Some Remarks on the Three-Dimensional Problems Concerned with the Isotropic and Anisotropic Elastic Solids.

By

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§ 1. Introduction.

In this paper some remarks on the solutions of problems for stresses in three-dimensions caused in isotropic and anisotropic elastic solids are stated for the sake of continuity with the earlier papers[1)] associated with thick plate problem. As aforementioned in the introduction of the second of these two papers, the method of solutions due to J. H. Michell and A. E. H. Love to a problem of thick plate is, strictly speaking, that for a moderately thick plate, and so, if one employs this method, one cannot but apply the reduced boundary conditions represented by the resultant forces and couples or displacements and their derivatives on the middle plane of the plate. Also one must apply in the case of free edge conditions Kirchhoff's theorem on torsional couple and vertical tangential force on the cylindrical surface of the plate. In the previous paper the author says that in view of this deficiency of accuracy of this method of solution to the three-dimensional problems, it will be pertinent to indicate the forms of solutions in the higher degree of accuracy to be furnished by the theory of the first order, that is, the solutions to be applicable to a sufficiently thick plate problem under general boundary conditions, and in a later report this complete solution of the problem will be presented.

In the present paper, solutions for a rectangular plate of sufficient thickness or a short column of square cross-section, which are to be referred to the rectangular cartesian coordinates, are obtained by both the procedures due to J. H. Michell[2)] and J. Boussinesq[3)] in the case of isotropy. By the way, the author discusses various modes of approach to the three-dimensional stress problems and asserts that most methods of solution to the three-dimensional problems for isotropic elastic solids should be equivalent to J. Boussinesq's method and thereby he extends the methods analogous to H. Neuber's[4)].

J. H. Michell[5)] says that the method of extension of his solution for a moderately thick, isotropic, elastic plate to anisotropic solid is perfectly obvious, but it seems to the present author that he does mean by his "anisotropic solid" an anisotropic solid possessing elastic symmetry equivalent to that of a crystal of the hexagonal system. And yet even for a moderately thick plate possessing transverse isotropy, if the axis of elastic symmetry lies in the middle plane of the plate, this extension would be impossible. In the present report the author extends to aeolotropic or orthotropic solid his solutions obtained by the method of series, referred to above, for a sufficiently thick, isotropic plate.

So far there have been published a number of analytical treatments for plane stress or strain problems in the theory of anisotropic elasticity, but investigations on three-dimensional problems for aeolotropic media seem comparatively few, and yet a mode of attack in the case of aeolotropy, corresponding to J. Boussinesq's approach or H. Neuber's, is deemed not known. Though it will seldom be necessary to deal with the aeolotropic elasticity problems which require twenty-one independent elastic constants, materials of construction which are essentially orthotropic or are to be regarded as such from the macroscopic viewpoint, will be numerous. Hence, it would be desirable that a general method of solution to the three-dimensional problem for orthotropic or anisotropic solid is derived, and the extensive studies on its application are undertaken.

This paper excludes the case of curvilinear anisotropy or orthotropy as found in Love's textbook on elasticity[6]. It will be inevitable to be content with investigating the cases, in which the so-called elastic constants are constant in true sense of the word or are not funtcions of positional coordinates with the exception of the case of curvilinear orthotropy relevant, for instance, to spherical or cylindrical coordinates.

A. S. Lodge[7] has recently shown that problems for generally anisotropic elastic solids can be transformed into those for isotropic ones by the use of an appropriate linear transformation of coordinates, or affine transformation, imposing specific conditions upon elastic constants. When these conditions for the validity of his method can be satisfied, his result serves to facilitate calculations, but, needless to say, transformation of any such kind without any restrictions upon elastic constants would be unattainable and such restrictions will be evidently undesirable. He has moreover extended the solutions for the case of transverse isotropy obtained by H. A. Elliott[8], and these extended solutions seem to the present author more noticeable, though Lodge says that this result will be applicable to a wider class of problems but whether his extended solutions are completely general or no is not known. By applying the general method of solution for anisotropic solids to be derived in this paper, the solutions due to A. S. Lodge can be verified to be completely general. It is to be added that J. H. Michell[9] presented in 1901 perfectly general solutions for transversely isotropic solids, but the forms of his solutions may be said to be awkward and rather hard to apply in practice. The subject of transversely isotropic solids is thought noticeable and it seems most accessible to us, so that problems associated with transverse isotropy will be treated at some length.

The main purpose of this paper is to describe the general methods of solution to the elasticity problems for sufficiently thick plate, which is subjected to surface tractions, in cases of both isotropy and aeolotropy under homogeneity restraint within the assumptions of the infinitesimal theory of elasticity for the sake of continuity with the author's earlier papers. Most notations to be used throughout this paper are Love's or are self-explaining and so, if not necessary, no pains will be taken to explain the implications of notations in detail.

§ 2. General Method of Solution to the Three-Dimensional Problems for Anisotropic Elastic Solids.

The writer will take here and throughout this paper the rectangular cartesian coordinate system such that stress-strain relation or strain energy function, referred to this system, can be expressed as simply as possible by considering every degree of elastic symmetry, and notations for elastic constants will be the same as Love's[10]. Now let a homogeneous, generally aeolotropic, elastic solid in the absence of volume force be considered within the scope of the infinitesimal theory of elasticty. Then needed stress-strain relations or the generalized Hooke's law in matrix notation are of the following forms:

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} & c_{46} \\ c_{51} & c_{52} & c_{53} & c_{54} & c_{55} & c_{56} \\ c_{61} & c_{62} & c_{63} & c_{64} & c_{65} & c_{66} \end{pmatrix} \cdot \begin{pmatrix} e_x \\ e_y \\ e_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{pmatrix}, \tag{2.1}$$

in which c_{rs} (r, $s=1, 2, 3 \cdots, 6$), are the elastic constants of the anisotropic solid and have symmetry relations of the type

$$c_{rs} = c_{sr} \tag{2.2}$$

Hence the solid possesses twenty-one independent elastic constants. As is well known, the relations (2.1) are equivalent to the following.

$$\sigma_x = \frac{\partial W}{\partial e_x}, \quad \sigma_y = \frac{\partial W}{\partial e_y}, \quad \sigma_z = \frac{\partial W}{\partial e_z}, \quad \tau_{yz} = \frac{\partial W}{\partial \gamma_{yz}},$$
$$\tau_{xz} = \frac{\partial W}{\partial \gamma_{xz}}, \quad \tau_{xy} = \frac{\partial W}{\partial \gamma_{xy}}, \tag{2.3}$$

in which $e_x = \frac{\partial u}{\partial x}, \quad e_y = \frac{\partial v}{\partial y}, \quad e_z = \frac{\partial w}{\partial z},$

$$\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}, \quad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \quad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}, \tag{2.4}$$

and W in (2.3) is a homogeneous quadratic function of strains containing the above 21 constants and of the form.

$$\begin{aligned}2W &= c_{11}e_x^2+c_{22}e_y^2+c_{33}e_z^2+c_{44}\gamma_{yz}^2+c_{55}\gamma_{zx}^2+c_{66}\gamma_{xy}^2+\\&+2c_{12}e_xe_y+2c_{13}e_xe_z+2c_{23}e_ye_z+2c_{45}\gamma_{yz}\gamma_{xz}+2c_{46}\gamma_{yz}\gamma_{xy}+\\&+2c_{56}\gamma_{zx}\gamma_{xy}+2c_{14}e_x\gamma_{yz}+2c_{15}e_x\gamma_{zx}+2c_{16}e_x\gamma_{xy}+\\&+2c_{24}e_y\gamma_{yz}+2c_{25}e_y\gamma_{xz}+2c_{26}e_y\gamma_{xy}+2c_{34}e_z\gamma_{yz}+\\&+2c_{35}e_z\gamma_{yz}+2c_{36}e_z\gamma_{xy}.\end{aligned} \tag{2.5}$$

The stress-equations of equilibrium are

$$\begin{aligned}&\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{xz}}{\partial z}=0\,,\\&\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{yz}}{\partial z}=0\,,\\&\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\sigma_z}{\partial z}=0\,.\end{aligned} \tag{2.6}$$

Since there is little prospect of obtaining solutions if one undertakes to solve these equations directly with the aid of the conditions of compatibility, a start will be made from the displacement-equations of equilibrium. In the subsequent, extensive use of the operational method will be made in order to facilitate much the process of calculation. By inserting the expressions for strains (2.4) in equations (2.6), one obtains the equations of equilibrium of the forms

$$A_{11}u + A_{12}v + A_{13}w = 0\,, \tag{2.7 a}$$

$$A_{21}u + A_{22}v + A_{23}w = 0\,, \tag{2.7 b}$$

$$A_{31}u + A_{32}v + A_{33}w = 0\,, \tag{2.7 c}$$

in which the operators A_{rs} (r, $s=1$, 2, 3) possess the symmetry property, namely

$$A_{rs} = A_{sr}\,, \tag{2.8}$$

and are expressed as

$$\begin{aligned}A_{11}&=(c_{11},\ c_{66},\ c_{55},\ c_{56},\ c_{15},\ c_{16})\left(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right)^2,\\A_{22}&=(c_{66},\ c_{22},\ c_{44},\ c_{24},\ c_{46},\ c_{26})\left(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right)^2,\\A_{33}&=(c_{55},\ c_{44},\ c_{33},\ c_{34},\ c_{35},\ c_{45})\left(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right)^2.\end{aligned} \tag{2.9}$$

$$\Lambda_{12} = \left(c_{16},\ c_{26},\ c_{45},\ \frac{1}{2}(c_{46}+c_{25}),\ \frac{1}{2}(c_{14}+c_{56}),\ \frac{1}{2}(c_{12}+c_{66})\right)\times \times\left(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right)^2,$$

$$\Lambda_{13} = \left(c_{15},\ c_{46},\ c_{35},\ \frac{1}{2}(c_{36}+c_{45}),\ \frac{1}{2}(c_{13}+c_{55}),\ \frac{1}{2}(c_{14}+c_{56})\right)\times \times\left(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right)^2,$$

$$\Lambda_{23} = \left(c_{56},\ c_{24},\ c_{34},\ \frac{1}{2}(c_{23}+c_{44}),\ \frac{1}{2}(c_{45}+c_{36}),\ \frac{1}{2}(c_{25}+c_{46})\right)\times \times\left(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right)^2,$$

wherein symbolic representation implies a formula such that

$$(d_{11},\ d_{22},\ d_{33},\ d_{23},\ d_{31},\ d_{12})\left(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right)^2 = = d_{11}\frac{\partial^2}{\partial x^2} + d_{22}\frac{\partial^2}{\partial y^2} + d_{33}\frac{\partial^2}{\partial z^2} + 2d_{23}\frac{\partial^2}{\partial y\partial z} + 2d_{31}\frac{\partial^2}{\partial x\partial z} + 2d_{12}\frac{\partial^2}{\partial x\partial y}\,. \tag{2. 10}$$

It appears self-evident that the operators constructed with differential coefficients only, referred to rectangular cartesian coordinates, have the commutative property or are subject to the fundamental laws of ordinary algebra. Now, by multiplying equations (2. 7b), (2. 7c) with Λ_{13} and Λ_{12} respectively and subtracting, one obtains

$$(\Lambda_{13}\Lambda_{22}-\Lambda_{12}\Lambda_{23})\,v+(\Lambda_{13}\Lambda_{23}-\Lambda_{12}\Lambda_{33})\,w = 0\,, \tag{2. 11a}$$

and similarly, by multiplying (2. 7a), (2. 7c) with Λ_{23} and Λ_{12} respectively and subtracting and thus eliminating v, it results that

$$(\Lambda_{11}\Lambda_{23}-\Lambda_{12}\Lambda_{13})\,u+(\Lambda_{13}\Lambda_{23}-\Lambda_{12}\Lambda_{33})\,w = 0\,. \tag{2. 11b}$$

Accordingly the desired relations can be arrived at

$$\Gamma^1 u = \Gamma^2 v = \Gamma^3 w\,, \tag{2. 12}$$

in which

$$\begin{aligned} \Gamma^1 &= \Lambda_{11}\Lambda_{23}-\Lambda_{12}\Lambda_{13}\,,\\ \Gamma^2 &= \Lambda_{22}\Lambda_{13}-\Lambda_{12}\Lambda_{23}\,,\\ \Gamma^3 &= \Lambda_{33}\Lambda_{12}-\Lambda_{13}\Lambda_{23}\,. \end{aligned} \tag{2. 13}$$

Since for the present the general case is being treated, one will be obliged to put for the displacements

$$u = \Gamma^2\Gamma^3\phi\,, \quad v = \Gamma^1\Gamma^3\phi\,, \quad w = \Gamma^1\Gamma^2\phi\,, \tag{2.14}$$

in which $\phi \equiv \phi(x, y, z)$,
from the relations (2.12), though one should obviously be cautious to utilize equations (2.12) to deduce the forms of solutions. Next one must look for the basic partial differential equation to be satisfied by function ϕ. When the expressions for displacements (2.14) in equation of equilibrium (2.7a) are inserted, this equation becomes

$$\Lambda_{12}\Lambda_{13}\Big\{\Lambda_{11}\Lambda_{22}\Lambda_{33}+2\Lambda_{23}\Lambda_{13}\Lambda_{12}-\Lambda_{11}(\Lambda_{23})^2+ \\ -\Lambda_{22}(\Lambda_{13})^2-\Lambda_{33}(\Lambda_{12})^2\Big\}\,\phi = 0\,. \tag{2.15}$$

This differential equation is of the 10-th order, and, by considering the general property of boundary value problem in three-dimensions, it is readily inferred that the differential equation satisfied by function ϕ is to be of the 6-th order. Further, solutions of the equations

$$\Lambda_{12}\phi = 0\,, \quad \Lambda_{13}\phi = 0 \tag{2.16}$$

will be easily seen to be trivial, by observing the form of equation (2.7a). Also from other equations of (2.7), the same operator as in the brace of equation (2.15) are obtained. Consequently, the basic partial differential equation to be satisfied by ϕ is found to be of the form.

$$\Big\{\Lambda_{11}\Lambda_{22}\Lambda_{23}+2\Lambda_{23}\Lambda_{13}\Lambda_{12}-\Lambda_{11}(\Lambda_{23})^2-\Lambda_{22}(\Lambda_{13})^2-\Lambda_{33}(\Lambda_{12})^2\Big\}\,\phi = 0\,. \tag{2.17}$$

The operator in the brace of this equation is a homogeneous function of operators $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$ of degree 6, and, if this operator can be resolved into three factors of degree 2 with real coefficients, analytical treatment will be much facilitated, but such cases are likely to be few, if any. For definiteness let the case be considered, wherein such resolution as stated above can be performed. Using symbolic representation (2.10), one may write

$$\begin{aligned}
&\Big\{\Lambda_{11}\Lambda_{22}\Lambda_{33}+2\Lambda_{23}\Lambda_{31}\Lambda_{12}-\Lambda_{11}(\Lambda_{23})^2-\Lambda_{22}(\Lambda_{13})^2-\Lambda_{33}(\Lambda_{12})^2\Big\}\,\phi = \\
&= \Big(a_{11},\ a_{12},\ a_{13},\ \frac{1}{2}a_{14},\ \frac{1}{2}a_{15},\ \frac{1}{2}a_{16}\Big)\Big(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\Big)^2\times \\
&\times \Big(a_{21},\ a_{22},\ a_{23},\ \frac{1}{2}a_{24},\ \frac{1}{2}a_{25},\ \frac{1}{2}a_{26}\Big)\Big(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\Big)^2\times \\
&\times \Big(a_{31},\ a_{32},\ a_{33},\ \frac{1}{2}a_{34},\ \frac{1}{2}a_{35},\ \frac{1}{2}a_{33}\Big)\Big(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\Big)^2\phi = 0\,,
\end{aligned} \tag{2.18}$$

and hence one has

$$\phi = \phi_1 + \phi_2 + \phi_3 \,, \tag{2.19}$$

in which

$$\left(a_{11},\ \cdots,\ \cdots,\ \frac{1}{2}a_{14},\ \cdots,\ \cdots\right)\left(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right)^2 \phi_1 = 0 \,,$$

$$\left(a_{21},\ \cdots,\ \cdots,\ \frac{1}{2}a_{24},\ \cdots,\ \cdots\right)\left(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right)^2 \phi_2 = 0 \,,$$

$$\left(a_{31},\ \cdots,\ \cdots,\ \frac{1}{2}a_{31},\ \cdots,\ \cdots\right)\left(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right)^2 \phi_3 = 0 \,.$$

The solutions for displacements (2.14), together with the basic differential equation (2.17), can be taken in all cases of aeolotropy, perhaps, with the exception of the case of transverse isotropy. It may be said that the method of solution, derived above for the aeolotropic elasticity problem, corresponds to J. Boussinesq's or H. Neuber's approach to the three-dimensional isotropic elasticity problems. If the resolution with real coefficients as aforementioned cannot be carried out, it will be troublesome to treat equation (2.17), and the above result, referred to other coordinates than rectangular cartesian, may be too intricate to be utilized in practice. And yet the resolution or factoring in this case cannot be enforced as it can be in a two-dimensional orthotropic elasticity theory. At any rate it will be rather difficult to obtain a reasonably correct factoring with real coefficients, if it is possible, taking account of the fact that values of elastic constants are, of course, determined to within some errors. Though the author is assured that the above result is useful, its form may be awkward and so it will be desirable to seek a more convenient one.

