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Title	On a Rectangular Elastic Solid Compressed by the Forces Applied on Its Two Opposite Boundaries
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Citation	Memoirs of the Faculty of Engineering, Hokkaido University, 10(3), 371-382
Issue Date	1957-09-30
Doc URL	http://hdl.handle.net/2115/37805
Type	bulletin (article)
File Information	10(3)_371-382.pdf

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On a Rectangular Elastic Solid Compressed by the Forces Applied on Its Two Opposite Boundaries

By

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(Received July 1, 1957)

Abstract

This paper presents one of the methods of solving a problem of rectangular block by treating as a case as one of plane stress. The elastic block in this case, however, is understood to be deformed by an external compressive force applied on its two opposite boundaries by rigid bodies.

1. Conditions and assumptions as to the elastic block.

Fig. 1 is a schematic presentation of the problem discussed in the present paper. The x- and the y-axes are taken parallel to the sides of the block and the z-axis is taken so that it is normal to the xy-plane passing point 0. h is the thickness and 2a, 2b are the lengths of the sides of block. This block is compressed by two rigid bodies A and B, and subjected to a compressive force P.

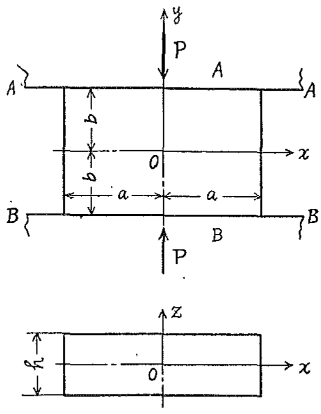

Fig. 1.

The following conditions are assumed:

(a) External force P is applied by two rigid bodies A and B, which have uniformly finished surfaces as AA and BB, respectively.

(b) The contact surfaces AA and BB between the elastic block and the rigid bodies are uniformly finished. In other words, the surfaces of the elastic block AA and BB are finished in the same degree, but the finishing grades of the two bodies are not necessarily the same.

(c) The deformation is symmetrical to the x- and the y-axis, and the contact surfaces AA and BB remain parallel to the xz-plane even after the deformation.

(d) The deformation in x-direction is restricted by a condition at the contact surfaces AA and BB. Therefore, the cases of no deformation and no limitation in the deformation being completely free, and the deformation between these two extremes can occur at the contact surfaces.

The solution of these kinds of problems involve various cases to be considered depending upon the conditions of the external force applied and the contact surfaces. Since the solutions of these problems become quite difficult sometimes, it was proposed to solve a case which can be practically expected under the conditions described above.

When the contact surfaces AA and BB of the rectangular block are perfectly flat and there are oil films upon them, so that their displacements in x-direction are not restricted, the problem can be deemed as one of the cases that would satisfy the above assumptions.

In this case, however, the external forces P are distributed over AA and BB surfaces, and the solution can easily be obtained. The details of this solution will be shown later.

Strictly speaking, the case of having no displacement on the surfaces of AA and BB would be impossible. However, the case when the two surfaces of the elastic body are riveted to the rigid bodies or cemented by a strong adhesive can be understood to satisfy this condition.

2. The conditions at the boundaries.

There are two cases in treating this problem as a two-dimensional case—cases of plane stress and plane strain. These two cases depend upon coefficients that include the Poisson's ratio ν. In this case, however, either one of these two can be applied in obtaining its fundamental solution.

A case of plane stress is handled in this paper; the stress function is represented by F.

The conditions at the boundaries give

$$\sigma_x = \tau_{xy} = 0 \qquad \text{at} \quad x = \pm a ,$$

hence

$$\frac{\partial^2 F}{\partial y^2} = 0$$

and

$$\frac{\partial^2 F}{\partial x \partial y} = 0$$

The first condition means that $F=0$ at $x=\pm a$, and the second condition results in $\frac{\partial F}{\partial x} = \text{const.}$

From the conditions shown in Fig. 1, one can easily obtain

$$\begin{aligned} \int_0^a \sigma_y dx &= \int_0^a \frac{\partial^2 F}{\partial x^2} dx \\ &= \left(\frac{\partial F}{\partial x}\right)_a - \left(\frac{\partial F}{\partial x}\right)_0 \\ &= -\frac{P}{2} \end{aligned}$$

Therefore the above two conditions can be rewritten as follows:

$$(F)_{x=a} = 0 \tag{1}$$

$$\left(\frac{\partial F}{\partial x}\right)_{x=a} - \left(\frac{\partial F}{\partial x}\right)_{x=0} = -\frac{P}{2} \tag{2}$$

