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Mathematical Investigation of the Flow in a Stream with a Finite Drop on the Free Surface

By

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Abstract

By means of conformal representation several examples of stream-lines with the free surface have been calculated by A. R. Richardson and Fr. Prášil.* The inclinations of the free surface were represented by certain functions of φ and the inclinations of the free surface at infinity were assumed to be zero. In this article, the inclinations are represented by a more simple function of φ. Hence the calculations are easier and the results are more adequate for discussion and applications.

I. Introduction

Let

$$\chi = \varphi\,(x, y) + i\psi\,(x, y) \tag{1}$$

be an analytical function of

$$z = x + iy\,, \tag{2}$$

where φ is velocity potential and ψ is stream function. Hence it follows

$$\frac{d\chi}{dz} = \frac{\partial \varphi}{\partial x} - i\,\frac{\partial \varphi}{\partial y} = v_x - iv_y\,,$$

$$\frac{dz}{d\chi} = \frac{1}{v^2}\,(v_x + iv_y)\,,$$

where v is velocity and v_x and v_y are the components of v with respect to x and y; the direction of y being taken to direct upward. For the

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* A. R. Richardson, Phil. Mag. 1920.

  Fr. Prášil, Technische Hydrodynamik, J. Springer 1936.

line element ds of the streamline and its components dx and dy one has the relation

$$\frac{dv}{ds} = \frac{dv_x}{dx} = \frac{dv_y}{dy} .$$

If the angle between the streamline and x-axis be denoted by α, one has on the stream line $v_x = v \cos \alpha,\ v_y = v \sin \alpha,$ \hfill (3)

$$v = \frac{d\varphi}{ds} = \frac{d\varphi}{dy} \cdot \frac{dy}{ds} = \frac{d\varphi}{dy} \cdot \frac{v_y}{v} = \frac{d\varphi}{dy} \sin \alpha .$$ \hfill (4)

On the other hand, one has Bernoulli's formula in the same flux of the flow

$$y + \frac{P}{\gamma} + \frac{v^2}{2g} = y_0 + \frac{P_0}{\gamma} + \frac{v_0^2}{2g} ,$$ \hfill (5)

where P is pressure of fluid, γ specific weight and g gravity constant, and the suffix 0 is the notation which shows the initial values of the quantities.

Now let it be assumed that $\varphi = 0$ and $\psi = 0$ at the origin $x = 0$ and $y = 0$, and $\psi = 0$ is the free surface on which it may be put $P = 0$. Hence Bernoulli's formula (5) will be written

$$y_f + \frac{v_f^2}{2g} = \frac{v_0^2}{2g} ,$$ \hfill (6)

where the suffix f is the notation which shows the values on the free surface.

Changing the unit of φ, x, y, t and v into φ', x', y', t' and v' as shown in the following

$$2g\varphi = \varphi',\quad 2gx = x',\quad 2gy = y',\quad 2gt = t' \text{ and } v = v',$$

one gets Bernoulli's formula expressed by

$$y_f + v_f^2 = v_0^2 .$$ \hfill (7)

Let the start be made from this formula; after the numerical calculation is completed, the original unit will be used again.

On the other hand, if one puts

$$\varphi = \varepsilon\varphi',\quad x = \eta x',\quad y = \eta y' \quad \text{and} \quad t = \eta t'$$

he obtains

$$v_x = \frac{\partial\varphi}{\partial x} = \frac{\varepsilon}{\eta} \frac{\partial\varphi'}{\partial x'} = \frac{\varepsilon}{\eta} v_x' .$$

Then Bernoulli's formula (6) is written in terms of the new unit,

$$\frac{\varepsilon^2}{\eta^2}\frac{v_f'^2}{2g} + \eta y_f' = \frac{\varepsilon^2}{\eta^2}\frac{v_0'^2}{2g} \ .$$

If $\eta = \varepsilon^{\frac{2}{3}}$ be taken, Bernoulli's formula remains invariable.   Hence it follows

$$x = \varepsilon^{\frac{2}{3}}x', \quad y = \varepsilon^{\frac{2}{3}}y' \quad \text{and} \quad v_f = \varepsilon^{\frac{1}{3}}v_f' \ .$$

Thus if one makes use of the initial velocity $\varepsilon^{\frac{1}{3}}$ times as large as the orginal one, the scale of x and y must be multiplied by the factor $\varepsilon^{\frac{2}{3}}$.

In this case besides the condition $v_0 = \varepsilon^{\frac{1}{3}}v_0'$, sin a must be a function of $\varepsilon\varphi$ instead of φ. Hence sin a is evidently different from the original one. It may be unnecessary to add that if sin a remains the same and the initial velocity alone differs from the original one the conformal representation can not be obtained by means of enlargement or contraction.