§ 3. General Solutions for Orthotropic Elastic Solids.

As a matter of fact the problem concerning a material with higher degree of aeolotropy as found in certain crystals may be insignificant and may be excluded, and at most the solids with orthotropic or orthorhombic symmetry or transverse isotropy will be worth noting. For an orthotropic elastic body stress-strain relations (2.1) reduce to

$$\sigma_x = c_{11} e_x + c_{12} e_y + c_{13} e_z \,,$$

$$\sigma_y = c_{12} e_x + c_{22} e_y + c_{23} e_z \,, \tag{3.1}$$

$$\sigma_z = c_{13}e_x + c_{23}e_y + c_{33}e_z\,,$$
$$\tau_{yz} = c_{44}\gamma_{yz}\,,\quad \tau_{xz} = c_{55}\gamma_{xz}\,,\quad \tau_{xy} = c_{66}\gamma_{xy}\,.$$

Accordingly operators in the foregoing section are much simplified. From formulae (2.9) one has

$$\begin{aligned}
\Lambda_{11} &= c_{11}\frac{\partial^2}{\partial x^2} + c_{66}\frac{\partial^2}{\partial y^2} + c_{55}\frac{\partial^2}{\partial z^2}\,,\\
\Lambda_{22} &= c_{66}\frac{\partial^2}{\partial x^2} + c_{22}\frac{\partial^2}{\partial y^2} + c_{44}\frac{\partial_2}{\partial z^2}\,,\\
\Lambda_{33} &= c_{55}\frac{\partial^2}{\partial x^2} + c_{44}\frac{\partial^2}{\partial y^2} + c_{33}\frac{\partial^2}{\partial z,^2}\,,\\
\Lambda_{23} &= \alpha^1\frac{\partial^2}{\partial y\partial z}\,,\quad \Lambda_{13} = \alpha^2\frac{\partial^2}{\partial x\partial z}\,,\quad \Lambda_{12} = \alpha^3\frac{\partial^2}{\partial x\partial y}\,,
\end{aligned}\tag{3.2}$$

in which

$$\alpha^1 = (c_{23}+c_{44})\,,\quad \alpha^2 = (c_{13}+c_{55})\,,\quad \alpha^3 = (c_{12}+c_{66})\,.\tag{3.3}$$

From (2.13) and (3.2) there are obtained

$$\Gamma^1 = \frac{\partial^2}{\partial y\partial z}\Gamma^1_B\,,\quad \Gamma^2 = \frac{\partial^2}{\partial x\partial z}\Gamma^2_B\,,\quad \Gamma^3 = \frac{\partial^2}{\partial x\partial y}\Gamma^3_B\,,\tag{3.4}$$

in which

$$\begin{aligned}
\Gamma^1_B &= (c_{11}\alpha^1-\alpha^2\alpha^3)\frac{\partial^2}{\partial x^2} + \alpha^1\left(c_{66}\frac{\partial^2}{\partial y^2}+c_{55}\frac{\partial^2}{\partial z^2}\right),\\
\Gamma^2_B &= (c_{22}\alpha^2-\alpha^1\alpha^3)\frac{\partial^2}{\partial y^2} + \alpha^2\left(c_{66}\frac{\partial^2}{\partial x^2}+c_{44}\frac{\partial^2}{\partial z^2}\right),\\
\Gamma^3_B &= (c_{33}\alpha^3-\alpha^1\alpha^2)\frac{\partial^2}{\partial z^2} + \alpha^3\left(c_{55}\frac{\partial^2}{\partial x^2}+c_{44}\frac{\partial^2}{\partial y^2}\right).
\end{aligned}\tag{3.5}$$

For convenience' sake one puts

$$\Gamma^2_B\Gamma^3_B = A\Pi_1\,,\quad \Gamma^1_B\Gamma^3_B = A\Pi_2\,,\quad \Gamma^1_B\Gamma^2_B = A\Pi_3\,,\tag{3.6}$$

in which

$$\begin{aligned}
A &= \alpha^1\alpha^2\alpha^3\,,\\
\alpha^1\Pi_1 &= (d_{11},\, d_{22},\, d_{33},\, d_{23},\, d_{31},\, d_{12})\left(\frac{\partial^2}{\partial x^2},\,\frac{\partial^2}{\partial y^2},\,\frac{\partial^2}{\partial z^2}\right)^2,\\
\alpha^2\Pi_2 &= (e_{11},\, e_{22},\, e_{33},\, e_{23},\, e_{31},\, e_{12})\left(\frac{\partial^2}{\partial x^2},\,\frac{\partial^2}{\partial y^2},\,\frac{\partial^2}{\partial z^2}\right)^2,\\
\alpha^3\Pi_3 &= (f_{11},\, f_{22},\, f_{33},\, f_{23},\, f_{31},\, f_{12})\left(\frac{\partial^2}{\partial x^2},\,\frac{\partial^2}{\partial y^2},\,\frac{\partial^2}{\partial z^2}\right)^2,
\end{aligned}\tag{3.7}$$

wherein

$$d_{11} = c_{55}c_{66},\ d_{22}=c_{44}\left(c_{22}-\frac{\alpha^1\alpha^3}{\alpha^2}\right),\ d_{33} = c_{44}\left(c_{33}-\frac{\alpha^1\alpha^2}{\alpha^3}\right),$$

$$2\,d_{23} = \left(c_{22}-\frac{\alpha^1\alpha^3}{\alpha^2}\right)\left(c_{23}-\frac{\alpha^1\alpha^2}{\alpha^3}\right)+c_{44}^{\ 2},\ 2d_{31}=c_{66}\left(c_{33}-\frac{\alpha^1\alpha^2}{\alpha^3}\right)+c_{44}c_{55},$$

$$2\,d_{12} = c_{55}\left(c_{22}-\frac{\alpha^1\alpha^3}{\alpha^2}\right)+c_{44}c_{66}, \tag{3.8a}$$

$$e_{11} = c_{55}\left(c_{11}-\frac{\alpha^2\alpha^3}{\alpha^1}\right),\ e_{22}=c_{44}c_{66},\ e_{33}=c_{55}\left(c_{33}-\frac{\alpha^1\alpha^2}{\alpha^3}\right),$$

$$2\,e_{23} = c_{66}\left(c_{33}-\frac{\alpha^1\alpha^2}{\alpha^3}\right)+c_{44}c_{55},\ 2e_{31}=\left(c_{11}-\frac{\alpha^2\alpha^3}{\alpha^1}\right)\left(c_{33}-\frac{\alpha^1\alpha^2}{\alpha^3}\right)+c_{55}^2,$$

$$2\,e_{12} = c_{44}\left(c_{11}-\frac{\alpha^2\alpha^3}{\alpha^1}\right)+c_{55}c_{66}, \tag{3.8b}$$

$$f_{11} = c_{66}\left(c_{11}-\frac{\alpha^2\alpha^3}{\alpha^1}\right),\quad f_{22}=c_{66}\left(c_{22}-\frac{\alpha^1\alpha^3}{\alpha^2}\right),\quad f_{33}=c_{44}c_{55},$$

$$2f_{23} = c_{55}\left(c_{22}-\frac{\alpha^1\alpha^3}{\alpha^2}\right)+c_{44}c_{66},\ 2f_{31}=c_{44}\left(c_{11}-\frac{\alpha^2\alpha^3}{\alpha^1}\right)+c_{55}c_{66},$$

$$2f_{12} = \left(c_{11}-\frac{\alpha^2\alpha^3}{\alpha^1}\right)\left(c_{22}-\frac{\alpha^1\alpha^3}{\alpha^2}\right)+c_{66}^2. \tag{3.8c}$$

The implications of symbolic representations in (3.7) are similar to those in (2.10), but for caution's sake it is indicated that

$$(d_{11},\ d_{22},\ d_{33},\ d_{23},\ d_{31},\ d_{12})\left(\frac{\partial^2}{\partial x^2},\ \frac{\partial^2}{\partial y^2},\ \frac{\partial^2}{\partial z^2}\right)^2$$
$$= d_{11}\frac{\partial^4}{\partial x^4}+d_{22}\frac{\partial^4}{\partial y^4}+d_{33}\frac{\partial^4}{\partial z^4}+2d_{23}\frac{\partial^4}{\partial y^2\partial z^2}+2d_{31}\frac{\partial^4}{\partial x^2\partial z^2}+2d_{12}\frac{\partial^4}{\partial x^2\partial y^2}. \tag{3.9}$$

Now, if ϕ is written in place of $\frac{\partial^3\phi}{\partial x\partial y\partial z}$, which is evidently permissible, from (2.14) and (3.4) the expressions for displacements are obtained

$$u = \frac{\partial}{\partial x}\Gamma^2_B\Gamma^3_B\phi,\quad v = \frac{\partial}{\partial y}\Gamma^1_B\Gamma^3_B\phi,\quad w = \frac{\partial}{\partial z}\Gamma^1_B\Gamma^2_B\phi, \tag{3.10}$$

or, by substituting the expressions (3.6) in these formulae and writing ϕ/A in stead of ϕ, which is likewise permissible, one gets

$$u = \frac{\partial}{\partial x}\Pi_1\phi,\quad v = \frac{\partial}{\partial y}\Pi_2\phi,\quad w = \frac{\partial}{\partial z}\Pi_3\phi. \tag{3.11}$$

The basic differential equation to be satisfied by function ϕ, appearing in (3.11) or (3.10), can be found from (2.17) and (3.2), namely

$$\Big[c_{11}c_{55}c_{66}\frac{\partial^6}{\partial x^6}+c_{22}c_{44}c_{66}\frac{\partial^6}{\partial y^6}+c_{33}c_{44}c_{55}\frac{\partial^6}{\partial z^6}+$$
$$+\big\{c_{11}c_{22}c_{33}+2c_{44}c_{55}c_{66}+2\alpha^1\alpha^2\alpha^3-c_{11}c_{23}(c_{23}+2c_{44})+$$
$$-c_{22}c_{13}(c_{13}+2c_{55})-c_{33}c_{12}(c_{12}+2c_{66})\big\}\frac{\partial^6}{\partial x^2\partial y^2\partial z^2}+$$
$$+\big\{c_{22}c_{44}c_{55}+c_{22}c_{33}c_{66}-c_{23}c_{66}(c_{23}+2c_{44})\big\}\frac{\partial^6}{\partial y^4\partial z^2}+$$
$$+\big\{c_{33}c_{44}c_{66}+c_{22}c_{33}c_{55}-c_{23}c_{55}(c_{23}+2c_{44})\big\}\frac{\partial^6}{\partial z^4\partial y^2}+$$
$$+\big\{c_{11}c_{44}c_{55}+c_{11}c_{33}c_{66}-c_{13}c_{66}(c_{13}+2c_{55})\big\}\frac{\partial^6}{\partial x^4\partial z^2}+$$
$$+\big\{c_{33}c_{55}c_{66}+c_{11}c_{33}c_{44}-c_{13}c_{44}(c_{13}+2c_{55})\big\}\frac{\partial^6}{\partial z^4\partial x^2}+$$
$$+\big\{c_{11}c_{44}c_{66}+c_{11}c_{22}c_{55}-c_{12}c_{55}(c_{12}+2c_{66})\big\}\frac{\partial^6}{\partial x^4\partial y^2}+$$
$$+\big\{c_{22}c_{55}c_{66}+c_{11}c_{22}c_{44}-c_{12}c_{44}(c_{12}+2c_{66})\big\}\frac{\partial^6}{\partial y^4\partial x^2}\Big]\phi=0\,. \tag{3.12}$$

The above solutions of displacement-equations of equilibrium regarding orthotropic elastic solids are manifestly completely general except for certain cases of the class of orthotropic solids. For orthorhombic solids the resolution of the operator in the bracket of equation (3.12) in the three real factors of degree two will be easily performed, if possible, since the operator concerned is virtually considered as that of degree 3, though that is needless to say.

In addition the above solutions can be derived in somewhat different way. When one puts for displacements

$$u=\frac{\partial u'}{\partial x},\quad v=\frac{\partial v'}{\partial y},\quad w=\frac{\partial w'}{\partial z}, \tag{3.13}$$

and integrates the reduced equations (2.11a) and (2.11b) with respect to x and y respectively, he obtains the relation formulae similar to (2.12).

$$\Gamma^1_B u'=\Gamma^2_B v'=\Gamma^3_B w'\,. \tag{3.14}$$

Hence the same expressions as in (3.10) are obtained and further the substitution of the expressions (3.10) in the equations of equilibrium (2.7a) yields the differential equation (3.12) multiplied by coefficients $(c_{13}+c_{55})(c_{12}+c_{66})$. Supposing that the differential equation (3.12) can be resolved into threer differential equations of the second oder with real coefficients, for form's sake equation (3.12) may be written in the form:

$$\begin{aligned} &\left(a_{11}\frac{\partial^2}{\partial x^2}+a_{12}\frac{\partial^2}{\partial y^2}+a_{13}\frac{\partial^2}{\partial z^2}\right)\phi_1=0\,,\\ &\left(a_{21}\frac{\partial^2}{\partial x^2}+a_{22}\frac{\partial^2}{\partial y^2}+a_{23}\frac{\partial^2}{\partial z^2}\right)\phi_2=0\,,\\ &\left(a_{31}\frac{\partial^2}{\partial x^2}+a_{32}\frac{\partial^2}{\partial y^2}+a_{33}\frac{\partial^2}{\partial z^2}\right)\phi_3=0\,,\\ &\phi=\phi_1+\phi_2+\phi_3\,. \end{aligned} \tag{3.15}$$

§ 4. Completely General Solutions for a Transversely Isotropic Solid.

As mentioned before, this class of solids is one of the most important particular cases of orthotropic solids. The stress-strain relations for a transversely isotropic solid contain five independent elastic constants and are expressed as

$$\begin{aligned} \sigma_x&=c_{11}e_x+c_{12}e_y+c_{13}e_z\,,\\ \sigma_y&=c_{12}e_x+c_{11}e_y+c_{13}e_z\,,\\ \sigma_z&=c_{13}e_x+c_{13}e_y+c_{33}e_z\,,\\ \tau_{yz}&=c_{44}\gamma_{yz}\,,\quad \tau_{xz}=c_{44}\gamma_{xz}\,,\quad \tau_{xy}=\frac{1}{2}(c_{11}-c_{12})\gamma_{xy}\,. \end{aligned} \tag{4.1}$$

These may be obtained from the relations (3.1) by putting

$$c_{11}=c_{22}\,,\quad c_{23}=c_{13}\,,\quad c_{44}=c_{55}\,,\quad c_{66}=\frac{1}{2}(c_{11}-c_{12})\,. \tag{4.2}$$

Of course, the z-axis is parallel to that of elastic symmetry. The solution for a transversely isotropic solid achieved by H. A. Elliott[8] and A. S. Lodge[7] seems convenient of application. The latter says that whether this solution is completely general or not is not known, and hence it is deemed necessary to investigate its generality. The subsequent discussion will assure one that the solution obtained by them is perfectly general and of most simplified form. Indeed, J. H. Michell[9] showed, though A. S. Lodge does not point it out, that the differential equations in three-dimensions for a solid, which is elastically equivalent to a crystal of the hexagonal system, can be expressed in the following form. By transforming Michell's notations into the ordinary ones, if required, they became

$$
\begin{aligned}
&\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\gamma_1\frac{\partial^2}{\partial z^2}\right)V_1=0\,,\\
&\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\gamma_2\frac{\partial^2}{\partial z^2}\right)V_2=0\,,\\
&\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\gamma_3\frac{\partial^2}{\partial z^2}\right)\omega_3=0\,,
\end{aligned}
\tag{4.3}
$$

in which

$$
\begin{aligned}
&V_\alpha=\left(\theta+q_\alpha\frac{dw}{dz}\right),\qquad \omega_3=\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right),\\
&\theta=e_x+e_y+e_z\,,
\end{aligned}
$$

and, furhter,

$$
\begin{aligned}
&q_\alpha=\frac{1}{c_{11}}\left\{c_{13}-c_{11}+c_{44}(p_\alpha+1)\right\},\\
&p_\alpha=\frac{(c_{11}\gamma_\alpha-c_{44})}{c_{13}+c_{44}}\,,\\
&\gamma_1\gamma_2=\frac{c_{33}}{c_{11}}\,,\quad \gamma_1+\gamma_2=\frac{c_{11}c_{33}-c_{13}^2-2c_{13}c_{44}}{c_{11}c_{44}}\,,\\
&\gamma_3=\frac{c_{44}}{c_{66}}\,,\quad (\alpha=1,\ 2)\,.
\end{aligned}
\tag{4.4}
$$

As described in the following, equations (4. 3) are utterly analogous to those due to A. S. Lodge, and in fact J. H. Michell certainly shows that there are needed three independent partial differential equations of the types of equations in (4. 3) for the three-dimensional problems concerning transversely isotropic solids, but his method of solution may be said to be harder to apply.

