Geophysical Consideration on a Characteristic Factor of a Natural River

By

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Abstract

This is an investigation of the problem of how the discharge of a river increases as the elevation of its water surface rises. The increasing rate of the discharge is proportional to the breadth and the mean velocity of the river. The proportional constant is given as a certain definite value when the river-bed is stable and is given as a larger value than the former one when the bed is unstable. These facts are introduced theoretically and the results are compared with the observed data.

1. Introduction

Amongst the hydraulic features of a river, discharge is the most important one. Many investigations of the methods of measurements in regard to the discharge have been reported so far. The methods of measurements may be generally divided into a direct method and an indirect method. The direct method is that the velocity distribution of the river is measured by current-meters or floats and the amount of discharge is calculated from the data thus obtained. Especially the method with currentmeters has seemed to be the most reliable; it is generally adopted at present. At flooding time, however, this method is usually impracticable; the float method is also rarely employed in Janan because of the intricate configuration of the rivers and the considerable expense.

The indirect method is chiefly adopted at flooding time nowadays. There are roughly two ways of the indirect methods: the first way is one based on Kutter's and Manning's formula etc., mean velocity is calculated and with this the discharge is estimated; the second way is one in which the relation between discharge and stage is investigated, and extending this characteristic curve to flood zone, the discharge is decided by reading the stage at flood time.

In practice, this method is generally employed as it is convenient. There are many reports on the relation between discharge and stage. The author has investigated especially the gradients of discharge-stage curves, compared them with the results observed at a few points along the River Tokachi, and concluded that the gradients of the curves are proportional to the hydraulic factors without stage,—viz., the mean velocity and the breadth of the river approximately. The proportional coefficient k is an interesting factor; its value is discussed from different angles.

2. Relation between discharge and stage

The discharge of a river increases or decreases as the stage increases or decreases, and there is a constant relation between the two factors.

Because of that constant relation discharge Q is treated as a monotonous increasing function of stage H and is generally represented by a quadratic function.
So,

$$Q = a + bH + cH^2 \tag{2.1}$$

or

$$Q = a(b+H)^n \tag{2.2}$$

where, n is fitted to be about 2.

These formulae are derived from data gathered from actual experience that discharge-stage curves are similar to a parabolic curve. But there also have been many attempts to represent the relation theoretically. When the stage rises, the slope of water surface also increases. But when it is assumed that the increasing rate of the stage is small, that is, variation of the surface slope can be neglected, the theoretical relation between Q and H can be introduced for various configurations of cross sections. Namely, if Chézy's formula is thought to hold good for the average velocity over the cross section, the next equations are derived(1).

For a parabolic section

$$Q = a(H \pm H_0)^2 \tag{2.3}$$

For a rectangular section

$$Q = a(H \pm H_0)^{1.5} \tag{2.4}$$

For a trapezoid section

$$Q = a[1+a_1(H\pm H_0)](H\pm H_0)^{\frac{3}{2}} \tag{2.5}$$

For a triangular section

$$Q = a(H\pm H_0)^{\frac{5}{2}} \tag{2.6}$$

where, a is a proportional constant and H_0 is the elevation of the river-bed from the cardinal surface. As configuration of cross section is usually intricate, the upper equations are not always consistent with experimental results. Furthermore, when a large flood wave passes, as the difference between the surface slopes before and behind the wave is fairly large at the same water stage, it is known for the $Q-H$ curve to make a partial loop. (Fig. 1) In the above case, $Q-H$ curve is regarded as a monotone increasing function of H, and the loop curve of $Q-H$ is thought to be less useful except when the other waves of the same scale and the same configuration have arrived. When $Q-H$ curves are drawn, there are some ways with which such loop curves are calibrated to one valued functional curve(2). However when the displacement of the stage is small and irregular, such loop curves are scarcely recognized and moreover, the $Q-H$ curves are able to be regarded approximately as straight lines. Fig. 3 shows the graphs of the $Q-H$ curves which were observed at the River Tokachi while the propagative velocities of flood waves were under investigation by the author and others(3).