Now from Eqs. (4) and (7) one gets

$$\sin a_f d\varphi = \sqrt{v_0^2 - y_f}\, dy_f \ .$$

Integrating both sides, one obtains

$$C - \int_0^\varphi \sin a_f d\varphi = \frac{2}{3}(v_0^2 - y_f)^{\frac{3}{2}} \ ,$$

where integral constant C is determined by the magnitude of the velocity at $\varphi = 0$ and $\psi = 0$; then one puts $\frac{3}{2}C = v_0^3$.   Therefore

$$v_0^3 - \frac{3}{2}\int_0^\varphi \sin a_f d\varphi = (v_0^2 - y_f)^{\frac{3}{2}} = v_f^3 \ ,$$

$$v_f^2 = \left[v_0^3 - \frac{3}{2}\int_0^\varphi \sin a_f d\varphi \right]^{\frac{2}{3}} \ . \tag{8}$$

From Eq. (3) it follows

$$z_f = \int_0^\varphi \frac{1}{v_f} e^{ia} d\varphi = \int_0^\varphi \frac{\exp(ia)\, d\varphi}{\left[v_0^3 - \frac{3}{2}\int_0^\varphi \sin a_f d\varphi \right]^{\frac{1}{3}}} \ . \tag{9}$$

Hence

$$x_f = \int_0^\varphi \frac{\cos a\, d\varphi}{\left[v_0^3 - \frac{3}{2}\int_0^\varphi \sin a_f d\varphi \right]^{\frac{1}{3}}} \ , \tag{10}$$

$$y_f = \int_0^\varphi \frac{\sin \alpha \, d\varphi}{\left[v_0^3 - \frac{3}{2} \int_0^\varphi \sin \alpha_f d\varphi \right]^{\frac{1}{3}}} \; . \tag{11}$$

If $\psi < 0$ be assumed to be the velocity field of the stream, the coordinate of a point in the inner stream is expressed by the following:

$$x = \text{Real part of} \int_0^\chi \frac{\exp\{i\alpha(\chi)\} \, d\chi}{\left[v_0^3 - \frac{3}{2} \int_0^\chi \sin \alpha(\chi) d\chi \right]^{\frac{1}{3}}} \; , \tag{12}$$

$$y = \text{Imaginary part of} \int_0^\chi \frac{\exp\{i\alpha(\chi)\} \, d\chi}{\left[v_0^3 - \frac{3}{2} \int_0^\chi \sin \alpha(\chi) d\chi \right]^{\frac{1}{3}}} \; . \tag{13}$$

II. Flow with a Finite Drop on the Free Surface

In natural phenomena it is often observed that there are flows in a stream with finite drop. The inclinations of the free surface will be expressed by various functions of φ. If it be assumed that the inclination at infinity is zero and there is only one drop in the stream, the inclination may be expressed by the function of e^φ and $e^{-\varphi}$. As the most simple function the author takes

$$\sin \alpha_f = \frac{-1}{e^\varphi + e^{-\varphi}} = \frac{-1}{2 \cosh \varphi} \; . \tag{14}$$

Since

$$\int_0^\varphi \sin \alpha_f d\varphi = -\tan^{-1} e^\varphi + \frac{\pi}{4} \; ,$$

it follows

$$v_f = \left[v_0^3 + \frac{3}{2} \tan^{-1} e^\varphi - \frac{3}{8} \pi \right]^{\frac{1}{3}} \; ,$$

$$v_0 - y_f = \left[v_0^3 + \frac{3}{2} \tan^{-1} e^\varphi - \frac{3}{8} \pi \right]^{\frac{2}{3}} \; ,$$

$$z_f = \int_0^\varphi \frac{\exp\{i\alpha\} \, d\varphi}{\left[v_0^3 + \frac{3}{2} \tan^{-1} e^\varphi - \frac{3}{8} \pi \right]^{\frac{1}{3}}} \; . \tag{15}$$

* Richardson assumed

$$v_f^2 = -\frac{3}{2\mu} \left(B - \tanh \varepsilon \varphi \right)^{\frac{2}{3}}$$

and Prášil assumed

$$v_f^2 = (C + \tanh \varepsilon \varphi)^2.$$

The values of $\sin \alpha_f$, v_f and y_f at the points $\varphi = -\infty$, 0 and $+\infty$ are shown in the following:

	$\varphi = -\infty$	$\varphi = 0$	$\varphi = +\infty$
$\sin \alpha_f$	0	$-\dfrac{1}{2}$	0
v_f	$\left[v_0^3 - \dfrac{3}{8}\pi\right]^{\frac{1}{3}}$	v_0	$\left[v_0^3 + \dfrac{3}{8}\pi\right]^{\frac{1}{3}}$
$v_0^2 - y_f$	$\left[v_0^3 - \dfrac{3}{8}\pi\right]^{\frac{2}{3}}$	v_0^2	$\left[v_0^3 + \dfrac{3}{8}\pi\right]^{\frac{2}{3}}$

As is easily seen, the velocity at infinity is uniform.