In the present case the deformation is assumed to be symmetrical to the x- and the z-axis, so that the second term of the left-hand side of Eq. (2) becomes zero. Consequently the condition obtained in Eq. (2) gives

$$\left(\frac{\partial F}{\partial x}\right)_{x=a} = -\frac{P}{2} \tag{2a}$$

Now, the conditions at the contact surfaces $y=\pm b$ will be considered. Since the displacements of the surfaces are parallel to xz-plane and they are not dependent upon x even after deformations, the displacement in y-direction v gives the following relations:

$$\frac{\partial v}{\partial x} = 0$$

or

$$\frac{\partial^2 v}{\partial x^2} = 0 \qquad \text{at} \quad y = \pm b \tag{3}$$

The relations between a displacement and a stress in the case of a plane-stress are:

$$\left.\begin{aligned}\frac{\partial u}{\partial x} &= \frac{1}{E}(\sigma_x - \nu\sigma_y) = \frac{1}{E}\left(\frac{\partial^2 F}{\partial y^2} - \nu\frac{\partial^2 F}{\partial x^2}\right)\\ \frac{\partial v}{\partial y} &= \frac{1}{E}(\sigma_y - \nu\sigma_x) = \frac{1}{E}\left(\frac{\partial^2 F}{\partial x^2} - \nu\frac{\partial^2 F}{\partial y^2}\right)\\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} &= \frac{2(1+\nu)}{E}\tau_{xy} = -\frac{2(1+\nu)}{E}\frac{\partial^2 F}{\partial x \partial y}\end{aligned}\right\} \tag{4}$$

When $\frac{\partial^2 v}{\partial x^2} = 0$ is computed after differentiating the first and the third relations with respect to y and x, respectively, one obtains

$$\left(-\frac{\partial^2 v}{\partial x^2}\right)_{y=\pm b} = \frac{1}{E}\left[\frac{\partial_3 F}{\partial y^3} + (2+\nu)\frac{\partial^3 F}{\partial x^2 \partial y}\right]_{y=\pm b}$$

Hence, Eq. (3) gives

$$\left[\frac{\partial^3 F}{\partial y^3} + (2+\nu)\frac{\partial^3 F}{\partial x^2 \partial y}\right]_{y=\pm b} = 0 \qquad \text{at} \quad y = \pm b \tag{5}$$

One more condition remains which must cause a certain displacement parallel to the x-axis as was assumed. For the purpose of considering a practically possible and the simplest case, the following relation is assumed to be valid:

$$u = \mu_0 x\,, \tag{6}$$

where μ_0 is the constant. Namely, the displacement in x-direction u is assumed to be proportional to x. If μ_0 is null, the displacement in x-direction is perfectly zero. When μ_0 takes the value of $\frac{\nu}{E}\frac{P}{2a}$, the block can deform perfectly freely as will be described later. Sinse μ_0 can take a value between these two values, μ_0 can generally be defined as

$$0 \leqq \mu_0 \leqq \frac{\nu}{E}\cdot\frac{P}{2a} \tag{7}$$

If the relation given in Eq. (6) is valid, one has

$$\left(\frac{\partial u}{\partial x}\right)_{y=\pm b} = \mu_0 \qquad \text{at} \quad y = \pm b \tag{6a}$$

The first relation in Eq. (4) also gives

$$\left(\frac{\partial^2 F}{\partial y^2} - \nu\frac{\partial^2 F}{\partial x^2}\right)_{y=\pm b} = \mu_0 E \qquad \text{at} \quad y = \pm b \tag{8}$$

Thus, the stress fuction F which satisfies the above four conditions (1), (2a), (5) and (8) must be found.

3. The stress function.

In order to solve the problem, the stress function F is assumed to be composed of two functions F_0 and F_1. Thus,

$$F = F_0 + F_1 \tag{9}$$

At first, F_0 is assumed to take the following form:

$$F_0 = \frac{Pa}{4}\left(1-\frac{x^2}{a^2}\right) \tag{10}$$

This form of the function F_0 obviously satisfies the following Laplace's differential equation:

$$\Delta\Delta\, F_0 = 0\,, \tag{11}$$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