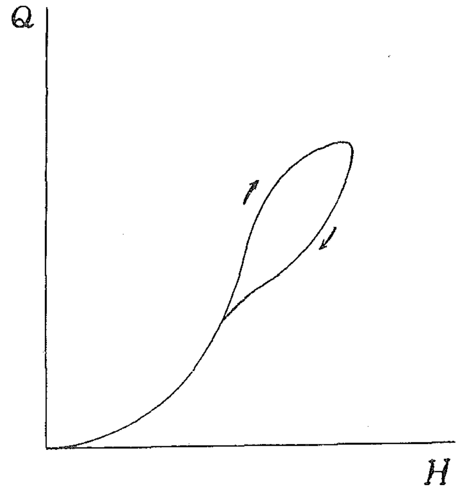


Fig. 1.

The observed flood waves were of small scale and the $Q-H$ curves were regarded as straight lines because the variations of stages was small.

3. Gradient of $Q-H$ curve

The values of the gradients of the $Q-H$ curves which were observed more than three times, are respectively different at every observing station and at every observing term. So the correlation of

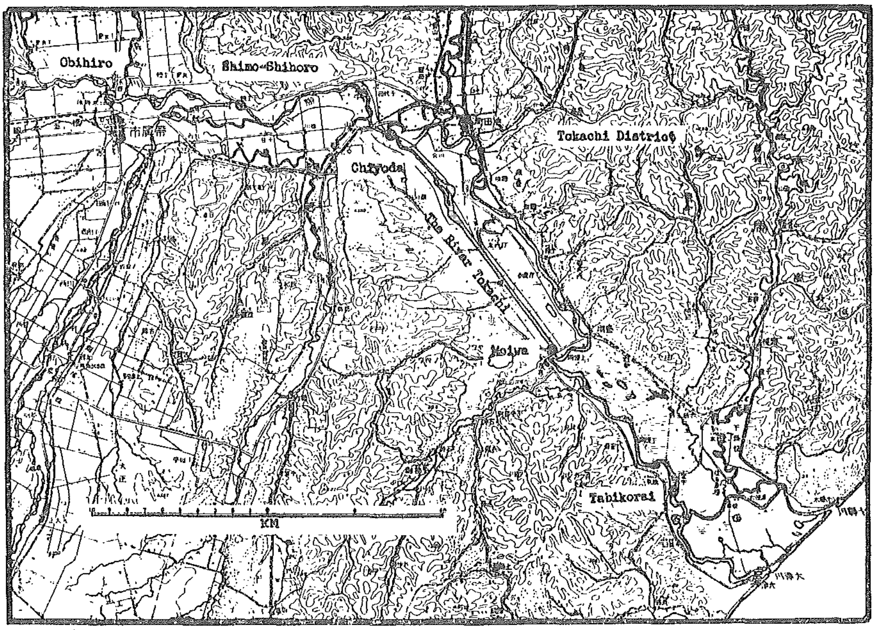


Fig. 2.

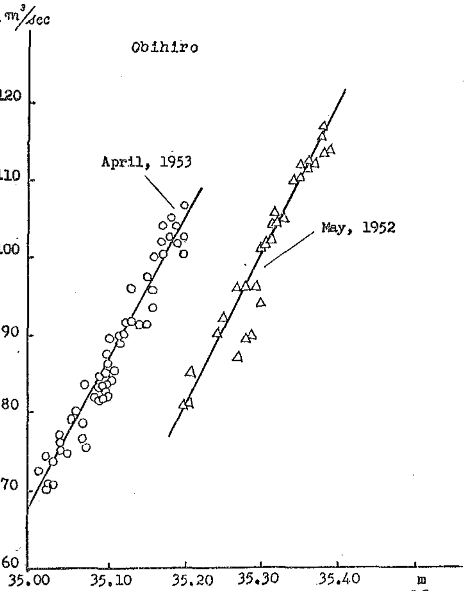


Fig. 3. (1)

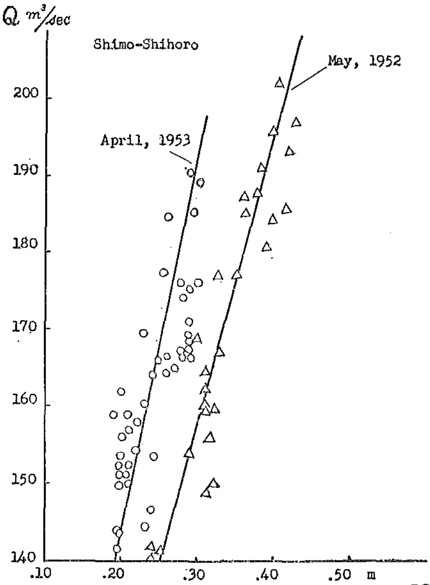


Fig. 3. (2)

