On the Thick Plate Problem

(First Report)

By

Kin-ichi HATA

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CONTENTS

§ I. Introduction.

The purpose of the present paper is to indicate an extension of the method of Michell and Love[1)] and to investigate the nature of this method and the problem concerned as well. The thick plate problem is far more difficult, being rather cumbersome, than the thin plate one and researches in this problem, especially those by Michell-Love's method, may be said to be very few compared with essentially two dimensional ones. Thick plate here discussed is supposed to be isotropic, homogeneous, and of uniform thickness and thus the problem is much simplified but the last two suppositions would be indispensable in this theory. The term "plate" means or admits certain approximations, the author thinks. If exactness is strictly required, though we are going to discuss in the scope of the theory of the first order, distinct differences could not be found between plate and column. Hence, even if solutions for moderately thick or thick plate are not three-dimensional in a strict sense and so contain some approximations, it would be unreasonable and too severe to say that approximations of this kind included herein would be unfavourable. When we observe the process of calculation to obtain the solutions, regardless of the kinds into which they are classified, by the use of the method of Michell and Love, the solutions seem to be accurate at first sight because of this method being elegant and systematic. However, in fact, the solutions are obtained in some approximations which may probably be ascribable to the way of integration and in consequence of which they can be classified into the two kinds of plane stress, namely plane stress and generalized plane stress, and a particular solution. Additionally we are obliged to impose some restrictions upon the manner in which the boundary conditions are satisfied, i. e., we cannot but use the reduced or simplified boundary conditions and apply Kirchhoff's theorem[1] regarding the boundary conditions. In any way this method of Michell and Love may be said to be ingenious and excellent for solving the thick plate problem, especially because of its easiness and plainness. It is, however, regrettable that many authors say Michell-Love's method is only suitable for or applicable to the case where thick plate is stressed by a simply distributed load and tacitly assert this method could not be employed in the case of generally distributed load. Nevertheless, there is no theoretical reason why this method should be of so limited application and the author presumes sufficient consideration should be given to this fact. The author has extended this method pertaining to this respect and the extension presented by him may not be the best, but it may serve the purpose adequately. The author surmises that stress problem on the basis of the thin plate theory is, indeed, much simplified but on the other hand certain mathematical difficulties come to be involved, which are avoided in the thick plate problem, probably because in the latter case all the equations of compatibility and equilibrium are considered. The uniqueness of solutions is guaranteed by the theory of elasticity itself but in the former case the uniqueness is substituted for a mere mathematical one and so the thick plate problem is worth being investigated in more detail. In the case of thin plate it is comparatively easy to solve the problem of plate under the variable normal load and, taking the initiative, Navier treated the case of a rectangular thin plate loaded with variable pressure. S. D. Conte[2] and others have recently discussed the problem concerned with the above, referred to various coordinate systems. Woinowsky and Krieger[3] treated the thick plate problem starting with the equations of equilibrium expressed in terms of displacements and positional coordinates and so dispensed with the conditions of compatibility, but they did not state in detail the manner in which the boundary conditions are sufficed. We shall discuss their results and method at a later date. At any rate their method is very suggestive. By the way Eric Reissner and others have solved the so-called thick plate problem using a method differing from those in common use. Then method is very instructive but its order of accuracy seems to be not so high, though the reason for it, being obvious, may not be stated here. The treatment of the problem by this approach is not attempted in this paper. The following treatment depends principally upon the contents of the paragraphs ranging from §299 to §311, i. e., Chapter XXII, of the book by A. E. H. Love. The present author adopts notations as many as possible used in Love's book suitable and therefore no detailed account of denotations which are in common use or referable to this book will be given but, if necessary, they will be explained without fail. Any historical statement about the researches in the thick plate problem is omitted here, for the reader can find an account in books and papers available near at hand. In the present paper only particular solutions are dealt with and the description of the manner to satisfy the boundary conditions are left for the second report. In Sec. II we record the results in the book by Love and show the relations between them and the solutions deduced by H. Neuber reassertaining the solutions by Love thus making it possible to utilize the solutions by Neuber[3], whether indirectly or not. In Sec. III we treat the thick plate, on the upper surface of which the normal load is applied, loading function being plane-harmonic. In this case we are able to follow directly the method of Michell and Love without recourse to the expansion of the loading function. In Sec. IV we deal with the case where the loading function satisfies the equation of Helmholtz with a view to making use of the results to obtain the particular solutions when the loading function is in one sense general. In Sec. V we treat the case in which the load is tangential and the loading function is the same as in Sec. IV. In Sec. VII we shall discuss briefly the cases in which the normal loading function satisfies the equation of Helmholtz

,referred to the cylindrical coordinates and the elliptic cylinder coordinates respectively. In the following we shall always make the assumption that no body forces are applied and only upper or lower surface normal or tangential tractions exist and the plate is elastically isotropic, homogeneous, and of uniform thickness as mentioned above. We shall use the rectilinear coordinates and cylinder coordinates and let the plate be located horizontally, the origin be on the middle surface, the z axis be drawn upwards, the thickness $2h$.

§ II. On the Plane Stress and Generalized Plane Stress.

For later convenience the results obtained by Prof. Love are quoted here pertaining to the plane stress and generalized plane stress but the process of calculation to get them are omitted, though it is of importance as the preliminary in order to gain the particular solutions. In the following we snall write down the solutions for plane stress in the plate and here omit the solutions for plane stress state produced in a particular way such that $\Theta_1=\beta$ (const.) and let them belong to generalized plane stress solutions.

$$\sigma_x=\frac{\partial^2\chi}{\partial y^2},\quad \sigma_y=\frac{\partial^2\chi}{\partial x^2},\quad \tau_{xy}=-\frac{\partial^2\chi}{\partial x\partial y},\qquad \cdots\cdots (2.1a)$$

$$T_1=\int_{-h}^{h}\sigma_x dz=\frac{\partial^2\chi''}{\partial y^2},\quad T_2=\int_{-h}^{h}\sigma_y dz=\frac{\partial^2\chi''}{\partial x^2},\quad S_1=\int_{h-}^{h}\tau_{xy}dz=\frac{-\partial^2\chi''}{\partial x\partial y},\qquad \cdots\cdots (2.1b)$$

$$\chi=\chi_0-\frac{1}{2}\frac{\nu}{1+\nu}z^2\Theta_0,\quad \chi''=2h\chi_0-\frac{1}{3}\frac{\nu}{1+\nu}h^3\Theta_0,\qquad \cdots\cdots (2.1c)$$

in which ν is Poisson's ratio,

$$N_1=N_2=0,\quad G_1=G_2=H_1=-H_2=0,\qquad \cdots\cdots (2.2a)$$

where

$$\left.\begin{aligned} &N_1=\int\tau_{xz}dz, \quad N_2=\int\tau_{yz}dz, \quad G_1=\int z\sigma_x dz,\\ &G_2=\int z\sigma_y dz, \quad H_1=\int -z\tau_{xy}dz, \quad H_2=\int z\tau_{xy}dz, \end{aligned}\right\}\qquad \cdots\cdots (2.2b)$$

$$\left.\begin{aligned} &\nabla_1^2\chi_0=\Theta_0=\sigma_x+\sigma_y+\sigma_z,\quad \nabla_1^2\Theta_0=0,\\ &\qquad\left(\nabla_1^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right). \end{aligned}\right\}\qquad \cdots\cdots (2.3)$$

The stress resultants N and couples G, H vanish in this case. Θ and χ_0 are plane harmonic and plane bi-harmonic functions respectively.

$$T=\left(\frac{\partial^2}{\partial s^2}+\frac{1}{\rho}\frac{\partial}{\partial\mu}\right)\chi'',\quad S=-\frac{\partial}{\partial\mu}\left(\frac{\partial}{\partial s}\chi''\right). \qquad (2.4)$$

These are the normal stress and shearing stress resultant concerned with any curve on the middle plane of the plate and μ denotes the direction of the outer normal to any curve in question drawn on the middle plane, s the direction of the tangent to this curve, ρ the radius of curvature of this curve.

The displacements are

$$\left.\begin{aligned} u&=\frac{1}{E}\left(\xi+\frac{1}{2}\nu z^2\frac{\partial\Theta_0}{\partial x}\right)-\frac{1+\nu}{E}\frac{\partial\chi_0}{\partial x},\\ v&=\frac{1}{E}\left(\eta+\frac{1}{2}\nu z^2\frac{\partial\Theta_0}{\partial y}\right)-\frac{1+\nu}{E}\frac{\partial\chi_0}{\partial y},\\ w&=-\frac{\nu}{E}z\Theta_0,\ \left(\frac{\partial\xi}{\partial x}=\frac{\partial\eta}{\partial y}=\Theta_0,\quad \frac{\partial\xi}{\partial y}=-\frac{\partial\eta}{\partial x}\right). \end{aligned}\right\} \qquad (2.5)$$

Excluding plane stress components, though $\Theta_1=\beta$ (const.) is considered, generalized plane stress solutions are

$$\Theta=z\Theta_1,\quad \nabla^2\Theta=0,\quad \nabla_1^2\Theta_1=0 \qquad (2.6)$$

$$\left.\begin{aligned} &\sigma_x=\frac{z}{1+\nu}\Theta_1+\frac{\partial^2\chi'}{\partial y^2},\quad \sigma_y=\frac{z}{1+\nu}\Theta_1+\frac{\partial^2\chi'}{\partial x^2},\\ &\tau_{xy}=-\frac{\partial^2\chi'}{\partial x\partial y},\qquad \sigma_z=0,\\ &\tau_{xz}=\frac{1}{2}\frac{1}{(1+\nu)}(h^2-z^2)\frac{\partial\Theta_1}{\partial x},\quad \tau_{yz}=\frac{1}{2(1+\nu)}(h^2-z^2)\frac{\partial\Theta_1}{\partial y}, \end{aligned}\right\} \qquad (2.7a)$$

$$\left.\begin{aligned} u&=\frac{-1}{E}\left\{(1+\nu)z\frac{\partial\chi_1'}{\partial x}+\frac{1}{6}(2-\nu)z^3\frac{\partial\Theta_1}{\partial x}\right\},\\ v&=\frac{-1}{E}\left\{(1+\nu)z\frac{\partial\chi_1'}{\partial y}+\frac{1}{6}(2-\nu)z^3\frac{\partial\Theta_1}{\partial y}\right\},\\ w&=\frac{1}{E}\left\{(1+\nu)\chi_1'+(h^2-\frac{1}{2}\nu z^2)\Theta_1\right\}, \end{aligned}\right\} \qquad (2.7b)$$

in which

$$\left.\begin{aligned}\chi' &= z\chi_1' + \frac{2-\nu}{6(1+\nu)} z^3\Theta_1\,, \quad \nabla_1^2\chi_1' = \frac{-(1-\nu)}{(1+\nu)}\,\Theta_1\,,\\ \chi_1' &= \frac{-1}{2}\frac{(1-\nu)}{(1+\nu)}\,x\int\Theta_1 dx + F_1\,, \quad \nabla_1^2 F_1 = 0\,, \quad \nabla^2 = \nabla_1^2 + \frac{\partial^2}{\partial z^2}\,.\end{aligned}\right\}\cdots (2.8)$$

The results in (2.7) can be rewritten into the following form:

$$\left.\begin{aligned}\sigma_x &= \frac{-Ez}{(1-\nu^2)}\nabla_1^2 w_0 + \frac{\partial^2}{\partial y^2}\chi'\,, \quad \sigma_y = \frac{-Ez}{(1-\nu^2)}\nabla_1^2 w_0 + \frac{\partial^2}{\partial x^2}\chi'\,,\\ \tau_{xy} &= \frac{-\partial^2}{\partial x\partial y}\chi'\,, \quad \chi' = \frac{E}{(1+\nu)} z w_0 + \frac{E}{(1-\nu^2)}\left\{h^2 z - \frac{1}{6}(2-\nu)z^3\right\}\nabla_1^2 w_0\,,\\ w_0 &= \frac{1}{E}\left\{h^2\Theta_1 + (1+\nu)\chi_1'\right\}, \nabla_1^4 w_0 = 0\,, \quad (\nabla_1^4 = \nabla_1^2\cdot\nabla_1^2)\,,\\ \Theta_1 &= \frac{-E}{(1-\nu)}\nabla_1^2 w_0, \quad \chi_1' = \frac{E}{(1+\nu)} w_0 + \frac{Eh^2}{1-\nu^2}\nabla_1^2 w_0\,. \quad (w_0 = w/_{z=0})\end{aligned}\right\} (2.9)$$

The stress resultants and stress couples are:

$$\left.\begin{aligned}&T_1 = T_2 = S_1 = S_2 = 0\,, \quad N_1 = -D\frac{\partial}{\partial x}\nabla_1^2 w_0, \quad N_2 = -D\frac{\partial}{\partial y}\nabla_1^2 w_0\\ &G_1 = -D\left(\frac{\partial^2 w_0}{\partial x^2} + \nu\frac{\partial^2 w_0}{\partial y^2}\right) + \frac{8+\nu}{10} Dh^2 \frac{\partial^2}{\partial y^2}\nabla_1^2 w_0\,,\\ &G_2 = -D\left(\frac{\partial^2 w_0}{\partial y^2} + \nu\frac{\partial^2 w_0}{\partial x^2}\right) + \frac{8+\nu}{10} Dh^2 \frac{\partial^2}{\partial x^2}\nabla_1^2 w_0\,,\\ &H_1 = -H_2 = D(1-\nu)\frac{\partial^2 w_0}{\partial x\partial y} + \frac{8+\nu}{10} Dh^2 \frac{\partial^2}{\partial x\partial y}\nabla_1^2 w_0\,,\end{aligned}\right\} (2.10)$$

in which we denote the flexural rigidity of the plate by $D = \frac{2Eh^3}{3(1-\nu^2)}$.
Now it will be useful to prove that there exists a complete correspondence between the above stated solutions and those which are derived by H. Neuber. This fact will on the one hand assure us that the former solutions are accurate to a certain degree and on the other hand indicate that we can make use of Neuber's method or solutions, if necessary, in course of calculation.

The expressions for stresses by Neuber are

$$\left.\begin{aligned}\sigma_x &= \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial x^2} + \alpha\left(\frac{\partial\Phi_1}{\partial x} - \frac{\partial\Phi_2}{\partial y} - \frac{\partial\Phi_3}{\partial z}\right), \text{ etc.,}\\ \tau_{yz} &= \frac{-\partial^2 F}{\partial x\partial y} + \alpha\left(\frac{\partial\Phi_1}{\partial y} + \frac{\partial\Phi_2}{\partial x}\right), \text{ etc.,}\end{aligned}\right\}\cdots (2.11)$$

$$F=\Phi_0+x\Phi_0+y\Phi_2+z\Phi_3\,, \quad \nabla^2\Phi_0=\nabla^2\Phi_1=\nabla^2\Phi_2=\nabla^2\Phi_3=0, \quad \alpha=2(1-\nu)\,.$$

These formulae may be transformed into the following forms:

$$\left.\begin{aligned}
\sigma_z &= \frac{\partial^2}{\partial x^2}(F-\alpha\varphi_1)+\frac{\partial^2}{\partial y^2}(F-\alpha\varphi_2)+\alpha\frac{\partial^2}{\partial z^2}\varphi_3\\
\sigma_x &= \frac{\partial^2}{\partial y^2}(F-\alpha\varphi_1-\alpha\varphi_2)+\frac{\partial^2}{\partial z^2}(F-\alpha\varphi_3-\alpha\varphi_1)\,.\\
\sigma_y &= \frac{\partial^2}{\partial x^2}(F-\alpha\varphi_1-\alpha\varphi_2)+\frac{\partial^2}{\partial z^2}(F-\alpha\varphi_3-\alpha\varphi_2)\,,\\
\tau_{yz} &= \frac{-\partial^2}{\partial y\partial z}(F-\alpha\varphi_3-\alpha\varphi_2)\,, \quad \tau_{xz}=\frac{-\partial^2}{\partial x\partial z}(F-\alpha\varphi_3-\alpha\varphi_1)\,,\\
\tau_{xy} &= \frac{-\partial^2}{\partial x\partial y}(F-\alpha\varphi_1-\alpha\varphi_2)\,,
\end{aligned}\right\} \quad (2.12)$$

where

$$\Phi_1=\frac{\partial\varphi_1}{\partial x}\,,\quad \Phi_2=\frac{\partial\varphi_2}{\partial y}\,,\quad \Phi_3=\frac{\partial\varphi_3}{\partial z}\,.$$

The plane stress solutions by H. Neuber can be quoted.

$$\left.\begin{aligned}
\Phi_0 &= \frac{4-4\alpha+\alpha^2}{\alpha(4-\alpha)}\left(z^2\frac{\partial\Phi_1'}{\partial x}-x\frac{\partial\Phi_1'}{\partial x}\right)+\frac{4}{4-\alpha}\Phi_1'+\Phi_0'\,,\\
\Phi_1 &= \frac{4}{\alpha(4-\alpha)}\frac{\partial\Phi_1'}{\partial x}\,,\quad \Phi_2=0\,,\quad \Phi_3=\frac{-4+2\alpha}{\alpha(4-\alpha)}z\frac{\partial^2\Phi_1'}{\partial x^2}\,,
\end{aligned}\right\} \quad (2.13)$$

where $\frac{2-\alpha}{4-\alpha}\frac{d^2}{12}\frac{\partial^2\Phi_1'}{\partial x^2}$ in his formula Φ_0 is omitted, because this and $\frac{4}{4-\alpha}\Phi_1'$ term are unnecessary to satisfy the condition $\sigma_z=0$, $\tau_{xz}=0$, $\tau_{yz}=0$, and may be contained in Φ_0', but the latter term is left as it is for arrangement or later convenience. The symbol d in the former term is the thickness of the plate according to Neuber's book. It will be easily seen that in the plane stress problem there is no reason why thickness should appear in the formulae, unless the mean is taken.