From Eq. (10), one can compute the values of σ_{0y}, σ_{0x}, τ_{0xy}, ε_{0x}, and ε_{0y} as follows:

$$\left.\begin{aligned}
\sigma_{0y} &= \frac{\partial^2 F_0}{\partial x^2} = -\frac{P}{2a} \\
\sigma_{0x} &= \frac{\partial^2 F_0}{\partial y^2} = 0 \\
-\tau_{0xy} &= \frac{\partial^2 F_0}{\partial x \partial y} = 0 \\
\varepsilon_{0x} &= \frac{\partial u_0}{\partial x} = \frac{\nu}{E}\frac{P}{2a} \\
\varepsilon_{0y} &= \frac{\partial v_0}{\partial y} = -\frac{1}{E}\frac{P}{2a}
\end{aligned}\right\}, \tag{12}$$

where the terms with suffix 0 mean the stresses or strains related to the stress function F_0. Namely, the state repressented by the stress function F_0 given in Eq. (10) means that the external force is uniformly distributed with the intensity of $\frac{P}{2a}$ over its working surface as will be seen in Eq. (12). Therefore, the contractions in y-direction and the elongations in x-direction are similar to those to be introduced by simple tension and compression.

For the purpose of obtaining u_0 and v_0 from Eq. (12), the use of the conditions of $u_0=0$ at $x=0$ and $v_0=0$ at $y=0$ gives

$$\left.\begin{aligned} u_0 &= \frac{\nu}{E}\,\frac{P}{2a}\,x \\ v_0 &= -\frac{1}{E}\,\frac{P}{2a}\,y \end{aligned}\right\} \tag{13}$$

Therefore, the value of μ_0 in Eq. (6) becomes $\dfrac{\nu P}{2aE}$. Then, the meaning of the value in the right-hand side of Eq. (7) becomes clear.

It can be easily obtained that Eq. (10) gives the following relations at the boundaries.

$$\left.\begin{aligned} &(F_0)_{x=a} = 0 \\ &\left(\frac{\partial F_0}{\partial x}\right)_{x=a} = -\frac{P}{2} \\ &\left[\frac{\partial^3 F_0}{\partial y^3} + (2+\nu)\frac{\partial^3 F_0}{\partial x^2 \partial y}\right]_{y=b} = 0 \\ &\left[\frac{\partial^2 F_0}{\partial y^2} - \nu\frac{\partial^2 F_0}{\partial x^2}\right]_{y=b} = \frac{\nu P}{2a} \\ &\text{where} \\ &0 \leqq \mu_0 \leqq \frac{\nu P}{2aE} \end{aligned}\right\} \tag{14}$$

As will be seen in these equations, the function F_0 satisfies the conditions in Eqs. (1), (2a) and (5), but the fourth condition in Eq. (14) is satisfied only in a special case ; i. e., F_0 gives a perfect solution when $\mu_0=1$. However, since a problem of arbitrary μ_0 is under the present discussion, a solution must be obtained by introducing the second stress function F_1, too.

Now the stress function F_1 defined by Eq. (9) must satisfy the following relations when Eq. (14) is taken into account:

$$\left.\begin{aligned} &(F_1)_{x=a} = 0 \\ &\left(\frac{\partial F_1}{\partial x}\right)_{x=a} = 0 \\ &\left[\frac{\partial^3 F_1}{\partial y^3} + (2+\nu)\frac{\partial^3 F_1}{\partial x^2 \partial y}\right]_{y=b} = 0 \end{aligned}\right\}, \tag{15}$$

$$\left[\frac{\partial^2 F_1}{\partial y^2} - \nu \frac{\partial^2 F_1}{\partial x^2}\right]_{y=b} = (\mu_1 - 1)\frac{\nu P}{2a}$$

where

$$\left.\begin{aligned} \mu_0 &= \mu_1 \frac{\nu P}{2aE} \\ \mu_1 &= 0 \sim 1 \end{aligned}\right\} \tag{16}$$

The stress function F_1 is selected so that the following equation would be satisfied:

$$F_1 = C_0 \sum_m \frac{1}{\cosh \alpha b} \cdot h(\alpha y) \cdot \cos \alpha x + C_0 \sum_m \frac{1}{\cosh \beta a} k(\beta x) \cdot \cos \beta y ,$$

where

$$\alpha = \frac{m\pi}{2a}, \qquad \beta = \frac{m\pi}{2b}$$

$$m = 1, 3, 5, \cdots\cdots$$

$$C_0 = \text{const.} = \frac{2Pa}{\pi^2} \tag{17}$$

Of course, Eq. (17) must satisfy the Laplace's differential equation of Eq. (11). The forms of $h(\alpha y)$ and $k(\beta x)$ can be assumed as follows when the condition of symmetry is taken into consideration:

$$\left.\begin{aligned} h(\alpha y) &= A_m \cosh \alpha y + B_m \frac{y}{b} \sinh \alpha y \\ k(\beta x) &= C_m \cosh \beta x + D_m \frac{x}{a} \sinh \beta x \end{aligned}\right\} \tag{18}$$

in which A_m, B_m, C_m, and D_m are unknown coefficients.