Fig. 3. (3)

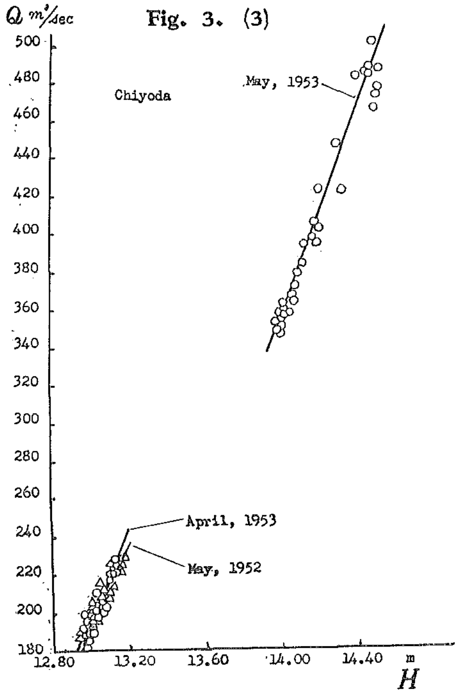


Fig. 3. (4)

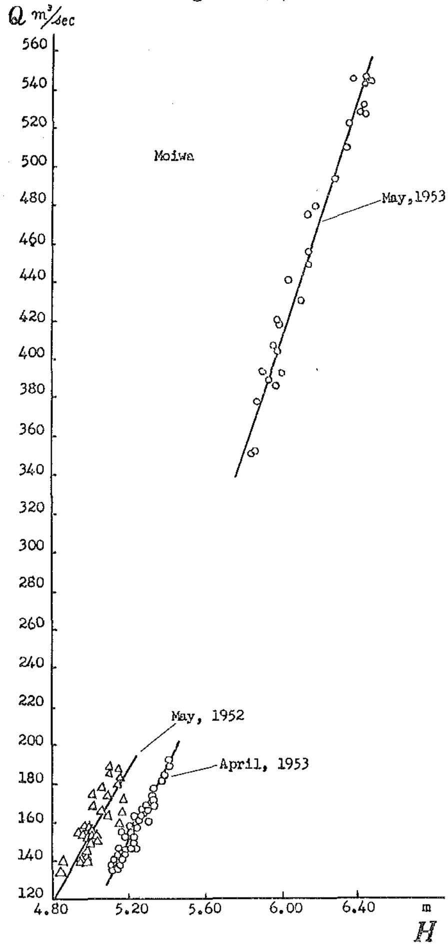
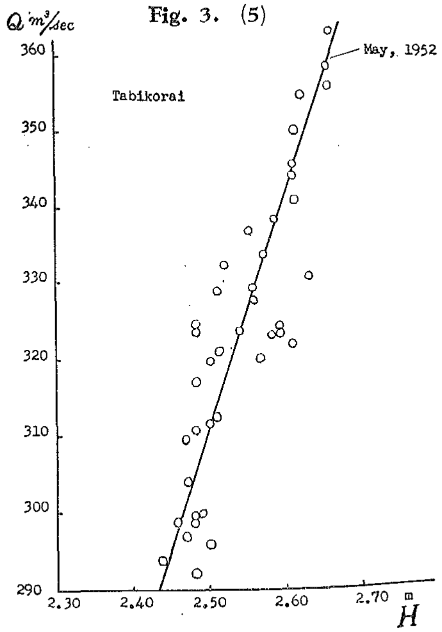


Fig. 3. (5)



those values are investigated with a brief theoretical consideration. It is assumed that discharge varies quasi-statically in keeping with the variation of stage with surface slope constant.