Neuber's three-dimensional stress function F is

$$F=\Phi_0'+x\frac{\partial\Phi_1'}{\partial x}+\frac{\alpha-2}{4-\alpha}z^2\frac{\partial^2\Phi_1'}{\partial x^2}+\frac{4}{4-\alpha}\Phi_1'\,,$$

and so

$$F-\alpha\varphi_3=\Phi_0'+x\frac{\partial\Phi_1'}{\partial x}+\frac{4}{4-\alpha}\Phi_1'=F'+\frac{4}{4-\alpha}\Phi_1' \quad (2.14)$$

$$F-\alpha\varphi_1=F'+\frac{\alpha-2}{4-\alpha}z^2\frac{\partial^2\Phi_1'}{\partial x^2}\,,\quad \nabla_1^2F=2\frac{\partial^2\Phi_1'}{\partial x^2}\,,\text{ and } \varphi=0.$$

From the above formulae it is evident that σ_z, τ_{xz}, and τ_{yz} are identically zero. And hence the plane stresses are expressible in the forms

$$\left.\begin{aligned}&\sigma_x=\frac{\partial^2\bar{F}}{\partial y^2}\,,\quad \sigma_y=\frac{\partial^2\bar{F}}{\partial x^2}\,,\quad \tau_{xy}=\frac{-\partial^2\bar{F}}{\partial x\partial y}\,,\\&\bar{F}=F-\alpha\varphi_1=F'+\frac{\alpha-2}{4-\alpha}z^2\frac{\partial^2\Phi_1'}{\partial x^2}\,.\end{aligned}\right\}\quad\cdots\cdots\cdots(2.15)$$

Then, considering the formulae by Love (2.1), the correspondence will be easily derived as follows:

$$\left.\begin{aligned}&\chi_0=F'\,,\quad \Theta_0=\nabla^2F'=\nabla_1^2F'=2\frac{\partial^2\Phi_1'}{\partial x^2}\\&\chi=\chi_0-\frac{1}{2}\frac{\nu}{1+\nu}z^2\Theta_0=\bar{F}\end{aligned}\right\}\quad\cdots\cdots\cdots(2.16)$$

In the case of generalized plane stress the correspondence can also be established with ease. As Neuber's four fundamental harmonic functions the following forms will be taken:

$$\left.\begin{aligned}&\Phi_0=Bz\frac{\partial^2\Phi_1''}{\partial x^2}+A\left(z^3\frac{\partial^2\Phi_1''}{\partial x^2}-3zx\frac{\partial\Phi_1''}{\partial x}\right),\quad \Phi_1=Cz\frac{\partial\Phi_1''}{\partial x}\,,\\&\Phi_2=0\,,\quad \Phi_3=D\left(z^2\frac{\partial^2\Phi_1''}{\partial x^2}-x\frac{\partial\Phi_1''}{\partial x}\right),\quad \nabla_1^2\Phi_1''=0\end{aligned}\right\}(2.17)$$

Let the formulae for σ_z, τ_{yz}, and τ_{xz} be obtained from these expressions by the use of the representations (2.11), and be compared with the formulae (2.7), then we can determine the coefficients A, B, C, D, when the relation $\Theta_1=\dfrac{\partial^2\Phi_1''}{\partial x^2}$ can be obtained.

$$A=\frac{1}{12}\frac{(1-2\nu)}{(1+\nu)}\,,\quad B=\frac{-h^2}{2(1+\nu)}\,,\quad C=0\,,\quad D=\frac{1}{4(1+\nu)}\,.\quad(2.18)$$

Therefore we may write

$$\left.\begin{aligned}&\sigma_x=\frac{\partial^2F}{\partial y^2}+\frac{\partial^2}{\partial z^2}(F-\alpha\varphi_3)\,,\quad \sigma_y=\frac{\partial^2F}{\partial x^2}+\frac{\partial^2}{\partial z^2}(F-\alpha\varphi_3)\,,\\&\tau_{xy}=\frac{-\partial^2F}{\partial x\partial y}\,,\quad F=\Phi_0+z\Phi_3\,.\end{aligned}\right\}(2.19)$$

By the way an arbitrary plane harmonic function Φ_0'' to be included in the expression for Φ_0 in formula (2.17) is left out, since this may be involved in Φ'_0 of equations (2.13). Now we have

$$\left.\begin{aligned} F &= \left(\frac{-h^2}{2(1+\nu)}\bar{\Theta}_1 - \frac{(1-\nu)}{2(1+\nu)}x\int\bar{\Theta}_1 dx\right)z + \frac{(2-\nu)}{6(1+\nu)}\bar{\Theta}_1 z^3 , \\ \frac{\partial^2}{\partial z^2}(F-\alpha\varphi_3) &= \frac{1}{1+\nu}\bar{\Theta}_1 z , \end{aligned}\right\} \quad (2.20)$$

where we put $\bar{\Theta}_1 = \dfrac{\partial^2 \Phi''_1}{\partial x^2}$. Then the correspondence will be seen that

$$\left.\begin{aligned} &\Theta_1 = \bar{\Theta}_1 , \quad X' = zX'_1 + \frac{2-\nu}{6(1+\nu)} z^3\Theta_1 = F , \\ &\text{where} \\ &X'_1 = -\frac{1}{2}\frac{(1-\nu)}{(1+\nu)}x\int\bar{\Theta}_1 dx + \frac{-h^2}{2(1+\nu)}\bar{\Theta}_1 . \end{aligned}\right\} \cdots\cdots\cdots (2.21)$$

But this verification or correspondence is not perfect, for X'_1 must have an indeterminate plane harmonic function F_1 as indicated in the solutions (2.7), (2.8). In order to remove this defect we have to add $A'\alpha fz$ to Φ_0, $A'f$ to Φ_3, in equations (2.17), where f is an arbitrary plane harmonic function. In consequence X'_1 acquires an additional term $A'(1+2)f$ and so the following relation is written:

$$F_1 = \frac{-h^2}{2(1+\nu)}\Theta_1 + A'(3-2\nu)f . \quad \cdots\cdots\cdots (2.22)$$

In the above description the forms for the shearing stress τ_{xz}, τ_{yz} in (2.7) are used in the process of comparison but, if this is to be avoided, a slight change in verification will be made as follows.

We determine the relations between the coefficients A, B, and D so as to make the formulae σ_z, τ_{xz}, and τ_{yz} gained from equations (2.17) satisfy the conditions $\sigma_z\equiv 0$; $\tau_{xz}=0$, $\tau_{yz}=0$ at $z=\pm h$.

$$D = \frac{-B}{2h^2} , \quad A = \frac{-B}{2h^2}\frac{(1-2\nu)}{3} .$$

Hence, $2\{3(D+A)-\alpha A\}$ the coefficient of $z\Theta_1$ in the expression for σ_x or σ_y becomes $\dfrac{-2B}{h^2}$ and, if we put $B=\dfrac{-h^2}{2(1+\nu)}B'$ and $\Theta_1=\bar{\bar{\Theta}}B'$, in which $\bar{\bar{\Theta}}_1\left(=\dfrac{\partial^2\Phi''_1}{\partial x^2}\right)$ corresponds to $\bar{\Theta}_1$ in the first proof, then all the other processes of verification will be carried out in the same way as before.

§ III. Particular Solutions for the Thick Plate Stressed by Normal Surface Load, the Distribution Function of Which is a Plane Harmonic Function.

We shall want to get particular solutions in the case where a variable load whose loading function is plane-harmonic is applied to the upper surface of the thick plate. Before commencing his study of this problem, the present author doubted if the particular solutions in the case of variable load could be obtained by the method of Michell and Love and so in the first place he investigated the case of harmonic loading function. We shall take the rectilinear coordinates and let the x- and y-axis lie on the middle plane of the plate. It will be unnecessary to designate the form of the edges of the plate for the present purpose. The author feels sure that the ingenious method of Love largely depends upon the assumption $\frac{\partial \sigma_z}{\partial z} = 0$ at $z = \pm h$. Indeed, Prof. Love states a certain nature of his method in note 312c of his book "on the theory of moderately thick plates". At any rate the above assumption would indirectly compel him to use a term of moderate thickness, the author thinks. In his treatise in 1900 Michell made some remarks about the order of accuracy of his method and assumed σ_z to be given or known. In the case of the method of Love, by virtue of the above assumption, we can determine the form of σ_z explicitly. In fact it is impossible that the normal stres: σ_z be given or known beforehand in a strict sense. This particular assumption, of course, leads to assumption with regard to the nature of the stress system. But this assumption regarding the derivative of σ_z with respect to z will be very reasonable considering the equations of equilibrium or from the practical point of view. In other words, if the theory of elasticity is consummate, it is very natural and rational that this derivative vanishes, passing to the surface of the plate. By this assumption, σ_z can be obtained comparatively easily and, once σ_z is known, other stresses come to be determined in succession leaving some plane harmonic functions contained in them undetermined till the boundary conditions are considered. But it is to be regretted that this systematic process cannot be continued, if we take other coordinate systems than rectangular coordinates, though this defect can be averted to some extent.

We write the fundamental equations to be employed as follows:

the equations of equilibrium—

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{xz}}{\partial z}=0\,,\ \frac{\tau\partial_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{yz}}{\partial z}=0\,,\ \frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\sigma_z}{\partial z}=0\,, \quad (3.1)$$

the conditions of compatibility—

$$\nabla^2\sigma_x+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial x^2}=0\,,\ \nabla^2\sigma_y+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial y^2}=0\,,\ \nabla^2\sigma_z+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial z_2}=0\,, \quad \cdots (3.2a)$$

$$\nabla^2\tau_{xy}++\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial x\partial y}=0,\ \nabla^2\tau_{xz}+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial x\partial z}=0,\ \nabla^2\tau_{yz}+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial y\partial z}=0, \quad (3.2b)$$

in which $\Theta=\sigma_x+\sigma_y+\sigma_z$, $\nabla^2\Theta=0$, $\left(\nabla^2=\nabla_1^2+\frac{\partial^2}{\partial z^2}\right)$, ... (3.3)

and each stress component satisfies the equation of the type $\nabla^4 f=0$ ($\nabla^4=\nabla^2\cdot\nabla^2$). Now we may put from the boundary condition

$$\sigma_z=-pf(x,y) \quad \text{at} \quad z=+h\,, \quad \cdots\cdots (3.4)$$

$\nabla_1^2 f(x,y)=0$, $pf(x,y)$ is the given, plane harmonic, loading function. ($p=$ const.)

Then the following may be taken:

$$\sigma_z=f(x,y)[\sigma_z]\,, \quad \cdots\cdots (3.5)$$

where

$$[\sigma_z]=\frac{p}{4h^3}(z+h)^2(z-2h) \quad \cdots\cdots (3.6)$$

is the solution in the case of uniform pressure p, for the form (3.5) may satisfy the equation stated above

$$\nabla^4\sigma_z=[\sigma_z]\nabla_1^4 f+2\nabla_1^2 f\cdot\frac{d^2}{dz^2}[\sigma_z]+f\frac{d^4}{dz^4}[\sigma_z]=0, (f\equiv f(x,y).)$$

and the boundary conditions

$$\left.\begin{aligned}&\sigma_z=-pf(x,y),\ \text{at}\ z=h\,;\ \sigma_z=0\,,\ \text{at}\ z=-h\,,\\ &\frac{\partial\sigma_z}{\partial z}=0\,,\ \text{at}\ z=\pm h\,.\end{aligned}\right\} \cdots\cdots (3.7)$$

Once σ_z is obtained, we shall be able to determine the form of Θ. From the last of equations (3.2a) it follows that

$$\frac{\partial^2\Theta}{\partial z^2}=-(1+\nu)\left(\nabla_1^2+\frac{\partial^2}{\partial z^2}\right)\sigma_z=-(1+\nu)\frac{3p}{2h^2}zf(x,y) \quad \cdots\cdots (3.8)$$

Hence

$$\Theta = -\frac{(1+\nu)}{4h^3} pf(x, y) z^3 + g(x, y) z + z\Theta_1 + \Theta_0 . \qquad \cdots\cdots\cdots\cdots \text{(3.9a)}$$

$z\Theta_1 + \Theta_0$ in (3.9) can be omitted for the present purpose to obtain particular solutions. Taking the condition $\nabla^2\Theta = 0$ into consideration, $g(x, y)$ is determined as

$$g(x, y) = \frac{3(1+\nu)p}{4h^3} x \int^x f(x, y)\, dx = \frac{3(1+\nu)}{8h^3} p \left(x \int^x f dx + y \int^y f dy\right). \quad \text{(3.9 b)}$$

For later convenience we shall take the symmetrical, last form of (3.9 b) hereafter, though the first form may suffice, in which it is supposed that $\int^x f dx$ and $\int^y f dy$ are adjusted so as to be plane harmonic functions. With this expression for $g(x,y)$, Θ is written

$$\Theta = -(1+\nu) \frac{K}{3} z \left\{ z^2 f - \frac{3}{2} \left(x \int^x f dx + y \int^y f dy\right) \right\}, \quad \cdots\cdots\cdots \text{(3.10)}$$

where $K = \dfrac{3p}{4h^3}$. Next we shall obtain the shear stress components τ_{zx}, τ_{yz} by means of the last of the equations (3.1), the last two of the equations (3.2b) and the boundary conditions $\tau_{zx} = \tau_{yz} = 0$ at $z = \pm h$. We have with the formula (3.6)

$$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = -\frac{\partial \sigma_z}{\partial z} = -Kf(x,y)(z^2 - h^2), \quad \cdots\cdots\cdots\cdots\cdots\cdots \text{(3.11)}$$

and with formula (3.10)

$$\nabla^2 \tau_{xz} = \frac{-1}{(1+\nu)} \frac{\partial^2 \Theta}{\partial z \partial x} = K\left(z^2 f_x - \frac{1}{2} \int^x f dx - \frac{1}{2} x f - \frac{1}{2} y \int^y f_x dx\right),$$

$$\nabla^2 \tau_{yz} = \frac{-1}{(1+\nu)} \frac{\partial^2 \Theta}{\partial y \partial z} = K\left(z^2 f_y - \frac{1}{2} \int^y f dy - \frac{1}{2} y f - \frac{1}{2} x \int^x f_y dx\right),$$

in which f_x and f_y denote the derivatives $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$.

Then we may get the integrals of the above differential equations after some tedious calculations as follows:

$$\tau_{xz} = K(z^2 - h^2) X, \quad \tau_{yz} = K(z^2 - h^2) Y, \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots \text{(3.13)}$$

in which X and Y are

$$X = A \int^x f dx + Bxf + Cf_x + Dy \int^x f_y dx + Ey \int^y f_x dy + Fz^2 f_x, \quad \cdots\cdots \text{(3.14a)}$$

$$Y = A' \int^y f dy + B'yf + C'f_y + D'x \int^y f_x dy + E'x \int^x f_y dx + F'z^2 f_y. \quad \text{(3.14b)}$$

The coefficients in these formulae are given by

$$A=A'=-\frac{1}{4}\,,\ B=B'=-\frac{1}{4}\,,\ C=C'=-\frac{h^2}{3}\,,\ F=F'=\frac{1}{6}\,,$$
$$\left(D-E=\frac{1}{4}\,,\ D'-E'=\frac{1}{4}\right).$$

Concerning the coefficients D, D', E, E', they may be taken to be $D=D'=0,\ E=E'=\dfrac{-1}{4}$, since the relation $\int^x f_y dx=-\int^y f_x dy$ may be utilized. The author expected X and Y to be functions of only x and y at first, because the shearing stresses τ_{xz}, τ_{yz} are supposed to vary according to a parabolic law in the generalized plane stress solution and particular solution in the case of uniform pressure, but it seemed impossible to find such solutions and so he was led to add Fz^2f_x to X and $F'z^2f_y$ to Y. In the above calculation we made frequent use of the relation

$$\nabla_1^2(xf(x,y))=2\frac{\partial f}{\partial x}=2f_x,\ \ (\nabla_1^2 f(x,y)=0). \quad\cdots\cdots\cdots\cdots (3.15)$$

Next we shall determine the forms of σ_x, σ_y and τ_{xy}. From the fundamental equations (3.1), (3.2) and (3.3), we may write, using the formulae σ_z, Θ, τ_{xz}, and τ_{yz} obtained above,

$$\left.\begin{aligned}\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}&=-\frac{\partial\tau_{xz}}{\partial z}=-K\frac{\partial}{\partial z}\left\{(z^2-h^2)\,X\right\},\\ \frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}&=-\frac{\partial\tau_{yz}}{\partial z}=-K\frac{\partial}{\partial z}\left\{(z^2-h^2)\,Y\right\},\end{aligned}\right\}\cdots\cdots\cdots (3.15a)$$

$$\left.\begin{aligned}\nabla^2\sigma_x&=\frac{-1}{(1+\nu)}\frac{\partial^2\Theta}{\partial x^2}=\frac{Kz}{3}\left\{z^2f_{xx}-\frac{3}{2}\left(2f+xf_x+y\int^y f_{xx}dy\right)\right\},\\ \nabla^2\sigma_y&=\frac{-1}{(1+\nu)}\frac{\partial^2\Theta}{\partial y^2}=\frac{Kz}{3}\left\{z^2f_{yy}-\frac{3}{2}\left(2f+yf_y+x\int^x f_{yy}dx\right)\right\},\\ \nabla^2\tau_{xy}&=\frac{-1}{(1+\nu)}\frac{\partial^2\Theta}{\partial x\partial y}=\frac{Kz}{3}\left\{z^2f_{xy}-\frac{3}{2}\left(xf_y+yf_x\right)\right\},\end{aligned}\right\}\ (3,.15b)$$

$$\sigma_x+\sigma_y=\Theta-\sigma_z=\frac{K}{3}\left[-z^3(2+\nu)f+3z\left\{\frac{(1+\nu)}{2}\left(x\int^x f\,dx+y\int^y f\,dy\right)+h^2f\right\}+2h^3f\right].\ \ (f(x,y)\equiv f)\cdots (3.15c)$$

If we put $\tau_{xy}=-\dfrac{\partial^2 F}{\partial x\partial y}$, $\cdots\cdots\cdots\cdots\cdots\cdots$ (3.15d)