Therefore the depths of the stream at $-\infty$ and $+\infty$ are inversely proportional to the velocity of the stream,

$$\frac{[\text{Depth at } +\infty]}{[\text{Depth at } -\infty]} = \frac{\left[v_0^3 - \dfrac{3}{8}\pi\right]^{\frac{1}{3}}}{\left[v_0^3 + \dfrac{3}{8}\pi\right]^{\frac{1}{3}}} \ .$$

If v_0^3 equals $\dfrac{3}{8}\pi$, v at $\varphi = -\infty$ is zero, and the depth at $\varphi = -\infty$ tends to infinity, while the depth at $\varphi = +\infty$ keeps finite.

III. Bed of the Flow

Even though the free surface of the flow appears to be smoothly curved to make a finite drop, the bed of the flow just below the drop makes various geometrical forms. In other words a slight difference in the form of the free surface causes considerable variations of the form of the bed.

For example, one may put $\sin \alpha = \dfrac{-1}{2\cosh \chi}$ in the general velocity field. Hence one has from (9),

$$z = \int_0^\chi \frac{\exp\left\{i \sin^{-1}\left(\dfrac{-1}{2\cosh\chi}\right)\right\} d\chi}{\left[v_0^3 - \dfrac{3}{2}\displaystyle\int_0^\chi \dfrac{d\chi}{2\cosh\chi}\right]^{\frac{1}{3}}} \ . \tag{16}$$

By separating the real and imaginary parts of $\sin \alpha$, and putting

$$\sin \alpha = R + iI \ , \quad \alpha = a + ib \ , \tag{17}$$

one obtains

$$R = \frac{-\cosh\varphi\cos\psi}{\cosh 2\varphi + \cos 2\psi} = \sin a \cosh b , \qquad (18)$$

$$I = \frac{\sinh\varphi\sin\psi}{\cosh 2\varphi + \cos 2\psi} = \cos a \sinh b . \qquad (19)$$

Since the magnitude and the direction of the velocity must change continuously in the stream, the singularity of the factor $\exp\{ia(\chi)\}$ and $\left[v_0^3 - \frac{3}{2}\int_0^\chi \sin a\, d\chi\right]^{-\frac{1}{3}}$ must be inspected.

First, the continuity of a will be investigated in general complex domain. From Eqs. (18) and (19) one has on the locus of $\varphi=0$

$$R = \frac{-1}{2\cos\psi} , \qquad (20)$$

$$I = 0 . \qquad (21)$$

Hence there are three possible cases.

1) $b = 0$, $a = -\dfrac{\pi}{2}$, consequently $R = -1$.

2) $b = 0$, $a \cdots\cdots$ arbitrary, consequently $R = \sin a$.

3) $b \neq 0$, $a = -\dfrac{\pi}{2}$, consequently $R = -\cosh b$.

Case 1) is where $\psi = -\dfrac{\pi}{3}$; 2) is the case where a limiting condition $|\psi| < \dfrac{\pi}{3}$ exists, because $\sin a < 1$; 3) is the case where the condition of continuity does not hold. In this case from (18) one has

$$\left|\frac{1}{2\cos\psi}\right| > 1$$

Consequently the interval where ψ exists is limited from $-\dfrac{\pi}{3}$ to $-\dfrac{\pi}{2}$.

Denote the domain

$$-\infty < \varphi < +\infty , \quad 0 > \psi > -\frac{\pi}{3}$$

by D_1, and the domain

$$-\infty < \varphi < +\infty , \quad -\frac{\pi}{3} > \psi > -\frac{\pi}{2}$$

by D_2. In D_1, b changes the sign on passing through $\varphi=0$, as $b=0$ on

the locus $\varphi=0$, while in D_2, b cannot change the sign. As $\cosh b$ is always positive, from (18) and (19) it follows that

$$\varphi < 0 \quad \sin a < 0 , \quad \cos a \sinh b > 0$$
$$\varphi > 0 \quad \sin a < 0 , \quad \cos a \sinh b < 0 .$$

Therefore on the locus $\psi=$const., a and b vary in the following way:

