) becomes as follows;

$$\int_0^{\frac{l}{2}} I(x)\left\{\alpha_n\left(\frac{n\pi}{l}\right)^4\sin^2\frac{n\pi x}{l}+\sum_m^{except\ n}\alpha_m n^2m^2\left(\frac{\pi}{l}\right)^4\sin\frac{m\pi x}{l}\sin\frac{n\pi x}{l}\right\}dx-\frac{kl}{4}\alpha_n=0. \quad (193)$$

If the variation of the moment of inertia is

$$I(x)=a+bx+cx^2+dx^3+ex^4,\quad \left(x=0\sim\frac{l}{2}\right) \qquad (194)$$

considering only a symmetrical shape of the oscillation, a system of homogeneous linear equations in a_1, a_3, a_5, $\cdots$, corresponding to Eq. (187) becomes as follows:

$$a_n \left\{ n^4 \pi^4 \left[\frac{a}{4l^3} + \frac{b}{4l^2} \left(\frac{1}{4} + \frac{1}{n^2\pi^2} \right) + \frac{c}{8l} \left(\frac{1}{6} + \frac{1}{n^2\pi^2} \right) + \frac{d}{8} \left(\frac{1}{16} + \frac{3}{4n^2\pi^2} - \frac{3}{n^4\pi^4} \right) \right.\right.$$

$$\left.\left. + \frac{el}{8} \left(\frac{1}{40} + \frac{1}{2n^2\pi^2} - \frac{3}{n^4\pi^4} \right) \right] - \frac{kl}{4} \right\} + \sum_{m}^{\frac{n+m}{2}:\text{odd}} a_m \frac{n^2 m^2 \pi^2}{(n+m)^2} \left\{ \frac{b}{l^2} + \frac{c\,(n^2+m^2)}{l\,(n-m)^2} \right.$$

$$\left. + 3d \left[\frac{n^2+m^2}{4\,(n-m)^2} - \frac{2}{(n+m)^2\,\pi^2} \right] + \frac{2el}{(n-m)^2} \left[\frac{n^2+m^2}{4} - \frac{6\,(n^4+6n^2m^2+m^4)}{(n^2-m^2)^2 \pi^2} \right] \right\}$$

$$- \sum_{m}^{\frac{n+m}{2}:\text{even}} a_m \frac{n^2 m^2 \pi^2}{(n-m)^2} \left\{ \frac{b}{l^2} + \frac{c\,(n^2+m^2)}{l\,(n+m)^2} + 3d \left[\frac{n^2+m^2}{4\,(n+m)^2} - \frac{2}{(n-m)^2 \pi^2} \right] \right.$$

$$\left. + \frac{2el}{(n+m)^2} \left[\frac{n^2+m^2}{4} - \frac{6\,(n^4+6n^2m^2+m^4)}{(n^2-m^2)^2\,\pi^2} \right] \right\} = 0 \qquad (195)$$

When the moment of inertia of any cross section at x is

$$I\,(x) = I_c \left(\frac{\lambda + x}{\lambda + \frac{l}{2}} \right)^n, \quad (n = 1,\ 2,\ 3,\ 4) \qquad (196)$$

the equations, from which the frequencies are to be approximately determined, become as follows, taking the two terms with the coefficients a_1 and a_3:

I. In the case of $n = 1$;

$$(\alpha + 1 - K) \left(9\alpha + 1 - \frac{K}{9} \right) - 9 = 0 \qquad (196)$$

in which

$$\alpha = \pi^2 \left(\frac{\lambda}{l} + \frac{1}{4} \right), \quad K = \frac{kl^3 \left(\lambda + \frac{l}{2} \right)}{\pi^2\, I_c}.$$

II. In the case of $n = 2$;

$$(\alpha + \beta - K) \left(9\alpha + \beta - \frac{K}{9} \right) - 9 \left(\beta + \frac{1}{8} \right)^2 = 0 \qquad (198)$$

in which

$$\alpha = \pi^2 \left(\frac{\lambda^2}{l^2} + \frac{\lambda}{2l} + \frac{1}{12} \right), \quad \beta = \frac{2\lambda}{l} + \frac{1}{2}, \quad K = \frac{kl^2 \left(\lambda + \frac{l}{2} \right)^2}{\pi^2\, I_c}.$$

III. In the case of $n = 3$;

$$(\alpha + \beta - \gamma - K) \left(9\alpha + \beta - \frac{\gamma}{9} - \frac{K}{9} \right) - 9\delta^2 = 0 \qquad (198)$$

in which

$$\alpha = \pi^2\left(\frac{\lambda^3}{l^3}+\frac{3}{4}\frac{\lambda^2}{l^2}+\frac{1}{4}\frac{\lambda}{l}+\frac{1}{32}\right), \quad \beta = \left(\frac{3\lambda^2}{l^2}+\frac{3}{2}\frac{\lambda}{l}+\frac{3}{8}\right)$$

$$\gamma = \frac{3}{2\pi^2}, \quad \delta = 3\frac{\lambda^2}{l^2}+\frac{15}{8}\frac{\lambda}{l}+\frac{3}{4}\left(\frac{5}{8}-\frac{2}{\pi^2}\right), \quad K = \frac{kl\left(\lambda+\frac{l}{2}\right)^3}{\pi^2 I_c}.$$