F being a so-called stress function, from (3.15a) σ_x and σ_y are given in the following forms:

$$\left.\begin{aligned}\sigma_x &= -2Kz\int^x X_1 dx + \frac{\partial^2 F}{\partial y^2} - \frac{K}{3} z(2z^2-h^2)f\,,\\ \sigma_y &= -2Kz\int^y Y_1 dy + \frac{\partial^2 F}{\partial x^2} - \frac{K}{3} z(2z^2-h^2)f\,,\end{aligned}\right\} \cdots (3.16)$$

in which

$$X_1 = X - \frac{1}{6} z^2 f_x,\ \ Y_1 = Y - \frac{1}{6} z^2 f_y\,.$$

Accordingly we get

$$\sigma_x + \sigma_y = -2Kz\left(\int^x X_1 dx + \int^y Y_1 dy\right) - \nabla_1^2 F - \frac{2}{3} Kz(2z^2-h^2)f\,. \quad \cdots (3.17)$$

Concerning the determination of F function it is regrettable that the process of integration to be employed to determine the form of F function is seemingly too complicated to be applied to this case and therefore we are compelled to proceed by making a slight modification in the method of Love. Through Eqs. (3.15b), (3.16) the necessary equations are as follows:

$$\begin{aligned}\nabla^2\sigma_x &= \frac{\partial^2}{\partial y^2}\nabla^2 F - 2K\nabla^2\left(z\int^x X_1 dx\right) - \frac{K}{3}\nabla^2\left\{z(2z^2-h^2)f\right\}\\ &= \frac{Kz}{3}\left\{z^2 f_{xx} - \frac{3}{2}\left(2f + xf_x + y\int^y f_{xx} dy\right)\right\}. \quad \cdots\cdots\cdots\cdots (3.18a)\end{aligned}$$

$$\begin{aligned}\nabla^2\sigma_y &= \frac{\partial^2}{\partial x}\nabla^2 F - 2K\nabla^2\left(z\int^y Y_1 dy\right) - \frac{K}{3}\nabla^2\left\{z(2z^2-h^2)f\right\}\\ &= \frac{Kz}{3}\left\{z^2 f_{yy} - \frac{3}{2}\left(2f + yf_y + x\int^x f_{yy} dx\right)\right\} \quad \cdots\cdots\cdots\cdots (3.18b)\end{aligned}$$

$$\nabla^2\tau_{xy} = \frac{\partial^2}{\partial x\partial y}(-\nabla^2 F) = \frac{Kz}{3}\left\{z^2 f_{xy} - \frac{3}{2}(xf_y + yf_x)\right\} \quad \cdots\cdots (3.18c)$$

From Eq. (3.15b) and Eq. (3.18c) we have

$$\nabla^2\tau_{xy} = \frac{-\partial^2}{\partial x\partial y}(\nabla^2 F) = \frac{-1}{(1+\nu)}\frac{\partial^2\Theta}{\partial x\partial y}\,, \quad \cdots\cdots\cdots\cdots (3.19)$$

then we integrate the equations (3.19)

$$\nabla^2 F = \frac{\Theta}{(1+\nu)} + F^1(x,z) + F^2(y,z)\,, \quad \cdots\cdots\cdots\cdots (3.20)$$

in which a function of z only is not contained for an evident reason.

Eq. (3.20) being substituted in Eq. (3.18a), for example, each term in $F^2_{yy}(y,z)$ contains $f(x,y)$ function and therefore $F^2(y,z)$ may be put equal to zero. Likewise, we may have $F^1(x,z)=0$.

But if it is needful to symmetrize the result of $\int^x X_1 dx$ in (3.16), for example, with respect to x and y by adding a function of only y, $F^2_{yy}(y,z)$ will takes the form $2K\frac{\partial^2 g}{\partial y^2}$, where $g(y)$ should be determined cautiously. And so we may write

$$F^2(y,z) = 2Kzg(y)+a(z)y+b(z)$$

As is usual, a linear function of x and y, i.e., the above type $a(z)y+$ $+b(z)$, may be taken to be zero. Hereafter a function of the type $2Kzg(y)$ shall be omitted, because it appears only in special case due to the property of $f(x,y)$, though the calculation to be continued does not become troublesome, even if we do not exclude it. Consequently $\nabla^2 F(x,y,z)$ may be seen to be equivalent to Θ.

Now we may write from (3.20)

$$\nabla_1^2 F+\frac{\partial^2 F}{\partial z^2}=\frac{\Theta}{1+\nu}=-\frac{Kz}{3}\left\{z^2 f-\frac{3}{2}\left(x\int^x f\,dx+y\int^y f\,dy\right)\right\} \quad \cdots (3.21)$$

Then we equate the right-hand side of (3.17) to the right-hand side of (3.15c), obtaining the expression for $\nabla_1^2 F$.

$$\nabla_1^2 F = 2Kz\left(\int^x X_1 dx+\int^y Y_1 dy\right)+\frac{2}{3}Kz(2z^2-h^2)f+$$
$$+\frac{K}{3}\left[-z^3(2+\nu)f+3z\left\{\frac{(1+\nu)}{2}\left(x\int^x f\,dx+y\int^y f dy\right)+h^2 f\right\}+2h^3 f\right]. \quad (3.22)$$

Hence we have from (3.21), (3.22), using the expressions for X_1 and Y_1,

$$F=\frac{(\nu-3)}{60}Kz^5 f+\frac{h^2}{6}Kz^3 f+Kz^3\frac{1}{12}\left\{(\nu-2)x\int^x f dx+(\nu-2)y\int^y f\,dy\right\}$$
$$-\frac{h^3}{3}Kz^2 f+zE^1(x,y)+E^2(x,y)\,. \quad \cdots\cdots (3.23)$$

Next we shall determine the forms of $E^1(x,y)$ and $E^2(x,y)$. Substituting formula (3.23) in (3.22) and equating the coefficients of the similar terms with respect to the power of z on both sides of the equation, we obtain $\nabla_1^2 E^1$ and $\nabla_1^2 E^2$.

$$\nabla_1^2 E^1(x,y)=K\left\{\frac{(\nu-1)}{2}\left(x\int^x f\,dx+y\int^y f\,dy\right)-h^2 f\right\}, \quad \cdots\cdots (3.24a)$$

$$\nabla_1^2 E^2(x,y) = K\frac{2}{3}h^3 f = \frac{p}{2}f . \qquad \cdots\cdots (3.24b)$$

Eqs. (3.24a), (3.24b) are integrated by the use of the following relations:

$$\left.\begin{aligned} &\nabla_1^2(x^2 f) = 2f + 4xf_x . \quad (\nabla_1^2 f = 0) \\ &\nabla_1^2(x\int^x x\int^x f\,dx^2) = 4x\int f\,dx , \ \nabla_1^2(xyf) = 2(yf_x + xf_y) , \\ &\nabla_1^2(yx\int^x\int^y f\,dx\,dy) = 2\,(x\int^x f\,dx + y\int^y f\,dy) , \end{aligned}\right\} \quad (3.25)$$

in which and throughout the following, $\int^x\int^y f dx^m dy^n$ is always supposed to be adjusted to satisfy the relation.

$$\nabla_1^2(\int\int f\,dx^m dy^n) = 0 . \qquad (m, n \text{ are integers})$$

We have

$$E^1(x,y) = K\Big[\frac{(\nu-1)}{16}\Big\{x\int^x x\int^x f\,dx^2 + 2xy\int^x\int^y f\,dx\,dy + y\int^y y\int^y f\,dy^2\Big\}$$
$$-\frac{h^2}{4}\,(x\int^x f\,dx + y\int^y f\,dy)\Big] , \qquad \cdots\cdots (3.26a)$$

$$E^2(x,y) = \frac{p}{8}(x\int^x f\,dx + y\int^y f\,dy) , \qquad \cdots\cdots (3.26b)$$

in which the forms of integrals are symmetrized. So we find, for example,

$$\left.\begin{aligned} E^1_{yy}(x,y) &= K\Big\{\frac{(\nu-1)}{16}(-x^2 f + y^2 f + 5x\int^x f\,dx + 3y\int^y f\,dy \\ &\quad + 2xy\int^x f_y\,dx) - \frac{h^2}{4}(-xf_x + yf_y + 2f)\Big\} , \\ E^2_{yy}(x,y) &= \frac{p}{8}(-xf_x + yf_y + 2f) . \end{aligned}\right\} \quad (3.27)$$

Then we have for σ_x from (3.16)

$$\sigma_x = K\Big[\frac{(\nu-3)}{60} f_{yy} z^2 + \frac{1}{12}\Big\{(2-\nu)\,-xf_x + yf_y) - 2(2+\nu)f$$
$$+ 2h^2 f_{yy}\Big\} z^3 - \frac{1}{3}h^3 f_{yy} z^2 + \frac{1}{16}\Big\{(5\nu+3)\,x\int^x f\,dx + (3\nu+5)\,y\int^y f\,dy$$
$$+ (\nu-1)(-x^2 f + y^2 f + 2xy\int^x f_y dx) - 4h^2(-xf_x - 2f + yf_y)\Big\} z$$
$$+ \frac{h^2}{6}(-xf_x + yf_y + 2f)\Big] . \qquad \cdots\cdots (3.28)$$

It can be shown that this expression for σ_x satisfies equation (3.18a). And σ_y are from (3.16), (3.26)

$$\sigma_y = K\Big[\frac{(\nu-3)}{60}f_{xx}z^5+\frac{1}{12}\Big\{(2-\nu)(xf_x-yf_y)-2(2+\nu)f+2h^2f_{xx}\Big\}z^3$$
$$-\frac{1}{3}h^3f_{xx}z^2+\frac{1}{16}\Big\{(5\nu+3)y\int^y f\,dy+(3\nu+5)x\int^x f\,dx+(\nu-1)\times$$
$$(x^2f-y^2f+2xy\int^y f_x dy)-4h^2(xf_x-yf_y-2f)\Big\}z+\frac{h^3}{6}(xf_x-yf_y+2f)\Big]. \qquad \cdots\cdots (3.29)$$

Also this expression for σ_y may be shown to satisfy equation (3.18b). For τ_{xy} we have the expression.

$$\tau_{xy}=-K\Big[\frac{(\nu-3)}{60}f_{xy}z^5+\frac{1}{12}\Big\{(2-\nu)(xf_x+yf_y)+2h^2f_{xy}\Big\}z^3-\frac{h^3}{3}f_{xy}z^2$$
$$-\frac{1}{16}\Big\{(1-\nu)(x^2\int^x f_y\,dx+y^2\int^y f_x dy+x\int^x f\,dx+y\int^y f\,dy+2xyf$$
$$+4\int^x\int^y f\,dxdy)+4h^2(xf_y+yf_x)\Big\}z+\frac{h^3}{6}(xf_y+yf_x)\Big]. \qquad \cdots\cdots (3.30)$$

Formula (3.30) can be proved to satisfy equation (3.18c).

Next we shall want to determine the forms of diplacements. First we write down the necessary fundamental equations.

$$\frac{\partial u}{\partial x}=\frac{1}{E}\{\sigma_x-\nu(\sigma_y+\sigma_z)\}, \qquad \cdots\cdots (3.31a)$$

$$\frac{\partial v}{\partial y}=\frac{1}{E}\{\sigma_y-\nu(\sigma_x+\sigma_z)\}, \qquad \cdots\cdots (3.31b)$$

$$\frac{\partial w}{\partial z}=\frac{1}{E}\{\sigma_z-\nu(\sigma_x+\sigma_y)\}, \qquad \cdots\cdots (3.31c)$$

$$\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}=\frac{1}{\mu}\tau_{xy}, \qquad \cdots\cdots (3.32a)$$

$$\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}=\frac{1}{\mu}\tau_{xz}, \qquad \cdots\cdots (3.32b)$$

$$\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}=\frac{1}{\mu}\tau_{yz}, \qquad \cdots\cdots (3.32c)$$

in which μ is the shear modulus.

By Eqs. (3.5), (3.16), (3.31a) the displacement u may be written as follows :

$$u=\frac{1}{E}\Big\{2Kz\Big(\nu\int^{x}\int^{y}Y_1dx\,dy-\int^{(x)}X_1dx^2\Big)+\Big(\int^{x}F_{yy}dx-\nu F_x\Big)$$

$$+\frac{K}{3}(\nu-1)z(2z^2-h^2)\int^{x}f\,dx-\frac{K}{3}\nu(z^3-3h^2z-2h^3)\int^{x}f\,dx\Big\}\,. \cdots (3.33)$$

Substituting the formulae for X_1, Y_1, and $F(x, y, z)$ indicated above in this equation, we have for u

$$u=\frac{K}{E}\Big[\frac{(\nu+1)(\nu-3)}{60}f_xz^5+\frac{(\nu+1)}{12}\Big\{(\nu-2)\Big(\int^{x}f\,dx+xf+y\int^{y}f_xdy\Big)$$

$$-2h^2f_x\Big\}z^3+\frac{(\nu+1)}{3}h^3f_xz^2+\frac{1}{16}\Big\{(1-\nu^2)\Big(3y\int^{x}\int^{y}f\,dxdy+(x^2-y^2)$$

$$\times\int^{x}f\,dx+2xy\int^{y}f\,dy\Big)+(1-\nu)(3+\nu)\int^{x}x\int^{x}f\,dx^2+4(\nu+1)h^2\Big(\int^{x}f dx$$

$$+xf+y\int^{y}f_xdy\Big)\Big\}z+\frac{(1+\nu)}{6}h^3\Big(-xf+3\int^{x}f\,dx-y\int^{y}f_xdy\Big)\Big]. \cdots (3.34)$$

In the same way we get the expression for v from (3.31b)

$$v=\frac{1}{E}\Big[2Kz\Big(\nu\int^{x}\int^{y}X_1dxdy-\int^{(y)}\int Y_1dy^2\Big)+\Big(\int^{y}F_{xx}dy-\nu F_y\Big)$$

$$+\frac{K}{3}\Big\{(\nu-2)z^3+(2\nu+1)zh^2+2\nu h^3\Big\}\int^{y}f\,dy\Big]. \quad\cdots\cdots\cdots\cdots\cdots\cdots (3.35)$$

This reduces to the form

$$v=\frac{K}{E}\Big[\frac{(\nu+1)(\nu-3)}{60}f_yz^5+\frac{(\nu+1)}{12}\Big\{(\nu-2)\Big(\int^{y}f\,dy+yf+x\int^{x}f_ydx\Big)$$

$$-2h^2f_y\Big\}z^3+\frac{(\nu+1)}{3}h^3f_yz^2+\frac{1}{16}\Big\{(1-\nu^2)\Big(3x\int^{x}\int^{y}f\,dxdy+(y^2-x^2)\int^{y}f dy$$

$$+2xy\int^{x}f\,dx\Big)+(1-\nu)(3+\nu)\int^{y}y\int^{y}f\,dy^2+4(\nu+1)h^2\Big(\int^{y}f\,dy+yf$$

$$+x\int^{x}f_ydx\Big)\Big\}z+\frac{(1+\nu)}{6}h^3\Big(-yf+3\int^{y}f dy-x\int^{x}f_ydx\Big)\Big]. \cdots\cdots\cdots (3.36)$$

The above solutions (3.34), (3.36) must, of course, satisfy equation (3.32a), together with (3.30), which can be proved after some calculation. By this proof it is also seen that solution (3.30) and others are correct.

By (3.31c) the displacement w is given the form

$$w=\frac{K}{E}\Big[\frac{1}{12}(z^4-6h^2z^2-8h^3z)f+\frac{\nu(2+\nu)}{12}z^4f-\frac{\nu}{2}z^2\times$$
$$\times\left\{\frac{(1+\nu)}{2}\Big(x\int^x f\,dx+y\int^y f\,dy\Big)+h^2f\right\}-\frac{2\nu}{3}h^3zf\Big]+W(x,y),\quad(3.37)$$

where $W(x,y)$ is an integration constant, the function of only x and y. Incidentally the relation $\Theta-\sigma_z=\sigma_x+\sigma_y$ is easily proved to be satisfied by formulae (3.5), (3.10), (3.28), (3.29).