In D_1

$$\varphi < 0 \qquad\qquad \varphi = 0 \qquad\qquad \varphi > 0$$

$$\begin{cases} 0 > a > -\dfrac{\pi}{2} \\[2mm] b > 0 \end{cases} \longrightarrow \begin{cases} a = -\dfrac{\pi}{2} \\[2mm] b = 0 \end{cases} \longrightarrow \begin{cases} 0 > a > -\dfrac{\pi}{2} \\[2mm] b < 0 \end{cases}$$

or

$$\begin{cases} 0 > -\pi - a > -\dfrac{\pi}{2} \\[2mm] b < 0 \end{cases} \longrightarrow \begin{cases} a = -\dfrac{\pi}{2} \\[2mm] b = 0 \end{cases} \longrightarrow \begin{cases} 0 > -\pi - a > -\dfrac{\pi}{2} \\[2mm] b > 0 \end{cases} \quad .$$

In D_2

$$\begin{cases} 0 > a > -\dfrac{\pi}{2} \\[2mm] b > 0 \end{cases} \longrightarrow \begin{cases} a = -\dfrac{\pi}{2} \\[2mm] b > 0 \end{cases} \longrightarrow \begin{cases} 0 > -\pi - a > -\dfrac{\pi}{2} \\[2mm] b > 0 \end{cases}$$

or

$$\begin{cases} 0 > -\pi - a > -\dfrac{\pi}{2} \\[2mm] b < 0 \end{cases} \longrightarrow \begin{cases} a = -\dfrac{\pi}{2} \\[2mm] b < 0 \end{cases} \longrightarrow \begin{cases} 0 > a > -\dfrac{\pi}{2} \\[2mm] b < 0 \end{cases} \quad .$$

So long as b varies continuously, in D_2 a varies into a different quadrant as φ increases from $-\infty$ to $+\infty$. Therefore there appears a discontinuous part of a on the locus $\psi = -\dfrac{\pi}{3}$. Since the discontinuity of physical quantities in the velocity field is excluded the domain $|\psi| > \dfrac{\pi}{3}$ must be excluded.

Every streamline may be assumed as bed of the stream, but as above shown, the domain of the existence of velocity field is limited. The lowest bed of the flow may be found in some cases. Next, it is needful to investigate the factor

$$v_0^3 - \frac{3}{2} \int_0^\chi \sin a \, d\chi$$

The zero of the factor may be the singular point of the velocity field.

Again another condition becomes available to use in obtaining the lowest bed of the flow.

IV. Example

When the inclination is given by a function of φ and the initial velocity by a certain value, the geometrical form of the streamline will be determined. For examples the following two cases are calculated:

$$v_0^3 = 2 \quad \text{and} \quad v_0^3 = \frac{3}{8}\pi \ .$$

Let the inclination be assumed

$$\sin \alpha = \frac{-1}{e^\varphi + e^{-\varphi}} \ .$$

1). Free surface : $\psi = 0$.

As shown in Eq. (15), the free surface is obtained by

$$x_1 = \int_0^\varphi \frac{\cos\left\{\sin^{-1}\left(\dfrac{-1}{2\cosh\varphi}\right)\right\}}{\left[v_0^3 + \dfrac{3}{2}\tan^{-1}e^\varphi - \dfrac{3}{8}\pi\right]^{\frac{1}{3}}} \, d\varphi \ ,$$

$$y_1 = \int_0^\varphi \frac{\dfrac{-1}{2\cosh\varphi}}{\left[v_0^3 + \dfrac{3}{2}\tan^{-1}e^\varphi - \dfrac{3}{8}\pi\right]^{\frac{1}{3}}} \, d\varphi \ .$$

The numerical results are tabulated in Table I for the above two exampeles.

2). The locus of $\varphi = 0$.

From Eqs. (20) and (21) one has

$$\alpha = a = \sin^{-1}\frac{-1}{2\cos\psi} \ .$$

Since

$$\int_0^\psi \frac{-i}{2\cos\psi} \, d\psi = -\frac{i}{2}\log\frac{1+\sin\psi}{\cos\psi} \ ,$$

by putting

$$v_0^3 - \frac{3}{2}\int_0^\psi \frac{-i\,d\psi}{2\cos\psi} = A + iB = [A^2 + B^2]^{\frac{1}{2}} e^{i\beta} \ , \qquad \beta = \tan^{-1}\frac{B}{A} \ ,$$

TABLE I.