Substituting Eq. (17) into the first and the fourth relations in Eq. (15), one obtains

$$C_m = -D_m \cdot \tanh \beta a , \tag{19}$$

and

$$C_0 \sum_m \alpha^2 \left\{ A_m(1+\nu) + B_m \left[\frac{2}{\alpha b} + (1+\nu) \tanh \alpha b \right] \right\} \cdot \cos \alpha x$$

$$= (\mu_1 - 1) \cdot \frac{\nu P}{2a}$$

or

$$\sum_m m^2\left\{A_m(1+\nu)+B_m\left[\frac{2}{\alpha b}+(1+\nu)\tanh\alpha b\right]\right\}\cdot\cos\alpha x$$
$$=\nu(\mu_1-1) \tag{20}$$

Expanding the right-hand side of this equation into a form of Fourier's series, one has

$$\nu(\mu_1-1)=\frac{4\nu(\mu_1-1)}{\pi}\sum_m\frac{\sin\frac{m\pi}{2}}{m}\cos mx \tag{21}$$

Substitution of Eq. (21) into Eq. (20) gives

$$A_m=-B_m\left[\frac{2}{(1+\nu)\alpha b}+\tanh\alpha b\right]-\frac{4\nu(1-\mu_1)}{\pi(1+\nu)}\cdot\frac{\sin\frac{m\pi}{2}}{m^3}$$
$$=-B_m\left[\frac{4}{\pi(1+\nu)\lambda m}+\tanh\frac{m\pi\lambda}{2}\right]-\frac{4\nu(1-\mu_1)}{\pi(1+\nu)}\cdot\frac{\sin\frac{m\pi}{2}}{m^3} \tag{22}$$

where

$$\lambda=\frac{b}{a} \tag{23}$$

Next, substituting Eq. (17) into the second and third relations in Eq. (15), one obtains

$$\frac{\pi}{2}\sum_m B_m\frac{m\sin\frac{m\pi}{2}}{\cosh\frac{m\pi\lambda}{2}}\left\{\left[\frac{4}{\pi(1+\nu)\lambda m}+\tanh\frac{m\pi\lambda}{2}\right]\cosh\alpha y-\frac{y}{b}\sinh\alpha y\right\}$$
$$+\sum_m D_m\left\{\tanh\frac{m\pi}{2\lambda}+\frac{m\pi}{2\lambda}\left(1-\tanh^2\frac{m\pi}{2\lambda}\right)\right\}\cos\beta y$$
$$+\frac{2\nu(1-\mu_1)}{\pi(1+\nu)}\sum_m\frac{1}{m^2\cosh\frac{m\pi\lambda}{2}}\cdot\cosh\alpha y=0\,, \tag{24}$$

and

$$\lambda^3\sum_m m^3\left\{B_m\left[\frac{6-2\nu}{m\pi\lambda}\tanh\frac{m\pi\lambda}{2}-(1+\nu)\left(1-\tanh^2\frac{m\pi\lambda}{2}\right)\right]\right.$$
$$\left.+\frac{4\nu(1-\mu_1)}{\pi}\cdot\frac{\sin\frac{m\pi}{2}\cdot\tanh\frac{m\pi\lambda}{2}}{m^3}\right\}\cos\alpha x$$

$$+\sum_m D_m \frac{m^3 \sin\frac{m\pi}{2}}{\cosh\frac{m\pi}{2\lambda}} \left\{ \left[(1+\nu)\tanh\frac{m\pi}{2\lambda} - \frac{\lambda(8+4\nu)}{m\pi} \right] \cosh\beta x \right.$$

$$\left. -(1+\nu)\frac{x}{a}\sinh\beta x \right\} = 0 \qquad (25)$$

Since the treatment of Eqs. (24) and (25) is rather difficult, the following computations may be also useful. Fourier's expanssions of $\cosh\alpha y$, $\frac{y}{b}\sinh\alpha y$, $\cosh\beta x$, and $\frac{x}{a}\sinh\beta x$ give