In order to determine the form of $W(x,y,)$ we shall use (3.32c) or (3.32b). Substituting the expressions (3.37) and (3.36) in equation (3.32c), we get the formula of $\frac{\partial W(x,y)}{\partial y}$ and integrate in with respect to y, proving that the net coefficients of various powers of z are equal to zero. Then we obtain the following result:

$$W=\frac{K}{E}\left\{\frac{(1+\nu)}{4}h^2\Big(x\int^x f\,dx+y\int^y f\,dy\Big)+\frac{2\,(1+\nu)}{3}h^4f+\frac{\nu(\nu-1)}{16}\Big(x\int^x x\int^x f\,dx^2\right.$$
$$\left.+2xy\int^x\int^y f\,dxdy+y\int^y y\int^y f\,dy^2\Big)\right\}+\frac{K}{E}\frac{(\nu-1)}{16}\Big\{x\int^x\int^y\int^y f\,dxdy^2$$
$$+2xy\int^x\int^y f\,dxdy-y\int^{(y)} f\,dy^3+(y^2-x^2)\int^{(y)}\int f\,dy^2\Big\}.\quad\cdots\cdots(3.38)$$

Instead of substituting Eq. (3.38) in Eq. (3.32b), we shall deduce the expression for $W(x,y)$ from Eq. (3.32b) in the same way as above. We have the following result:

$$W=\frac{K}{E}\left\{\frac{(1+\nu)}{4}h^2\Big(x\int^x f\,dx+y\int^y f\,dy\Big)+\frac{2}{3}(1+\nu)\,h^4f+\frac{\nu(\nu-1)}{16}\times\right.$$
$$\left.\times\Big(x\int^x x\int^x f\,dx^2+2xy\int^x\int^y f\,dxdy+y\int^y y\int^y f\,dy^2\Big)\right\}+\frac{K}{E}\frac{(\nu-1)}{16}\times$$
$$\times\Big\{y\int^x\int^x\int^y f\,dx^2y+2xy\int^x\int^y f\,dxdy-x\int^{(x)}\iint f\,dx^3+(x^2-y^2)\int^{(x)}\int f\,dx^2\Big\}.$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\quad(3.39)$$

By virtue of the relation $\int^{(y)} f\,dy^2=-\int^{(x)} f\,dx^2$, which will hold assuredly, if integration constants of these integrations are disposed of so that they are plane harmonic functions, the fact that the result (3.39) is identically equal to (3.38) will be readily observed, this, of course, indicating the solution obtained to be correct. Symmetrizing the form of $W(x,y)$ obtained above, we may write

$$W(x,y)=\frac{K}{E}\Big[\frac{(\nu-1)}{32}\Big\{(2\nu+1)(x^2\int f\,dx^2+y^2\int f dy^2-x\int f\,dx^3+$$

$$-y\int f\,dy^3)+x\int f\,dxdy^2+y\int f\,dx^2dy+4(\nu+1)\,xy\int f\,dx\,dy$$

$$-(y^2\int f\,dx^2+x^2\int f\,dy^2)\Big\}+\frac{2}{3}(1+\nu)\,h^4f+\frac{(1+\nu)}{4}h^2(x\int f\,dx+y\int f\,dy)\Big].$$

$$\cdots\cdots\cdots\cdots \quad (3.40)$$

As a consequence the expression for w is

$$w=\frac{K}{E}\Big[\frac{\nu(\nu+2)+1}{12}fz^4-\frac{(1+\nu)}{2}\Big\{h^2f+\frac{\nu}{2}(x\int f\,dx+y\int f\,dy)\Big\}z^2$$

$$-\frac{2}{3}(1+\nu)h^3fz+\frac{(1+\nu)}{4}h^2(x\int f\,dx+y\int f\,dy)+\frac{2}{3}(1+\nu)h^4f$$

$$+\frac{(\nu-1)}{32}\Big\{2(\nu+1)(y^2\int f\,dy^2+x^2\int f\,dx^2)-(2\nu+1)(y\int f\,dy^3+x\int f\,dx^3)$$

$$+x\int f\,dxdy^2+y\int f\,dx^2dy+4(\nu+1)xy\int f\,dxdy\Big\}\Big]. \quad \cdots\cdots \quad (3.41)$$

And herewith we let the stress components τ_{xz}, τ_{yz} be rewritten into the following forms:

$$\tau_{xz}=K(h^2-z^2)\Big\{\frac{1}{4}(\int f\,dx+xf+y\int f_x dy)+\frac{1}{6}(2h^2-z^2)f_x\Big\} \quad \cdots \quad (3\cdot 13a')$$

$$\tau_{yz}=K(h^2-z^2)\Big\{\frac{1}{4}(\int f\,dy+yf+x\int f\,dx)+\frac{1}{6}(2h^2-z^2)f_y\Big\}, \cdots \quad (3.13b')$$

in which $\left(K=\dfrac{3p}{4h^3}\right)$.

Thus we have been able to get the particular solutions for the problem of thick plate under the load with plane harmonic distribution function and therefore, if the normal pressure is distributed according to a given plane harmonic function, however complicated it may be, its particular solutions can be had with ease. But it is to be noted that without investigating cautiously the property of a given plane harmonic function this function should not be substituted directly in the above final results, because in case where this given plane harmonic is so simple that in course of calculation to obtain the last results it may vanish or we are forced to dispose particularly of integrations, for example, $\int^x X_1\,dx$, in order to symmetrize solutions or integrals, this substitution would be erroneous. In this case we could not but calculate from the beginning following the process as indicated above. At any rate it may be said that owing to the simple and convenient property of plane harmonic function we could manage to solve the differential equations, but it will be well-nigh impossible to solve the equations in the case of arbitrary loading function following the same manner as mentioned above without expanding this loading function in series, though, needless to say, it is desirable to avoid expansion. Further, it will be readily seen that in the case of plane harmonic loading function σ_z varies along the thickness according to the same law as in the uuiform load case, and further, stress and displacement components have at most five or four powers of z terms, which is similar to the solutions in the latter case. Though the solutions obtained in the former case certainly satisfy the equations of equilibrium and the conditions of compatibility, they are not strictly correct because of their loss of generality, as stated above. Therefore they cannot satisfy generally any arbitrary boundary conditions and yet, as a matter of course, they can when the boundary conditions are constrained to be given by the terms of the resultant forces and couples. It is to the author's regret that he has been obliged to abridge much in describing the procedure to obtain the solutions.

§ IV Particular Solutions for the Thick Plate under Normal Surface Loal, the Distribution Function of Which Satisfies the Equation of Helmholtz.

In this section we shall undertake to get the particular solutions for the case of a given variable load whose loading function is such that

$$\sigma_z = -pv(x, y) \quad \text{at} \quad z = +h\,, \qquad \cdots\cdots (4.1a)$$

$$\nabla_1^2 v + k^2 v = 0\,, \quad (v(x, y) \equiv v) \qquad \cdots\cdots (4.1b)$$

in which p and k are given constants. We shall, from now on, call a function which satisfies the equation of Helmholtz a v-function. This function is seemingly usable because of the simple equation (4.1b). In fact, each term of a double Fourier series expressed in the cartesian coordinates may be a v-function and so the thick plate under an arbitrary load will be led to be treated, if the double Fourier series associated with the distribution function of this load exists and the plate problem for the case of v-loading function can be solved. Inthis section we only select the cartesian coordinates. Now we begin by obtaining the expression for the stress component σ_z using equation (4.1b). We proceed according to the mode of attack as described in the preceding section. For the form of σ_z we may put

$$\sigma_z = p\, v(x, y) w(z) \,. \quad \cdots\cdots \quad (4.2)$$

From the boundary conditions for σ_z those for $w(z)$ will be deduced

$$w(z) = -1 \quad \text{at} \quad z = h\,, \quad \frac{\partial w(z)}{\partial z} = 0 \quad \text{at} \quad z = \pm h\,, \quad \cdots (4.3)$$
$$w(z) = 0 \quad \text{at} \quad z = -h\,.$$

And the following equation must hold

$$\nabla^4(\sigma_z) = p\left\{w(z)\nabla_1^4 v + 2\frac{\partial^2 w(z)}{\partial z^2}\nabla_1^2 v + v\frac{\partial^4 w(z)}{\partial z^4}\right\} = 0\,.$$

Then we obtain by means of equation (4.1b)

$$\frac{\partial^4 w(z)}{\partial z^4} - 2k^2\frac{\partial^2 w(z)}{\partial z^2} + k^4 w(z) = 0\,. \quad \cdots\cdots \quad (4.4)$$

Therefore as the solution for the differential equation we may have

$$w(z) = A\cosh kz + B\sinh kz + Cz\cosh kz + Dz\sinh kz\,, \qquad (4.5)$$

in which the coefficients are determined considering the boundary conditions (4.3) as follows:

$$A = \bar{A}/\delta\,, \quad B = \bar{B}/\delta\,, \quad C = \bar{C}/\delta\,, \quad D = \bar{D}/\delta\,,$$

$$\bar{A} = \begin{vmatrix} -1 & \sinh kh & h\cosh kh & h\sinh kh \\ 0 & -\sinh kh & -h\cosh kh & h\sinh kh \\ 0 & k\cosh kh & \cosh kh + kh\sinh kh & \sinh kh + kh\cosh kh \\ 0 & k\cosh kh & \cosh kh + kh\sinh kh & -\sinh kh - kh\cosh kh \end{vmatrix}$$
$$= kh(-\cosh 2kh\sinh kh + \sinh kh) + 2k^2h^2\cosh kh - \sinh kh\sinh 2kh\,,$$

$$\delta = \begin{vmatrix} \cosh kh & \sinh kh & k\cosh kh & h\sinh kh \\ \cosh kh & -\sinh kh & -k\cosh kh & h\sinh kh \\ k\sinh kh & k\cosh kh & \mathrm{chsh}\, kh + kh\sinh kh & \sinh kh + kh\cosh kh \\ -k\sinh kh & k\cosh kh & \cosh kh + kh\sinh kh & -\sinh kh - kh\cosh kh \end{vmatrix}$$
$$= \sinh^2 2kh - 4k^2h^2\,.$$

$$\bar{B} = -kh(\cosh 2kh \cosh kh + \cosh kh) - 2k^2h^2 \sinh kh - \cosh kh \sinh 2kh ,$$

$$\bar{C} = k(\cosh kh \sinh 2kh + 2kh \cosh kh) ,$$

$$\bar{D} = k(\sinh kh \sinh 2kh - 2kh \sinh kh) .$$

The expression for $w(z)$ may be reduced to

$$w(z) = \frac{1}{(\sinh^2 2kh - 4k^2h^2)}\Big[-(kh\cosh 2kh + \sinh 2kh)\sinh k(z+h) - kh \sinh k(z-h) + 2k^2h^2 \cosh k(z-h) + kz\Big\{\sinh 2kh \cosh k(z+h) + 2kh \cosh k(z-h)\Big\}\Big]. \quad \cdots\cdots (4.6)$$

Thus σ_z may be put into the following form:

$$\sigma_z = Kv(x,y)[\sigma_z] , \quad \cdots\cdots (4.7)$$

where

$$[\sigma_z] = -(kh\cosh 2kh + \sinh 2kk)\sinh k(z+h) - kh\sinh k(z-h) + 2k^2h^2 \cosh k(z-h) + kz\Big\{\sinh 2kh \cosh k(z+h) + 2kh \cosh k(z-h)\Big\} , \quad \cdots\cdots (4.8)$$

$$K = p\Big/(\sinh^2 2kh - 4k^2h^2) . \quad \cdots\cdots (4.9)$$

This expression, however, is not so simple as in the case of harmonic loading function.

By the last equation of (3.2a) we have

$$\frac{\partial^2\Theta}{\partial z^2} = -(1+\nu)Kv(x,y)\Big\{-k^2[\sigma_z] + \frac{\partial^2[\sigma_z]}{\partial z^2}\Big\} . \quad \cdots\cdots (4.10)$$

Therefore Θ is given through integration

$$\Theta = -2(1+\nu)K\Big\{\sinh 2kh \sinh k(z+h) + 2kh \sinh k(z-h)\Big\}v(x,y) + g(x,y) + z\Theta_1 + \Theta , \quad \cdots\cdots (4.11)$$

in which

$$\nabla_1^2 g(x,y) \neq 0 , \quad \nabla_1^2\Theta_1 = \nabla_1^2\Theta_0 = 0 , \quad \nabla^2\Theta = 0 .$$

From the form of Θ, $g(x,y)$ is obviously taken to be zero and both Θ_1 and Θ_0 are omitted hereafter.

In the next place we shall determine the stress components τ_{yz} and τ_{xz}. The necessary equations and conditions are

$$\frac{\partial \tau_{zx}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}=-\frac{\partial \sigma_z}{\partial z}=-Kv(x, y)\frac{\partial}{\partial z}[\sigma_z], \quad \cdots\cdots (4.12a)$$

$$\nabla^2\tau_{xz}=\frac{-1}{(1+\nu)}\frac{\partial^2\Theta}{\partial z\partial x}=2Kk\Big\{\sinh 2kh\cosh k(z+h)+ + 2kh\cosh k(z-h)\Big\}\frac{\partial v}{\partial x}, \quad \cdots\cdots (4.12b)$$

$$\nabla^2\tau_{yz}=\frac{-1}{(1+\nu)}\frac{\partial^2\Theta}{\partial z\partial y}=2Kk\Big\{\sinh 2kh\cosh k(z+h)+ + 2kh\cosh k(z-h)\Big\}\frac{\partial v}{\partial y}, \quad \cdots\cdots (4.12c)$$

$$\tau_{yz}=\tau_{zx}=0 \quad \text{at} \quad z=\pm h. \quad \cdots\cdots (4.12d)$$

Then the solutions for these shear stresses may be taken as follows:

$$\tau_{zx}=\frac{K}{k^2}\frac{\partial[\sigma_z]}{\partial z}\frac{\partial v}{\partial x}, \quad \cdots\cdots (4.13a)$$

$$\tau_{yz}=\frac{K}{k^2}\frac{\partial[\sigma_z]}{\partial z}\frac{\partial v}{\partial y}. \quad \cdots\cdots (4.13b)$$

Solutions (4.13) evidently satisfy Eqs. (4.12a), (4.12b), (4.12c) by the relation (4.1b) and the boundary conditions (4.12d) by virtue of the property of $w(z)$, i.e., $\frac{\partial \sigma_z}{\partial z}=0$ at $z=\pm h$.

And thereby it will be easily understood that the conditions $\left(\frac{\partial \tau_{zx}}{\partial z}=\frac{\partial \tau_{yz}}{\partial z}=0 \text{ at } z=\pm h\right)$ could not be imposed besides the conditions (4.12d).

Now we can determine the expressions for σ_x, σ_y and τ_{xy} from the following equations:
from Eqs. (3.1)

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}=\frac{-\partial \tau_{xz}}{\partial z}=\frac{-K}{k^2}\frac{\partial v}{\partial x}\frac{\partial^2[\sigma_z]}{\partial z^2}, \quad \cdots\cdots (4.14a)$$

$$\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}=\frac{-\partial \tau_{yz}}{\partial z}=\frac{-K}{k^2}\frac{\partial v}{\partial y}\frac{\partial^2[\sigma_z]}{\partial z^2}. \quad \cdots\cdots (4.14b)$$

from the conditions (3.2)

$$\nabla^2\sigma_x=\frac{-1}{(1+\nu)}\frac{\partial^2\Theta}{\partial x^2}=2KZ\frac{\partial^2 v}{\partial x^2}, \quad \cdots\cdots (4.15a)$$

$$\nabla^2\sigma_y=\frac{-1}{(1+\nu)}\frac{\partial^2\Theta}{\partial y^2}=2KZ\frac{\partial^2 v}{\partial y^2}, \quad \cdots\cdots (4.15b)$$

$$\nabla^2\tau_{xy} = \frac{-1}{(1+\nu)}\frac{\partial^2\Theta}{\partial x\partial y} = 2KZ\frac{\partial^2 v}{\partial x\partial y}, \qquad \cdots\cdots (4.15c)$$

in which

$$Z = \sinh 2kh \sinh k(z+h) + 2kh \sinh k(z-h), \qquad \cdots\cdots (4.15d)$$

also

$$\sigma_x + \sigma_y = \Theta - \sigma_z = -2(1+\nu)KZv(x,y) - K[\sigma_z]v(x,y). \qquad (4.16)$$

In the first place we put, in the same way as before,

$$\tau_{xy} = -\frac{\partial^2 F(x,y,z)}{\partial x\partial y}, \qquad \cdots\cdots (4.17)$$

where $F(x,y,z)$ is a three dimentional complementary stress function. Substituting formula (4.17) in the equations (4.14) and integrating the equations thus obtained with respect to x and y respectively, we have

$$\sigma_x = \frac{-K}{k^2}v(x,y)\frac{\partial^2[\sigma_z]}{\partial z^2} + \frac{\partial^2 F}{\partial y^2}, \qquad \cdots\cdots (4.18a)$$

$$\sigma_y = \frac{-K}{k^2}v(x,y)\frac{\partial^2[\sigma_z]}{\partial z^2} + \frac{\partial^2 F}{\partial x^2}. \quad (F(x,y,z)\equiv F) \qquad \cdots\cdots (4.18b)$$

Adding these equations, equation (4.16) is written

$$\frac{-2K}{k^2}\frac{\partial^2[\sigma_z]}{\partial z^2}v(x,y) + \nabla_1^2 F = -Kv(x,y)\left\{2(1+\nu)Z + [\sigma_z]\right\}. \quad (4.19)$$

Next we may take for $F(x,y,z)$ from Eqs. (4.15c), (4.17)

$$F(x,y,z) = \frac{\Theta}{1+\nu} + F^1(x,z) + F^2(y,z) \qquad \cdots\cdots (4.20)$$

For the same reason as in the preceding section $F^1(x,z)$ and $F^2(y,z)$ can be discarded. In fact, for example, if we substitute the expression (4.20) in Eq. (4.18a) and the fromula thus obtained in Eq. (4.15a), we get $\frac{\partial^2 F^2(y,z)}{\partial y^2} = 0$, therefore, $F^2(y,z)$ is a linear fnnction of y and so can be abandoned as usual.

In consequence we have the equation

$$\nabla^2 F = \nabla_1^2 F + \frac{\partial^2}{\partial z^2}F = \frac{\Theta}{1+\nu} = 2KZv(x,y). \qquad \cdots\cdots (4.21)$$

Substituting the representation $\nabla_1^2 F$ obtainable from Eq. (4.19) in Eq. (4.21), $\frac{\partial^2 F}{\partial z^2}$ becomes

$$\frac{\partial^2 F}{\partial z^2} = Kv(x,y)\Big[\Big\{(2\nu-3)\sinh 2kh + kh\cosh 2kh\Big\}\sinh k(z+h)$$
$$+(4\nu-7)\sinh k(z-h) - 2k^2h^2\cosh k(z-h) - kz\times$$
$$\times\Big\{\sinh 2kh\cosh k(z+h) + 2kh\cosh k(z-h)\Big\}\Big]. \quad\cdots (4.22)$$

Then Eq. (4.22) comes to be integrated as

$$F = \frac{K}{k^2}v(x,y)\Big[\Big\{(2\nu-1)\sinh 2kh + kh\cosh 2kh\Big\}\sinh k(z+h)$$
$$+(4\nu-3)kh\sinh k(z-h) - 2k^2h^2\cosh k(z-h) - kz\times$$
$$\times\Big\{\sinh 2kh\cosh k(z+h) + 2kh\cosh k(z-h)\Big\}\Big] + zE^1(x,y) + E^2(x,y).$$
$$\cdots\cdots (4.23)$$

In order to determine the forms of $E^1(x,y)$ and $E^2(x,y)$ we substitute this expression for $F(x,y,z)$ in Eq. (4.19) and get the equation

$$z\nabla_1^2 E^1(x,y) + \nabla_1^2 E^2(x,y) = 0 .$$

Thus $E^1(x,y)$ and $E^2(x,y)$ are found to be plane harmonic functions but in determining the forms of E^1, E^2 plane harmonic fuuction can be excluded always. Hence we may write

$$E^1(x,y) = E^2(x,y) = 0 . \quad\cdots\cdots (4.24)$$

Finally we obtain the expressions for the stress components σ_x, σ_y, τ_{xy} from Eqs. (4.18a), (4.18b) and (4.17), using formula (4.23), respectively.

$$\sigma_x = \frac{-K}{k^2}\frac{\partial^2[\sigma_z]}{\partial z^2}v(x,y) + \frac{\partial^2 F}{\partial y^2} = \frac{K}{k^2}\Big[\Big(2\nu\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2}\Big)\sinh 2kh\times$$
$$\times\sinh k(z+h) + \Big(4\nu\frac{\partial^2 v}{\partial y^2} + 3\frac{\partial^2 v}{\partial x^2}\Big)kh\sinh k(z-h)$$
$$+\frac{\partial^2 v}{\partial x^2}\Big\{-kh\cosh 2kh\sinh k(z+h) + 2k^2h^2\cosh k(z-h)$$
$$+kz(\sinh 2kh\cosh k(z+h) + 2kh\cosh k(z-h))\Big\}\Big]. \quad\cdots (4.25)$$

This form of σ_x is easily confirmed to satisfy Eq. (4.15a).