	a) $\quad v_0^3 = 2 \quad \varphi = \pm \dfrac{n\pi}{30}$					b) $\quad v_0^3 = \dfrac{3}{8}\pi \quad \varphi = \pm \dfrac{n\pi}{30}$			
	$\varphi > 0$		$\varphi < 0$			$\varphi > 0$		$\varphi < 0$	
n	x_l	y_l	x_l	y_l	n	x_l	y_l	x_l	y_l
	$\times 10^{-2}$	$\times 10^{-2}$	$\times 10^{-2}$	$\times 10^{-2}$		$\times 10^{-2}$	$\times 10^{-2}$	$\times 10^{-2}$	$\times 10^{-2}$
0	0.	0.	0.	0.	0	0.	0.	0.	0.
1	0.3652	− 0.2100	− 0.3699	0.2129	1	0.4337	− 0.2497	− 0.4434	0.2553
2	0.7274	− 0.4155	− 0.7462	0.4263	2	0.8608	− 0.4920	− 0.8991	0.5137
3	1.0878	− 0.6143	− 1.1305	0.6381	3	1.2827	− 0.7246	− 1.3696	0.7780
4	1.4472	− 0.8045	− 1.5238	0.8461	4	1.7003	− 0.9456	− 1.8572	1.0310
5	1.8071	− 0.9853	− 1.9270	1.0486	5	2.1163	− 1.1546	− 2.3640	1.2853
6	2.1678	− 1.1557	− 2.3409	1.2440	6	2.5311	− 1.3505	− 2.8915	1.5344
7	2.5296	− 1.3149	− 2.7656	1.4310	7	2.9451	− 1.5328	− 3.4415	1.7764
8	2.8418	− 1.4629	− 3.2014	1.6085	8	3.3591	− 1.7016	− 4.0152	2.0100
9	3.2568	− 1.5996	− 3.6478	1.7760	9	3.7724	− 1.8568	− 4.6136	2.2345
10	3.6220	− 1.7253	− 4.1046	1.9331	10	4.1858	− 1.9991	− 5.2374	2.4491
11	3.9879	− 1.8404	− 4.5710	2.0798	11	4.5986	− 2.1289	− 5.8875	2.6532
12	4.3547	− 1.9454	− 5.0466	2.2160	12	5.0115	− 2.2471	− 6.5644	2.8470
13	4.7223	− 2.0412	− 5.5308	2.3420	13	5.4246	− 2.3547	− 7.2688	3.0302
14	5.0903	− 2.1281	− 6.0226	2.4581	14	5.8375	− 2.4522	− 8.0012	3.2030
15	5.4586	− 2.2069	− 6.5216	2.5650	15	6.2499	− 2.5405	− 8.7625	3.3658
16	5.8270	− 2.2782	− 7.0276	2.6629	16	6.6619	− 2.6203	− 9.5533	3.5188
17	6.1955	− 2.3427	− 7.5396	2.7525	17	7.0734	− 2.6923	−10.3740	3.6624
18	6.5639	− 2.4010	− 8.0572	2.8343	18	7.4843	− 2.7573	−11.2259	3.7969
19	6.9322	− 2.4535	− 8.5799	2.9089	19	7.8948	− 2.8159	−12.1098	3.9230
20	7.3002	− 2.5009	− 9.1071	2.9768	20	8.3047	− 2.8688	−13.0260	4.0409
21	7.6681	− 2.5437	− 9.6384	3.0385	21	8.7139	− 2.9164	−13.9760	4.1512
22	8.0357	− 2.5822	−10.1735	3.0946	22	9.1225	− 2.9592	−14.9605	4.2543
23	8.4030	− 2.6169	−10.7120	3.1455	23	9.5306	− 2.9978	−15.9809	4.3506
24	8.7700	− 2.6481	−11.2536	3.1916	24	9.9382	− 3.0325	−17.0381	4.4405
25	9.1368	− 2.6763	−11.7987	3.2334	25	10.3453	− 3.0638	−18.1334	4.5246
26	9.5033	− 2.7016	−12.3464	3.2712	26	10.7519	− 3.0919	−19.2683	4.6029
27	9.8697	− 2.7245	−12.8956	3.3054	27	11.1581	− 3.1172	−20.4439	4.6759
28	10.2358	− 2.7450	−13.4468	3.3363	28	11.5638	− 3.1400	−21.6614	4.7444
29	10.6016	− 2.7632	−13.9997	3.3643	29	11.9691	− 3.1605	−22.9138	4.8077
30	10.9674	− 2.7802	−14.5544	3.3895	30	12.3743	− 3.1789	−24.2117	4.8667

one has

$$A = v_0^3, \quad B = \frac{3}{4} \log \frac{1+\sin \psi}{\cos \psi} \, ,$$

$$\tan \beta = \frac{3}{4 v_0^3} \log \frac{1+\sin \psi}{\cos \psi} \, .$$

It follows

$$x_2 = - \int_0^\psi \frac{\sin \left(\alpha - \dfrac{\beta}{3} \right)}{\left[A^2 + B^2 \right]^{\frac{1}{6}}} \, d\psi \, ,$$