In the next place, let the method of solution deduced in the foregoing section be applied to a problem for the class of solid concerned. From (3. 5) and (4. 1) one has

$$
\Gamma_B^1=\Gamma_B^2=(c_{13}+c_{44})\left(c_{66}\frac{\partial^2}{\partial x^2}+c_{66}\frac{\partial^2}{\partial y^2}+c_{44}\frac{\partial^2}{\partial z^2}\right),
\tag{4.5a}
$$

$$\Gamma_B^3 = \left[c_{44}(c_{12}+c_{66})\frac{\partial^2}{\partial x^2} + c_{44}(c_{12}+c_{66})\frac{\partial^2}{\partial y^2} + \right.$$
$$\left. + \left\{ c_{33}(c_{12}+c_{66}) - (c_{13}+c_{44})^2 \right\} \frac{\partial^2}{\partial z^2} \right], \tag{4.5b}$$

and the basic differential equation (3.12) with (4.2) can be readily resolved into three factors with real coefficients as follows:

$$\frac{c_{44}}{c_{11}} \left[c_{11}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) + \left\{(c_{13}+c_{44})\,k_1 + c_{44}\right\} \frac{\partial^2}{\partial z^2} \right] \times$$
$$\times \left[c_{11}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) + \left\{(c_{13}+c_{44})\,k_2 + c_{44}\right\} \frac{\partial^2}{\partial z^2} \right] \times$$
$$\times \left[c_{66}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) + c_{44}\frac{\partial^2}{\partial z^2} \right] \phi = 0\,, \tag{4.6}$$

in which k_1 and k_2 are the roots in k of the equation.

$$c_{44}(c_{13}+c_{44})\,k^2 + \left\{(c_{13}+c_{44})^2 - c_{11}c_{33} + c_{44}^2\right\} k + c_{44}(c_{13}+c_{44}) = 0\,. \tag{4.7}$$

Of course, there seems to exist no definite reason why the coefficients or k_α appearing in factors of (4.6) should be real, but for almost every transversely isotropic solid this will be the case. Then, considering equations (3.15), one can put for the forms of displacements by means of the expressions (3.10) with formulae (4.5), except the case of equation, appearing in (4.6),

$$\left\{ c_{66}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) + c_{44}\frac{\partial^2}{\partial z^2} \right\} \phi_3 = 0\,. \tag{4.8}$$

This is easily seen from the fact that the operator in the brace of this equation is equivalent to that of (4.5a), so that the expressions (3.10) cannot be used in this case. Thus, excluding function ϕ_3, and deleting the operator Γ_B^1 or Γ_B^2 in each expression in (3.10), since Γ_B^1 equals Γ_B^2, one can write for displacements

$$u_\alpha = \frac{\partial}{\partial x}\Gamma_B^3\,\phi_\alpha\,,\quad v_\alpha = \frac{\partial}{\partial y}\Gamma_B^3\,\phi_\alpha\,,\quad w_\alpha = \frac{\partial}{\partial z}\Gamma_B^1\,\phi_\alpha\,, \qquad (\alpha = 1,\ 2) \tag{4.9}$$

in which $\alpha=1$ and $\alpha=2$ correspond to k_1 and k_2 in (4.7) respectively. ϕ_α satisfies the equation

$$\left[c_{11}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) + \left\{(c_{13}+c_{44})\,k_\alpha + c_{44}\right\} \frac{\partial^2}{\partial z^2} \right] \phi_\alpha = 0\,. \tag{4.18}$$

As stated in the preceding, the basic partial differential equation of the sixth order (2.17) or (3.12) contains all needed basic differential equations for the three-dimensional problem considered, and hence it becomes needful to seek a solution relevant to the third basic equation (4.8).

Now let a beginning be made by writing for displacements the following:

$$u = A^1\phi\,,\quad v = A^2\phi\,,\quad w = A^3\phi\,, \tag{4.11}$$

in which $A^\nu(\nu=1, 2, 3)$ is the operator to be determined and is taken to be subject to the laws of ordinary algebra for the present. Then, by referring to the relations (2.12) with formulae (3.4), or the relations (3.14), and formulae (4.5), it follows that

$$\Gamma_B^1 \frac{\partial u}{\partial y} - \Gamma_B^1 \frac{\partial v}{\partial x} = 0\,, \tag{4.12a}$$

$$\Gamma_B^1 \frac{\partial v}{\partial z} - \Gamma_B^3 \frac{\partial w}{\partial y} = 0\,. \tag{4.12b}$$

The substitution of the expressions (4.11) in equations (4.12a) and (4.12b) yields the equations

$$\Gamma_B^1\left(A^1\frac{\partial}{\partial y} - A^2\frac{\partial}{\partial x}\right)\phi = 0\,, \tag{4.13a}$$

$$\left(\Gamma_B^1 A^2\frac{\partial}{\partial z} - \Gamma_B^3 A^3\frac{\partial}{\partial y}\right)\phi = 0\,, \tag{4.13b}$$

respectively. Hence, on condition that $\Gamma_B^1\phi \neq 0$, one gets by (4.13a)

$$A^1 = A\frac{\partial}{\partial x}\,,\quad A^2 = A\frac{\partial}{\partial y}\,, \tag{4.14}$$

in which

$$A\phi \neq 0\,,$$

and, further, one has

$$\Gamma_B^1\phi = 0\,. \tag{4.15}$$

From (4.14) and (4.13b) it is, therefore, obtained that

$$A = \bar{A}\Gamma_B^3\,,\quad A^3 = \bar{A}\Gamma_B^1\frac{\partial}{\partial z}\,, \tag{4.16}$$

in which $\bar{A}\phi \neq 0$. As the operator $\bar{A}$ can be merged in function ϕ, the expressions (4.11) with (4.14) and (4.16) evidently agree with those of (4.9). Next, ϕ appearing in (4.15) should be originally denoted by ϕ_3, observing equation (4.8), then from (4.15) and (4.13b) one obtains

$$A^3 = 0\,, \tag{4.17}$$

namely displacement w vanishes for ϕ_3. Thus, for a transversely isotropic solid three displacement equations of equilibrium reduce to a simple equation:

$$\left(A^1\frac{\partial}{\partial x} + A^2\frac{\partial}{\partial y}\right)\phi_3 = 0\,. \tag{4.18}$$

Accordingly, it is obtained from (4.18) that

$$A^1 = A'\frac{\partial}{\partial y}\,,\quad A^2 = -A'\frac{\partial}{\partial x}\,,\qquad (A'\phi_3 \neq 0) \tag{4.19}$$

and one can obviously do without the operator A'. In consequence one gets the third solution corresponding to function ϕ_3, namely,

$$u^3 = \frac{\partial}{\partial y}\phi_3\,,\quad v_3 = -\frac{\partial}{\partial x}\phi_3\,,\quad w_3 = 0\,, \tag{4.20}$$

in which ϕ_3 satisfies equation (4.8) or (4.15). Adding solutions (4.9) and (4.20), the complete three-dimensional solutions are arrived at for a transversely isotropic solid by the use of the method of solution proposed by the present author, showing one case which needs some slight modifications in this approach.

Now let the solution due to H. A. Elliott and A. S. Lodge be cited for the case of transverse isotropy and let it be shown that the above solutions perfectly agree with the result to be cited in the following. Solutions due to Lodge and Elliott are

$$u = \sum_i u_i\,,\quad v = \sum_i v_i\,,\quad w = \sum_i w_i\,,\qquad (i=1,\ 2,\ 3)\,.$$

$$u_\alpha = \frac{\partial\phi_\alpha}{\partial x}\,,\quad v_\alpha = \frac{\partial\phi_\alpha}{\partial y}\,,\quad w_\alpha = \frac{\partial}{\partial z}k_\alpha\,\phi_\alpha\,,\qquad (\alpha=1,\ 2)\,. \tag{4.21a}$$

$$u_3 = \frac{\partial\phi_3}{\partial y}\,,\quad v_3 = -\frac{\partial}{\partial x}\phi_3\,,\quad w_3 = 0\,, \tag{4.21b}$$

and $\phi_i\,(i=1,\ 2,\ 3)$ satisfies the equations

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \nu_i\frac{\partial^2}{\partial z^2}\right)\phi_i = 0\,, \tag{4.22}$$

wherein ν_1 and ν_2 are the roots in ν of the equation

$$c_{11}c_{44}\nu^2 + (c_{13} - c_{11}c_{33} + 2c_{13}c_{44})\nu + c_{33}c_{44} = 0\,. \tag{4.23a}$$

$$\nu_3 = \frac{c_{44}}{c_{66}} = \frac{2c_{44}}{c_{11}-c_{12}}, \tag{4.23b}$$

and

$$k_\alpha = \frac{(c_{11}\nu_\alpha - c_{44})}{c_{13}+c_{44}}, \qquad (\alpha = 1,\ 2) \tag{4.24a}$$

and let the relation be added.

$$\nu_\alpha = \frac{(c_{13}+c_{44})k_\alpha + c_{44}}{c_{11}} = \frac{c_{33}k_\alpha}{(c_{13}+c_{44})+c_{44}k_\alpha}. \tag{4.24b}$$

By (4.24b) it may be seen that k_α satisfies the equation (4.7). Hence, (4.24a) are the same as those which appear in equation (4.6), and differential equations (4.8) and (4.10) can be rewritten into equations (4.22), that is, the symbols ν_i can be used for the solutions obtained by the author. In order that the conformity of the expressions for displacements (4.9) with those in (4.21a) may be confirmed, it must be shown that the following equations hold:

$$\Gamma^1_B\phi_\alpha = k_\alpha \Gamma^3_B\phi_\alpha, \qquad (\alpha = 1,\ 2). \tag{4.25}$$

By means of equations (4.7) or (4.23a) it is readily ascertained that equations (4.25) are equivalent to equations (4.22) relevant to ν_1 and ν_2. Thus from (4.25) ϕ_α and $k_\alpha\phi_\alpha$ can be substituted in place of $\Gamma^3_B\phi_\alpha$ and $\Gamma^1_B\phi_\alpha$ respectively, which appear in the expressions (4.9). By the above shortened proof the completeness of the solutions for a transversely isotropic solid due to H. A. Elliott and A. S. Lodge is established in the opinion of the present writer.

In addition, reference will be made briefly to J. H. Michell's approach. From an inspection of his result and of A. S. Lodge's, it follows that $\gamma_i\,(i=1,\ 2,\ 3)$ equals $\nu_i\,(i=1,\ 2,\ 3)$ and so $p_\alpha\,(\alpha=1,\ 2)$ agrees with k_α $(\alpha=1,\ 2)$. Though Michell's differential equations (4.3) are certainly equal to those in (4.22), it is to be regretted that Lodge's solution or equivalent one was not reached by J. H. Michell. Since A. S. Lodge's solution for a transversely isotropic elastic solid is expedient and simplified in form, and available for use without any apprehension about its completeness as verified above, it will, of course, be advisable to employ this solution.

It will be noted that the solutions for the case of transverse isotropy may be regarded as a simple extension of the solution for plane stress or plane strain in an orthotropic material to three-dimensional one, and that function ϕ_i in (4.22) is of the harmonic type and can be readily determined. For instance, let an elastic plate of moderate thickness be considered. Take the z-axis to be in the direction of thickness of the plate and positive upwards, and the origin of coordinates to lie on the middle plane, assuming that the plate has transverse isotropy about the x-axis. Then, solutions can be written by virtue of the solutions (4.21) to (4.24). Stress-strain relations are

$$\begin{aligned} \sigma_x &= a_{11} e_x + a_{12} e_y + a_{12} e_z , \\ \sigma_y &= a_{12} e_x + a_{22} e_y + a_{23} e_z , \\ \sigma_z &= a_{12} e_x + a_{23} e_y + a_{22} e_z , \\ \tau_{yz} &= a_{44} \gamma_{yz} , \quad \tau_{zx} = a_{55} \gamma_{zx} , \quad \tau_{xy} = a_{55} \gamma_{xy} , \end{aligned} \tag{4.26}$$

in which, using notations for elastic constants $c_{\alpha\beta}$, one can write

$$a_{11} = c_{33} , \quad a_{12} = c_{13} , \quad a_{22} = c_{11} , \quad a_{23} = c_{12} ,$$

$$a_{44} = \frac{1}{2}(a_{22} - a_{23}) = c_{66} = \frac{1}{2}(c_{11} - c_{12}) , \quad a_{55} = c_{44} .$$

Solutions for displacements are as follows:

$$\begin{aligned} (u_\alpha , v_\alpha , w_\alpha) &= \left(k_\alpha \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \phi_\alpha , \qquad (\alpha = 1, 2) \\ (u_3, v_3, w_3) &= \left(0, \frac{\partial}{\partial z}, -\frac{\partial}{\partial y} \right) \phi_3 , \end{aligned} \tag{4.27}$$

in which

$$\left(\nu_i \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \phi = 0 , \qquad (i = 1, 2, 3) \tag{4.28}$$

and ν_1, ν_2 are the roots of the equation

$$a_{22} a_{55} \nu^2 + (a_{12}^2 - a_{11} a_{22} + 2 a_{12} a_{55}) \nu + a_{11} a_{55} = 0 .$$

Further,

$$\nu_3 = \frac{a_{55}}{a_{44}} = \frac{2 a_{55}}{a_{22} - a_{23}} , \quad k_\alpha = \frac{\nu_\alpha c_{22} - c_{55}}{c_{12} + c_{55}} , \tag{4.29}$$

and ν_i and k_α are evidently the same as those in (4.23) and (4.24). When this plate is in plane-stress state, the relations (4.26) are, of course, as follows:

$$\sigma_z = \tau_{yz} = \tau_{xz} = 0 ,$$

or

$$\begin{aligned} e_x &= \alpha_{11}\sigma_x + \alpha_{12}\sigma_y\,, \\ e_y &= \alpha_{12}\sigma_x + \alpha_{22}\sigma_y\,, \\ e_z &= \alpha_{13}\sigma_x + \alpha_{23}\sigma_y\,, \\ \gamma_{xy} &= \alpha_{55}\tau_{xy}\,. \end{aligned} \tag{4.30}$$

Forms of solutions (4.27)–(4.29) will undergo no change, and it will be obvious that for an orthotropic thin plate, i.e., an orthotropic plate of zero thickness, the solution derived from that for transversely isotropic solid is available, though it will lead to a final result which is slightly different from what is to be obtained by an ordinary approach. Yet, from the following simple explantion it will readily be inferred that, when the thickness of the plate, possessing such a transverse isotropy as stated above, is moderate, there does not exist any correct solution for plane stress state which corresponds to what is obtained by A. E. H. Love[17] for the case of isotropy. Thence also exact solution for this plate in the generalized plane stress state could by no means be achieved. Of course, this generalized plane stress does not mean the mean value, taken through the thickness of the plate, of the stress in the middle plane of the plate. In order to satisfy the first two stress-equations of equilibrium for plane stress state even in the broadened meaning, it will suffice to take the stresses of the following forms:

$$\sigma_x = \frac{\partial^2 F}{\partial y^2}\,, \quad \sigma_y = \frac{\partial^2 F}{\partial x^2}\,, \quad \tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y}\,, \tag{4.31}$$

Since ϕ_1, ϕ_2 and ϕ_3 are independent of one another, they can be treated separately. For instance, from (4.27), (4.30) and (4.31) there are obtained.

$$\begin{aligned} e_x &= k_1 \frac{\partial^2 \phi_1}{\partial x^2} = \left(\alpha_{11}\frac{\partial^2}{\partial y^2} + \alpha_{12}\frac{\partial^2}{\partial x^2}\right)F\,, \\ e_y &= \frac{\partial^2 \phi_1}{\partial y^2} = \left(\alpha_{12}\frac{\partial^2}{\partial y^2} + \alpha_{22}\frac{\partial^2}{\partial x^2}\right)F\,, \\ e_z &= \frac{\partial^2 \phi_1}{\partial z^2} = \left(\alpha_{13}\frac{\partial^2}{\partial y^2} + \alpha_{23}\frac{\partial^2}{\partial x^2}\right)F\,, \end{aligned} \tag{4.32}$$

and

$$\gamma_{xy} = (k_1+1)\frac{\partial^2 \phi_1}{\partial x \partial y} = -\alpha_{55}\frac{\partial^2 F}{\partial x \partial y}\,. \tag{4.33}$$

Equation (4.33) gives elastically

$$F = -\frac{(k_1+1)}{a_{55}}\phi_1, \tag{4.34}$$

and, hence, in order that the solution for a plate of moderate thickness, i. e., one not in the limit case of zero thickness, can be obtained and expressed as finite power series in z, the insertion of the expresion (4.34) in equations (4.32) must yield the basic differential equation (4.28) with ν_1, but it does not do so certainly. In the next place, if the expressions (4.30), together with (4.31), be substituted in one of the equations of compatibility

$$\frac{\partial^2 e_x}{\partial y^2} + \frac{\partial^2 e_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}, \tag{4.35}$$

the following differential equation for F is obtained:

$$\left\{a_{22}\frac{\partial^4}{\partial x^4} + (2a_{12}+a_{55})\frac{\partial^4}{\partial x^2 \partial y^2} + a_{11}\frac{\partial^4}{\partial y^4}\right\}F = 0. \tag{4.36}$$

This can be rewritten into the form

$$\left(\beta_1\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(\beta_2\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)F = 0, \tag{4.37}$$

in which

$$\beta_1\beta_2 = \frac{a_{11}a_{22}-a_{12}^2}{a_{22}^2-a_{23}^2}, \quad \beta_1+\beta_2 = \frac{-2a_{12}a_{55}+a_{11}a_{22}-2a_{12}^2}{a_{22}a_{55}}. \tag{4.38}$$

Thus, by the inspection of (4.29) and (4.38) it is easily seen that only, if the elastic constants a_{23} and a_{12} be ignored, β_α can be thought equal to ν_α in (4.28). These facts show that the solutions (4.30) and (4.27)-(4.29) can be applied to the orthotropic thin plate problems, and, further, that correct solutions for a moderately thick plate of transverse isotropy and, of other aeolotropies, needless to say, could not exist.