For σ_y we have

$$\sigma_y = \frac{-K}{k^2}\frac{\partial^2}{\partial z^2}[\sigma_z]v(x,y) + \frac{\partial^2 F}{\partial x^2} = \frac{K}{k^2}\Big[\Big(2\nu\frac{\partial^2 v}{\partial x} + \frac{\partial^2 v}{\partial y^2}\Big)\sinh 2kh$$
$$\times\sinh k(z+h) + \Big(4\nu\frac{\partial^2 v}{\partial x^2} + 3\frac{\partial^2 v}{\partial y^2}\Big)kh\sinh k(z-h) + \frac{\partial^2 v}{\partial y^2}\times$$

$$\times \Big\{ -kh \cosh 2kh \sinh k(z+h) + 2k^2h^2 \cosh k(z-h)$$
$$+ kz\Big(\sinh 2kh \cosh k(z+h) + 2kh \cosh k(z-h)\Big)\Big\}\Big]. \quad \cdots\cdots \quad (4.26)$$

This solution is readily seen to satisfy Eq. (4.15b). And τ_{xy} is of the following form:

$$\tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y} = \frac{-K}{k^2}\Big[\Big\{(2\nu-1)\sinh 2kh + kh\cosh 2kh\Big\}\sinh k(z+h)$$
$$+ (4\nu-3)\, kh \sinh k(z-h) - 2k^2h^2 \cosh k(z-h) - kz \times$$
$$\times \Big\{\sinh 2kh \cdot \cosh k(z+h) + 2kh \cosh k(z-h)\Big\}\Big]\frac{\partial^2 v}{\partial x \partial y} \quad (4.27)$$

We can verify that Eq. (4.15c) is satisfied by the formula (4.27) and further Eq. (4.16) by Eqs. (4.25), (4.26). Herewith we rewrite the formulae of τ_{zx} and τ_{yz}.

$$\tau_{zx} = \frac{K}{k}\overline{Z}\frac{\partial v}{\partial x}, \quad \tau_{yz} = \frac{K}{k}\overline{Z}\frac{\partial v}{\partial y}, \quad \cdots\cdots \quad (4.28a)$$

in which

$$\overline{Z} = -kh\cosh 2kh \cosh k(z+h) + 2k^2h^2 \sinh k(z-h) + kh\cosh k(z-h)$$
$$+ kz\Big\{\sinh 2kh \sinh k(z+h) + 2kh \sinh k(z-h)\Big\}. \quad \cdots\cdots \quad (4.28b)$$

Now we shall want to obtain the representations for displacements by the use of Eqs. (3.31), (3.32) and the above solutions for stress oomponents. Through Eq. (3.31a) displacement u is given in the following form:

$$\frac{\partial u}{\partial x} = \frac{1}{E}\Big\{\sigma_x - \nu(\sigma_y+\sigma_z)\Big\} = \frac{K}{\mathrm{k}^2}\frac{1}{E}Z'\left(\frac{\partial^2 v}{\partial y^2} - \nu\frac{\partial^2 v}{\partial x^2} + k\, v(x,y)\right)$$
$$= \frac{-(1+\nu)}{E}\frac{K}{k^2}Z'\frac{\partial^2 v}{\partial x^2}, \quad \cdots\cdots \quad (4.29a)$$

and so

$$u = \frac{-(1+\nu)}{E}\frac{K}{k^2}Z'\frac{\partial v}{\partial x}, \quad \cdots\cdots \quad (4.29b)$$

in which

$$Z' = (2\nu-1)\sinh 2kh \sinh k(z+h) + kh\cosh 2kh \sinh k(z+h)$$
$$+ (4\nu-3)\, kh \sinh k(z-h) - 2k^2h^2\cosh k(z-h)$$
$$- kz\Big\{\sinh 2kh \cosh k(z+h) + 2kh \cosh k(z-h)\Big\}. \quad \cdots\cdots \quad (4.29c)$$

And in the same way we have for v by Eq. (3.31b)

$$v=\frac{-(1+\nu)}{E}\frac{K}{k^2}Z'\frac{\partial v}{\partial y}. \qquad (4.30)$$

We can make certain that Eq. (3.32a) is satisfied by Eqs. (4.27), (4.29a) and (4.29b).

From Eq. (3.31c) the form of w can be obtained as follows:

$$w=\frac{(1+\nu)}{E}\frac{K}{k}\Big[\Big\{2(\nu-1)\sinh 2kh-kh\cosh 2kh\Big\}\cosh k(z+h) + (4\nu-3)kh\cosh k(z-h)+2k^2h^2\sinh k(z-h)+ + kz\Big\{\sinh 2kh\sinh k(z+h)+2kh\sinh k(z-h)\Big\}\Big]v(x,y)+W(x,y). \qquad (4.31)$$

Substituted in Eq. (3.32c), the expression for w (4.31) yields $\frac{\partial W(x,y)}{\partial y}=0$ and so we may take $W(x,y)$ to be zero. Next instead of substituting the formula (4.31) in Fq. (3.32b) we handle Eqs. (4.31), (3.32b) in the same manner as before and we obtain $\frac{\partial W(x,y)}{\partial x}=0$. Therefore we can certainly put the integration constant $W(x,y)$ equal to zero.

$$W(x,y)=0. \qquad (4.32)$$

Thus we have obtained the solutions for the plate under the pressure with v-loading function. Compared with the case of plane harmonic loading function, computation in this case is relatively simple owing to the property of v-function and hyperbolic function of z. Further, for the sake of simplification the author has heretofore dealt with the thick plate, to the upper surface of which a given pressure is applied. But, when the plate is stressed by the load applied to the lower surface or both to the upper aud lower surface, furthermore, solutions for the lower surface case are needed and in order to obtain these solutions it seems unadvisable to employ the transformation of the coordinates, namely the rotation of the coordinates, and we had better change the sign of h, the thickness, in the solutions obtained above, i.e., we can attain the object by the reflection of the z axis. For the solutions in the latter oase we have only to apply the principle of superposition in the theory of the first order, upon which we depend in this paper.

In the next place we shall write here the resultant forces and couples for later convenience to satisfy the boundary conditions.

From Eqs. (4.25), (4.26), (4.27) we get the following formulae:

$$T_1=\int_{-h}^{h}\sigma_x dz=\frac{K}{k^3}2\nu(\cosh 2kh-1)(\sinh 2kh-2kh)\frac{\partial^2 v}{\partial y^2}, \quad (4.33\text{ a})$$

$$T_2=\int_{-h}^{h}\sigma_y dz=\frac{K}{k^3}2\nu(\cosh 2kh-1)(\sinh 2kh-2kh)\frac{\partial^2 v}{\partial x^2}, \quad (4.33\text{b})$$

$$S_1=-S_2=\int_{-h}^{h}\tau_{xy}dz=\frac{-K}{k^3}2\nu(\cosh 2kh-1)(\sinh 2kh-2kh)\frac{\partial^2 v}{\partial x\partial y}. \quad \cdots\cdots (4.33\text{c})$$

From equations (4.28) we obtain:

$$N_1=\int_{-h}^{h}\tau_{xz}dz=\frac{-K}{k^2}(\sinh^2 2kh-4k^2h^2)\frac{\partial v}{\partial x}=\frac{-p}{k^2}\frac{\partial v}{\partial x}, \quad \cdots (4.34\text{a})$$

$$N_2=\int_{-h}^{h}\tau_{yz}dz=\frac{-p}{k^2}\frac{\partial v}{\partial y}. \quad \cdots\cdots (4.34\text{b})$$

By Eq. (4.27) the torsional couples H_1 and H_2 are

$$H_1=-H_2=\int_{-h}^{h}-z\tau_{xy}dz=\frac{K}{k^4}\Big\{-(2\nu+1)\sinh^2 2kh+2\nu kh\times$$

$$\times\sinh 2kh(\cosh 2kh-1)+4k^2h^2(1+\nu+\nu\cosh 2kh)\Big\}\frac{\partial^2 v}{\partial x\partial y} \quad \cdots (4.35)$$

By Eq. (4.25) the bending moment G_1 is

$$G_1=\int_{-h}^{h}z\sigma_x dz=\frac{K}{k^2}\Big[\frac{1}{k^2}\frac{\partial^2 v}{\partial y^2}\Big\{2\nu kh\sinh 2kh(\cosh 2kh-1)$$

$$+4\big(\nu+1-(2-\nu)\cosh 2kh\big)k^2h^2-2(\nu+1)\sinh^2 2kh\Big\}$$

$$+v(x,y)\Big\{4(1-2\cosh 2kh)k^2h^2-2\sinh^2 2kh\Big\}\Big]. \quad \cdots\cdots (4.36)$$

The expression for $G_2=\int_{-h}^{h}z\sigma_y dz$ is obtained from Eq. (4.36) by changing $\frac{\partial^2}{\partial y^2}$ into $\frac{\partial^2}{\partial x^2}$.

Next we shall want to seek the solutions for the case of tangential surface load, though in the plate problem this case has been scarcely treated. It will be of some significance to endeavour to obtain the solutions for this case.

§ V Particular Solutions for the Thick Plate under the Tangential Surface Load, Whose Loading Function Satisfies the Equation of Helmholtz.

We let the given tangential surface load or traction be decomposed into the two components in x- and y-direction and this x-component be denoted by $Av(x,y)$. At first we treat the case in which only the tangential load, whose loading function is $Av(x,y)$, is applied to the upper surface of the plate. The solutions for the case where only the y-component tangential load is applied will be obtained by the use of the transformation of the coordinates.

Then the boundary conditions for the stress component σ_z, τ_{yz}, τ_{zx} at the upper and lower surface may be written as follows:

$$\sigma_z = 0 \quad \text{at} \quad z = \pm h, \quad \cdots\cdots \quad (5.1a)$$

$$\frac{\partial \sigma_z}{\partial z} = -A\frac{\partial v}{\partial x} \quad \text{at} \quad z = h, \quad \cdots\cdots \quad (5.1b)$$

$$\frac{\partial \sigma_z}{\partial z} = 0 \quad \text{at} \quad z = -h, \quad \cdots\cdots \quad (5.1c)$$

$$\tau_{yz} = 0 \quad \text{at} \quad z = \pm h, \quad \cdots\cdots \quad (5.2a)$$

$$\tau_{xz} = Av(x,y) \quad \text{at} \quad z = h, \qquad \tau_{xz} = 0 \text{ at } z = -h, \quad \cdots\cdots \quad (5.2b)$$

in which $v(x,y)$ is a v-function, i. e., a function which satisfies the equation $\nabla_1^2 v + k^2 v = 0$. The condition (5.1b) might be deduced from the third equation of (3.1). Though Prof. Love did not refer to the tangential load case, the author is convinced that the presumption regarding this condition will be pertinent.

Now we may put for σ_z, as before,

$$\sigma_z = A\frac{\partial v}{\partial x} w(z), \qquad (v(x,y) \equiv v), \quad \cdots\cdots \quad (5.3)$$

where, needless to say, $\dfrac{\partial v}{\partial x}$ is a v-function. Considering Eq. $\nabla^4 \sigma_z = 0$, we get the same differential equation as Eq. (4.4) and hence the solution of the same type

$$w(z) = A' \cosh kz + B' \sinh kz + C'z \cosh kz + D'z \sinh kz. \quad (5.4)$$

By the conditions derived from (5.1) the coefficients in this equation are determined in the following forms:

$$\left.\begin{aligned} A' &= 2h \sinh kh\,(\sinh kh \text{ csoh } kh - kh)\,/\,\delta'\,, \\ B' &= 2h \cosh kh\,(\sinh kh \cosh kh + kh)\,/\,\delta'\,, \\ C' &= -\sinh kh\,(\sinh 2kh + 2kh)\;/\;\delta'\,, \\ D' &= -\cosh kh\,(\sinh 2kh - 2kh)\;/\;\delta'\,, \quad \text{and} \\ \delta' &= \sinh^2 2kh - 4k^2h^2\,. \end{aligned}\right\} \quad \cdots\cdots\cdots (5.5)$$

Finally we have for σ_z.

$$\sigma_z = A \frac{\partial v}{\partial x} w(z) = K \frac{\partial v}{\partial x} [\sigma_z]'\,, \quad \cdots\cdots\cdots (5.6a)$$

in which

$$K = A \Big/ (\sinh^2 2kh - 4k^2h^2)\,, \quad \cdots\cdots\cdots (5.6b)$$

$$[\sigma_z]' = \sinh 2kh \cdot (h-z) \sinh k\,(z+h) + 2kh\,(z+h) \sinh k\,(z-h)\,. \quad \cdots\cdots\cdots (5.6c)$$

Therewith we can get the form of Θ, integrating the third equation of (3.2a) with respect to z, considering the relation $\nabla^2\Theta = 0$.

$$\Theta = 2\,(1+\nu) \frac{K}{k} \frac{\partial v}{\partial x} \left\{\sinh 2kh \cosh k\,(z+h) - 2kh \cosh k\,(z-h)\right\}. \quad \cdots\cdots\cdots (5.7)$$

Next we shall determine the forms of shear stresses τ_{xz} and τ_{yz} by means of conditions (5.2) and of the following equations:

$$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = -\frac{\partial \sigma_z}{\partial z} = -K \frac{\partial v}{\partial x} \frac{\partial}{\partial z} [\sigma_z]'\,, \quad \cdots\cdots\cdots (5.8a)$$

$$\nabla^2 \tau_{zx} = \frac{-1}{(1+\nu)} \frac{\partial^2 \Theta}{\partial z\, \partial x} = -2K \frac{\partial^2 v}{\partial x^2} H\,, \quad \cdots\cdots\cdots (5.8b)$$

$$\nabla^2 \tau_{yz} = \frac{-1}{(1+\nu)} \frac{\partial^2 \Theta}{\partial z\, \partial y} = -2K \frac{\partial^2 v}{\partial x\, \partial y} H\,, \quad \cdots\cdots\cdots (5.8c)$$

in which $H = \sinh 2kh \sinh k\,(z+h) - 2kh \sinh (kz-h)$.