$$\left.\begin{aligned}
\cosh\frac{m\pi y}{2a} &= \frac{4}{\pi}\sum_n \frac{n\sin\frac{n\pi}{2}\cosh\frac{m\pi\lambda}{2}}{\lambda^2 m^2+n^2}\cos\frac{n\pi y}{2b} \\
\frac{y}{b}\sinh\frac{m\pi y}{2a} &= \frac{4}{\pi}\sum_n \left[\frac{n\sin\frac{n\pi}{2}\sinh\frac{m\pi\lambda}{2}}{\lambda^2 m^2+n^2} \right. \\
&\qquad \left. -\frac{4\lambda}{\pi}\frac{mn\sin\frac{n\pi}{2}\cosh\frac{m\pi\lambda}{2}}{(\lambda^2 m^2+n^2)^2} \right]\cos\frac{n\pi y}{2b} \\
\cosh\frac{m\pi x}{2b} &= \frac{4}{\pi}\sum_n \frac{n\sin\frac{n\pi}{2}\cosh\frac{m\pi}{2\lambda}}{\lambda^{-2} m^2+n^2}\cos\frac{n\pi x}{2a} \\
\frac{x}{a}\sinh\frac{m\pi x}{2b} &= \frac{4}{\pi}\sum_n \left[\frac{n\sin\frac{n\pi}{2}\sinh\frac{m\pi}{2\lambda}}{\lambda^{-2} m^2+n^2} \right. \\
&\qquad \left. -\frac{4}{\pi\lambda}\frac{mn\sin\frac{n\pi}{2}\cosh\frac{m\pi}{2\lambda}}{(\lambda^{-2} m^2+n^2)^2} \right]\cos\frac{n\pi x}{2a}
\end{aligned}\right\} \quad (26)$$

where n takes the values of 1, 3, 5, ······ for each value of m. Application of these relations to Eq. (24) results

$$\frac{8}{\pi}\sum_m\sum_n B_m \left[\frac{1}{\lambda(1+\nu)} \frac{n\sin\frac{m\pi}{2}\sin\frac{n\pi}{2}}{\lambda^2 m^2+n^2} \right.$$

$$\left. + \frac{\lambda m^2 n\sin\frac{m\pi}{2}\sin\frac{n\pi}{2}}{(\lambda^2 m^2+n^2)^2} \right]\cos\frac{n\pi y}{2b}$$

$$+\sum_m D_m\left[\tanh\frac{m\pi}{2\lambda}+\frac{m\pi}{2\lambda}\left(1-\tanh^2\frac{m\pi}{2\lambda}\right)\right]\cos\frac{m\pi y}{2b}$$

$$=-\frac{8\nu(1-\mu_1)}{\pi^2(1+\nu)}\sum_m\sum_n\frac{n\sin\frac{n\pi}{2}}{m^2(\lambda^2m^2+n^2)}\cos\frac{n\pi y}{2b}$$

One can easily rewrite this equation as

$$\frac{8}{\pi}\sum_m\sum_n B_n\left[\frac{1}{\lambda(1+\nu)}\frac{m\sin\frac{m\pi}{2}\sin\frac{n\pi}{2}}{m^2+\lambda^2n^2}\right.$$

$$\left.+\frac{\lambda mn^2\sin\frac{m\pi}{2}\sin\frac{n\pi}{2}}{(m^2+\lambda^2n^2)^2}\right]\cos\frac{m\pi y}{2b}$$

$$+\sum_m D_m\left[\tanh\frac{m\pi}{2\lambda}+\frac{m\pi}{2\lambda}\left(1-\tanh^2\frac{m\pi}{2\lambda}\right)\right]\cos\frac{m\pi y}{2b}$$

$$=-\frac{8\nu(1-\mu_1)}{\pi^2(1+\nu)}\sum_m\sum_n\frac{m\sin\frac{m\pi}{2}}{n^2(m^2+\lambda^2n^2)}\cos\frac{m\pi y}{2b}$$

or

$$\sum_n B_n\left[\frac{m}{m^2+\lambda^2n^2}+\frac{\lambda^2(1+\nu)mn^2}{(m^2+\lambda^2n^2)^2}\right]\sin\frac{m\pi}{2}\sin\frac{n\pi}{2}$$

$$+D_m\frac{\pi\lambda(1+\nu)}{8}\left[\tanh\frac{m\pi}{2\lambda}+\frac{m\pi}{2\lambda}\left(1-\tanh^2\frac{m\pi}{2\lambda}\right)\right]$$

$$=-\frac{\lambda\nu(1-\mu_1)}{\pi}\sum_n\frac{m\sin\frac{m\pi}{2}}{n^2(m^2+\lambda^2n^2)} \tag{27}$$