Hereupon, it becomes needful to point out that between the implications of a generalized plane stress state which is termed frequently in conjunction with thin plate problems and with problems concerning plate of moderate thickness there is an utter difference. That is, mean values, taken through the thickness of the plate, of the generalized plane stresses may be said to be nothing but the values of plane stresses relevant to the plate of zero thickness.

Accordingly, concerning aeolotropic plate problems, one can only deal either with the case of zero thickness by drastic assumptions or with the case of sufficient thickness exactly by the three-dimensional approach as mentioned before.

§ 5. On the Method of Solution to the Three-Dimensional Problem for an Isotropic Solid.

It will be of considerable importance to recall in the following discussion that in order to ensure the completeness of the solution for a transversely isotropic solid there must exist the displacement-solution of the vector form

$$(u_3,\ v_3,\ w_3) = \mathrm{rot}\,(0,\ 0,\ \phi_3)\,, \tag{5.1}$$

in which

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \nu_3 \frac{\partial^2}{\partial z^2}\right)\phi_3 = 0\,,$$

as shown in the preceding section, when there is transverse isotropy about the z-axis of the reference system.

In the first place let an isotropic solid as the particular case of aeolotropic ones be considered. In the case of isotropy, by putting

$$\begin{aligned} &c_{11} = c_{22} = c_{33}\,, \quad c_{23} = c_{13} = c_{12}\,, \\ &\qquad c_{44} = c_{55} = c_{66} = \frac{1}{2}(c_{11} - c_{12})\,, \end{aligned} \tag{5.2}$$

and any other elastic constants equal to zero, one gets from (2.9) or (3.2) and (2.17)

$$\begin{aligned} A_{11} &= c_{11}\frac{\partial^2}{\partial x^2} + c_{66}\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right), \quad A_{23} = (c_{11} - c_{66})\frac{\partial^2}{\partial y \partial z}\,, \\ A_{22} &= c_{11}\frac{\partial^2}{\partial y^2} + c_{66}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right), \quad A_{31} = (c_{11} - c_{66})\frac{\partial^2}{\partial x \partial z}\,, \\ A_{33} &= c_{11}\frac{\partial^2}{\partial z^2} + c_{66}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right), \quad A_{12} = (c_{11} - c_{66})\frac{\partial^2}{\partial x \partial y}\,, \end{aligned} \tag{5.3}$$

and

$$c_{11}c_{66}^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)^3\phi = c_{11}c_{66}^2(\nabla^2)^3\phi = 0\,, \tag{5.4}$$

in which

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\,. \tag{5.5}$$

Next, by means of (2.13) and (5.3) one obtains

$$\Gamma^1 = c_{66}(c_{11}-c_{66})\nabla^2\frac{\partial^2}{\partial y\partial z}\,, \quad \Gamma^2 = c_{66}(c_{11}-c_{66})\nabla^2\frac{\partial^2}{\partial x\partial z}\,,$$

$$\Gamma^3 = c_{66}(c_{11}-c_{66})\nabla^2\frac{\partial^2}{\partial x\partial y}\,, \tag{5.6}$$

and, by putting

$$c_{66}^2(c_{11}-c_{66})^2\nabla^4\frac{\partial^3\phi}{\partial x\partial y\partial z} = \varphi(x,\ y,\ z)/2G \equiv \varphi/2G\,, \tag{5.7}$$

in which G denotes shear modulus, one can reach a solution of the vector form

$$2G(u_1,\ v_1,\ w_1) = \left(\frac{\partial}{\partial x}\,,\ \frac{\partial}{\partial y}\,,\ \frac{\partial}{\partial z}\right)\varphi \equiv \operatorname{grad}\varphi\,, \tag{5.8}$$

in which

$$\nabla^2\varphi = 0\,.$$

Now it becomes needful to seek solutions of other kinds. Again, write the expressions for displacements as in (4.11).

$$u = A^1\phi\,, \quad v = A^2\phi\,, \quad w = A^3\phi\,, \tag{5.9}$$

in which the operators A^1, A^2 and A^3 are commutative or non-commutative as the case may require, that is, they cannot be assumed beforehand to be constructed with $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ only. Then, from equations (2.7) and formulae (5.3), (5.9), the equations of equilibrium can be written as follows:

$$\left\{c_{66}\nabla^2A^1+(c_{11}-c_{66})\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}A^1+\frac{\partial}{\partial y}A^2+\frac{\partial}{\partial z}A^3\right)\right\}\phi = 0\,, \tag{5.10a}$$

$$\left\{c_{66}\nabla^2A^2+(c_{11}-c_{66})\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}A^1+\frac{\partial}{\partial y}A^2+\frac{\partial}{\partial z}A^3\right)\right\}\phi = 0\,, \tag{5.10b}$$

$$\left\{c_{66}\nabla^2A^3+(c_{11}-c_{66})\frac{\partial}{\partial z}\left(\frac{\partial}{\partial x}A^1+\frac{\partial}{\partial y}A^2+\frac{\partial}{\partial z}A^3\right)\right\}\phi = 0\,. \tag{5.10c}$$

By means of formulae (5.6) and (5.9) the relation (2.12) become

$$\nabla^2\left(\frac{\partial}{\partial y}A^1-\frac{\partial}{\partial x}A^2\right)\phi = 0\,, \tag{5.11a}$$

$$\nabla^2\left(\frac{\partial}{\partial z}A^2-\frac{\partial}{\partial y}A^3\right)\phi = 0\,, \tag{5.11b}$$

$$\nabla^2 \left(\frac{\partial}{\partial z} A^1 - \frac{\partial}{\partial x} A^3 \right) \phi = 0 \,, \tag{5.11c}$$

and further, considering the general property of partial differential equations (2.17), and observing the reduced equation (5.4), it is easily seen that function ϕ, appearing in (5.9)-(5.11), is a harmonic one.

By inspecting the forms of the operators in the parentheses of equations (5.11), it readily occurs to one that the following expressions for displacements may serve the purpose:

$$u = \frac{\partial}{\partial x} A\phi \,, \quad v = \frac{\partial}{\partial y} A\phi \,, \quad w = \frac{\partial}{\partial z} A\phi \,, \tag{5.12}$$

or the expressions (5.9) with operators

$$A^1 = \frac{\partial}{\partial x} A \,, \quad A^2 = \frac{\partial}{\partial y} A \,, \quad A^3 = \frac{\partial}{\partial z} A \,. \tag{5.12'}$$

In fact equations (5.11) are satisfied by (5.12) or (5.12′) with any function ϕ and any operator A. By substituting the expressions for the operators (5.12′) in equations of equilibrium (5.10), these equations reduce to the following:

$$\begin{aligned} &c_{11}\nabla^2 \frac{\partial}{\partial x} A\cdot\phi = 0 \,, \\ &c_{11}\nabla^2 \frac{\partial}{\partial y} A\cdot\phi = 0 \,, \\ &c_{11}\nabla^2 \frac{\partial}{\partial z} A\cdot\phi = 0 \,. \end{aligned} \tag{5.13}$$

When A is taken to be constant, equations (5.13) are satisfied, since ϕ is a harmonic function, as mentioned above, and solutions (5.12) with constant A apparently correspond to those in (5.8).

Next, if operators A^i $(i=1, 2, 3)$, ∇^2, $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$ are commutative with one another, equations of equilibrium reduce to a single equation

$$\left(\frac{\partial}{\partial x} A^1 + \frac{\partial}{\partial y} A^2 + \frac{\partial}{\partial z} A^3 \right) \phi = 0 \,, \tag{5.14}$$

and the relations (5.11) are satisfied, for any operators A^i, since $\nabla^2\phi=0$. Also, from equation (5.14) it is easily seen that solution (5.8) can be achieved by taking A in (5.12′) to be constant. Further, by analogy with the case of transverse isotropy, A^3 can be taken to be zero and, putting the operator in parenthese of (5.14) equal to zero, one has

$$A^1 = \frac{\partial}{\partial y} B\,, \quad A^2 = -\frac{\partial}{\partial x} B\,, \quad A^3 = 0\,. \tag{5.15}$$

Using harmonic function ϑ_3/G in place of $B\phi$ appearing in (5.9), the following solution can be reached by means of (5.9) and (5.15).

$$G(u,\ v,\ w) = \left(\frac{\partial}{\partial y}\,,\ -\frac{\partial}{\partial x}\,,\ 0\right)\vartheta_3 \equiv \mathrm{rot}(0,\ 0,\ \vartheta_3)\,. \tag{5.16}$$

From reason of symmetry, by taking A^2 and A^1 to be zero in succession, one obtains solutions

$$\mathrm{rot}(0,\ \vartheta_2,\ 0),\quad \mathrm{rot}(\vartheta_1,\ 0,\ 0), \tag{5.17}$$

with harmonic functions ϑ_2 and ϑ_1 respectively, or, adding these, he has

$$\begin{aligned} G(u_2,\ v_2,\ w_2) &= \mathrm{rot}(\vartheta_1,\ \vartheta_2,\ \vartheta_3)\,, \\ &\equiv \mathrm{rot}\,\vartheta, \end{aligned} \tag{5.18}$$

in which

$$\nabla^2 \vartheta = 0\,.$$

Now let it be undertaken to obtain the third solution. By scrutinizing equations (5.13) and (5.10) and taking account of the relation

$$\nabla^2 (xf) = 2\frac{\partial f}{\partial x}\,, \quad (f \equiv f(x,\ y,\ z),\quad \nabla^2 f = 0) \tag{5.19}$$

one can reasonably suppose that

$$A = x\,, \tag{5.20}$$

and the insertion of (5.20) in the left-hand sides of equations of (5.13) yields the residuals $2c_{11}\frac{\partial^2\phi}{\partial x^2}$, $2c_{11}\frac{\partial^2\phi}{\partial x\partial y}$ and $2c_{11}\frac{\partial^2\phi}{\partial x\partial z}$, which can be evidently removed by the use of the constant operator A^1, which appears in parentheses of equations (5.10). Thus the removing of these residuals demands that

$$2c_{11} + (c_{11} - c_{66})\,A^1 = 0\,,$$

and hence it results that

$$A \equiv \bar{A}^1 = -\frac{4c_{11}}{c_{11}+c_{12}} = -4(1-\nu)\,, \tag{5.21}$$

in which ν denotes Poisson's ratio and $\nu = \dfrac{c_{12}}{c_{11}+c_{12}}$.
Accordingly, it is obtained that for operators A_i

$$A^1 = \frac{\partial}{\partial x}x + \bar{A}^1 = \frac{\partial}{\partial x}x - 4(1-\nu)$$
$$A^2 = \frac{\partial}{\partial y}x, \quad A^3 = \frac{\partial}{\partial z}x, \tag{5.22}$$

and for displacements by (5.9), using harmonic function $\lambda_1/2G$ in place of ϕ,

$$2G(u,\ v,\ w) = \mathrm{grad}\,(x\lambda_1) - 4(1-\nu)(\lambda_1,\ 0,\ 0). \tag{5.23}$$

And, furthermore, from symmetry reason one gets immediately

$$2G(u,\ v,\ w) = \mathrm{grad}\,(y\lambda_2) - 4(1-\nu)(0,\ \lambda_2,\ 0),$$
$$2G(u,\ v,\ w) = \mathrm{grad}\,(z\lambda_3) - 4(1-\nu)(0,\ 0,\ \lambda_3). \tag{5.24}$$

and, summing these up,

$$2G\boldsymbol{u}_3 = \mathrm{grad}\,(\boldsymbol{r}\cdot\boldsymbol{\lambda}) - 4(1-\nu)\boldsymbol{\lambda}, \tag{5.25}$$

in which

$$\boldsymbol{u}_3 = (u_3,\ v_3,\ w_3), \quad \boldsymbol{\lambda} = (\lambda_1,\ \lambda_2,\ \lambda_3),$$
$$\boldsymbol{r} = (x,\ y,\ z), \quad \nabla^2\boldsymbol{\lambda} = 0. \tag{5.26}$$

The above obtained solutions (5.8), (5.18), (5.25) will be referred to as basic solutions 1, 2 and 3 respectively, and they are, needless to say, basic solutions due to J. Boussinesq. Thus, it has been shown that, if some caution be exercised in utilizing the relations (2.11), it is possible to arrive at the correct solution.

Hereon the writer wishes to insist that the second basic solution (5.18) has been obtained by a reasonably general procedure of calculation and should be an indispensable one as the solution (5.1) is so for a transversely isotropic solid.

Although by the above derivation it is evident that basic solutions by J. Boussinesq are perfectly general, the writer will undertake to derive them in the manner customary in the theory of isotropic elasticity, and to extend some approaches. As mentioned above the case of the absence of volume force is being discussed. When investigation at some length is made of the methods of solution to the three-dimensional stress problems for a solid of finite extent and of the derivation of the basic solutions from the equations of equilibrium by the various methods of integration, it will be easily shown that methods of solutions analogous to the method of H. Neuber may be said to be not general enough, that is, H. Neuber's basic solutions are devoid of one important solution which corresponds to the second basic one due to J. Boussinesq (5. 18). Since the evidence of the marked effectiveness of the three-functions approaches is well known, caution should be exercised in applying H. Neuber's approach to the three-dimensional stress problems, if the above stated drawback of his approach or of similar ones is credible. It will be undertaken to show in a later section that, if the method of H. Neuber be applied without the use of some particular device, it will be impossible to construct the exact solutions to the problem of a short column of rectangular cross-section, to which surface tranctions are applied.

§6. On Some Three-Dimensional Approaches for an Isotropic Solid.*

First of all, let the basic solutions of J. Boussinesq be transformed by the use of H. Neuber's notations. Without violating generality of the solutions, φ and λ can manifestly be put in the following forms:

$$\varphi = -\Phi_0, \quad \lambda = -\bar{\Phi} = -(\Phi_1, \Phi_2, \Phi_3), \tag{6.1}$$

in which

$$\nabla^2(\Phi_0) = 0, \quad \nabla^2 \bar{\Phi} = 0.$$

By writing F as

$$F = \Phi_0 + \boldsymbol{r}\cdot\bar{\Phi}, \tag{6.2}$$

J. Boussinesq's solutions (5. 8), (5. 18), (5. 25) assume the forms

$$2G\boldsymbol{u} = -\operatorname{grad} F + 4(1-\nu)\bar{\Phi} + 2\operatorname{rot}\boldsymbol{\vartheta}, \tag{6.3}$$

in which

$$\boldsymbol{u} \equiv (u, v, w) = \boldsymbol{u}_1 + \boldsymbol{u}_2 + \boldsymbol{u}_3.$$
$$\boldsymbol{u}_1 \equiv (u_1, v_1, w_1), \quad \boldsymbol{u}_2 \equiv (u_2, v_2, w_2), \quad \boldsymbol{u}_3 \equiv (u_3, v_3, w_3).$$

Indeed, the method of solution (6. 3) is evidently J. Boussinesq's and might be said to be a variant of H. Neuber's method or the extended method of H. Neuber. Though solution (6. 3) is obtained formally from J. Boussinesq's, this process surely indicates the difference between these two approaches. The third term $2\operatorname{rot}\boldsymbol{\vartheta}$ in (6. 3), which is J. Boussinesq's basic solution 2, may be indispensable in solving the three-dimensional problems as shown later, but, of course, this third term can be dispensed with in the case of a moderately thick plate or torsion-free, axisymmetrical stress state.