TABLE II.

a)	$v_0^3 = 2$ $\quad \varphi = 0$ $\quad \psi = -\dfrac{n\pi}{60}$		b)	$v_0^3 = \dfrac{3}{8}\pi$ $\quad \varphi = 0$ $\quad \psi = -\dfrac{n\pi}{60}$	
n	x_2	y_2	n	x_2	y_2
	$\times 10^{-2}$	$\times 10^{-2}$		$\times 10^{-2}$	$\times 10^{-2}$
0	0.	0.	0	0.	0.
1	-0.1055	-0.1838	1	-0.1255	-0.2199
2	-0.2100	-0.3682	2	-0.2485	-0.4403
3	-0.3139	-0.5530	3	-0.3699	-0.6622
4	-0.4174	-0.7377	4	-0.4893	-0.8842
5	-0.5209	-0.9226	5	-0.6076	-1.1066
6	-0.6246	-1.1071	6	-0.7255	-1.3291
7	-0.7290	-1.2910	7	-0.8429	-1.5510
8	-0.8343	-1.4741	8	-0.9602	-1.7719
9	-0.9410	-1.6561	9	-1.0781	-1.9918
10	-1.0496	-1.8366	10	-1.1969	-2.2097
11	-1.1606	-2.0153	11	-1.3178	-2.4260
12	-1.2746	-2.1917	12	-1.4408	-2.6393
13	-1.3923	-2.3651	13	$-1 5673$	-2.8495
14	-1.5146	-2.5348	14	-1.6979	-3.0551
15	-1.6424	-2.6997	15	-1.8342	-3.2556
16	-1.7770	-2.8583	16	-1.9786	-3.4484
17	-1.9207	-3.0077	17	-2.1326	-3.6311
18	-2.0750	-3.1450	18	-2.2979	-3.8010
19	-2.2413	-3.2641	19	-2.4780	-3.9510
20	-2.4289	-3.3331	20	-2.6857	-4.0464

$$y_2 = \int_0^\psi \frac{\cos\left(\alpha - \dfrac{\beta}{3}\right)}{[A^2 + B^2]^{\frac{1}{6}}} \, d\psi \ .$$

The numerical results are shown in Table II.

3). The extreme bed line: $\psi = -\dfrac{\pi}{3}$.

Again let it be put

$$v_0^3 - \frac{3}{2} \int_0^{\varphi - i\frac{\pi}{3}} \sin \alpha \, d\chi = A' + iB' \ .$$

Now

$$\int_0^{\varphi - i\frac{\pi}{3}} \sin \alpha \, d\chi = \int_0^{-i\frac{\pi}{3}} \sin \alpha \, d\chi + \int_{-i\frac{\pi}{3}}^{\varphi - i\frac{\pi}{3}} \sin \alpha \, d\chi \ .$$

The first integral is already calculated; it equals -0.9877. On the locus $\psi = -\dfrac{\pi}{3}$, $\sin \alpha$ can be separated into real and imaginary parts as shown in the following

$$R' = \frac{-\cosh \varphi}{2 \cosh 2\varphi - 1} = \frac{-\cosh \varphi}{4 \sinh^2 \varphi + 1} \ ,$$

$$I' = \frac{-\sqrt{3} \, \sinh \varphi}{2 \cosh 2\varphi - 1} = \frac{-\sqrt{3} \, \sinh \varphi}{4 \cosh^2 \varphi - 3} \ .$$

Since

$$\int_0^\varphi R' \, d\varphi = -\frac{1}{2} \tan^{-1} (2 \sinh \varphi) \ ,$$

$$\int_0^\varphi I' \, d\varphi = -\frac{1}{4} \log \frac{2 \cosh \varphi - \sqrt{3}}{2 \cosh \varphi + \sqrt{3}} + \frac{1}{2} \log (2 - \sqrt{3}) \ ,$$

it follows

$$\int_{0 - i\frac{\pi}{3}}^{\varphi - i\frac{\pi}{3}} \sin \alpha \, d\chi = \int_0^\varphi (R' + iI') \, d\varphi \ ,$$

$$A' = v_0 - \frac{3}{4} \tan^{-1} (2 \sinh \varphi) \ ,$$

$$B' = \frac{3}{8} \log \frac{2 \cosh \varphi - \sqrt{3}}{2 \cosh \varphi + \sqrt{3}}$$