By inspection we may put for the shear stress components:

$$\tau_{xz} = \frac{K}{k^2} \frac{\partial^2 v}{\partial x^2} \frac{\partial}{\partial z} [\sigma_z]' - \frac{A}{k^2} \frac{\partial^2 v}{\partial y^2} \gamma(z)\,, \quad \cdots\cdots\cdots (5.9a)$$

$$\tau_{yz} = \frac{K}{k^2} \frac{\partial^2 v}{\partial x\, \partial y} \frac{\partial}{\partial z} [\sigma_z]' + \frac{A}{k^2} \frac{\partial^2 v}{\partial x\, \partial y} \gamma(z)\,, \quad \cdots\cdots\cdots (5.9b)$$

where $\gamma(z)$ is determined so as to satisfy the equation $\nabla^2 \big(v(x,y) \cdot \gamma(z)\big) = 0$, namely $\frac{\partial^2 \gamma(z)}{\partial z^2} - k^2 \gamma(z) = 0$, and the conditions $\gamma(z) = 1$ at $z = h$; $\gamma(z) = 0$ at

$z=-h$. Also it will be readily shown that the solutions obtained above suffice the required equations and boundary conditions. $\gamma(z)$ has the form

$$\gamma(z)=\frac{1}{\sinh 2kh}\cdot\sinh k(z+h)\,. \qquad \cdots\cdots \quad (5.9\text{ c})$$

Now we shall solve the following equations:

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}=-\frac{\partial\tau_{zx}}{\partial z}=\frac{-K}{k^2}\frac{\partial^2 v}{\partial x^2}\frac{\partial^2[\sigma_z]'}{\partial z^2}+\frac{A}{k^2}\frac{\partial\gamma(z)}{\partial z}\frac{\partial^2 v}{\partial y^2}\,, \quad \cdots \quad (5.10\text{a})$$

$$\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}=-\frac{\partial\tau_{yz}}{\partial z}=\frac{-K}{k^2}\frac{\partial^2 v}{\partial x\partial y}\frac{\partial^2[\sigma]'}{\partial z^2}-\frac{A}{k^2}\frac{\partial\gamma(z)}{\partial z}\frac{\partial^2 v}{\partial x\partial y}\,, \qquad (5.10\text{b})$$

$$\nabla^2\sigma_x=\frac{-1}{(1+\nu)}\frac{\partial^2\Theta}{\partial x^2}=\frac{-2K}{k}\frac{\partial^3 v}{\partial x^3}F(z)\,, \qquad \cdots\cdots \quad (5.11\text{a})$$

$$\nabla^2\sigma_y=\frac{-1}{(1+\nu)}\frac{\partial^2\Theta}{\partial y^2}=\frac{-2K}{k}\frac{\partial^3 v}{\partial x\partial y^2}F(z)\,, \qquad \cdots\cdots \quad (5.11\text{b})$$

$$\nabla^2\tau_{xy}=\frac{-1}{(1+\nu)}\frac{\partial^2\Theta}{\partial x\partial y}=\frac{-2K}{k}\frac{\partial^3 v}{\partial x^2\partial y}F(z)\,, \qquad \cdots\cdots \quad (5.11\text{c})$$

$$\sigma_x+\sigma_y=\Theta-\sigma_z\,, \qquad \cdots\cdots \quad (5.12)$$

in which

$$F(z)=\sinh 2kh\cosh k(z+h)-2kh\cosh k(z-h)\,.$$

We put the same manner as above

$$\tau_{xy}=-\frac{\partial^2 F}{\partial x\partial y}\,, \qquad \big(F\equiv F(x,y,z)\big)\,. \qquad \cdots\cdots \quad (5.13)$$

Then from Eqs. (5.10) we may write

$$\sigma_x=\frac{-K}{k^2}\frac{\partial v}{\partial x}\frac{\partial^2[\sigma_z]'}{\partial z^2}+\frac{A}{k^2}\frac{\partial\gamma(z)}{\partial z}\int^x\frac{\partial^2 v}{\partial y^2}dx+\frac{\partial^2 F}{\partial y^2}\,, \quad \cdots\cdots \quad (5.14\text{a})$$

$$\sigma_y=\frac{-K}{k^2}\frac{\partial v}{\partial x}\frac{\partial^2[\sigma_z]'}{\partial z^2}-\frac{A}{k^2}\frac{\partial\gamma(z)}{\partial z}\frac{\partial v}{\partial x}+\frac{\partial^2 F}{\partial x^2}\,. \qquad \cdots\cdots \quad (5.14\text{b})$$

Next we have for $F(x,y,z)$ in the same way as in the preceding:

$$F(x,y,z)=\frac{\Theta}{(1+\nu)}+F^1(y,z)+F^2(x,z)\,.$$

Substituting Eq. (5.14a) in Eq. (5.11a), and Eq. (5.14b) in Eq. (5.11b), we may find that $\frac{\partial^2 F^1}{\partial y^2}$ and $\frac{\partial^2 F^2}{\partial x^2}$ are equal to zero. Accordingly we can put $F^1(y,z)$ and $F^2(x,z)$ equal to zero for the same reason as before. Therefore we have

$$\nabla^2 F(x,y,z) = \nabla_1^2 F(x,y,z) + \frac{\partial^2}{\partial z^2} F(x,y,z) = \frac{\Theta}{1+\nu} \quad \cdots\cdots\cdots (5.15)$$

Adding Eqs. (5.14a) and (5.14b) and using Eqs. (5.12), (5.15), we may arrive at the following result:

$$F(x,y,z) = \frac{-K}{k^3}\frac{\partial v}{\partial x}\Big\{2(\nu-1)\sinh 2kh\cosh k(z+h) + k\sinh 2kh \times$$
$$\times (h-z)\sinh k(z+h) + 4(1-\nu)kh\cosh k(z-h) + 2k^2h(z+h)\times$$
$$\times \sinh k(z-h)\Big\} + \frac{A}{k^4}\frac{\partial \gamma(z)}{\partial z}\left(\int^x \frac{\partial^2 v}{\partial y^2}dx - \frac{\partial v}{\partial x}\right) + zE^1(x,y) + E^2(x,y)\,.$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (5.16)$$

Substituting the formula (5.16) in the relation obtainable from Eqs. (5.14) and (5.12), we have

$$\nabla_1^2 E^1(x,y) = \nabla_1^2 E^2(x,y) = 0\,.$$

and in this case we can also put $E^1(x,y)$ and $E^2(x,y)$ equal to zero.

Then $F(x,y,z)$ thus determined yields the expressions for σ_x, σ_y and τ_{xy} through Eqs. (5.14a), (5.14b) and (5.13) respectively.

$$\sigma_x = \frac{-K}{k}\frac{\partial v}{\partial x}\Big\{-2\sinh 2kh\cdot\cosh k(z+h) + k\sinh 2kh\cdot(h-z)\times$$
$$\times\sinh k(z+h) + 4kh\cosh k(z-h) + 2k^2h(z+h)\sinh k(z-h)\Big\} +$$
$$\frac{-K}{k^3}\frac{\partial^3 v}{\partial x\partial y^2}\Big\{2(\nu-1)\sinh 2kh\cosh k(z+h) + k\sinh 2kh\cdot(h-z)\sinh k(z+h)$$
$$+4(1-\nu)kh\cosh k(z-h) + 2k^2h(z+h)\sinh k(z-h)\Big\} + \frac{-2A}{k^4}\frac{\partial \gamma(z)}{\partial z}\frac{\partial^3 v}{\partial x\partial y^2}\,.$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (5.17a)$$

$$\sigma_y = \frac{-K}{k}\frac{\partial v}{\partial x}\Big\{-2\sinh 2kh\cosh k(z+h) + k\sinh 2kh\times$$
$$\times(h-z)\sinh k(z+h) + 4kh\cosh k(z-h) + 2k^2h(z+h)\sinh k(z-h)\Big\} +$$
$$\frac{-K}{k^3}\frac{\partial^3 v}{\partial x^3}\Big\{2(\nu-1)\sinh 2kh\cosh k(z+h) + k\sinh 2kh\cdot(h-z)\sinh k(z+h)$$
$$+4(1-\nu)kh\cosh k(z-h) + 2k^2h(z+h)\sinh k(z-h)\Big\} + \frac{2A}{k^4}\frac{\partial \gamma(z)}{\partial z}\frac{\partial^3 v}{\partial x\partial y^2}\,.$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (5.17b)$$

$$\tau_{xy} = \frac{K}{k^3}\frac{\partial^3 v}{\partial x^2\partial y}\Big\{2(\nu-1)\sinh 2kh\cosh k(z+h) +$$
$$+k\sinh 2kh\cdot(h-z)\sinh k(z+h) + 4(1-\nu)kh\cosh k(z-h) +$$

$$+ 2k^2h(z+h)\sinh k(z-h)\Big\} + \frac{A}{k^4}\frac{\partial\gamma(z)}{\partial z}\left(\frac{\partial^3 v}{\partial x^2\partial y} - \frac{\partial^3 v}{\partial y^3}\right). \quad \cdots (5.17c)$$

It will be easily verified that formulae (5.17) satisfy equations (5.11) and also the formulae (5.17a) and (5.17b) are subject to the condition (5.12). Further, if we substitute the formulae (5.17) in Eq. (5.10), though it is unnecessary, it will be seen that Eqs. (5.10) are satisfied.

Here we rewrite formulae (5.9) into the forms

$$\tau_{zx} = \frac{K}{k^2}\frac{\partial^2 v}{\partial x^2} I - \frac{A}{k^2}\frac{\sinh k(z+h)}{\sinh 2kh}\frac{\partial^2 v}{\partial y^2}, \quad \cdots\cdots\cdots\cdots\cdots\cdots (5.9a)'$$

$$\tau_{yz} = \frac{K}{k^2}\frac{\partial^2 v}{\partial x\partial y} I + \frac{A}{k^2}\frac{\sinh k(z+h)}{\sinh 2kh}\frac{\partial^2 v}{\partial x\partial y}, \quad \cdots\cdots\cdots\cdots\cdots\cdots (5.9b)'$$

$$I = -\sinh 2kh \sinh k(z+h) + k\sinh 2kh\cdot(h-z)\cosh k(z+h) + $$
$$+ 2kh\sinh k(z-h) + 2k^2h(z+h)\cosh k(z-h). \quad \cdots\cdots\cdots (5.9c)'$$

Now we can obtain the solutions for displacements u, v and w. From Eq. (3.31a) we have for u

$$u = \frac{(1+\nu)}{E}\Big[\frac{K}{k^3}\Big\{-2(1-\nu)\sinh 2kh\cosh k(z+h) + k\sinh 2kh \times$$
$$\times (h-z)\sinh k(z+h) + 4(1-\nu)kh\cosh k(z-h) +$$
$$+ 2k^2h(z+h)\sinh k(z-h)\Big\}\frac{\partial^2 v}{\partial x^2} - \frac{2A}{k^4}\frac{\partial\gamma(z)}{\partial z}\frac{\partial^2 v}{\partial y^2}\Big], \quad \cdots (5.18a)$$

and for v from Eq. (3.31b)

$$v = \frac{(1+\nu)}{E}\Big[\frac{K}{k^3}\Big\{-2(1-\nu)\sinh 2kh\cosh k(z+h) +$$
$$+ k\sinh 2kh\cdot(h-z)\sinh k(z+h) + 4(1-\nu)kh\cosh k(z-h) +$$
$$+ 2k^2h(z+h)\sinh k(z-h)\Big\}\frac{\partial^2 v}{\partial x\partial y} + \frac{2A}{k^4}\frac{\partial\gamma(z)}{\partial z}\frac{\partial^2 v}{\partial x\partial y}\Big]. \quad \cdots (5.18b)$$

It will be confirmed that the formulae (5.18) satisfy Eq. (3.32a).

By the use of Eq. (3.31c) the displacement w is expressible in the form

$$w = \frac{(1+\nu)}{E}\frac{K}{k^2}\frac{\partial v}{\partial x}\Big[k\sinh 2kh\cdot(h-z)\cosh k(z+h) +$$
$$+ (1-2\nu)\sinh 2kh\cdot\sinh k(z+h) + 2k^2h\cdot(z+h)\cosh k(z-h) +$$
$$+ 2(2\nu-1)kh\sinh k(z-h)\Big] + W(x,y). \quad \cdots\cdots\cdots\cdots\cdots\cdots (5.19)$$

Substituting the formula of w (5.19) in Eqs. (3.32c) and (3.32b), we obtain the results

$$\frac{\partial W(x,y)}{\partial y}=0\,,\qquad \frac{\partial W(x,y)}{\partial x}=0\,.$$

Therefore we may take for $W(x,y)$

$$W(x,y)=0\,. \qquad (5.19)'$$

In the next place we shall want to obtain the solutions for the case where only the tangential, upper surface load exists, which has no component in the x-direction and the loading function of which is $A'v'(x,y)$, by utilizing the transformation of the coordinates as already stated. That is to say, we shall obtain immediately the results to be sought from the solutions mentioned above by the rotation of the coordinate system heretofore employed about the axis passing through the origin vertically by the angle $-\frac{\pi}{2}$ radian, namely change of the independent variables x, y into x', y', which are subject to the relations $x=y'$ and $y=-x'$. Then the necessary transformation relation between quantities and operators referred to the (x,y) coordinate system and the (x',y') system will be as follows:

$$\left.\begin{aligned}
&\frac{\partial}{\partial x}=\frac{\partial}{\partial y'}\,,\qquad \frac{\partial}{\partial y}=-\frac{\partial}{\partial x'}\,,\\
&\sigma_{x'}=\sigma_y\,,\quad \sigma_{y'}=\sigma_x\,,\quad \sigma_{z'}=\sigma_z\,,\\
&\tau_{x'y'}=-\tau_{xy}\,,\quad \tau_{x'z'}=-\tau_{yz}\,,\quad \tau_{y'z'}=\tau_{zx}\,,\\
&u'=-v\,,\quad v'=u,\quad w'=w\,.
\end{aligned}\right\} \qquad (5.20)$$

Hence we can immediately write down the required results from the solutions obtained above by virtue of the relation (5.20), effacing the symbols of "dash" in the solutions expressed in the (x',y') coordinate system and noticing that

$$\left.\begin{aligned}
&Av(x,y)=Av(y',-x')=Av'(x',y'),\rightarrow A'v'(x,y),\\
&\nabla_1^2 v'(x,y)+k'^2v'(x,y)=0\,,\\
&K'=A'\Big/\sinh^2 2k'h-4k'^2h^2\,.
\end{aligned}\right\} \cdots (5.21)$$

For the stress components in this case from formulae (5.6), (5.17) and (5.9) we have the following forms:

$$\sigma_z=K'\frac{\partial v'}{\partial y}[\sigma_z]''\,,\qquad (v'\equiv v'(x,y))\,, \qquad (5.22a)$$

in which

$$[\sigma_z]'' = \sinh 2k'h\,(h-z)\sinh k'(z+h) + 2k'h(z+h)\sinh k'(z-h)\,,$$

$$\sigma_x = \frac{-K'}{k'}\frac{\partial v'}{\partial y}H' - \frac{K'}{k'^3}\frac{\partial^3 v'}{\partial y^3}I' + \frac{2A'}{k'^4}\frac{\partial \gamma'(z)}{\partial z}\frac{\partial^3 v'}{\partial y\partial x^2}, \quad \cdots (5.22\text{b})$$

$$\sigma_y = \frac{-K'}{k'}\frac{\partial v'}{\partial y}H' - \frac{K'}{k'^3}\frac{\partial^3 v'}{\partial x^2\partial y}I' - \frac{2A'}{k'^4}\frac{\partial \gamma'(z)}{\partial z}\frac{\partial^3 v'}{\partial x^2\partial y}, \quad \cdots\cdots (5.22\text{ c})$$

in the latter two formulae

$$H' = -2\sinh 2k'h\cosh k'(z+h) + k'\sinh 2k'h\cdot(h-z)\sinh k'(z+h) +$$
$$+4k'h\cosh k'(z-h) + 2k'^2h(z+h)\sinh k'(z-h)\,,$$

$$I' = 2(\nu-1)\sinh 2k'h\cosh k'(z+h) + k'\sinh 2k'h\cdot(h-z)\sinh k'(z+h) +$$
$$+4(1-\nu)k'h\cosh k'(z-h) + 2k'^2h(z+h)\sinh k'(z-h)\,,$$

$$\gamma'(z) = \frac{\sinh k'(z+h)}{\sinh 2k'h}\,,$$

$$\tau_{xz} = \frac{K'}{k'^2}\frac{\partial^2 v'}{\partial x\partial y}J + \frac{A'}{k'^2}\gamma'(z)\frac{\partial^2 v'}{\partial x\partial y}, \quad \cdots\cdots (5.23\text{a})$$

$$\tau_{yz} = \frac{K'}{k'^2}\frac{\partial^2 v'}{\partial y^2}J + \frac{-A'}{k'^2}\gamma'(z)\frac{\partial^2 v'}{\partial x^2}, \quad \cdots\cdots (5.23\text{b})$$

$$J = -\sinh 2k'h\sinh k'(z+h) + k'\sinh 2k'h\,(h-z)\cosh k'(z+h) +$$
$$+2k'h\sinh k'(z-h) + 2k'^2h(z+h)\cosh k'(z-h)\,,$$

$$\tau_{xy} = \frac{K_{\prime}}{k'^3}\frac{\partial^3 v'}{\partial y^2\partial x}I' + \frac{A'}{k'^4}\frac{\partial \gamma'(z)}{\partial z}\left(\frac{\partial^3 v'}{\partial x\partial y^2} - \frac{\partial^3 v'}{\partial x^3}\right). \quad \cdots\cdots (5.23\text{c})$$

From the formulae (5.18) and (5.19) the components of displacement are expressible in the following forms:

$$u = \frac{(1+\nu)}{E}\left[\frac{K'}{k'^3}I'\frac{\partial^2 v'}{\partial x\partial y} + 2\frac{A'}{k'^4}\frac{\partial \gamma'(z)}{\partial z}\frac{\partial^2 v'}{\partial y\partial x}\right], \quad \cdots\cdots (5.24\text{a})$$

$$v = \frac{(1+\nu)}{E}\left[\frac{K'}{k'^3}I'\frac{\partial^2 v'}{\partial y^2} - 2\frac{A'}{k'^4}\frac{\partial \gamma'(z)}{\partial z}\frac{\partial^2 v'}{\partial x}\right] \quad \cdots\cdots (5.24\text{b})$$

and

$$w = \frac{(1+\nu)}{E}\frac{K'}{k'^2}\frac{\partial v'}{\partial y}\Big[k'\sinh 2k'h\cdot(h-z)\cosh k'(z+h) +$$
$$+(1-2\nu)\sinh 2k'h\sinh k'(z+h) + 2k'^2h(z+h)\cosh k'(z-h) +$$
$$+2(2\nu-1)k'h\sinh k'(z-h)\Big]. \quad \cdots\cdots (5.24\text{c})$$

The author would add that the boundary condtion $\frac{\partial \sigma_z}{\partial z} = 0$ at $z = \pm h$

is more reasonable, someone may think, but this condition would result in non-existence of the stress component σ_z and, furthermore, it would seem impossible to obtain the solutions for shear stress components τ_{xz} and τ_{yz}, for example. A discussion of why it seems impossible is omitted here and, if the reader would calculate, the point could be easily understood. The boundary condition (5.1b) was inferred from the equation of equilibrium, considering the stress state in the vicinity of the upper surface and this seems to the author quite rational. In addition, to consider the condition in regard to the derivatives of shear stress components τ_{zx} and τ_{yz} with respect to z will be unadvisable, the author thinks.

It will be needless to say that the solutions for the case of tangential load applied to the lower surface may be determined in the same manner as described above in Sec. IV. In these tangential load cases the function $\Theta_1 z+\Theta_0$ to be added to Θ has been omitted for the sufficient reason which was partly described above. Though the solutions for the case in which the distribution function of a given tangential load is plane-harmonic ought to be sought, we let them be left not obtained in this paper, because such does not seem to impair the discussion that follows.

§ VI. On the Particular Solutions for the Thick Plate under a Given and Generally Distributed Load.