Now, it will be shown briefly that the diverse kinds of integration of the following displacement-equations of equilibrium for an isotropic solid essentially yield the third term $2\,\mathrm{rot}\,\boldsymbol{\vartheta}$ appearing in (6. 3).

$$\nabla^2 \boldsymbol{u} = \frac{-1}{1-2\nu}\,\mathrm{grad}\ \mathrm{div}\,\boldsymbol{u}\,. \tag{6. 4}$$

From this vector equation one manifestly gets

$$\nabla^2\,\mathrm{div}\,\boldsymbol{u} = 0\,, \quad \nabla^2\nabla^2\,\boldsymbol{u} = 0\,. \tag{6. 5}$$

By considering equation (6. 5) and inspecting the form of equation (6. 4), P. F. Papkovitch[12] puts the solution in the form

$$\boldsymbol{u} = \boldsymbol{B} + \mathrm{grad}\,F\,, \tag{6. 6}$$

in which $\boldsymbol{B}$ is a harmonic vector and F denotes a biharmonic function. The author is of the opinion, however, that $\boldsymbol{B}$, which appears in equation (6. 6), is to be superseded by

$$\boldsymbol{B} + \frac{1}{G}\,\mathrm{rot}\,\boldsymbol{\vartheta}\,, \tag{6. 7}$$

since the following relation should be taken into account:

$$\mathrm{div}\cdot\mathrm{rot}\,\boldsymbol{A} = 0\,, \quad \boldsymbol{A} = (A_1,\ A_2,\ A_3)\,. \tag{6. 8}$$

Then the insertion of expression (6. 6) modified with (6. 7) in equation (6. 4) leads to

$$\nabla^2 F = \frac{-1}{2(1-\nu)}\,\mathrm{div}\,\boldsymbol{B}\,, \tag{6. 9}$$

and, by utilizing the relation (5. 19) in integrating this equation, one readily gets the solution.

$$\boldsymbol{u} = \frac{-1}{4(1-\nu)}\,\mathrm{grad}\,(\boldsymbol{r}\cdot\boldsymbol{B}+\varphi_0) + \boldsymbol{B} + \frac{1}{G}\,\mathrm{rot}\,\boldsymbol{\vartheta}\,. \quad (\nabla^2\varphi_0=0)\,. \tag{6. 10}$$

This solution, exclusive of the third term $\frac{1}{G}\,\mathrm{rot}\,\boldsymbol{\vartheta}$, is P. F. Papkovitch's basic solution, and, by rewriting $\boldsymbol{B}$ and φ_0 in the forms

$$\boldsymbol{B} = \frac{1}{G} 2(1-\nu)\bar{\boldsymbol{\Phi}}, \quad \varphi_0 = \frac{1}{G} 2(1-\nu)\Phi_0, \tag{6.11}$$

solution (6.3) is apparently obtained from (6.10).

Next, let the method of integration due to R. D. Mindlin be followed.[13] One can put generally for $\boldsymbol{u}$

$$\boldsymbol{u} = \operatorname{grad} \phi + \operatorname{rot} \boldsymbol{S}, \qquad (\operatorname{div} \boldsymbol{S} = 0), \tag{6.12}$$

and the substitution of this expression in the equation of equilibrium yields

$$\nabla^2 \left\{ \frac{2(1-\nu)}{1-2\nu} \operatorname{grad} \phi + \operatorname{rot} \boldsymbol{S} \right\} = 0. \tag{6.13}$$

Then, one can put from (6.13)

$$\frac{2(1-\nu)}{1-2\nu} \operatorname{grad} \phi + \operatorname{rot} \boldsymbol{S} = \frac{2(1-\nu)}{G} \bar{\boldsymbol{\Phi}} + \frac{1}{G} \operatorname{rot} \boldsymbol{\vartheta}, \tag{6.14}$$

and, when one performs the operator of divergence on both sides of this equation, he gets

$$\nabla^2 \phi = \frac{(1-2\nu)}{G} \operatorname{div} \bar{\boldsymbol{\Phi}}, \tag{6.15}$$

and, thus, it is obtained that

$$\phi = \frac{(1-2\nu)}{2G} (\boldsymbol{r} \cdot \bar{\boldsymbol{\Phi}} + \Phi_0), \qquad (\nabla^2 \Phi_0 = \nabla^2 \bar{\boldsymbol{\Phi}} = 0). \tag{6.16}$$

Accordingly, by (6.12), (6.14), (6.16), solution (6.3) or (6.10) can be reached. Prof. Mindlin disregards the second term $\frac{1}{G} \operatorname{rot} \boldsymbol{\vartheta}$ on the right-hand side of equation (6.14).

Again, from (6.13) the writer will derive the basic solution of B. Galerkin[14], and show that the correct solution relevant to the form of his basic solution also could not be destitute of the term, which corresponds to J. Boussinesq's basic solution 2.

From the condition in (6.12), namely,

$$\operatorname{div} \boldsymbol{S} = 0, \tag{6.17}$$

$\boldsymbol{S}$ may be taken to be of the form

$$\boldsymbol{S} = -2(1-\nu) \operatorname{rot} \boldsymbol{W} + \frac{1}{2G} \left\{ 2\boldsymbol{\vartheta} - \operatorname{grad} (\boldsymbol{r} \cdot \boldsymbol{\vartheta}) \right\}, \tag{6.18}$$

in which

$$\boldsymbol{W}=(X,\ Y,\ Z)\,,\quad \nabla^2\boldsymbol{\vartheta}=0\,,\quad (\nabla^2)^2\boldsymbol{W}=0\,.$$

Prof. Mindlin does not use the term within the brace in (6.18). By applying the operator of rotation to formula (6.18), one obtains

$$\text{rot}\,\boldsymbol{S}=-2(1-\nu)(\text{grad}\cdot\text{div}\,\boldsymbol{W}-\nabla^2\boldsymbol{W})+\frac{1}{G}\text{rot}\,\boldsymbol{\vartheta}\,,\qquad (6.19)$$

utilizing the relation

$$\text{rot}\cdot\text{grad}\,\varphi=0\,.\qquad (6.20)$$

Then, equation (6.13), substituted by the expression for rot $\boldsymbol{S}$ in (6.19), reduces to

$$\text{grad}\,\nabla^2\left\{\phi-(1-2\nu)\,\text{div}\,\boldsymbol{W}\right\}=0\,.\qquad (6.21)$$

From this equation one obtains, elastically,

$$\phi=(1-2\nu)\,\text{div}\,\boldsymbol{W}+\phi_0\,,\qquad (\nabla^2\phi_0=0)\,,\qquad (6.22)$$

in which ϕ_0 can be evidently dispensed with. Consequently, by formulae (6.12), (6.18), (6.22) one arrives at the solution of the vector form

$$\boldsymbol{u}=-\text{grad}\cdot\text{div}\,\boldsymbol{W}+2(1-\nu)\nabla^2\boldsymbol{W}+\frac{1}{G}\text{rot}\,\boldsymbol{\vartheta}\,,\qquad (6.23)$$

in which $(\nabla^2)^2\,\boldsymbol{W}=0\,.$

This solution, exclusive of the third right-hand term $\frac{1}{G}$ rot $\boldsymbol{\vartheta}$, is of the form due to B. Galerkin, and it is well known that B. Galerkin's basic solution is equivalent to H. Neuber's. Hence, it is verified also that B. Galerkin's solution should be equivalent to J. Boussinesq's.

Yet, it is to be noticed that the harmonic vector $\frac{1}{G}$ rot $\boldsymbol{\vartheta}$ can be merged in other appopriate harmonic vector formally, or the expressions (6.3), (6.10), (6.23) can be transformed into the original forms, but whether this harmonic vector 2 rot $\boldsymbol{\vartheta}$ or $\frac{1}{G}$ rot $\boldsymbol{\vartheta}$ is of profound importance or not will be another thing. Now, let solution (6.3) be considered and put as

$$4(1-\nu)\bar{\boldsymbol{\Phi}}+2\,\text{rot}\,\boldsymbol{\vartheta}=4(1-\nu)\bar{\boldsymbol{\Phi}}'\,,\quad \bar{\boldsymbol{\Phi}}'\equiv(\Phi_1',\ \Phi_2',\ \Phi_3')\,.\qquad (6.24)$$

By virtue of the relation

$$\nabla^2(\boldsymbol{r}\cdot\mathrm{rot}\,\boldsymbol{A})=0\,.\quad(\nabla^2\boldsymbol{A}=0)\,,\tag{6.25}$$

one can write

$$\Phi_0-\frac{1}{2(1-\nu)}\,\boldsymbol{r}\cdot\mathrm{rot}\,\boldsymbol{\vartheta}=\Phi_0'\,,\tag{6.26 a}$$

and put

$$F'=\Phi_0'+\boldsymbol{r}\cdot\bar{\boldsymbol{\Phi}}'\,.\tag{6.26 b}$$

Hence, solution (6.13) becomes reduced to the form

$$2G\boldsymbol{u}=-\mathrm{grad}\,F'+4(1-\nu)\,\bar{\boldsymbol{\Phi}}'\,,\tag{6.27}$$

and, of course, the dashes appearing in (6.27) can be effaced without loss of generality. It will be needless to discuss the case of the solution (6.10).

Next, for the case of the extended basic solution due to B. Galerkin (6.23), let a beginning be made by putting not as

$$\begin{aligned}\boldsymbol{S}=-2(1-\nu)\,\mathrm{rot}\,\boldsymbol{W}+\frac{1}{2G}\left\{2\,\boldsymbol{\vartheta}-\mathrm{grad}\,(\boldsymbol{r}\cdot\boldsymbol{\vartheta})\right\}=\\=-2(1-\nu)\,\mathrm{rot}\,\boldsymbol{W}'\,,\end{aligned}$$

but as

$$2(1-\nu)\,\nabla^2\boldsymbol{W}+\frac{1}{G}\mathrm{rot}\,\boldsymbol{\vartheta}=2(1-\nu)\,\nabla^2\boldsymbol{W}'\,,\tag{6.28}$$

in which

$$\boldsymbol{W}'=(X',\ Y',\ Z'),\quad(\nabla^2)^2\,\boldsymbol{W}'=0\,.$$

By integrating equation (6.28) elastically, one gets

$$\boldsymbol{W}=\boldsymbol{W}'-\frac{1}{2(1-\nu)G}\,[\boldsymbol{r}\cdot\boldsymbol{\vartheta}]\,,\tag{6.29}$$

in which $[\boldsymbol{r}\cdot\boldsymbol{\vartheta}]$ denotes vector product of $\boldsymbol{r}$ and $\boldsymbol{\vartheta}$.
From (6.23), (6.28), (6.29) it is found that

$$\begin{aligned}\boldsymbol{u}=-\mathrm{grad}\cdot\mathrm{div}\,\boldsymbol{W}'+2(1-\nu)\,\nabla^2\boldsymbol{W}'+\\+\mathrm{grad}\left\{\frac{1}{2(1-\nu)G}\,\mathrm{div}\,[\boldsymbol{r}\cdot\boldsymbol{\vartheta}]\right\}.\end{aligned}\tag{6.30}$$

If account is taken of the relation

$$\begin{aligned}\nabla^2\,\mathrm{div}\,[\boldsymbol{r}\cdot\boldsymbol{\vartheta}]=\nabla^2(\boldsymbol{\vartheta}\cdot\mathrm{rot}\,\boldsymbol{r}-\boldsymbol{r}\cdot\mathrm{rot}\,\boldsymbol{\vartheta})=\\=-\nabla^2(\boldsymbol{r}\cdot\mathrm{rot}\,\boldsymbol{\vartheta})=0\,,\quad\text{by (6.25)}\end{aligned}\tag{6.31}$$

it is easily seen that

$\frac{1}{2(1-\nu)G}\operatorname{div}[\boldsymbol{r}\cdot\boldsymbol{\vartheta}]$ can be merged in an appropiate part of $\operatorname{div}\boldsymbol{W}'$, which is a harmonic function, and hence one obtains the solution of the original form, effacing dashes.

Although these processes seem plausible, they might be merely equivalent to the neglect of the solution

$$2\operatorname{rot}\boldsymbol{\vartheta}\,,\quad \frac{1}{G}\operatorname{rot}\boldsymbol{\vartheta}\,,$$

or of the relation (6. 8), or of the relation formula

$$\operatorname{div}\left\{2\,\boldsymbol{\vartheta}-\operatorname{grad}(\boldsymbol{r}\cdot\boldsymbol{\vartheta})\right\}=0\,,\qquad (\nabla^2\boldsymbol{\vartheta}=0)\,. \tag{6. 32}$$

Nevertheless, the methods of solutions equivalent to H. Neuber's may be said to be completely general on condition that, if required, a basic harmonic function, consisting of biharmonic functions, should be employed, as is readily inferred from the fact that $\boldsymbol{r}\cdot\operatorname{rot}\boldsymbol{\vartheta}$, which appears in (6. 26a), and div $[\boldsymbol{r}\cdot\boldsymbol{\vartheta}]$, appearing in solution (6. 30), are certainly the functions which satisfy the above requirement, that is, they are harmonic functions which consist of biharmonic functions. And this remark will be elucidated a little later. Hereon it will be recalled that Profs. Sadowsky and Sternberg say that H. Neuber achieves complete symmetry of the three basic solutions at the expense of computational facility. When the method of the type of H. Neuber's is applied, one cannot but have recourse to such a cumbersome manipulation as to need particular harmonic functions stated above, and, of course, it is not customary to use such harmonic functions. Hence, it will be more advantageous to apply the methods analogous to J. Boussinesq's or the methods of the type of H. Neuber's extended by considering $2\operatorname{rot}\boldsymbol{\vartheta}$ or $\frac{1}{G}\operatorname{rot}\boldsymbol{\vartheta}$ additionally.

§ 7. Problems of Given Surface Tractions or Displacements for an Isotropic Solid.

In the first place it is desired to get the solutions to the problems of given surface tractions or displacements for a rectangular parallelepiped or a short column of rectangular cross-section without resorting to three-functions approaches described in the preceding section. Let the bounding surfaces of this rectangular parallelopiped or rectangular plate of sufficient thickness be given by

$$x=0\,,\ 2a\,;\quad y=0\,,\ 2b\,;\quad z=0\,,\ 2h\,, \tag{7.1}$$

and let the z-axis be vertical and positive upwards.
By taking into account that it will be necessary to expand functions, representing given surface tractions or displacements, into double Fourier series, and that functions, denoting stresses, must be biharmonic functions, and also by referring to equations of equilibrium (2.6), one can readily put for stresses the expressions as follows:
for normal stresses

$$\begin{aligned}\sigma_x = &\sum_n\sum_s \cos\beta_s y\,\cos k_n z\left\{\begin{bmatrix}A^1_{ns}\cosh\\ \bar{A}^1_{ns}\sinh\end{bmatrix} l_{ns}x + x\begin{bmatrix}B^1_{ns}\sinh\\ \bar{B}^1_{ns}\cosh\end{bmatrix} l_{ns}x\right\} +\\ &+\sum_n\sum_r \cos k_n z\,\cos\alpha_r x\left\{\begin{bmatrix}C^1_{nr}\cosh\\ \bar{C}^1_{nr}\sinh\end{bmatrix} m_{nr}y + y\begin{bmatrix}D^1_{nr}\sinh\\ \bar{D}^1_{nr}\cosh\end{bmatrix} m_{nr}y\right\} +\\ &+\sum_r\sum_s \cos\alpha_r x\,\cos\beta_s y\left\{\begin{bmatrix}E^1_{rs}\cosh\\ \bar{E}^1_{rs}\sinh\end{bmatrix}\gamma_{rs}z + z\begin{bmatrix}F^1_{rs}\sinh\\ \bar{F}^1_{rs}\cosh\end{bmatrix}\gamma_{rs}z\right\}, \end{aligned}\tag{7.2a}$$

$$\begin{aligned}\sigma_y = &\sum_n\sum_s \cos\beta_s y\,\cos k_n z\left\{\begin{bmatrix}A^2_{ns}\cosh\\ \bar{A}^2_{ns}\sinh\end{bmatrix} l_{ns}x + x\begin{bmatrix}B^2_{ns}\sinh\\ \bar{B}^2_{ns}\cosh\end{bmatrix} l_{ns}x\right\} +\\ &+\sum_n\sum_r \cos k_n z\,\cos\alpha_r x\left\{\begin{bmatrix}C^2_{nr}\cosh\\ \bar{C}^2_{nr}\sinh\end{bmatrix} m_{nr}y + y\begin{bmatrix}D^2_{nr}\sinh\\ \bar{D}^2_{nr}\cosh\end{bmatrix} m_{nr}y\right\} +\\ &+\sum_r\sum_s \cos\alpha_r x\,\cos\beta_s y\left\{\begin{bmatrix}E^2_{rs}\cosh\\ \bar{E}^2_{rs}\sinh\end{bmatrix}\gamma_{rs}z + z\begin{bmatrix}F^2_{rs}\sinh\\ \bar{F}^2_{rs}\cosh\end{bmatrix}\gamma_{rs}z\right\}, \end{aligned}\tag{7.2b}$$