Rewriting the condition given by Eq. (25) in the manner shown above, one has

$$\lambda^3\sum_m m^3\left\{B_m\left[\frac{6-2\nu}{m\pi\lambda}\tanh\frac{m\pi\lambda}{2}-(1+\nu)\left(1-\tanh^2\frac{m\pi\lambda}{2}\right)\right]\right.$$

$$\left.+\frac{4\nu(1-\mu_1)}{\pi}\frac{\sin\frac{m\pi}{2}\tanh\frac{m\pi\lambda}{2}}{m^3}\right\}\cos\frac{m\pi x}{2a}$$

$$+\frac{4(1+\nu)}{\pi}\sum_m\sum_n D_m\left[-\frac{(8+4\nu)\lambda}{\pi(1+\nu)}\frac{m^2n}{\lambda^{-2}m^2+n^2}\right.$$

$$\left.+\frac{4}{\pi\lambda}\frac{m^4n}{(\lambda^{-2}m^2+n^2)^2}\right]\sin\frac{m\pi}{2}\sin\frac{n\pi}{2}\cos\frac{n\pi x}{2a}=0$$

or

$$\lambda^3 \sum_m m^3 \Bigg\{ B_m \Bigg[\frac{6-2\nu}{\pi\lambda} \frac{\tanh \frac{m\pi\lambda}{2}}{m} - (1+\nu)\Big(1-\tanh^2 \frac{m\pi\lambda}{2}\Big)\Bigg] + \frac{4\nu(1-\mu_1)}{\pi} \frac{\sin\frac{m\pi}{2} \tanh\frac{m\pi\lambda}{2}}{m^3} \Bigg\} \cos\frac{m\pi x}{2a}$$

$$+ \frac{4(1+\nu)}{\pi} \sum_m \sum_n D_n \Bigg[-\frac{(8+4\nu)\lambda}{\pi(1+\nu)} \frac{m\,n^2}{m^2+\lambda^{-2}n^2} + \frac{4}{\pi\lambda} \frac{m\,n^4}{(m^2+\lambda^{-2}n^2)} \Bigg] \sin\frac{m\pi}{2} \sin\frac{n\pi}{2} \cos\frac{m\pi x}{2a} = 0$$

or

$$B_m \Bigg[\frac{6-2\nu}{\pi\lambda} \frac{\tanh \frac{m\pi\lambda}{2}}{m} - (1+\nu)\Big(1-\tanh^2 \frac{m\pi\lambda}{2}\Big)\Bigg]$$

$$+ \frac{4(1+\nu)}{\pi\lambda^3} \sum_n D_n \Bigg[-\frac{(8+4\nu)\lambda}{\pi(1+\nu)} \frac{n^2}{m^2(m^2+\lambda^{-2}n^2)} + \frac{4}{\pi\lambda} \frac{n^4}{m^2(m^2+\lambda^{-2}n^2)^2} \Bigg] \sin\frac{m\pi}{2} \sin\frac{n\pi}{2}$$

$$= -\frac{4\nu(1-\mu_1)}{\pi} \frac{\sin\frac{m\pi}{2} \tanh\frac{m\pi\lambda}{2}}{m^3} \tag{28}$$

Accordingly, one can easils obtain the values of B_m and D_m by solving Eqs. (24) and (25) or Eqs. (27) and (28) simultaneously. The values of A_m and C_m can also be obtained by substituting the values of B_m and D_m in Eqs. (19) and (22), and the solution of this problem is obtainable.

Though μ_1 can take any value between 0 and 1, the value of μ_1 must be determined by the conditions at the boundaries of AA and BB. If the value of μ_1 is once determined, the stresses on any arbitrary section can be readily computed by means of the following relations:

$$\left.\begin{aligned} \sigma_x &= \frac{\partial^2 F}{\partial y^2} \\ \sigma_y &= \frac{\partial^2 F}{\partial x^2} \\ \tau_{xy} &= -\frac{\partial^2 F}{\partial x\,\partial y} \end{aligned}\right\} \tag{29}$$

Another device of solution must be made when the conditions at the surface of AA and BB are difficult and the state of contact surface is asymmetrical to the y-axis, but the solution is not necessarily difficult.

The discussions of these cases will be presented on another occasion.