Now notations are given as follows;

Q: discharge (m^3/sec)
V: mean velocity (m/sec)
A: area of cross section (m^2)
H: depth measured from the deepest bottom (m)
L: breadth of a river (m)
R: hydraulic depth (m)

When the configuration of the cross section is taken as rectangular and it is assumed that $L \gg H$ and $R \fallingdotseq H$,

$$\begin{aligned} Q &= VA \\ &= CL\sqrt{IH^3} \end{aligned} \tag{3.1}$$

therefore,

$$\frac{dQ}{dH} = \frac{3}{2} LC\sqrt{HI} = \frac{3}{2} LV \tag{3.2}$$

In the above equation, the gradient of $Q-H$ curve, $\frac{dQ}{dH}$ is proportional to the breadth and the mean velocity.

Now, it is taken that

$$\frac{dQ}{dH} = kLV \tag{3.3}$$

Therefore, the above theoretical calculation shows that $k=1.5$.

The observed values of $\frac{dQ}{dH}$, L and V at the Tokachi River, and k's which are computed from the above factors are shown as follows. (Table I)

Every value of this Table is the mean over each observation term. Table II shows the value of k at every observing station.

From these values it is found that k of each station remains approximately constant, but there are not all points where $k \fallingdotseq 1.5$. The values of Obihiro, Shimo-Shihoro and Moiwa take larger than twice the theoretically derived value,-1.5. It is a remarkable fact that k of each station remains constant. Of course, there may be some errors of measurements but it may be considered certain that the values of

TABLE I.

May, 1952

	L (m)	V (m/sec)	LV (m²/sec)	dQ/dH (m²/sec)	k (dimensionless)
Obihiro	57	1.06	60.4	198	3.3
Shimo-Shihoro	100	1.40	140.0	376	2.7
Chiyoda	103	1.32	136.0	209	1.5
Moiwa	80	0.61	48.8	185	3.8
Tabikorai	185	0.98	181.3	320	1.7
April, 1953					
Obihiro	56	1.02	57.1	187	3.3
Shimo-Shihoro	113	1.33	150.3	488	3.2
Chiyoda	108	1.31	141.5	240	1.7
Moiwa	85	0.75	63.8	196	3.1
May, 1953					
Chiyoda	122	1.67	192.7	308	1.6
Moiwa	86	0.88	75.7	287	3.8

TABLE II.

Values of k

	1	2	3
Obihiro	3.3	3.3	—
Shimo-Shihoro	2.7	3.2	—
Chiyoda	1.5	1.7	1.6
Moiwa	3.1	3.8	3.1
Tabikorai	1.7	—	—

Obihiro, Shimo-Shihoro and Moiwa are different distinctly from the values of Chiyoda and Tabikorai.

The physical meaning of k is next examined in more detail.

4. Physical meaning of k

The configuration of the section is supposed as shown in Fig. 4, and the function of wetted periphery is assumed as

$$x = f(y) \tag{4.1}$$

Furthermore, this equation is divided into two parts,

$$f(y) = f_1(y), \quad x > 0$$
$$f(y) = f_2(y), \quad x < 0 \tag{4.2}$$

in which,

$$f_1(0) = f_2(0)$$

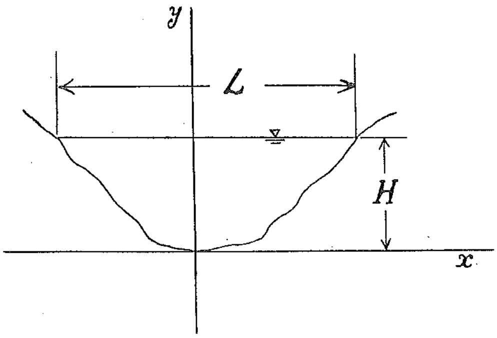


Fig. 4.

Giving the length of the wetted periphery P, the area of the cross section A and the hydraulic depth R, the next equations are presented,

$$A = \int_0^H \{f_1(y) - f_2(y)\}\, dy \tag{4.3}$$

$$P = \int_0^H \left\{\sqrt{1+f_1'(y)^2} + \sqrt{1+f_2'(y)^2}\right\} dy \tag{4.4}$$

$$R = \frac{A}{P} = \frac{\int_0^H \{f_1(y) - f_2(y)\}\, dy}{\int_0^H \left\{\sqrt{1+f_1'(y)^2} + \sqrt{1+f_2'(y)^2}\right\} dy} \tag{4.5}$$

As $Q = VA$.