and

$$\frac{B'}{A'} = \tan \beta' \ .$$

Next it is neccessary to separate α into real and imaginary parts

$$\alpha = a' + ib'$$

If one puts $e^{-b'} = s$, then s is always positive and he has

$$\sin a' \cosh b' = \frac{1}{2} \sin a' \left(\frac{1}{s} + s \right) = R'$$

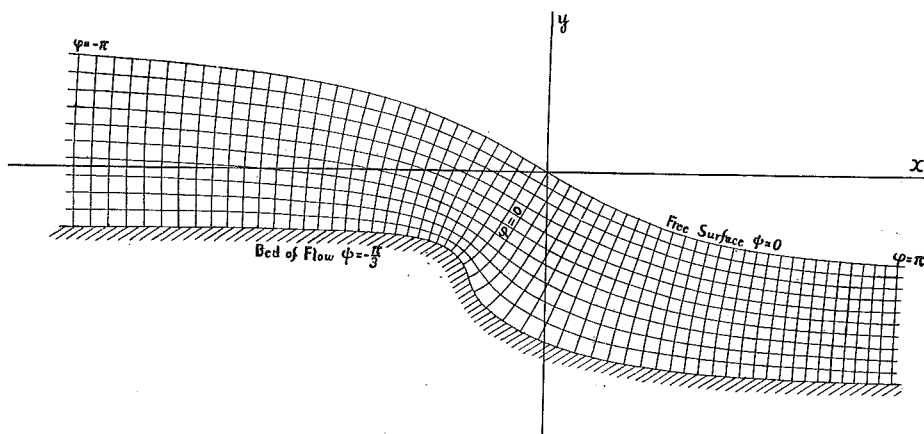$$\cos a' \sinh b' = \frac{1}{2} \cos a' \left(\frac{1}{s} - s \right) = I'$$
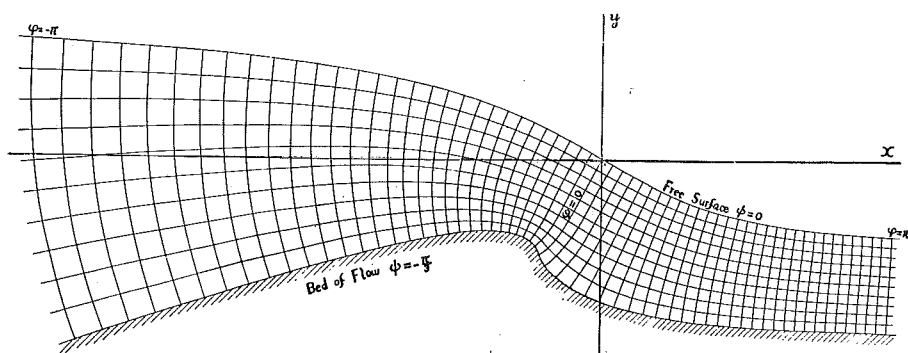

Plate A. $v_0^3 = 2$.

Plate B. $v_0^3 = \frac{3}{8}\pi$.

TABLE III.