$$\begin{aligned}\sigma_z = &\sum_n\sum_s \cos\beta_s y\,\cos k_n z\left\{\begin{bmatrix}A^3_{ns}\cosh\\ \bar{A}^3_{ns}\sinh\end{bmatrix} l_{ns}x + x\begin{bmatrix}B^3_{ns}\sinh\\ \bar{B}^3_{ns}\cosh\end{bmatrix} l_{ns}x\right\} +\\ &+\sum_n\sum_r \cos k_n z\,\cos\alpha_r x\left\{\begin{bmatrix}C^3_{nr}\cosh\\ \bar{C}^3_{nr}\sinh\end{bmatrix} m_{nr}y + y\begin{bmatrix}D^3_{nr}\sinh\\ \bar{D}^3_{nr}\cosh\end{bmatrix} m_{nr}y\right\} +\\ &+\sum_r\sum_s \cos\alpha_r x\,\cos\beta_s y\left\{\begin{bmatrix}E^3_{rs}\cosh\\ \bar{E}^3_{rs}\sinh\end{bmatrix}\gamma_{rs}z + z\begin{bmatrix}F^3_{rs}\sinh\\ \bar{F}^3_{rs}\cosh\end{bmatrix}\gamma_{rs}z\right\}, \end{aligned}\tag{7.2c}$$

for tangential stresses

$$\tau_{yz} = \sum_n\sum_s \sin\beta_s y \sin k_n z \left\{ \begin{bmatrix} G^1_{ns}\cosh \\ \bar{G}^1_{ns}\sinh \end{bmatrix} l_{ns}x + x \begin{bmatrix} H^1_{ns}\sinh \\ \bar{H}^1_{ns}\cosh \end{bmatrix} l_{ns}x \right\} +$$

$$+ \sum_n\sum_r \sin k_n z \cos\alpha_r x \left\{ \begin{bmatrix} I^1_{nr}\sinh \\ \bar{I}^1_{nr}\cosh \end{bmatrix} m_{nr}y + y \begin{bmatrix} J^1_{nr}\cosh \\ \bar{J}^1_{nr}\sinh \end{bmatrix} m_{nr}y \right\} +$$

$$+ \sum_r\sum_s \cos\alpha_r x \sin\beta_s y \left\{ \begin{bmatrix} K^1_{rs}\sinh \\ \bar{K}^1_{rs}\cosh \end{bmatrix} \gamma_{rs}z + z \begin{bmatrix} L^1_{rs}\cosh \\ \bar{L}^1_{rs}\sinh \end{bmatrix} \gamma_{rs}z \right\}, \qquad (7.3\text{a})$$

$$\tau_{xz} = \sum_n\sum_s \cos\beta_s y \sin k_n z \left\{ \begin{bmatrix} G^2_{ns}\sinh \\ \bar{G}^2_{ns}\cosh \end{bmatrix} l_{ns}x + x \begin{bmatrix} H^2_{ns}\cosh \\ \bar{H}^2_{ns}\sinh \end{bmatrix} l_{ns}x \right\} +$$

$$+ \sum_n\sum_r \sin k_n z \sin\alpha_r x \left\{ \begin{bmatrix} I^2_{nr}\cosh \\ \bar{I}^2_{nr}\sinh \end{bmatrix} m_{nr}y + y \begin{bmatrix} J^2_{nr}\sinh \\ \bar{J}^2_{nr}\cosh \end{bmatrix} m_{nr}y \right\} +$$

$$+ \sum_r\sum_s \sin\alpha_r x \cos\beta_s y \left\{ \begin{bmatrix} K^2_{rs}\sinh \\ \bar{K}^2_{rs}\cosh \end{bmatrix} \gamma_{rs}z + z \begin{bmatrix} L^2_{rs}\cosh \\ \bar{L}^2_{rs}\sinh \end{bmatrix} \gamma_{rs}z \right\}, \qquad (7.3\text{b})$$

$$\tau_{xy} = \sum_n\sum_s \sin\beta_s y \cos k_n z \left\{ \begin{bmatrix} G^3_{ns}\sinh \\ \bar{G}^3_{ns}\cosh \end{bmatrix} l_{ns}x + x \begin{bmatrix} H^3_{ns}\cosh \\ \bar{H}^3_{ns}\sinh \end{bmatrix} l_{ns}x \right\} +$$

$$+ \sum_n\sum_r \cos k_n z \sin\alpha_r x \left\{ \begin{bmatrix} I^3_{nr}\sinh \\ \bar{I}^3_{nr}\cosh \end{bmatrix} m_{nr}y + y \begin{bmatrix} J^3_{nr}\cosh \\ \bar{J}^3_{nr}\sinh \end{bmatrix} m_{nr}y \right\} +$$

$$+ \sum_r\sum_s \sin\alpha_r x \sin\beta_s y \left\{ \begin{bmatrix} K^3_{rs}\cosh \\ \bar{K}^3_{rs}\sinh \end{bmatrix} \gamma_{rs}z + z \begin{bmatrix} L^3_{rs}\sinh \\ \bar{L}^3_{rs}\cosh \end{bmatrix} \gamma_{rs}z \right\}, \qquad (7.3\text{c})$$

In these expressions, for instance, the formula

$$A^1_{ns}\cosh l_{ns}x + \bar{A}^1_{ns}\sinh l_{ns}x\,,$$

is denoted by the symbol

$$\begin{bmatrix} A^1_{ns}\cosh \\ \bar{A}^1_{ns}\sinh \end{bmatrix} l_{ns}x\,. \qquad (7.4)$$

In order that the expressions for stresses in (7.2), (7.3) may be biharmonic functions, the following relations are required:

$$l^2_{ns} = k^2_n + \beta^2_s\,, \quad m^2_{nr} = k^2_n + \alpha^2_r\,, \quad \gamma^2_{rs} = \alpha^2_r + \beta^2_s\,, \qquad (7.5)$$

in which

$$\alpha_r = \frac{r\pi}{2a}\,, \quad \beta_s = \frac{s\pi}{2b}\,, \quad k_n = \frac{n\pi}{2h}\,. \qquad (r,\ s,\ n = 0,\ 1,\ 2,\ 3\cdots)$$

Clearly, when the expressions for surface tractions or displacements are symmetrical about the plane $z=h$, such alteration may be made as to let the origin of the coordinate system be transferred to the point ($x=0$, $y=0$, $z=h$) and the coefficients $\bar{E}^{\nu}_{rs}$, $\bar{F}^{\nu}_{rs}$, $\bar{K}^{\nu}_{rs}$, $\bar{L}^{\nu}_{rs}$ with any affixes be deleted and to replace $k_n=\frac{n\pi}{2h}$ by $k_n=\frac{n\pi}{h}$. Of course, there is no need of the barred coefficients, if these expressions are symmetrical about the three planes $x=a$, $y=b$ and $z=h$. Now, since the linear independence of the above expressions with double summation symbols $\sum_n\sum_s$, $\sum_n\sum_r$ and $\sum_r\sum_s$ in (7.2), (7.3) is apparent, it will herein suffice to treat only the expressions with symbol $\sum_n\sum_s$, excluding coefficients with a bar, for instance. To avoid ambiguity the expressions or solutions with symbols $\sum_n\sum_s$, $\sum_n\sum_r$ and $\sum_r\sum_s$ shall be referred to as solutions 1, 2 and 3 respectively.

Let the equations be sought, relating coefficients with specific summation indexes n, s to one another, by means of equations of compatibility of the forms

$$\nabla^2\sigma_x + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial x^2} = 0\,, \tag{7.6a}$$

$$\nabla^2\sigma_y + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial y^2} = 0\,, \tag{7.6b}$$

$$\nabla^2\sigma_z + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial z^2} = 0\,, \tag{7.6c}$$

$$\nabla^2\tau_{yz} + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial y\partial z} = 0\,, \tag{7.7a}$$

$$\nabla^2\tau_{xz} + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial x\partial z} = 0\,, \tag{7.7b}$$

$$\nabla^2\tau_{xy} + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial x\partial y} = 0\,, \tag{7.7c}$$

in which

$$\Theta = \sigma_x + \sigma_y + \sigma_z\,, \tag{7.8}$$

Then, if solution 1 for σ_z in (7.2c) be inserted in equation (7.6c) and the resulting equation be integrated with respect to coordinate z, one obtains elastically

$$\Theta = (1+\nu)\sum_n\sum_s \frac{2l_{ns}}{k_n^2} B_{ns}^3 \cos k_n z \cos \beta_s y \cosh l_{ns} x . \tag{7.9}$$

Next, the substitution of the expressions for stresses of (7.2), (7.3) and of the expression for Θ (7.9) in equations of compatibility, except equation (7.6c), leads to the relations

$$B_{ns}^1 = \frac{-l_{ns}^2}{k_n^2} B_{ns}^3, \quad B_{ns}^2 = \frac{\beta_s^2}{k_n^2} B_{ns}^3,$$
$$H_{ns}^1 = \frac{-\beta_s}{k_n} B_{ns}^3, \quad H_{ns}^2 = \frac{l_{ns}}{k_n} B_{ns}^3, \quad H_{ns}^3 = \frac{\beta_s l_{ns}}{k_n^2} B_{ns}^3,$$
$$\text{for any } n \text{ and } s. \tag{7.10}$$

Equations of equilibrium (2.6), substituted by the solutions 1 for stresses, gives the relations

$$\begin{aligned} l_{ns} A_{ns}^1 + \beta_s G_{ns}^3 + k_n G_{ns}^2 + B_{ns}^1 &= 0, \\ l_{ns} G_{ns}^3 - \beta_s A_{ns}^2 + k_n G_{ns}^1 + H_{ns}^3 &= 0, \\ l_{ns} G_{ns}^2 + \beta_s G_{ns}^1 - k_n A_{ns}^3 + H_{ns}^2 &= 0. \end{aligned} \tag{7.11}$$

If formula (7.8) be taken into cosideration, it follows that

$$A_{ns}^1 + A_{ns}^2 + A_{ns}^3 = \frac{2(1+\nu)}{k_n^2} l_{ns} B_{ns}^3, \quad \text{for any } n \text{ and } s. \tag{7.12}$$

Thus, all needed relations are obtained. Next, let these nine algebraic equations (7.10), (7.11), (7.12) be solved for nine coefficients (B_{ns}^1, B_{ns}^2, B_{ns}^3), (G_{ns}^1, G_{ns}^2, G_{ns}^3), (H_{ns}^1, H_{ns}^2, H_{ns}^3), obtaining

$$\begin{aligned} B_{ns}^1 &= \frac{-1}{2(1+\nu)} l_{ns} A_{ns}^0, \quad B_{ns}^2 = \frac{1}{2(1+\nu)} \frac{\beta_s^2}{l_{ns}} A_{ns}^0, \quad B_{ns}^3 = \frac{1}{2(1+\nu)} \frac{k_n^2}{l_{ns}} A_{ns}^0, \\ G_{ns}^1 &= \frac{1}{2(1+\nu)k_n\beta_s}\left\{\nu l_{ns}^2 A_{ns}^1 + \left((1+\nu)\beta_s^2 - l_{ns}^2\right) A_{ns}^2 + (\nu k_n^2 - \beta_s^2) A_{ns}^3\right\}, \\ G_{ns}^2 &= \frac{1}{2(1+\nu)l_{ns}k_{ns}}\left\{\left(\beta_s^2 - (1+\nu) l_{ns}^2\right) A_{ns}^1 - \nu\beta_s^2 A_{ns}^2 + (\nu k_n^2 + l_{ns}^2) A_{ns}^3\right\}, \\ G_{ns}^3 &= \frac{1}{2(1+\nu)l_{ns}\beta_s}\left\{-(\beta_s^2 + \nu l_{ns}^2) A_{ns}^1 + (\nu\beta_s^2 + l_{ns}^2) A_{ns}^2 - \nu k_n^2 A_{ns}^3\right\}, \\ H_{ns}^1 &= \frac{-1}{2(1+\nu)} \frac{k_n\beta_s}{l_{ns}} A_{ns}^0, \\ H_{ns}^2 &= \frac{1}{2(1+\nu)} k_n A_{ns}^0, \quad H_{ns}^3 = \frac{1}{2(1+\nu)} \beta_s A_{ns}^0, \end{aligned} \tag{7.13}$$

in which

$$A^0_{ns} = A^1_{ns} + A^2_{ns} + A^3_{ns}.$$

It will be obvious that one can get immediately the desired formulae for coefficients $(\bar{B}^1_{ns}, \bar{B}^2_{ns}, \bar{B}^3_{ns})$, $(\bar{H}^1_{ns}, \bar{H}^2_{ns}, \bar{H}^3_{ns})$, $(\bar{G}^1_{ns}, \bar{G}^2_{ns}, \bar{G}^3_{ns})$, by attching a bar over coefficients to be determined from formulae (7. 13). Accordingly, for solutions 1, or for solutions of type 1, there exist six independent sequences of coefficients $(A^1_{ns}, A^2_{ns}, A^3_{ns})$, $(\bar{A}^1_{ns}, \bar{A}^2_{ns}, \bar{A}^3_{ns})$, and hence there are eighteen sequences of coefficients in all for the solutions to the problem considered. This fact is, doubtless, the necessary and sufficient condition for the validity of the solutions of the forms indicated in (7. 2) and (7. 3). In this way general solutions have been obtained, appropriate to the problems of given surface tractions or displacements, without applying three-functions approaches.

Incidentally, there will be given here the solutions of type 1 for displacements, which are to be obtained by integrating stress-strain relations.

$$u = \frac{1}{2E}\sum_n\sum_s \cos\beta_s y \cos k_n z\left[\frac{1}{l_{ns}}\left\{\begin{matrix}\{3A^0_{ns}+(1-2\nu)(A^2_{ns}+A^3_{ns})\}\sinh \\ \{3\bar{A}^0_{ns}+(1-2\nu)(\bar{A}^2_{ns}+\bar{A}^3_{ns})\}\cosh\end{matrix}\right. l_{ns}x + \right.$$
$$\left. - x\begin{bmatrix}A^0_{ns}\cosh \\ \bar{A}^0_{ns}\sinh\end{bmatrix} l_{ns}x\right], \qquad (7.\ 14\text{a})$$

$$v = \frac{1}{E}\sum_n\sum_s \sin\beta_s y \cos k_n z\left[\frac{1}{\beta_s}\left\{\begin{matrix}\{A^2_{ns}-\nu(A^1_{ns}+A^3_{ns})\}\cosh \\ \{\bar{A}^2_{ns}-\nu(\bar{A}^1_{ns}+\bar{A}^3_{ns})\}\sinh\end{matrix}\right. l_{ns}x + \right.$$
$$\left. + \frac{1}{2}\frac{\beta_s}{l_{ns}}x\begin{bmatrix}A^0_{ns}\sinh \\ \bar{A}^0_{ns}\cosh\end{bmatrix} l_{ns}x\right], \qquad (7.\ 14\text{b})$$

$$w = \frac{1}{E}\sum_n\sum_s \cos\beta_s y \sin k_n z\left[\frac{1}{k_n}\left\{\begin{matrix}\{A^3_{ns}-\nu(A^1_{ns}+A^2_{ns})\}\cosh \\ \{\bar{A}^3_{ns}-\nu(\bar{A}^1_{ns}+\bar{A}^2_{ns})\}\sinh\end{matrix}\right. l_{ns}x + \right.$$
$$\left. + \frac{1}{2}\frac{k_n}{l_{ns}}x\begin{bmatrix}A^0_{ns}\sinh \\ \bar{A}^0_{ns}\cosh\end{bmatrix} l_{ns}x\right]. \qquad (7.\ 14\text{c})$$

In the next place, application will be made, for instance, of the method of H. Neuber or extended H. Neuber's method (6. 3) to the problem under consideration. Thereby it will be shown that the third right side term $2\,\text{rot}\,\vartheta$ in (6. 3) or $\frac{1}{G}$ rot ϑ in (6. 10), (6. 23) is of great moment and, in case this term could not be used, one would be compelled to resort to some particular mathematical manipulation. Now it will be easily seen that the solutions similar to those of type 1, containing three sequences of coefficients $\{A^1_{ns}\}$, $\{A^2_{ns}\}$ and $\{A^3_{ns}\}$ only in (7.2) and (7.3), are obtainable from the basic harmonic functions of the forms

$$\begin{aligned}
\Phi_0 &= \sum_n \sum_s A'_{ns} \cos\beta_s y \cos k_n z \cosh l_{ns} x\,, \\
\bar{\Phi} &= (\Phi_1,\ 0,\ 0)\,, \\
&\quad \Phi_1 = \sum_n \sum_s C'_{ns} \cos\beta_s y \cos k_n z \sinh l_{ns} x\,, \\
\vartheta &= (\vartheta_1,\ 0,\ 0)\,, \\
&\quad \vartheta_1 = \sum_n \sum_s B'_{ns} \sin\beta_s y \sin k_n z \cosh l_{ns} x\,.
\end{aligned} \qquad (7.15)$$

By substituting these harmonic functions in formula (6.3) and comparing the resulting formulae with the solutions (7.14), one finds the following relations:

$$\begin{aligned}
A^1_{ns} &= -l^2_{ns} A'_{ns} + 2(1-\nu) l_{ns} C'_{ns}\,, \\
A^2_{ns} &= \beta^2_s A'_{ns} + 2k_n \beta_s B'_{ns} + 2\nu l_{ns} C'_{ns}\,, \\
A^3_{ns} &= k^2_n A'_{ns} - 2k_n \beta_s B'_{ns} + 2\nu l_{ns} C'_{ns}\,, \\
A^0_{ns} &= 2(1+\nu) l_{ns} C'_{ns}\,.
\end{aligned} \qquad (7.16)$$

Hence, when one employs A'_{ns}, B'_{ns} and C'_{ns}, expressed in terms of A^1_{ns}, A^2_{ns} and A^3_{ns} in accordance with the relations (7.16), in the basic functions (7.15), solutions will be achieved of precisely the same forms as (7.14) or (7.2) and (7.3) with (7.13). It appears apparent, however, that it seems well-nigh impossible to construct the solutions to the problem concerned, if it is not permitted to make use of the third term $2\,\mathrm{rot}\,\vartheta$ in (6.3). Therefore, as stated earlier, it may be said that the term $2\,\mathrm{rot}\,\vartheta$ or $\dfrac{1}{G}\,\mathrm{rot}\,\vartheta$ is assuredly an indispensable one.