$$\frac{dQ}{dH} = A\frac{dV}{dH} + V\frac{dA}{dH} \tag{4.6}$$

And from (4.3)

$$\frac{dA}{dH} = f_1(H) - f_2(H) = L \tag{4.7}$$

so,

$$\frac{dQ}{dH} = A\frac{dV}{dH} + LV \tag{4.8}$$

Therefore, in order to get the value of dQ/dH, dV/dH must be studied for various cases. The mean velocity is generally thought to be a function of the hydraulic depth R, the slope of the water surface I and the roughness coefficient of the bottom n,—viz.

$$V = V(n, I, R)$$

Moreover, the change of H influences not only R, but also, n and I, so V may be thought to be a function of H only.

However, as I varies with a parameter of time t, the representation of $I(H)$ is very complicated and as n which takes various values according to the sorts of the bed material also varies with the flow over the critical velocity, $n(H)$ is given as an intricate form too.

Therefore, first, it is assumed that the mean velocity is varied quasi-statically and is a function of R only, while n and I are constant. So it is taken that $V = V(R)$.

Now, it follows, as V is represented to be the m'th power of R, that

$$V = CR^m \tag{4.9}$$

where C and m are both constant.

Minding (4.4) and (4.9)

$$\begin{aligned} \frac{dV}{dH} &= V\frac{m}{R}\,\frac{P\,dA/dH - A\,dP/dH}{P^2} \\ &= mV\left(\frac{L}{A} - \frac{1}{P}\,\frac{dP}{dH}\right) \end{aligned} \tag{4.10}$$

Furthermore, substituting (4.10) into (4.8),

$$\frac{dQ}{dH} = kLV,$$

where

$$k = m + 1 - \frac{mR}{L}\,\frac{dP}{dH}. \tag{4.11}$$

dP/dH represents the length of P wetted for unit height of the water surface and may take the values from 1 to ∞ mathematically, so it seems that k of (4.11) can take negative, but it never happens actually.

It should be thought that (4.9) does not hold good for the river where dP/dH is extremely large at the bank. For the practical measurements for $m = \frac{1}{2} \sim \frac{2}{3}$, it is shown that,

$$\frac{mR}{L}\,\frac{dP}{dH} < \frac{1}{10}.$$

Therefore, it follows that

$$k \doteqdot m+1 \tag{4.12}$$

As $m=1/2$ for Chézy's formula and $m=2/3$ for Manning's formula, it follows that

$$k = 1.5 \sim 1.7 \tag{4.13}$$

Now, the practical k's of Chiyoda and Tabikorai lie quite in the region of (4.13) but the k's of the other stations are far larger than that shown in (4.13). Then, it is expected that the value of k is effected by not only R, but also by the other factors. The mean velocity has been treated as a function of R only but is actually varied with I, so it must follow that

$$V = V(I, R)\,. \tag{4.14}$$

Therefore,

$$\frac{dV}{dH} = \frac{\partial V}{\partial H} + \frac{\partial V}{\partial I}\,\frac{\partial I}{\partial t}\Big/\frac{\partial H}{\partial t}\,. \tag{4.15}$$

As $\dfrac{\partial I}{\partial t}\Big/\dfrac{\partial H}{\partial t}$ changes its value every moment, the $Q-H$ curve of a large flood wave appears to draw a loop. When the loop is rearranged to a quadratic function by the least square method, it must differ by $\left(\overline{\dfrac{\partial V}{\partial I}\,\dfrac{\partial I}{\partial t}\Big/\dfrac{\partial H}{\partial t}}\right)$ from the curve corrected by the before-mentioned ways[2].

As the actual calculation shows that $\left(\overline{\dfrac{\partial V}{\partial I}\,\dfrac{\partial I}{\partial t}\Big/\dfrac{\partial H}{\partial t}}\right)$ is nearly about $1/100\ VL$, it can be neglected against the value of $\partial V/\partial H$.

Then, the value of calculated k is not made yet very different from (4.13) even when I is also variable. Hereupon, there must be taken up the residual factor, viz. the roughness coefficient n. Hitherto, there have been many methods for representation of roughness coefficient, and in Manning's formula, it is adopted as n as also in the next equation.

$$V = \frac{1}{n}\,I^{\frac{1}{2}}R^{\frac{2}{3}}\,.$$

n takes experimentally various values adequate to the material of the river-bed. But when there is sweeped bed load, the apparent roughness coefficient does not keep its value; there are some experimental reports of the variable coefficient with bottom sand flowing. Also it seems that n of Manning's formula is observed as a monotone decreasing function of V[4].