	a) $v_0^3 = 2$		$\varphi = \pm\dfrac{n\pi}{30}$			b) $v_0^3 = \dfrac{3}{8}\pi$		$\varphi = \pm\dfrac{n\pi}{30}$	
	$\varphi > 0$		$\varphi < 0$			$\varphi > 0$		$\varphi < 0$	
n	x_3	y_3	x_3	y_3	n	x_3	y_3	x_3	y_3
	$\times 10^{-2}$	$\times 10^{-2}$	$\times 10^{-2}$	$\times 10^{-2}$		$\times 10^{-2}$	$\times 10^{-2}$	$\times 10^{-2}$	$\times 10^{-2}$
0	-2.4289	-3.3331	-2.4289	-3.3331	0	-2.6857	-4.0464	-2.6857	-4.0464
1	-2.2143	-3.7780	-2.5413	-2.9010	1	-2.4265	-4.5316	-2.8454	-3.6985
2	-1.8357	-4.2398	-2.7224	-2.6979	2	-1.9487	-5.0225	-3.0755	-3.4867
3	-1.3418	-4.6423	-2.9357	-2.5398	3	-1.3697	-5.4429	-3.3398	-3.3332
4	-0.7947	-4.9729	-3.1740	-2.4143	4	-0.7419	-5.7837	-3.6337	-3.2220
5	-0.2352	-5.2372	-3.4342	-2.3132	5	-0.1014	-6.0531	-3.9561	-3.1470
6	0.3265	-5.4464	-3.7168	-2.2316	6	0.5347	-6.2648	-4.3076	-3.1015
7	0.8780	-5.6132	-4.0219	-2.1653	7	1.1561	-6.4321	-4.6877	-3.0827
8	1.4148	-5.7469	-4.3469	-2.1122	8	1.7591	-6.5632	-5.0964	-3.0883
9	1.9362	-5.8556	-4.6908	-2.0689	9	2.3430	-6.6719	-5.5347	-3.1169
10	2.4418	-5.9449	-5.0530	-2.0342	10	2.9084	-6.7597	-6.0025	-3.1669
11	2.9331	-6.0193	-5.4326	-2.0066	11	3.4567	-6.8316	-6.5005	-3.2368
12	3.4107	-6.0816	-5.8280	-1.9842	12	3.9893	-6.8923	-7.0290	-3.3260
13	3.8765	-6.1341	-6.2382	-1.9663	13	4.5079	-6.9433	-7.5882	-3.4337
14	4.3311	-6.1795	-6.6627	-1.9525	14	5.0140	-6.9867	-8.1790	-3.5597
15	4.7760	-6.2183	-7.0999	-1.9413	15	5.5088	-7.0240	-8.8020	-3.7031
16	5.2122	-6.2520	-7.5494	-1.9326	16	5.9937	-7.0566	-9.4576	-3.8633
17	5.6408	-6.2816	-8.0096	-1.9260	17	6.4698	-7.0847	-10.1458	-4.0403
18	6.0627	-6.3071	-8.4800	-1.9204	18	6.9879	-7.1092	-10.8673	-4.2347
19	6.4785	-6.3300	-8.9596	-1.9163	19	7.3991	-7.1311	-11.6234	-4.4459
20	6.8887	-6.3499	-9.4479	-1.9133	20	7.8541	-7.1500	-12.4147	-4.6740
21	7.2943	-6.3678	-9.9433	-1.9107	21	8.3034	-7.1673	-13.2412	-4.9189
22	7.6953	-6.3831	-10.4463	-1.9092	22	8.7479	-7.1821	-14.1045	-5.1806
23	8.0923	-6.3974	-10.9555	-1.9076	23	9.1879	-7.1954	-15.0050	-5.4587
24	8.4862	-6.4096	-11.4708	-1.9066	24	9.6240	-7.2076	-15.9443	-5.7546
25	8.8770	-6.4209	-11.9912	-1.9061	25	10.0564	-7.2184	-16.9233	-6.0673
26	9.2647	-6.4306	-12.5167	-1.9056	26	10.4856	-7.2275	-17.9417	-6.3974
27	9.6499	-6.4397	-13.0463	-1.9051	27	10.9122	-7.2357	-19.0004	-6.7449
28	10.0331	-6.4474	-13.5800	-1.9049	28	11.3362	-7.2434	-20.1009	-7.1102
29	10.4142	-6.4545	-14.1172	-1.9047	29	11.7579	-7.2500	-21.2447	-7.4933
30	10.7938	-6.4607	-14.6575	-1.9047	30	12.1775	-7.2561	-22.4330	-7.8954

Hence

$$\cos^2 a' = \frac{1}{2} \left\{ -(R'^2 + I'^2 - 1) + \sqrt{(R'^2 + I'^2 - 1)^2 + 4I'^2} \right\}$$

$$= \frac{\sinh \varphi \ (2 \sinh \varphi + \sqrt{4 \sinh^2 \varphi + 3})}{4 \sinh^2 \varphi + 1},$$

$$s = \frac{-I' + \sqrt{I'^2 + \cos^2 a'}}{\cos a'}.$$

Thus one obtains

$$x_3 = x_0' + \int_0^\varphi \frac{e^{-b}}{[A'^2 + B'^2]^{\frac{1}{6}}} \cos \left(a' - \frac{\beta'}{3} \right) d\varphi,$$

$$y_3 = y_0' + \int_0^\varphi \frac{e^{-b}}{[A'^2 + B'^2]^{\frac{1}{6}}} \sin \left(a' - \frac{\beta'}{3} \right) d\varphi,$$

where x_0' and y_0' are the values of x_2 and y_2 at $\varphi = 0$, $\psi = -\frac{\pi}{3}$. The numerical results are shown in Table III and the two curves of the comformal representations are shown in plates A and B.

Above values show the actual measures by means of dividing the numerical values in calculation by $2g$; the adopted unit in length is metre. In the use of such a unit, when $v_0 = 1.26$ m/sec, the depth of the stream is only several centimetres. Such a stream can be realized in the laboratory or observed only in the shallow part of a brook. If, however, a new unit as $v_0 = 1.26 \times 10$ m/sec $= 12.6$ m/sec is employed the depth of the lowest bed of the stream is about 6 metres. For a laminar flow such a case can hardly exist. The medium cases where the initial velocity v_0 lies between these two values are observed in nature.

In this first report the author clears up only the basic part of the problem. It is expected that extension and applications of this result will be published later.