IV. In the case of $n=4$;

$$(\alpha+\beta-\gamma-K)\left(9\alpha+\beta-\frac{\gamma}{9}-\frac{K}{9}\right)-9\delta^2=0 \qquad (200)$$

in which

$$\alpha = \pi^2\left(\frac{\lambda^4}{l^4}+\frac{\lambda^3}{l^3}+\frac{\lambda^2}{2l^2}+\frac{\lambda}{8l}+\frac{1}{80}\right), \quad \beta = \frac{4\lambda^3}{l^3}+\frac{3\lambda^2}{l^2}+\frac{3}{2}\frac{\lambda}{l}+\frac{1}{4}$$

$$\gamma = \frac{3}{2\pi^2}\left(4\frac{\lambda}{l}+1\right), \quad \delta = \frac{4\lambda^3}{l^3}+\frac{15}{4}\frac{\lambda^2}{l^2}+3\frac{\lambda}{l}\left(\frac{5}{8}-\frac{2}{\pi^2}\right)+\frac{1}{8}\left(\frac{5}{2}-\frac{51}{4\pi^2}\right), \quad K = \frac{k\left(\lambda+\frac{l}{2}\right)^4}{\pi^2 I_c}.$$

Solving the above equations, several values of the fundamental freqency are given as shown in Table 4. In the case where the ratio I_0/I_c approaches zero, the values of frequency were, of course, calculated taking the three or more terms.

Table 4. Values of Frequency for Symmetrical Bars with Variable Crossff Section. Coeff : $\frac{\pi}{2l^2}\sqrt{\frac{EI_c g}{q}}$

n \ I_0/I_c	0.01	0.02	0.04	0.06	0.08	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	0.783	0.788	0.796	0.803	0.810	0.817	0.847	0.868	0.895	0.915	0.934	0.952	0.969	0.985
2	0.630	0.652	0.687	0.712	0.732	0.749	0.809	0.849	0.880	0.909	0.929	0.949	0.968	0.985
3	0.543	0.589	0.641	0.675	0.700	0.722	0.794	0.845	0.875	0.903	0.928	0.949	0.968	0.985
4	0.500	0.555	0.616	0.656	0.685	0.707	0.787	0.836	0.872	0.902	0.927	0.948	0.968	0.985

VII. APPLICATION OF FOURIER SINE TRANSFORMATION METHOD TO THE STUDY OF BENDING OF RECTANGULAR PLATES.

Taking, as an example of the simplest cases, a rectangular plate simply supported along all its edges, the well known differential equation of the deflection curve due to the normal pressures is

$$D\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)-p(x)=0 \qquad (201)$$

in which the following notations are used:

$D = \dfrac{Eh^3}{12\,(1-\sigma^2)}$; $\qquad h$ = thickness of plate;

σ = Poisson's ratio; $p\,(x)$ = load per unit area at any point;

w = deflection of plate at any point.

Let the axis of x anb y be taken along one pair of edges and the other pair be $x = a$, $y = b$. Then the deflection curve in this case can be represented in the form of a double infinite sine series:

$$w = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}. \tag{202}$$

Each term of the series satisfies the boundary conditions, since each term, together with its second derivative, becomes zero at the edges of plate.

Substituting the value of w given by Eq. (202), multiplying by $\sin \dfrac{m\pi x}{a} \sin \dfrac{n\pi y}{b}$ and then integrating over the whole plate, it follows that

$$a_{mn}\, D\pi^4 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2 \int_0^a \int_0^b \sin^2 \frac{m\pi x}{a} \sin^2 \frac{n\pi y}{b}\, dx\, dy$$

$$- \int_0^a \int_0^b p\,(x) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}\, dx\, dy = 0 \tag{203}$$

since every term in the integral vanishes except the two terms having the coefficients a_{mn} and $p\,(x)$. This equation is also derived by energy method by a more labourious process.

Using the formula:

$$\int_0^a \int_0^b \sin^2 \frac{m\pi x}{a} \sin^2 \frac{n\pi x}{b}\, dx\, dy = \frac{ab}{4}$$

the coefficient a_{mn} is finally obtained as follows:

$$a_{mn} = \frac{\displaystyle\int_0^a \int_0^b p\,(x) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}\, dx\, dy}{\displaystyle\frac{ab}{4} D\pi^4 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2} \tag{204}$$

and consequently

$$w = \frac{4}{abD\pi^4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\displaystyle\int_0^a \int_0^b p\,(x) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}\, dx\, dy}{\left(\dfrac{m^2}{a^2} + \dfrac{n^2}{b^2}\right)^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}. \tag{205}$$

Considering, for an example, the particular case of a uniform load fully applied over the whole plate, it follows that

$$\int_0^a \int_0^b p \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dx \, dy = p \frac{4ab}{mn\pi^2} \text{ when either } m \text{ or } n \text{ is odd,}$$

$$= 0 \qquad \text{when either } m \text{ or } n \text{ is even.}$$

Therefore the coefficient a_{mn} and the deflection w become respectively as follows:

$$a_{mn} = \frac{16\, p}{\pi^6 Dmn \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2} \qquad m, n = \text{odd,} \tag{206}$$

$$w = \frac{16p}{\pi^6 D} \sum_{m=1,3,5\cdots}^{\infty} \sum_{n=1,3,5\cdots}^{\infty} \frac{1}{mn \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} . \tag{207}$$

In the cases of partial distributed load and a single concentrated load, the results can be also found by a process similar to that used for a beam.

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