Having been referred to the rectilinear coordinate system, the results obtained in Sec. IV or V are convenient to satisfy the boundary conditions for a rectangular thick plate. Therefore as the first step we deal with the rectangular plate. We take the coordinate system (x,y,z) such that the edges of the plate are given by $x=2a, 0$ and $y=2b, 0$. Now a given, general loading function is assumed to be expansionalbe in a duble Fourier series and hence this function $F(x,y)$ may be put as follows:

$$F(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin\frac{m\pi}{2a}x\sin\frac{n\pi}{2b}y\,, \quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (6.1)$$

in which

$$A_{mn}=\frac{1}{ab}\int_0^{2a}\int_0^{2b}F(x,y)\sin\frac{m\pi}{2a}x\sin\frac{n\tau}{2b}y\,dxdy \quad\cdots\cdots\cdots\cdots \quad (6.2)$$

Then each term in this series, for example, has the property similar to that of $pv(x,y)$ in the foregoing. If we rewrite this term $A_{mn} \sin \frac{m\tau}{2a} x \sin \frac{n\pi}{2b} y$ into the form

$$p_{mn} \sin \alpha_m x \sin \beta_n y = p_{mn} v_{mn}(x,y), \qquad \cdots\cdots\cdots\cdots (6.3)$$

in which $\alpha_m = \frac{m\pi}{2a}$, $\beta_n = \frac{n\pi}{2b}$, $p_{mn} = A_{mn}$, and observe that this partial loading function $p_{mn} v(x,y)$ satisfies the equation $\nabla_1^2 f + k_{mn}^2 f = 0$, in which $k_{mn}^2 = \alpha_m^2 + \beta_n^2$, the above similarity becomes more apparent.

Therefore it follows that in order to obtain particular solutions for the case of a general loading function, be it either normal or tangential, we merely have to take solutions corresponding to terms in such series as in (6.1) from the results in Secs. IV and V and superpose them. Of course, there are various ways in which general loading function is expanded into a double Fourier series, but that indicated in (6.1) seems most convenient. Next we must treat the plate with other forms of the bounding curve, which is subjected to a general load. The circumstance is not so simple and yet it seems to be essential to expand the general loading function into series in such a manner that each term is a v-function as stated above. Needless to say, this v-function is to be represented in the cylinder coordinates so as to be useful in satisfying the boundary conditions. As is well known, for a circular plate to be treated in the cylindrical coordinates the Fourier-Bessel expansion serves the purpose. In the next section we shall discuss some detailed treatment of plates with other forms of the bounding curve than rectangular.

§ VII. Particular Solutions for Circular and Elliptical Thick Plates under the Normal Surface Load, Whose Loading Function Satisfies the Equation of Helmholtz.

As mentioned above, it is advantageous to take such a coordinate system as can be utilized to express the boundary conditions in terms of functions of only one coordinate or variable. As regards the thin plate theory which is based upon the well-known equation $D\nabla_1^4 w = -p(x,y)$ it may be said that severe difficulties are not encountered even in the case of general normal load. Solutions for the rectangular thin plate under variable pressure are due to Navier and those for thin circular and elliptic plate were obtained by Jen and Perry respectively. Circular plate with eccentric hole under arbitrary load was treated by S. D. Conte[2]. But it seems very difficult to follow the process of calculation similar to theirs, as is easily inferred from the fact that in the thick plate problem calculational difficulties to be surmounted in satisfying the equations and conditions may be far greater and so in the case of thick plate it will be useless to seek some ingenious method such as available in the thin plate problem. At any rate, if we begin by determining the stress component σ_z following the assumption laid down by Love, we could not but utilize the v-functions already stated, i.e., we shall have to expand a given loading function in series, terms of which are v-functions. The equation of Helmholtz, which v-function is to satisfy, is familiar in applied mathematics and hence the expansion for loading function in terms of this function may not be said to be unadvisable. Yet it is regrettable that, if we take other coordinates than the rectangular cartesian coordinates, it seems well-nigh impossible to proceed from the start, employing the method of Michell and Love owing to the complication of the forms of equations of compatibility represented in these coordinates.

Further, some difficulties are encountered, if we attempt to expand a given loading function in terms of v-functions represented in coordinate systems, except for a few ones, which are suitable for satisfying the boundary conditions, because in these cases mathematical properties of the functions which are generated by putting the v-functions in the product forms using the method of the separation of variables are not well investigated and numerical tabulations of these functions are in general not available. But it may be evident that there exists no other way than to have recourse to the v-function, if we employ the method of Love, and therefore we are forced to proceed to get over the difficulties or to make some approximations.

Now we shall undertake to state the derivation of the particular solutions for circular plate. As mentioned before, in this case it is inconvenient to follow the method of Michell and Love without making some alteration of it and hence we here make use of the solutions by H. Neuber. As well known, his solutions satisfy the conditions of compatibility, though he did not in particular endeavour to get the solutions to suffice these conditions. His solutions are, in fact, excellent and compact and hence usable but the determination of Neuber's Φ harmonic functions is not so easy as is ostensibly expected. Yet in this case we can determine them correspondingly readily. The continuation process between Love's and Neuber's particular solutions is carried out with the particular solutions σ_z obtained by both methods. The correspondence or continuation between the two kinds of solutions for the plane or generalized plane stress has been briefly described in Sec. II. The uniqueness of the Φ harmonic functions thus obtained by this process proposed by H. Neuber will be guaranteed by the uniqueness of the solutions in the theory of elasticity. Then we shall indicate the continuation process regarding particular solutions.

We may write the stress solutions of H. Neuber in the cylindrical coordinates as follows:
the normal stresses are

$$\sigma_r = \frac{-\partial^2 F}{\partial r^2} + 2\alpha\left(\cos\theta\frac{\partial\Phi_1}{\partial r} + \sin\theta\frac{\partial\Phi_2}{\partial r}\right) + \left(1-\frac{\alpha}{2}\right)\nabla^2 F, \quad \cdots \text{(7. 1a)}$$

$$\sigma_\theta = -\left(\frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial\theta^2}\right) + 2\alpha\left(\frac{-\sin\theta}{r}\frac{\partial\Phi_1}{\partial\theta} + \frac{\cos\theta}{r}\frac{\partial\Phi_2}{\partial\theta}\right)$$
$$+\left(1-\frac{\alpha}{2}\right)\nabla^2 F, \quad \cdots\cdots \text{(7. 1b)}$$

and

$$\sigma_z = -\frac{\partial^2 F}{\partial z^2} + 2\alpha\frac{\partial\Phi_3}{\partial z} + \left(1-\frac{\alpha}{2}\right)\nabla^2 F, \quad \cdots\cdots \text{(7. 1c)}$$

in which

$$F \equiv F(r,\theta,z), \quad \alpha = 2(1-\nu), \quad F = \Phi_0 + r\cos\theta\cdot\Phi_1 + r\sin\theta\cdot\Phi_2 + z\Phi_3, \quad \cdots\cdots \text{(7. 1d)}$$

and Φ's are harmonic functions.

$$\left(\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial\theta^2} + \frac{\partial}{\partial z^2}\right).$$

We have the relation as is easily shown

$$\nabla^2 F = 2\left(\frac{\partial\Phi_1}{\partial x} + \frac{\partial\Phi_2}{\partial y} + \frac{\partial\Phi_3}{\partial z}\right) = 2\left\{\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\Phi_1\right.$$
$$\left.+\left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)\Phi_2 + \frac{\partial}{\partial z}\Phi_3\right\}. \quad \cdots\cdots \text{(7. 1e)}$$

The shearing stresses are

$$\tau_{r\theta} = \frac{-\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F}{\partial\theta}\right) + \alpha\left(-\sin\theta\frac{\partial\Phi_1}{\partial r} + \frac{\cos\theta}{r}\frac{\partial\Phi_1}{\partial\theta} +\right.$$
$$\left.+\cos\theta\frac{\partial\Phi_2}{\partial r} + \frac{\sin\theta}{r}\frac{\partial\Phi_2}{\partial\theta}\right), \quad \cdots\cdots \text{(7. 2a)}$$

$$\tau_{rz} = -\frac{\partial^2 F}{\partial r \partial z} + \alpha\left(\cos\theta \frac{\partial \Phi_1}{\partial z} + \sin\theta \frac{\partial \Phi_2}{\partial z} + \frac{\partial \Phi_3}{\partial r}\right) \cdots\cdots\cdots\cdots (7.2b)$$

$$\tau_{\theta z} = \frac{-\partial^2 F}{r \partial\theta \partial z} + \alpha\left(-\sin\theta \frac{\partial \Phi_1}{\partial z} + \cos\theta \frac{\partial \Phi_2}{\partial z} + \frac{1}{r}\frac{\partial \Phi_3}{\partial \theta}\right). \cdots\cdots (7.2c)$$

In the first place we shall determine the form of the normal stress σ_z produced in the thick plate under the surface normal load, loading function of which is a v-function, by the method of Michell and Love. It is easily observed that the equation (4.4) remains unchanged, if we take the cylindrical coordinates, and, needless to say, the boundary conditions associated with this equation can be used here and therefore the function of only z coordinate contained in the solution for σ_z can be taken. We shall take the cylindrical coordinates (r,θ,z) as is usual. The boundary conditions for σ_z are as follows:

$$\sigma_z = -pv(r,\theta) \quad \text{at} \quad z = +h\,, \qquad \frac{\partial \sigma_z}{\partial z} = 0 \quad \text{at} \quad z = \pm h$$

$$= 0 \qquad\qquad \text{at} \quad z = -h\,,$$

in which p is a constant and $v(r,\theta)$ is $J_m(k,r)\sin m\theta$, which is obviously a v-function, and $J_m(k,r)$ is a Bessel function of the first kind, k and m being given and taken as constant. The necessary differential equation is

$$\nabla^4 \sigma_z = 0 \qquad \text{and therefore} \quad \frac{\partial^4 w}{\partial z^4} - 2k^2 \frac{\partial^2 w}{\partial z^2} + k^4 w = 0\ ,$$

and hence the solution for σ_z may be put

$$\sigma_z = K[\sigma_z]\,v(r,\theta)\ , \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (7.3)$$

where K and $[\sigma_z]$ have the same forms as the formulae (4.9), (4.8).

In order to obtain the form of F function according to H. Neuber, namely the Φ harmonic functions, it is to be considered that the expression (7.1c) to be determined should equal the solution (7.3) and the formulae (7.2b), (7.2c) are to vanish at the upper and lower surface and further the solution (7.1c) is to satisfy the third equation of (3.2a), the form of which is maintained even when it is represented in the cylindrical coordinates. Now we separate the expression (7.1c) into two kinds, one containing only solid harmonic functions and the other only solid bi-harmonic functions, and we also treat expression (7.3) in the same manner.

First we have

$$\sigma_z = \left[\frac{-\partial^2 \Phi_0}{\partial z^2} + \alpha \frac{\partial \Phi_3}{\partial z} + 2\nu \left\{ \left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r} \frac{\partial}{\partial \theta}\right) \Phi_1 + \left(\sin\theta \frac{\partial}{\partial r} + \right.\right.\right.$$

$$\left.\left.\left. + \frac{\cos\theta}{r} \frac{\partial}{\partial \theta}\right) \Phi_2 \right\}\right] - \left[r \cos\theta \frac{\partial^2 \Phi_1}{\partial z^2} + r \sin\theta \frac{\partial^2 \Phi_2}{\partial z^2} + z \frac{\partial^2 \Phi_3}{\partial z^2}\right]. \qquad (7.4)$$

And hence, if we put the first part of this expression equal to the part of the same kind in (7.3) which contains only solid harmonic functions and do in the same manner for the second sort of ones, though it is a mere trial, we get two relations

$$Kv(r,\theta) \left\{ -(kh \cosh 2kh + \sinh 2kh) \sinh k(z+h) + \right.$$

$$\left. - kh \sinh k(z-h) + 2k^2h^2 \cosh k(z-h) \right\} = - \frac{\partial^2 \Phi_0}{\partial z^2} + \alpha \frac{\partial \Phi_3}{\partial z} +$$

$$+ 2\nu \left\{ \left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r} \frac{\partial}{\partial \theta}\right) \Phi_1 + \left(\sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r} \frac{\partial}{\partial \theta}\right) \Phi_2 \right\}, \qquad (7.5a)$$

and

$$Kv(r,\theta) kz \left\{ \sinh 2kh \cosh k(z+h) + 2kh \cosh k(z-h) \right\}$$

$$= -\left(r \cos\theta \frac{\partial^2 \Phi_1}{\partial z^2} + r \sin\theta \frac{\partial^2 \Phi_2}{\partial z^2} + z \frac{\partial^2 \Phi_3}{\partial z^2}\right). \qquad (7.5b)$$

Having examined any and every possibility to choose the Φ harmonic functions which are to be inferred from these relations, we may assume Φ_1 and Φ_2 to be zero and have for Φ_3 and Φ_0 through integration from (7.5b)

$$\Phi_3 = \frac{-K}{k} v(r,\theta) \left\{ \sinh 2kh \cosh k(z+h) + 2kh \cosh k(z-h) \right\}, \quad (7.6a)$$

and from (7.5a) and (7.6a)

$$\Phi_0 = \frac{-K}{k^2} v(r,\theta) \left[-\left\{ kh \cosh 2kh + (1-\alpha) \sinh 2kh \right\} \sinh k(z+h) + \right.$$

$$\left. + (2\alpha - 1) kh \sinh k(z-h) + 2k^2h^2 \cosh k(z-h) \right]. \qquad (7.6b)$$

Whence we may write for the three dimensional stress function F $(F \equiv F(r,\theta,z))$

$$F = \Phi_0 + z\Phi_3 = \frac{-K}{k} v(r,\theta) \left[\frac{1}{k} \left\{ -(kh \cosh 2kh + (1-\alpha) \sinh 2kh) \times \right.\right.$$

$$\left. \times \sinh k(z+h) + (2\alpha - 1) kh \sinh k(z-h) + 2k^2h^2 \cosh k(z-h) \right\} +$$

$$+z\left\{\sinh 2kh\cosh k(z+h)+2kh\cosh k(z-h)\right\}\Big],\quad\cdots\cdots\quad(7.7a)$$

where

$$\Phi_1=\Phi_2=0\,.\quad\cdots\cdots\quad(7.7b)$$

It will be easily comfirmed that the formula (7.1c) with the expressions (7.7), (7.6a) thus obtained yields the expression in the right hand side of (7.3). Now we may have for the shear stresses τ_{rz}, $\tau_{\theta z}$ from (7.2b) and (7.2c)

$$\tau_{rz}=-\frac{\partial^2 F}{\partial r\partial z}+\alpha\frac{\partial\Phi_3}{\partial r}=\frac{K}{k}\frac{\partial v(r,\theta)}{\partial r}\Big[kz\left\{\sinh 2kh\sinh k(z+h)+\right.$$
$$\left.+2kh\sinh k(z-h)\right\}-kh\cosh 2kh\cosh k(z+h)+kh\cosh k(z-h)+$$
$$+2k^2h^2\sinh k(z-h)\Big],\quad\cdots\cdots\quad(7.8a)$$

and

$$\tau_{\theta z}=\frac{-\partial^2 F}{r\partial\theta\partial z}+\alpha\frac{1}{r}\frac{\partial\Phi_3}{\partial\theta}=\frac{K}{k}\frac{1}{r}\frac{\partial v(r,\theta)}{\partial\theta}\Big[kz\left\{\sinh 2kh\sinh k(z+h)+\right.$$
$$\left.+2kh\sinh k(z-h)\right\}-kh\cosh 2kh\cosh k(z+h)+$$
$$+kh\cosh k(z-h)+2k^2h^2\sinh k(z-h)\Big].\quad\cdots\cdots\quad(7.8b)$$

The boundary conditions for τ_{rz}, $\tau_{\theta z}$ are

$$\tau_{z\theta}=\tau_{rz}=0\quad\text{at}\quad z=\pm h\quad\cdots\cdots\quad(7.9)$$

and the function of only z given in the square bracket of (7.8) is easily seen to be the derivative of $[\sigma_z]$ in (7.3) divided by k with respect to z and consequently owing to the condition imposed on the formula (7.3) evidently satisfy the conditions (7.9).

By (7.1a) σ_r is

$$\sigma_r=(2-\alpha)\frac{\partial\Phi_3}{\partial z}-\frac{\partial^2 F}{\partial r^2}=\frac{-2\nu K}{k}v(r,\theta)L+\frac{K}{k}\frac{\partial^2 v(r,\theta)}{\partial r^2}M,\quad(7.10a)$$

and by (7.1b) σ_θ is

$$\sigma_\theta=\frac{\partial^2 F}{\partial r^2}+\frac{\partial^2 F}{\partial z^2}-\alpha\frac{\partial\Phi_3}{\partial z}=\frac{-2\nu K}{k}v(r,\theta)L+\frac{-K}{k}\left(\frac{\partial^2 v}{\partial r^2}+k^2 v\right)M,$$
$$\cdots\cdots\quad(7.10b)$$

in which

$$\left.\begin{aligned} L &= k\sinh 2kh\sinh k(z+h)+2k^2h\sinh k(z-h),\\ M &= z\left\{\sinh 2kh\cosh k(z+h)+2kh\cosh k(z-h)\right\}+\\ &+\frac{1}{k}\left\{-(kh\cosh 2kh+(1-\alpha)\sinh 2kh)\sinh k(z+h)+\right.\\ &\left.+(2\alpha-1)kh\sinh k(z-h)+2k^2h^2\cosh k(z-h)\right\}. \end{aligned}\right\} \quad (7.10c)$$

Hence we find the expression for Θ, adding the formulae (7.3), (7.10a) and (7.10b),

$$\Theta=\sigma_r+\sigma_\theta+\sigma_z=-2(1+\nu)Kv(r,\theta)\left\{\sinh 2kh\sinh k(z+h)\right.$$

$$\left.+2kh\sinh k(z-h)\right\}=-2(1+\nu)\frac{K}{k}v(r,\theta)L\,. \quad \cdots\cdots\cdots (7.11)$$

This formula is equal to (4.11) with regard to the form and therefore the third equation of (3.2a) will be seen to be satisfied by the formulae (7.3) and (7.11). Thus a mere trial has been shown to be successful, that is, the appropriate selection of Neuber's Φ functions has been made.