Nevertheless, if the following mathematical manipulation is performed, the purpose can be attained without applying the third term $2\,\mathrm{rot}\,\vartheta$ in (6.3) as stated before. Also in this case let the solutions of type 1 be considered, corresponding to the solutions (7.14), in which barred coefficients are deleted. First, to cite, for instance, the relations

$$\nabla^2(\boldsymbol{r}\cdot\mathrm{grad}\,f) = 0\,. \qquad (7.17\,\mathrm{a})$$

$$\nabla^2(\boldsymbol{r}\cdot\mathrm{rot}\,\boldsymbol{H}) = 0\,, \qquad (7.17\,\mathrm{b})$$

in which

$$\nabla^2 f = \nabla^2 \boldsymbol{H} = 0, \qquad \left(\boldsymbol{H} \equiv (H_1,\ H_2,\ H_3)\right)$$

$\boldsymbol{r}\cdot\mathrm{grad}\, f$ or $\boldsymbol{r}\cdot\mathrm{rot}\, \boldsymbol{H}$ is evidently a harmonic function which consists of biharmonic functions.
One may put in this case

$$f = \sum_n \sum_s V_{ns} \cos k_n z \cos \beta_s y \cosh l_{ns} x, \tag{7.18 a}$$

$$\boldsymbol{H} = (0,\ 0,\ H_3),$$

$$H_3 = \sum_n \sum_s V'_{ns} \sin \beta_s y \cos k_n z \sinh l_{ns} x. \tag{7.18 b}$$

If the expression for f in (7.18a) be taken, and if reference be made to the relations (7.17a), it will be pertinent to write functions Φ_i ($i=0$, 1, 2, 3), which are H. Neuber's basic harmonic functions, in the following forms:

$$\begin{aligned}
\Phi_0 &= \Phi_0^1 + \Phi_0^2, \\
\Phi_0^1 &= \sum_n \sum_s A'_{ns} \cos \beta_s y \cos k_n z \cosh l_{ns} x, \\
\Phi_0^2 &= \boldsymbol{r}\cdot\mathrm{grad}\, f \\
&= \sum_n \sum_s V_{ns} (l_{ns} x \cos \beta_s y \cos k_n z \sinh l_{ns} x + \\
&\quad - \beta_s y \sin \beta_s y \cos k_n z \cosh l_{ns} x + \\
&\quad - k_n z \cos \beta_s y \sin k_n z \cosh l_{ns} x),
\end{aligned} \tag{7.19 a}$$

$$\Phi_1 = \sum_n \sum_s C'_{ns} \cos \beta_s y \cos k_n z \sinh l_{ns} x, \tag{7.19 b}$$

$$\Phi_2 = \sum_n \sum_s D'_{ns} \sin \beta_s y \cos k_n z \cosh l_{ns} x, \tag{7.19 c}$$

$$\Phi_3 = \sum_n \sum_s E'_{ns} \cos \beta_s y \sin k_n z \cosh l_{ns} x. \tag{7.19 d}$$

Then, when the expression for $\boldsymbol{H}$ (7.18b) is taken, functions Φ_i are to be of the forms

$$\begin{aligned}
\Phi_0 &= \Phi_0^1 + \Phi_0^2, \\
\Phi_0^1 &= \sum_n \sum_s A'_{ns} \cos \beta_s y \cos k_n z \cosh l_{ns} x, \\
\Phi_0^2 &= \boldsymbol{r}\cdot\mathrm{rot}\,(0,\ 0,\ H_3) = \left(x \frac{\partial H_3}{\partial y} - y \frac{\partial H_3}{\partial x}\right) \\
&= \sum_n \sum_s V'_{ns} (\beta_s x \cos \beta_s y \cos k_n z \sinh l_{ns} x + \\
&\quad - l_{ns} y \sin \beta_s y \cos k_n z \cosh l_{ns} x),
\end{aligned} \tag{7.20 a}$$

$$\Phi_1 = \sum_n \sum_s C'_{ns} \cos \beta_s y \cos k_n z \sinh l_{ns} x, \tag{7.20 b}$$

$$\Phi_2 = \sum_n \sum_s D'_{ns} \sin\beta_s y \cos k_n z \cosh l_{ns} x\,, \qquad (7.20\,c)$$

$$\Phi_3 = 0\,, \qquad (7.20\,d)$$

When the expressions (7.19) are inserted in formula for u in the vector formula, of (6.3), exclusive of the term $2\,\mathrm{rot}\,\vartheta$, and both coefficients of the biharmonic functions $y\sin\beta_s y\cos k_n z\sinh l_{ns}x$ and $z\cos\beta_s y\sin k_n z \times \times \sinh l_{ns}x$ are put equal to zero, there are obtained

$$D'_{ns} = \beta_s V_{ns}\,, \quad E'_{ns} = k_n V_{ns}\,, \qquad (7.21\,a)$$

and

$$u = \frac{1}{2G}\left\{-l_{ns}(A'_{ns} + V_{ns}) + (3-4\nu)\,C'_{ns}\right\}\cos\beta_s y\cos k_n z\sinh l_{ns}x + \\ + \frac{-1}{2G} l_{ns}(l_{ns}V_{ns} + C'_{ns})\,x\cos\beta_s y\cos k_n z\cosh l_{ns}x\,. \qquad (7.21\,b)$$

By equating similar terms in the expressions (7.21b) and (7.14a), the relations are obtained

$$\begin{aligned} A^1_{ns} &= l_{ns}\left\{-l_{ns}\left(A'_{ns} + 2(1-\nu)V_{ns}\right) + 2(1-\nu)C'_{ns}\right\}, \\ A^0_{ns} &= A^1_{ns} + A^2_{ns} + A^3_{ns} = 2(1+\nu)\,l_{ns}(l_{ns}V_{ns} + C'_{ns})\,, \end{aligned} \qquad (7.22)$$

and similarly, from the expressions for v and w in (7.14) and those which are obtained by means of H. Neuber's method, using functions Φ_i (7.19), it is found that

$$\begin{aligned} A^2_{ns} &= \beta_s^2 A'_{ns} + 2\left\{2(1-\nu)\beta_s^2 + \nu l_{ns}^2\right\}V_{ns} + 2\nu l_{ns}C'_{ns}\,, \\ A^3_{ns} &= k_n^2 A'_{ns} + 2\left\{2(1-\nu)k_n^2 + \nu l_{ns}^2\right\}V_{ns} + 2\nu l_{ns}C'_{ns}\,. \end{aligned} \qquad (7.23)$$

It will be obvious that also by the use of the functions Φ_i in (7.20) the same object can be attained, or relations similar to those in (7.22) and (7.23) can be gotten.

Thus, it has been shown that by utilizing the above harmonic functions Φ_0^2, which are constructed with biharmonic functions, one can achieve, applying the method of H. Neuber, the exact three-dimensional solutions which are to be obtained without employing three-functions approaches.

As a consequence, once it is known that it is possible to use particular mathematical manipulation as described just above, it would be too much to say that methods of solution equivalent to H. Neuber's, destitute of the term $2\,\mathrm{rot}\,\vartheta$, are incomplete. But in any event it will be advisable to modify H. Neuber's method of solution so as to contain the term $2\,\mathrm{rot}\,\vartheta$, because the integration of equations of equilibrium yields this term essentially, and the modified method or J. Boussinesq's method is expedient to construct the solutions, involving all needed sequences of coefficients, if the method of series be applied.

§ 8. Solutions to the Problems of Given Surface Tractions for an Anisotropic Elastic Solid.

In this section also a rectangular parallelepiped will be considered. In the first place let the bounding surfaces of the solid concerned be given by

$$x=\pm a, \quad y=\pm b, \quad z=\pm h, \tag{8.1}$$

and let the coordinate z be positive upwards. For simplicity let it be supposed that the expressions for given surface tractions or displacements are symmetrical about the planes $x=0$, $y=0$ and $z=0$, and, further, that in the first place the solid is orthotropic and, of course, the axes of elastic symmetry are parallel to the coordinate axes. When the basic differential equation for ϕ in (3.12) can be resolved into three differential equations of the second order with real coefficients as equations in (3.15), the solutions analogous to those in (7.2), (7.3) could be found as easily as in the case of isotropy. However, when this is not the case, it becomes necessary to proceed to solve the problems under consideration in the following manner.

Firstly let stress function ϕ be written in the form.

$$\phi = A\exp(\beta_s iy + k_n iz + \delta x), \tag{8.2}$$

in which A is constant or coefficient and i denotes imaginary unit, and δ is a constant to be determined from differential equation (3.12)

$$k_n = \frac{n\pi}{h}, \quad \beta_s = \frac{s\pi}{b}. \qquad (n,\ s=0,\ 1,\ 2\cdots)$$

Then, by substituting exponential function ϕ of (8.2) in the basic differential equation (3.12) and solving the resulting algebraic equation for δ, one gets the solutions of the following forms, as examples, for the case of topaz or barytes.

$$_1\delta_{ns} = \pm\,{}_1l_{ns}, \quad {}_2\delta_{ns} = \pm\,({}_2l_{ns} + i\,{}_2\lambda_{ns}),$$
$${}_3\delta_{ns} = \pm\,({}_3l_{ns} + i\,{}_3\lambda_{ns}), \tag{8.3}$$

in which ${}_{\nu}l_{ns}$ and ${}_{\nu}\lambda_{ns}$ $(\nu=1,\ 2,\ 3)$ are real constants.
Accordingly, function ϕ can be written in the forms

$$\phi_{1(ns)} = \sum_n\sum_s A^1_{ns}\cos\beta_s y\cos k_n z\cosh{}_1l_{ns}x\,,$$
$$\phi_{2(ns)} = \sum_n\sum_s A^2_{ns}\cos\beta_s y\cos k_n z\cos{}_2\lambda_{ns}x\cosh{}_2l_{ns}x\,,$$
$$\phi_{3(ns)} = \sum_n\sum_s A^3_{ns}\cos\beta_s y\cos k_n z\cos{}_3\lambda_{ns}x\cosh{}_3l_{ns}x\,,$$

and

$$\phi_{(ns)} = \phi_{1(ns)} + \phi_{2(ns)} + \phi_{3(ns)}\,. \tag{8.4}$$

The solution to be derived from this function $\phi_{(ns)}$ may be referred to as solutions of type 1 by analogy with the solutions 1 in (7.2) and (7.3).

Similarly on may write

$$\phi = A\exp(\alpha_r ix + k_n iz + \delta y)\,, \tag{8.5}$$

in which

$$\alpha_r = \frac{r\pi}{a}. \qquad (r=0,\ 1,\ 2\cdots)\,.$$

The solutions of equation (3.12) for δ, in which the expression for ϕ (8.5) is substituted, are of the forms:

$$ {}_1\delta_{nr} = \pm\,{}_1m_{nr}\,,\quad {}_2\delta_{nr} = \pm\,({}_2m_{nr}+i\,{}_2\mu_{nr})\,,$$
$$ {}_3\delta_{nr} = \pm\,({}_3m_{nr}+i\,{}_3\mu_{nr})\,. \tag{8.6}$$

Hence, one gets for $\phi_{\nu(nr)}$

$$\phi_{1(nr)} = \sum_n\sum_r B^1_{nr}\cos\alpha_r x\cos k_n z\cosh{}_1m_{nr}y\,,$$
$$\phi_{2(nr)} = \sum_n\sum_r B^2_{nr}\cos\alpha_r x\cos k_n z\cos{}_2\mu_{nr}\cosh{}_2m_{nr}y\,,$$
$$\phi_{3(nr)} = \sum_n\sum_r B^3_{nr}\cos\alpha_r x\cos k_n z\cos{}_3\mu_{nr}\cosh{}_3m_{nr}y\,,$$

and

$$\phi_{(nr)} = \phi_{1(nr)} + \phi_{2(nr)} + \phi_{3(nr)}\,. \tag{8.7}$$

And in like manner the roots in δ of equations (3.12), into which is inserted

$$\phi = A\exp(\alpha_r ix+\beta_s iy+\delta z)\,, \tag{8.8}$$

are of the forms

$$ {}_1\delta_{rs} = \pm\,{}_1\gamma_{rs}\,,\quad {}_2\delta_{rs} = \pm\,({}_2\gamma_{rs}+i\,{}_2\nu_{rs})\,,$$
$$ {}_3\delta_{rs} = \pm\,({}_3\gamma_{rs}+i\,{}_3\nu_{rs})\,. \tag{8.9}$$

Thus, there are obtained for $\phi_{\nu(rs)}$

$$\phi_{1(rs)} = \sum_r \sum_s C^1_{rs} \cos \alpha_r x \cos \beta_s y \cosh {}_1\gamma_{rs} z ,$$

$$\phi_{2(rs)} = \sum_r \sum_s C^2_{rs} \cos \alpha_r x \cos \beta_s y \cos {}_2\nu_{rs} z \cosh {}_2\gamma_{rs} z ,$$

$$\phi_{3(rs)} = \sum_r \sum_s C^3_{rs} \cos \alpha_r x \cos \beta_s y \operatorname{cso} {}_3\nu_{rs} z \cosh {}_3\gamma_{rs} z ,$$

and

$$\phi_{(rs)} = \phi_{1(rs)} + \phi_{2(rs)} + \phi_{3(rs)} . \tag{8.10}$$

It is evident that

$$\phi = \phi_{(ns)} + \phi_{(nr)} + \phi_{(rs)} . \tag{8.11}$$

Solutions to the problem concerned, which are to be derived from function ϕ of (8.7) and (8.10), may be denoted by solutions of types 2 and 3 respectively by analogy with the solutions 2 and 3 for isotropic rectangular parallelepiped in (7.2) and (7.3) with double summation labels $\sum_n \sum_r$ and $\sum_r \sum_s$.

In this way it is possible to get the desired forms of stress functions, although it may be troublesome to solve the equations (3.12), substituted by the above exponential functions for many different specific indices r, s and n. This difficulty is, needless to say, presented by the fact that the differential equation in question (3.12) cannot be resolved into three differential equations of the second order with real coefficients.