As the present author has never observed the roughness coefficient, he has no experimental data concerning the fine structure of roughness.

But it is sufficiently recognized that there have been changes of the bed over the observation period of three times. Especially, the bed at Moiwa in May, 1953 was remarkably lower than it was in May of the previous year.

It seems that this is due to the large scale flood which occured in the Tokachi in April—May, 1953.

In the above case, it is possible that there were variations of the apparent roughness with movement of the bed. Therefore, it is thought that n is a mediate function of H, so

$$n = n(H)\,. \tag{4.16}$$

Considered I being constant which is not so contributive to k, as abovementioned,

$$\frac{dV}{dH} = \frac{\partial V}{\partial n}\frac{dn}{dH} + \frac{\partial V}{\partial R}\frac{dR}{dH} \tag{4.17}$$

is obtained.

Assuming that there comes for the mean velocity,

$$V = \frac{1}{n} I^{\frac{1}{2}} R^{m} \tag{4.18}$$

it is obtained that

$$\frac{dV}{dH} = \left(\frac{1}{1+\dfrac{V}{n}\dfrac{dn}{dV}}\right)\frac{\partial V}{\partial R}\frac{dR}{dH}\,. \tag{4.19}$$

As $\dfrac{\partial V}{\partial R}\dfrac{dR}{dH}$ is quite the same as (4.10), the next equation is given.

$$\frac{dV}{dH} = \left(\frac{1}{1+\dfrac{V}{n}\dfrac{dn}{dV}}\right) mV\left(\frac{L}{A} - \frac{1}{P}\frac{dP}{dH}\right)\,. \tag{4.20}$$

Substituting (4.20) into (4.8), it follows that

$$\frac{dQ}{dH} = kLV$$

where

$$k = \frac{m}{1+\dfrac{V}{n}\dfrac{dn}{dV}} + 1 - \frac{mR}{\left(1+\dfrac{V}{n}\dfrac{dn}{dV}\right)L}\frac{dP}{dH} . \tag{4.21}$$

Neglecting the third term of (4.21) as treated in (4.12)

$$k \doteqdot \frac{m}{1+\dfrac{V}{n}\dfrac{dn}{dV}} + 1 . \tag{4.22}$$

When it is regarded that n is a monotonous decreasing function of V, as it decreases with increase of V, it follows that $dn/dV<0$ and the value of (4.22) has to be greater than that of (4.12).

The universal relation between the roughness and the velocity is not obvious theoretically, and there is no experimental data on the matter in the hands of the author, so (4.22) can not be developed any more. But with the other observed results[4] and $m=2/3$, k at Moiwa is given as about 3.7 which is in good agreement with the observed value.

But at Obihiro and Shimo-Shihoro, the values of k's become rather larger than the observed ones. This is considered to be due to the fact that there is gravel on the river bed at Obihiro and Shimo-Shihoro, while there is sand at Moiwa. It is obvious that as the applied data for k at Moiwa are those for sand, they are not adoptable for the different bed material. But, it may be sure that the value of k is dependent on the sorts of bed material and the state of the flow. Further it may be considered certain that k settles on 1.5–1.7 when the bed is stable.

Furthermore, it is thought too on a certain river that the value of k is 1.5–1.7 when the flow is slow, and suddenly becomes larger when the velocity rises over a certain critical value, in other words, it is probable that the gradient of the $Q-H$ curve varies discontinuously when variation in the bed occurs.

5. Conclusion

Concerning the gradient of a $Q-H$ curve, the author ascertained that $dQ/dH=kLV$, where k is 1.5–1.7 as the bed is stable and that it becomes greater than that value when the bed begins to shift. The point will be more concisely confirmed by future work as the present data which support this theory are not abundant. If the value of k at a certain station is known from the material of the bed, dQ/dH at this station is calculated easily from the measured L and V, even though there is no level gauge.

dQ/dH seems to be a fairly important factor of natural rivers, for the rising rate of the stage may be predicted when the increasing rate of the discharge at the upper station is known.

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