From (7.2a) we obtain the solution for $\tau_{r\theta}$

$$\tau_{r\theta}=\frac{-\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)F=\frac{K}{k}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial v}{\partial\theta}\right)M. \quad (v\equiv v(r,\theta) \qquad (7.12)$$

in which M was explained in (7.10c).

In general the determination of the Φ functions of H. Neuber in the three-dimensional problem cannot be easily treated but fortunately in this case we could obtain these functions under the idea basic in the method of Love for solving the thick plate problem. As mentioned above, the results thus obtained satisfy the equations of equilibrium and of compatibility by the property of Neuber's solutions and also the boundary conditions for σ_z, τ_{rz}, and $\tau_{\theta z}$ and in this way all the necessary equations and conditions to determine particular solutions are satisfied.

Then the displacements u, v, w are given by means of Neuber's solutions in the following forms:

$$u=\frac{1}{2\mu}\left\{\frac{-\partial F}{\partial r}+2\alpha(\Phi_1\cos\theta+\Phi_2\sin\theta)\right\}=\frac{-1}{2\mu}\frac{\partial F}{\partial r}=\frac{1}{2\mu}\frac{K}{k}\frac{\partial v}{\partial r}M, \quad \cdots\cdots (7.13a)$$

$$v=\frac{1}{2\mu}\left\{-\frac{1}{r}\frac{\partial F}{\partial\theta}+2\alpha(-\Phi_1\sin\theta+\Phi_2\cos\theta)\right\}=\frac{1}{2\mu}\frac{K}{k}\frac{1}{r}\frac{\partial v}{\partial\theta}M, \quad \cdots\cdots (7.13b)$$

$$w = \frac{1}{2\mu}\left(\frac{-\partial F}{\partial z} + 2\alpha\Phi_3\right) = \frac{K}{k} v(r,\theta)\left\{zL - (kh\cosh 2kh + \right.$$
$$+ \alpha \cdot \sinh 2kh)\cosh k(z+h) + (1-2\alpha)\, kh \cosh k(z-h) +$$
$$\left. + 2k^2h^2 \sinh k(z-h)\right\}. \qquad (7.13c)$$

In a later report we shall have an occasion to solve the problems considering the complementary solutions pertaining to plane stress and generalized plane stress and the boundary conditions, using the results obtained above.

Next we shall describe briefly the derivation of the particular solutions for an elliptic plate under the surface normal load. As usual we take the elliptic cylinder coordinates (α, β, z) such that the origin lies at the middle point of the plate and the major axis of the ellipse is the x axis, the x and y axes being on the middle plane of the plate. The relation between the two coordinate systems (α, β, z), (x, y, z) is

$$\left.\begin{aligned} x &= c\cosh\alpha\cos\beta, \\ y &= c\sinh\alpha\sin\beta, \\ z &= z, \end{aligned}\right\} \qquad (7.14)$$

in which $2c$ is the interfocal distance of the ellipse. We let the parametric curve which describes the elliptical boundary be $\alpha = \bar{\alpha}$ (= const). Then, if we take such regions as $\bar{\alpha} \geqq \alpha > 0$ and $\pi \geqq \beta > -\pi$, the one-to-one correspondence between each point in the rectangular coordinates (x, y, z), and elliptic cylinder coordinates (α, β, z) will be established. According to H. Neuber, when expressed in the orthogonal curvilinear coordinates (α, β, γ), the stresses are of the types:

$$\sigma_\alpha = -\frac{\partial^2 F}{\partial n_\alpha^2} + \frac{2\alpha}{h_\alpha^2}\left(\frac{\partial\Phi_1}{\partial\alpha}\frac{\partial x}{\partial\alpha} + \frac{\partial\Phi_2}{\partial\alpha}\frac{\partial y}{\partial\alpha} + \frac{\partial\Phi_2}{\partial\alpha}\frac{\partial z}{\partial\alpha}\right) + \left(1 - \frac{\alpha'}{2}\right)\Delta F, \qquad (7.15a)$$

$$\tau_{\alpha\beta} = \frac{-\partial^2 F}{\partial n_\alpha \partial n_\beta} + \frac{\alpha'}{h_\alpha h_\beta}\left(\frac{\partial\Phi_1}{\partial\alpha}\frac{\partial x}{\partial\beta} + \frac{\partial\Phi_1}{\partial\beta}\frac{\partial x}{\partial\alpha} + \frac{\partial\Phi_2}{\partial\alpha}\frac{\partial y}{\partial\beta} + \frac{\partial\Phi_2}{\partial\beta}\frac{\partial y}{\partial\alpha}\right.$$
$$\left. + \frac{\partial\Phi_3}{\partial\alpha}\frac{\partial z}{\partial\beta} + \frac{\partial\Phi_3}{\partial\beta}\frac{\partial z}{\partial\alpha}\right), \qquad (7.15b)$$

in which

$$\frac{\partial^2}{\partial n_\alpha^2} = \frac{1}{h_\alpha}\frac{\partial}{\partial\alpha}\left(\frac{1}{h_\alpha}\frac{\partial}{\partial\alpha}\right) + \frac{1}{h_\alpha h_\beta^2}\frac{\partial h_\alpha}{\partial\beta}\frac{\partial}{\partial\beta} + \frac{1}{h_\alpha h_\gamma^2}\frac{\partial h_\alpha}{\partial\gamma}\frac{\partial}{\partial\gamma}, \qquad (7.16a)$$

$$\alpha' = 2(1-\nu),$$

$$\frac{\partial^2}{\partial n_\alpha \partial n_\beta} = \frac{1}{h_\beta}\frac{\partial}{\partial \beta}\left(\frac{1}{h_\alpha}\frac{\partial}{\partial \alpha}\right) - \frac{1}{h_\alpha h_\beta^2}\frac{\partial h_\beta}{\partial \alpha}\frac{\partial}{\partial \beta} = \frac{\partial^2}{\partial n_\beta \partial n_\alpha} \quad (7.16b)$$

$$h_\alpha^2 = \left(\frac{\partial x}{\partial \alpha}\right)^2 + \left(\frac{\partial y}{\partial \alpha}\right)^2 + \left(\frac{\partial z}{\partial \alpha}\right)^2, \ h_\beta^2 = \left(\frac{\partial x}{\partial \beta}\right)^2 + \left(\frac{\partial y}{\partial \beta}\right)^2 + \left(\frac{\partial z}{\partial \beta}\right)^2, \quad \cdots\cdots (7.16c)$$

etc.

Therefore in this case we have

$$\bar{h}^2 = h_\alpha^2 = h_\beta^2 = \frac{c^2}{2}(\cosh 2\alpha - \cos 2\beta), \quad h_z^2 = 1, \quad \cdots\cdots (7.17a)$$

$$\nabla_1^2 = \frac{1}{\bar{h}^2}\left(\frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \beta^2}\right), \quad \nabla_1^2 + \frac{\partial^2}{\partial z^2} = \nabla^2, \quad \cdots\cdots (7.17b)$$

$$\sigma_z = \frac{-\partial^2 F}{\partial z^2} + 2\alpha\frac{\partial \Phi_3}{\partial z} + \left(1 - \frac{\alpha'}{2}\right)\nabla^2 F. \quad \cdots\cdots (7.17c)$$

$$\left(\nabla^2 F = 2\frac{\partial \Phi_3}{\partial z}\right).$$

Now if a given loading function $\bar{F}(\alpha,\beta)$ is such that it can be expanded in a series of the type

$$\left.\begin{aligned} \bar{F}(\alpha,\beta) = & \sum_m\sum_n P_{mn} C_{em}(\alpha, q_{mn})\, c_{em}(\beta, q_{mn}) \\ & + \sum_m\sum_n Q_{mn} C_{em}(\alpha, q'_{mn})\, s_{em}(\beta, q'_{mn}) \\ & + \sum_m\sum_n R_{mn} S_{em}(\alpha, \bar{q}_{mn})\, c_{em}(\beta, \bar{q}_{mn}) \\ & + \sum_m\sum_n T_{mn} S_{em}(\alpha, \bar{q}'_{mn})\, s_{em}(\beta, \bar{q}'_{mn}), \end{aligned}\right\} \quad \cdots\cdots (7.18)$$

where the functions c_{em}, s_{em} are Mathieu functions of the m-th order, and the functions C_{em}, S_{em} are modified Mathieu functions, then we can get the particular solutions for the elliptic thick plate under the general load, whose loading function is $\bar{F}(\alpha,\beta)$, when we determine the solutions for each single term of the series (7.18), which is a v-function, in the same manner as explained before.

And here we shall seek the particular solutions for the elliptic plate under the variable load, the loading function of which is $pv(\alpha,\beta) = p_{mn} C_{em}(\alpha, q_{mn})\, c_{em}(\beta, q_{mn})$, for example. As is well known, we have the following relations:

$$\nabla_1^2 v(\alpha,\beta) + k^2 v(\alpha,\beta) = 0, \quad \frac{\partial^2 C_{em}}{\partial \alpha^2} - (a + 16 q_{mn} \cosh 2\alpha)\, C_{em} = 0, \quad \cdots\cdots (7.19)$$

$$\frac{\partial^2 c_{em}}{\partial\beta^2}+(a+16q_{mn}\cos 2\beta)\,c_{em}=0\,,$$

in which

$$q_{mn}=-\frac{c^2k^2}{32}\,,\quad a=A-\frac{c^2k^2}{2}{}^{5)}\,,$$

$$C_{em}\equiv C_{em}(\alpha,q_{mn}) \text{ and } c_{em}\equiv c_{em}(\beta,q_{mn})\,.$$

In this case we get the solution for σ_z, the same form as (7.3), because we employ the same forms of the boundary conditions and of the differential equation for σ_z.

$$\sigma_z=K[\sigma_z]v(\alpha,\beta)\,, \quad\cdots\cdots (7.20)$$

in which

$$K=p_{mn}\Big/(\sinh^2 2kh-4k^2h^2)\,.$$

As the fact that the equations (7.1c), and (7.17c) are of the same forms is suggestive of the applicability of the equations (7.6), we shall take their forms also in this case and examine their legitimacy.

From (7.15a) and (7.15b) we have for the forms of σ_α, σ_β

$$\sigma_\alpha=-\left\{\frac{1}{\bar{h}}\frac{\partial}{\partial\alpha}\left(\frac{1}{\bar{h}}\frac{\partial}{\partial\alpha}\right)+\frac{c^2\sin 2\beta}{2h^4}\frac{\partial}{\partial\beta}\right\}F(\alpha,\beta,z)+\left(1-\frac{\alpha'}{2}\right)\nabla^2F,$$

$$\cdots\cdots (7.21a)$$

$$\sigma_\beta=-\left\{\frac{1}{\bar{h}}\frac{\partial}{\partial\beta}\left(\frac{1}{\bar{h}}\frac{\partial}{\partial\beta}\right)+\frac{c^2\sinh 2\alpha}{2h^4}\frac{\partial}{\partial\alpha}\right\}F(\alpha,\beta,z)+\left(1-\frac{\alpha'}{2}\right)\nabla^2F,$$

$$(F\equiv F(\alpha,\beta,z)) \quad\cdots\cdots (7.21b)$$

and for $\tau_{\alpha z}$, $\tau_{\beta z}$,

$$\tau_{\alpha z}=\frac{-1}{\bar{h}}\frac{\partial^2}{\partial z\partial\alpha}F+\frac{2(1-\nu)}{\bar{h}}\frac{\partial\Phi_3}{\partial\alpha}\,, \quad\cdots\cdots (7.22a)$$

$$\tau_{\beta z}=\frac{-1}{\bar{h}}\frac{\partial^2}{\partial z\partial\beta}F+\frac{2(1-\nu)}{\bar{h}}\frac{\partial\Phi_3}{\partial\beta}\,. \quad\cdots\cdots (7.22b)$$

Hence we obtain Θ, adding the formulae (7.21a), (7.21b) and (7.20),

$$\Theta=\sigma_\alpha+\sigma_\beta+\sigma_z=-2(1+\nu)K\{\sinh 2kh\sinh k(z+h)+$$
$$+2kh\sinh k(z-h)\}\,v(\alpha,\beta)\,. \quad\cdots\cdots (7.23)$$

We can find the expression of the above form for Θ in Sec. IV and so it is readily understood that the normal stresses obtained by the above continuation-process satisfy the third equation of (3.2a). In addition, by comparison between Eqs. (7.8), (7.22) it is easily shown that

$$\tau_{\alpha z} = \frac{K}{k}\frac{1}{h}\frac{\partial v(\alpha,\beta)}{\partial \alpha}\frac{1}{k}\frac{\partial[\sigma_z]}{\partial z}, \quad \cdots\cdots\cdots\cdots \text{(7.24a)}$$

$$\tau_{\beta z} = \frac{K}{k}\frac{1}{h}\frac{\partial v(\alpha,\beta)}{\partial \beta}\frac{1}{k}\frac{\partial[\sigma_z]}{\partial z}, \quad \cdots\cdots\cdots\cdots \text{(7.24b)}$$

where $[\sigma_z]$ was explained in (7.20), (4.8) and so the boundary conditions concerning these shear stresses are satisfied.

And hereon we would point out that the solutions indicated in Sec. IV can also be obtained by the method of Neuber using the continuation-process explained above in this section. Indeed, it is in a certain degree tiresome to follow the method of Michell and Love and it will be convenient, if we could use the method of H. Neuber as much as possible. At any rate the above fact which we pointed out will assure us that this continuation-process is correspondingly reliable. The fact that in a good approximation of Michell and Love in the thick plate problem the expressions for the Φ harmonic tunctions in (7.6) and (7.7b) can be used in general, namely when it is necessary to take general orthogonal cylinder coordinates, will be easily inferred.

In this case, noticing the relation $h_\alpha = h_\beta$, this remark may be confirmed in the same manner as in the case of elliptic plate and so the verification concerned shall be omitted herein. In this way, if the boundary of the plate in question is described by a papametric curve expressed in the orthogonal cylinder coordinates properly taken, a treatment of this thick plate problem may be said to be theoretically feasible, once some other mathematical difficulties are surmounted.

We shall continue to state a little more regarding the elliptic plate. According to H. Neuber the form of displacement is of the following type:

$$u = u_\alpha = \frac{1}{2\mu h_\alpha}\left\{\frac{-\partial F}{\partial \alpha} + 2\alpha'\left(\Phi_1\frac{\partial x}{\partial \alpha} + \Phi_2\frac{\partial y}{\partial \alpha} + \Phi_3\frac{\partial z}{\partial \alpha}\right)\right\}, \quad \cdots \text{(7.25)}$$

in which μ is shear modulus.

Hence we have for displacements $u = u_\alpha$, $v = u_\beta$, $w = u_z$,

$$u = \frac{1}{2\mu h}\left(\frac{-\partial F}{\partial \alpha}\right), \quad v = \frac{1}{2\mu h}\left(\frac{-\partial F}{\partial \beta}\right), \quad w = \frac{1}{2\mu}\left(\frac{-\partial F}{\partial z} + 2\alpha'\Phi_3\right),$$

$$\cdots\cdots\cdots\cdots \text{(7.26)}$$

in which $F(\alpha,\beta,z)$ is by (7.6), (7.7c)

$$F(\alpha,\beta,z) \equiv F = \frac{-K}{k^2} v(\alpha,\beta)\Big[-(kh\cosh 2kh+(1-\alpha')\sinh 2kh)\times$$

$$\times\sinh k(z+h)+(2\alpha'-1)kh\sinh k(z-h)+2k^2h^2\cosh k(z-h)+$$

$$+kz\Big\{\sinh 2kh\cosh k(z+h)+2kh\cosh k(z-h)\Big\}\Big] = \Phi_0+z\Phi_3\,, \qquad (7.27)$$

and from (7.15b)

$$\tau_{\alpha\beta} = -\frac{\partial^2 F}{\partial n_\alpha \partial n_\beta} = -\left\{\frac{1}{\bar{h}^2}\frac{\partial^2}{\partial\alpha\partial\beta}-\frac{c^2}{2\bar{h}^4}\left(\sin 2\beta\frac{\partial}{\partial\alpha}+\sinh 2\alpha\frac{\partial}{\partial\beta}\right)\right\}F\,. \quad (7.28)$$

Next some remarks shall be offered for the case of tangential surface load concerned with the continuation-process. The present author has attempted in vain to apply the continuation process also in this case. It will be apparent that the results described in Sec. V cannot be obtained by the method of H. Neuber through this continuation process utilizing the Φ harmonic functions to be determined in a similar manner to that described when we obtained the expressions in (7.6). This is true since the shearing stress solutions (5.9) have the second terms, of which the expressions (4.13) are void. Furthermore we can decompose a solid harmonic function of the forms $x\Phi$, $y\Phi$, $z\Phi$, $r^2\Phi$, where $r^2=x^2+y^2+z^2$ and Φ is a harmonic function, into harmonic and bi-harmonic functions to some extent arbitrarily. Therefore we shall be obliged to obtain the required solutions by the transformation of the rectangular coodinates (x,y,z) to the desired cylinder coordinates from the solutions in Sec. V, if we want in this case such results as in that of normal surface load as described above.

§VIII. Conclusion.

The author has the intention to describe the ways to satisfy all the boundary conditions i.e., how to solve the problem prefectly, considering both the complementary and particular solutions, in a later report. In the present paper particular solutions were obtained for the thick plate under the variable load by the use of the method of Michell and Love and, when some mathematical difficulties occured, by the aid of the method of H. Neuber. The above obtained solutions are accurate in the approximation of Michell and Love. At some future time the author proposes to describe the solutions with a higher degree of accuracy. In order to investigate the thick plate problem in this respect it is necessary to remove the restriction imposed on the derivative of normal stress component σ_z with respect z. As a matter of course, in this higher degree of accuracy we shall be unable to get plane stress and generalized stress solutions, since these two kinds of solutions may be said to be begotten in some approximation. The results in Sec. IV will be suggestive of the direct extension of the solutions by Navier. At any rate the author believes that the results obtained in this paper may be of some significance in investigating the thick plate problem, though it is to his regret that he could not give the complete and pertinent description.

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