For the case of cubic crystals it is found that λ, μ, ν in (8.3), (8.6), (8.9) vanish, and so for the sake of simplification let, for the moment, such materials be considered that every δ is real, that is, λ, μ and ν vanish. Then, by means of the expressions (3.11) and of the formulae (3.7), (3.8), the expressions for displacements similar to those in (7.14) may be readily obtained from functions $\phi_{(ns)}$, $\phi_{(nr)}$ and $\phi_{(rs)}$ derived just above:

$$\begin{aligned} u = u_{(ns)} + u_{(nr)} + u_{(rs)} &= \frac{\partial}{\partial x} \Pi_1 \phi \\ &= \sum_{\nu=1}^{3} \sum_n \sum_s \cos \beta_s y \cos k_n z \sinh {}_\nu l_{ns} x \cdot {}_u A^\nu_{ns} + \\ &+ \sum_{\nu=1}^{3} \sum_n \sum_r \sin \alpha_r x \cos k_n z \cosh {}_\nu m_{nr} y \cdot {}_u B^\nu_{nr} + \\ &+ \sum_{\nu=1}^{3} \sum_r \sum_s \sin \alpha_r x \cos \beta_s y \cosh {}_\nu \gamma_{rs} z \cdot {}_u C^\nu_{rs} , \end{aligned} \tag{8.12}$$

in which

$$
\begin{aligned}
{}_uA^{\nu}_{ns} &= \frac{1}{\alpha^1}\, {}_\nu l_{ns}\cdot A^{\nu}_{ns}\Big\{(d_{22}\beta_s^4+d_{33}k_n^4+2d_{23}\beta_s^2k_n^2)+ \\
&\qquad + {}_\nu l_{ns}^2\,(d_{11}\cdot {}_\nu l_{ns}^2-2d_{12}\beta_s^2-2d_{31}k_n^2)\Big\}\,, \\
{}_uB^{\nu}_{nr} &= \frac{1}{\alpha^1}\,\alpha_r B^{\nu}_{nr}\Big\{-(d_{11}\alpha_r^4+d_{33}k_n^4+2d_{31}\alpha_r^2k_n^2+ \\
&\qquad + {}_\nu m_{nr}^2\,(-d_{22}\cdot {}_\nu m_{nr}^2+2d_{12}\alpha_r^2+2d_{23}k_n^2)\Big\}\,, \\
{}_uC^{\nu}_{rs} &= \frac{1}{\alpha^1}\,\alpha_r C^{\nu}_{rs}\Big\{-(d_{11}\alpha_r^4+d_{22}\beta_s^4+2d_{12}\alpha_r^2\beta_s^2)+ \\
&\qquad + {}_\nu \gamma_{rs}^2\,(-d_{33}\cdot {}_\nu\gamma_{rs}^2+2d_{23}\beta_s^2+2d_{31}\alpha_r^2)\Big\}\,; \qquad (8.13)
\end{aligned}
$$

and

$$
\begin{aligned}
v &= \frac{\partial}{\partial y}\,\Pi_2\phi \\
&= \sum_{\nu=1}^{3}\sum_{n}\sum_{s} \sin\beta_s y\,\cos k_n z\,\cosh {}_\nu l_{ns}x\cdot {}_vA^{\nu}_{ns}+ \\
&\quad + \sum_{\nu=1}^{3}\sum_{n}\sum_{r} \cos\alpha_r x\,\cos k_n z\,\sinh {}_\nu m_{nr}y\cdot {}_vB^{\nu}_{nr}+ \\
&\quad + \sum_{\nu=1}^{3}\sum_{r}\sum_{s} \cos\alpha_r x\,\sin\beta_s y\,\cosh {}_\nu\gamma_{rs}z\cdot {}_vC^{\nu}_{rs}\,, \qquad (8.14)
\end{aligned}
$$

in which

$$
\begin{aligned}
{}_vA^{\nu}_{ns} &= \frac{1}{a^2}\,\beta_s A^{\nu}_{ns}\Big\{-(e_{22}\beta_s^4+e_{33}k_n^4+2e_{23}\beta_s^2k_n^2)+ \\
&\qquad + {}_\nu l_{ns}^2(-e_{11}\cdot {}_\nu l_{ns}^2+2e_{12}\beta_s^2+2e_{31}k_n^2)\Big\}\,, \\
{}_vB^{\nu}_{nr} &= \frac{1}{a^2}\,{}_\nu m_{nr}B^{\nu}_{nr}\Big\{(e_{11}\alpha_r^4+e_{33}k_n^4+2e_{31}\alpha_r^2k_n^2)+ \\
&\qquad + {}_\nu m_{nr}^2\,(e_{22}\cdot {}_\nu m_{nr}^2-2e_{12}\alpha_r^2-2e_{23}k_n^2)\Big\}\,, \\
{}_vC^{\nu}_{rs} &= \frac{1}{a^2}\,\beta_s C^{\nu}_{rs}\Big\{-(e_{11}\alpha_r^4+e_{22}\beta_s^4+2e_{12}\alpha_r^2\beta_s^2)+ \\
&\qquad + {}_\nu\gamma_{rs}^2\,(-e_{33}\cdot {}_\nu\gamma_{rs}^2+2e_{23}\beta_s^2+2e_{31}\alpha_r^2)\Big\}\,; \qquad (8.15)
\end{aligned}
$$

and

$$
\begin{aligned}
w &= \frac{\partial}{\partial z} \Pi_3 \phi \\
&= \sum_{\nu=1}^{3} \sum_n \sum_s \cos\beta_s y \sin k_n z \cosh {}_\nu l_{ns} x \cdot {}_w A_{ns}^\nu + \\
&+ \sum_{\nu=1}^{3} \sum_n \sum_r \cos\alpha_r x \sin k_n z \cosh {}_\nu m_{nr} y \cdot {}_w B_{nr}^\nu + \\
&+ \sum_{\nu=1}^{3} \sum_r \sum_s \cos\alpha_r x \cos\beta_s y \sinh {}_\nu \gamma_{rs} z \cdot {}_w C_{rs}^\nu , \qquad (8.16)
\end{aligned}
$$

in which

$$
\begin{aligned}
{}_w A_{ns}^\nu &= \frac{1}{\alpha^3} k_n A_{ns}^\nu \Big\{ -(f_{22}\beta_s^4 + f_{33}k_n^4 + 2f_{23}k_n^2\beta_s^2) + \\
&\qquad + {}_\nu l_{ns}^2 (-f_{11} \cdot {}_\nu l_{ns}^2 + 2f_{12}\beta_s^2 + 2f_{31}k_n^2) \Big\} , \\
{}_w B_{nr}^\nu &= \frac{1}{\alpha^3} k_n B_{nr}^\nu \Big\{ -(f_{11}\alpha_r^4 + f_{33}k_n^4 + 2f_{31}k_n^2\alpha_r^2) + \\
&\qquad + {}_\nu m_{nr}^2 (-f_{22} \cdot {}_\nu m_{nr}^2 + 2f_{12}\alpha_r^2 + 2f_{23}k_n^2) \Big\} , \\
{}_w C_{rs}^\nu &= \frac{1}{\alpha^3} {}_\nu \gamma_{rs} C_{rs}^\nu \Big\{ (f_{11}\alpha_r^4 + f_{22}\beta_s^4 + 2f_{12}\alpha_r^2\beta_s^2) + \\
&\qquad + {}_\nu \gamma_{rs}^2 (f_{33} \cdot {}_\nu \gamma_{rs}^2 - 2f_{23}\beta_s^2 - 2f_{31}\alpha_r^2) \Big\} . \qquad (8.17)
\end{aligned}
$$

Accordingly, by formulae (2.4), the expressions are obtained for stresses from the stress-strain relations (3.1).

For instance, one has

$$
\begin{aligned}
\sigma_x &= \sigma_{x(ns)} + \sigma_{x(nr)} + \sigma_{x(rs)} \\
&= \sum_{\nu=1}^{3} \sum_n \sum_s \cos\beta_s y \cos k_n z \cosh {}_\nu l_{ns} \cdot {}_x A_{ns}^\nu + \\
&+ \sum_{\nu=1}^{3} \sum_n \sum_r \cos\alpha_r x \cos k_n z \cosh {}_\nu m_{nr} y \cdot {}_x B_{nr}^\nu + \\
&+ \sum_{\nu=1}^{3} \sum_r \sum_s \cos\alpha_r x \cos\beta_s y \cosh {}_\nu \gamma_{rs} z \cdot {}_x C_{rs}^\nu , \qquad (8.18)
\end{aligned}
$$

in which

$$
\begin{aligned}
{}_x A_{ns}^\nu &= c_{11}\, {}_\nu l_{ns} \cdot {}_u A_{ns}^\nu + c_{12}\beta_s \cdot {}_v A_{ns}^\nu + c_{13} k_n \cdot {}_w A_{ns}^\nu , \\
{}_x B_{nr}^\nu &= c_{11}\alpha_r \cdot {}_u B_{nr}^\nu + c_{12}\, {}_\nu m_{nr} \cdot {}_v B_{nr}^\nu + c_{13} k_n \cdot {}_w B_{nr}^\nu , \\
{}_x C_{rs}^\nu &= c_{11}\alpha_r \cdot {}_u C_{rs}^\nu + c_{12}\beta_s \cdot {}_v C_{rs}^\nu + c_{13}\gamma_{rs} \cdot {}_w C_{rs}^\nu , \qquad (8.19)
\end{aligned}
$$

$$\tau_{yz} = \sum_{\nu=1}^{3}\sum_{n}\sum_{s} \sin\beta_s y \sin k_n z \cosh {}_\nu l_{ns} x \cdot {}_{yz}A^{\nu}_{ns} +$$
$$+ \sum_{\nu=1}^{3}\sum_{n}\sum_{r} \cos\alpha_r x \sin k_n z \sinh {}_\nu m_{nr} y \cdot {}_{yz}B^{\nu}_{nr} +$$
$$+ \sum_{\nu=1}^{3}\sum_{r}\sum_{s} \cos\alpha_r x \sin\beta_s y \sinh {}_\nu \gamma_{rs} z \cdot {}_{yz}C^{\nu}_{rs}\,, \tag{8.20}$$

in which

$$\begin{aligned} {}_{yz}A^{\nu}_{ns} &= -c_{44}(k_n \cdot {}_vA^{\nu}_{ns} + \beta_s \cdot {}_wA^{\nu}_{ns})\,, \\ {}_{yz}B^{\nu}_{nr} &= c_{44}(-k_n \cdot {}_vB^{\nu}_{nr} + {}_\nu m_{nr} \cdot {}_wB^{\nu}_{nr})\,, \\ {}_{yz}C^{\nu}_{rs} &= c_{44}(\gamma_{rs} \cdot {}_vC^{\nu}_{rs} - \beta_s \cdot {}_wC^{\nu}_{rs})\,. \end{aligned} \tag{8.21}$$

Since the linear independence of three kinds of stress functions $\phi_{(ns)}$, $\phi_{(nr)}$ and $\phi_{(rs)}$ may be obvious and, for instance, the solutions of type 1 contain three sequences of coefficients (A^1_{ns}, A^2_{ns}, A^3_{ns}), one can determine these coefficients so that the boundary conditions may be satisfied. It goes without saying that even in the case of general loading functions the solutions can be obtained by analogy with the solutions for the case of isotropic rectangular thick plate.

Namely, when given surface tractions are general, one may write, for instance, $\phi_{(ns)}$ in the following form, moving the origin of the coordinate system to the corner ($x=-a$, $y=-b$, $z=-h$).

$$\phi_{(ns)} = \sum_{\nu=1}^{3}\sum_{n}\sum_{s} \cos\beta_s y \cos k_n z \cos {}_\nu\lambda_{ns} x\,(A^{\nu}_{ns}\cosh {}_\nu l_{ns} x + \bar{A}^{\nu}_{ns}\sinh {}_\nu l_{ns} x)\,,$$

in which

$$ {}_1\lambda_{ns} \equiv 0, \quad \beta_s = \frac{s\pi}{2b}\,, \quad k_n = \frac{n\pi}{2h} \qquad (s,\ n=0,\ 1,\ 2,\ 3\cdots)\,. \tag{8.22}$$

Thus, by referring to the expressions obtained for the case of symmetrical loading function, one can immediately obtain the desired solutions.

Additional remarks on the case, wherein the rectangular cartesian coordinate system, the axes of which are parallel to the edge lines of the rectangular parallelopiped concerned, can be derived by a rotation of the coordinate system to which the stress-strain relations (2.1) or (3.1) are referred, will be stated. Now let the former coordinate system be taken to be (x', y', z') and the latter to be (x, y, z), and suppose that they are connected with each other by the transformation scheme

$$\begin{array}{c|ccc} & x & y & z \\ \hline x' & l_1 & m_1 & n_1 \\ y' & l_2 & m_2 & n_2 \\ z' & l_3 & m_3 & n_3 \end{array} \tag{8.23}$$

which is to mean the rotation of the coordinate system.

First, let the case of stress-strain relations for a transversely isotropic solid (4.26) be considered. When the edge lines of the rectangular parallelepied under consideration are parallel to the x, y, z-axes, for instance, stress function $\phi_{(ns)}$, corresponding to the solutions of type 1, as stated before, may be written in the form

$$\phi_{(ns)} = \sum_{i=1}^{3}\sum_{n}\sum_{s} \cos\beta_s y \cos k_n z \cosh l_{ns}\mu^i x \cdot A_{ns}^i$$

in which $l_{ns}^2 = k_n^2 + \beta_s^2$ and μ^i denotes $\frac{1}{\sqrt{\nu_i}}$ ($i=1, 2, 3$), and which is the solution of the basic differential equation

$$\left(\nu_i \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi_i = 0 . \tag{4.28'}$$

Then, when the coordinate system (x', y', z') does not agree with the system (x, y, z), and further the transformation scheme (8.23) is general, the differential equation (4.28′) cannot obviously be invariant under the transformation of the coordinates in accordance with the scheme (8.23). However, since one can deal with the differential equations of the second order only in this case, the solutions may be readily obtained, though the solid in question may be virtually regarded as more highly anisotropic. In fact, under the rotation about the z-axis by angle θ, that is, the transformation in accordance with scheme

$$\begin{array}{c|ccc} & x & y & z \\ \hline x' & \cos\theta & \sin\theta & 0 \\ y' & -\sin\theta & \cos\theta & 0 \\ z' & 0 & 0 & 1 \end{array} \; . \tag{8.24}$$

the stress-strain relation for a generally orthotropic solid becomes that of the matrix form :

$$\begin{pmatrix}\sigma_{x'}\\ \sigma_{y'}\\ \sigma_{z'}\\ \tau_{y'z'}\\ \tau_{x'z'}\\ \tau_{x'x'}\end{pmatrix}=\begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13} & 0 & 0 & \alpha_{16}\\ \alpha_{12} & \alpha_{22} & \alpha_{23} & 0 & 0 & \alpha_{26}\\ \alpha_{13} & \alpha_{23} & \alpha_{33} & 0 & 0 & \alpha_{36}\\ 0 & 0 & 0 & \alpha_{44} & \alpha_{45} & 0\\ 0 & 0 & 0 & \alpha_{45} & \alpha_{55} & 0\\ \alpha_{16} & \alpha_{26} & \alpha_{36} & 0 & 0 & \alpha_{66}\end{pmatrix}\cdot\begin{pmatrix}e_{x'}\\ e_{y'}\\ e_{z'}\\ \gamma_{y'z'}\\ \gamma_{z'x'}\\ \gamma_{x'y'}\end{pmatrix}, \tag{8.25}$$

which contains 13 constants. Furthermore, when the transformation scheme to be used is general, the stress-strain relation for an orthoropic solid, referred to the coordinates (x', y', z') obviously comes to contain 21 constants. Hence, if the differential equation (3.12) cannot be resolved into three differential equations of the second order with real coefficients, it virtually becomes necessary to deal with the elastic solid, possessing 21 independent elastic constants. In this case one would have to treat basic differential equation of the type (2.17) directly.

First, in this case let the surfaces of a rectangular parallelepiped be taken to be given by

$$x' = 0,\ 2a, \quad y' = 0,\ 2b, \quad z' = 0,\ 2h, \tag{8.26}$$

as before, and, by putting, for instance, as

$$\phi = A \exp(k_n i z' + \beta_s i y' + i\delta x'), \tag{8.27}$$

in which i is imaginary unit and

$$k_n = \frac{n\pi}{2h}, \quad \beta_s = \frac{s\pi}{2b}, \qquad (n,\ s=0,\ 1,\ 2\cdots),$$

and by substituting this expression in equation (2.17), referred to the coordinate system (x', y', z'), one gets algebraic equation in δ of degree six with real coefficients. Then, solutions of the type 1 are gotten, but, of course, this process is tedious. At any rate, if the axes of elastic symmetry for an orthotropic solid are not parallel to the edge lines of a rectangular parallelepiped, the process of calculation obviously becomes complicated to an appreciable degree. It will be apparent that the above method, associated with the expression (8.27), applies to the case, where a rectangular parallelopiped is made of generally anisotropic material.

§9. Conclusion.

This paper, continued from the previous papers regarding the thick plate problem, has dealt with the isotropic and aeolotropic elasticity problems for a rectangular plate of sufficient thickness or a rectangular parallelepiped. The previous papers are concerned with a moderately thick plate, whereas it is found that in the case of orthotropy the exact solutions could not be obtained for a moderately thick plate, which corresponds to an orthotropic elastic plate of zero thickness. However, this fact seems quite immaterial, and it will not be inadequate that we can solve completely the elasticity problems of given surface tractions for a sufficiently thick plate in both cases of isotropy and anisotropy by the method of series as shown above in Secs. 7 and 8. The solutions obtained in this paper as governed by the boundary conditions are obviously exact within the infinitesimal theory of elasticity and, of course, those for the case of anisotropy do not involve any restriction on elastic constants.

Since no general method of solving three-dimensional problems for an anisotropic elastic solid of finite extent within the theory of the first order seems to have been developed as yet, the author has attempted to seek a three-dimensional approach to those problems. Though many of the anisotropic elasticity problems might be beyond the scope of the herein presented method of analysis, and the method obtained in this paper may be awkward in many respects, the result would be of some significance in studying the problems of anisotropic elasticity. It is to the author's regret that he could not resolve even the differential equation (3.12), not substituted by any numerical values of elastic constants, into three differential equations of the second order except for the case of transverse isotropy, and it is, of course, desirable to surmount the difficulty involved in resolving the differential equation for ϕ.

Further, in conjunction with the three-dimensional solutions for isotropic solids, obtained as the particular case of anisotropic elastic solids by the use of the general method for anisotropic solids explained in Sec. 2, the brief discussion of three-functions approaches for isotropic solids was presented. The remarkable effectiveness of the term $2 \operatorname{rot} \vartheta$ or $\frac{1}{G} \operatorname{rot} \vartheta$ was fully explained with illustration, and moreover, it was indicated that the correct integration of displacement-equations of equilibrium yielded essentially this term. In any case one should exercise some caution in applying H. Neuber's method of solution or any other method equivalent to H. Neuber's, though there will be no loss of generality in discarding the third term $2 \operatorname{rot} \vartheta$ in vector formula

(6. 3), by virtue of the existence of the above-described mathematical manipulation. However, it will be very obvious that the application of J. Boussinesq's approach or of any other approach equivalent to H. Neuber's, which is modified so as to include the solution as $\frac{1}{G}$ rot ϑ, is advisable.

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